J. Math. Kyoto Univ. (JMKYAZ)
45-2 (2005), 287–306

Homological invariants associated to semi-dualizing bimodules

By

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Abstract

Cohen-Macaulay dimension for modules over a commutative ring has been defined by A. A. Gerko. That is a homological invariant sharing many properties with projective dimension and Gorenstein dimension. The main purpose of this paper is to extend the notion of Cohen-Macaulay dimension for modules over commutative noetherian local rings to that for bounded complexes over non-commutative noetherian rings.

1. Introduction

Cohen-Macaulay dimension for modules over a commutative noetherian local ring has been defined by A. A. Gerko [10]. That is to be a homological invariant of a module which shares a lot of properties with projective dimension and Gorenstein dimension. The aim of this paper is to extend this invariant of modules to that of chain complexes, even over non-commutative rings. We try to pursue it in the most general context possible.

The key role will be played by semi-dualizing bimodules, which we introduce in this paper to generalize semi-dualizing modules in the sense of Christensen [5]. The advantage to consider an (R, S)-bimodule structure on a semi-dualizing module C is in the duality theorem. Actually we shall show that $\mathrm{Hom}_R(-, C)$ (resp. $\mathrm{Hom}_S(-, C)$) gives a duality between subcategories of R-mod and mod-S. We take such an idea from non-commutative ring theory, in particular, Morita duality and tilting theory.

In Section 2 we present a precise definition of a semi-dualizing bimodule and show several properties. Associated to a semi-dualizing (R, S)-bimodule C, of most importance is the notion of the G_C-dimension of an R-module and the full subcategory $\mathcal{R}_R(C)$ of R-mod consisting of all R-modules of finite G_C-dimensions. Under some special conditions the G_C-dimension will coincide with the Cohen-Macaulay dimension of a module.

In Section 3 we extend these notions to the derived category, hence to chain complexes. We introduce the notion of the trunk module of a complex, and

Received September 16, 2004

as one of the main results of this paper, we shall show that the G_C-dimension of a complex is essentially given by that of its trunk module (Theorem 3.1). By virtue of this theorem, we can show that many of the assertions concerning G_C-dimensions of modules will hold true for G_C-dimensions of complexes.

In Section 4 we shall show that a semi-dualizing bimodule, more generally a semi-dualizing complex of bimodules, yields a duality between subcategories of the derived categories. This second main result (Theorem 4.1) of this paper gives the advantage from considering the bimodule structure of a semi-dualizing module.

In Section 5 we shall apply the theory to the case where the base rings are commutative. Surprisingly, if both rings R, S are commutative, then we shall see that a semi-dualizing (R, S)-bimodule is nothing but a semi-dualizing R-module, and actually $R = S$ (Lemma 5.1). In this case, we are able to apply this theory to the subcategories $\mathcal{R}_R(C)$ and the Gorenstein dimension of the dualizing complex to obtain (in Corollary 5.1) a new characterization of Gorenstein rings.

2. G_C-dimensions for modules

*Throughout the present paper, we assume that R is a left noetherian ring. Let R-*mod *denote the category of finitely generated left R-modules. We also assume that S is a right noetherian ring and* mod-S *denotes the category of finitely generated right S-modules. When we say simply an R-module* (*resp. an S-module*), *we mean a finitely generated left R-module* (*resp. a finitely generated right S-module*).

In this section, we shall define the notion of G_C-dimension of a module, and study its properties. For this purpose, we begin with defining a semi-dualizing bimodule.

Definition 2.1. We call an (R, S)-bimodule C a *semi-dualizing bimodule* if the following conditions hold.

(1) The right homothety S-bimodule morphism $S \to \mathrm{Hom}_R(C, C)$ is a bijection.

(2) The left homothety R-bimodule morphism $R \to \mathrm{Hom}_S(C, C)$ is a bijection.

(3) $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$.

(4) $\mathrm{Ext}^i_S(C, C) = 0$ for all $i > 0$.

In the rest of this section, C always denotes a semi-dualizing (R, S)-bimodule.

Definition 2.2. We say that an R-module M is *C-reflexive* if the following conditions are satisfied.

(1) $\mathrm{Ext}^i_R(M, C) = 0$ for all $i > 0$.

(2) $\mathrm{Ext}^i_S(\mathrm{Hom}_R(M, C), C) = 0$ for all $i > 0$.

(3) The natural morphism $M \to \mathrm{Hom}_S(\mathrm{Hom}_R(M, C), C)$ is a bijection.

One can of course consider the same for right S-modules by symmetry.

Definition 2.3. If the following conditions hold for $N \in$ mod-S, we say that N is C-reflexive.
(1) $\mathrm{Ext}_S^i(N, C) = 0$ for all $i > 0$.
(2) $\mathrm{Ext}_R^i(\mathrm{Hom}_S(N, C), C) = 0$ for all $i > 0$.
(3) The natural morphism $N \to \mathrm{Hom}_R(\mathrm{Hom}_S(N, C), C)$ is a bijection.

Example 2.1.
(1) Both the ring R and the semi-dualizing module C are C-reflexive R-modules. Similarly, S and C are C-reflexive S-modules.
(2) Let R be a finite dimensional algebra. Then every tilting R-module is a semi-dualizing module (cf. [13, (4.1)]).
(3) Let R be a left and right noetherian ring. Then the ring R itself is a semi-dualizing (R, R)-bimodule and the R-reflexive modules coincide with the modules whose G-dimension is equal to 0 (cf. [1, Proposition 3.8]).
(4) Let R be a commutative Cohen-Macaulay local ring with dualizing module K. Then K is a semi-dualizing module and the K-reflexive modules coincide with the maximal Cohen-Macaulay modules (cf. [4, Theorem 3.3.10]).

Theorem 2.1.
(1) *Let $0 \to L_1 \to L_2 \to L_3 \to 0$ be a short exact sequence either in R-*mod *or in* mod-S. *Assume that L_3 is C-reflexive. Then, L_1 is C-reflexive if and only if so is L_2.*
(2) *If L is a C-reflexive module, then so is any direct summand of L. In particular, any projective module is C-reflexive.*
(3) *The functors* $\mathrm{Hom}_R(-, C)$ *and* $\mathrm{Hom}_S(-, C)$ *yield a duality between the full subcategory of R-*mod *consisting of all C-reflexive R-modules and the full subcategory of* mod-S *consisting of all C-reflexive S-modules.*

Proof. (1) Let $0 \to L_1 \to L_2 \to L_3 \to 0$ be a short exact sequence in R-mod. Suppose that L_3 is C-reflexive. Applying the functor $\mathrm{Hom}_R(-, C)$ to this sequence, we see that the sequence

$$0 \to \mathrm{Hom}_R(L_3, C) \to \mathrm{Hom}_R(L_2, C) \to \mathrm{Hom}_R(L_1, C) \to 0$$

is exact, and $\mathrm{Ext}_R^i(L_2, C) \cong \mathrm{Ext}_R^i(L_1, C)$ for $i > 0$. Now applying the functor $\mathrm{Hom}_S(-, C)$, we will have an exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(L_1, C), C) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(L_2, C), C) \\ & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(L_3, C), C) & \longrightarrow & \mathrm{Ext}_S^1(\mathrm{Hom}_R(L_1, C), C) \\ & \longrightarrow & \mathrm{Ext}_S^1(\mathrm{Hom}_R(L_2, C), C) & \longrightarrow & 0 \end{array}$$

and the isomorphisms $\mathrm{Ext}_S^i(\mathrm{Hom}_R(L_1, C), C) \cong \mathrm{Ext}_S^i(\mathrm{Hom}_R(L_2, C), C)$ for $i > 1$. It is now easy to see from the diagram chasing that L_1 is C-reflexive if and only if L_2 is C-reflexive.

(2) Trivial.

(3) It is straightforward to see that both functors send C-reflexive modules to C-reflexive modules (over the respective rings), and that the compositions of them are the identity functors (for the respective categories). $\square$

Example 2.2. Let R be a left and right noetherian ring. We denote $\mathcal{G}_R$ the full subcategory of R-mod consisting of all R-reflexive (left) R-modules and denote $\mathcal{G}_{R^{op}}$ the full subcategory of mod-R consisting of all R-reflexive (right) R-modules. In this situation, Theorem 2.1.(3) says that $\mathrm{Hom}_R(-,R)$ gives a duality between $\mathcal{G}_R$ and $\mathcal{G}_{R^{op}}$.

Lemma 2.1. *The following conditions are equivalent for $M \in R$-mod and $n \in \mathbb{Z}$.*

(1) *There exists an exact sequence*

$$0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^0 \to M \to 0$$

such that each X^i is a C-reflexive R-module.

(2) *For any projective resolution*

$$P^\bullet \ : \ \cdots \to P^{-m-1} \to P^{-m} \to \cdots \to P^0 \to M \to 0$$

of M and for any $m \geq n$, we have that $\mathrm{Coker}(P^{-m-1} \to P^{-m})$ *is a C-reflexive R-module.*

(3) *For any exact sequence*

$$\cdots \to X^{-m-1} \to X^{-m} \to \cdots \to X^0 \to M \to 0$$

with each X^i being a C-reflexive R-module, and for any $m \geq n$, we have that $\mathrm{Coker}(X^{-m-1} \to X^{-m})$ *is a C-reflexive R-module.*

Proof. (1) $\Rightarrow$ (2) : Since $P^\bullet$ is a projective resolution of M, there is a morphism $\sigma^\bullet : P^\bullet \to X^\bullet$ of complexes over R :

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & P^{-n+1} & \longrightarrow & \cdots & \longrightarrow & P^0 \\
 & & {\scriptstyle\sigma^{-n-1}}\big\downarrow & & {\scriptstyle\sigma^{-n}}\big\downarrow & & {\scriptstyle\sigma^{-n+1}}\big\downarrow & & & & {\scriptstyle\sigma^{0}}\big\downarrow \\
 & & 0 & \longrightarrow & X^{-n} & \longrightarrow & X^{-n+1} & \longrightarrow & \cdots & \longrightarrow & X^0
\end{array}$$

Taking the mapping cone of $\sigma^\bullet$, we see that there is an exact sequence

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P^{-m} & \longrightarrow & \cdots & \longrightarrow & P^{-n} \\
 & \longrightarrow & P^{-n+1} \oplus X^{-n} & \longrightarrow & P^{-n+2} \oplus X^{-n+1} & \longrightarrow & \cdots \\
 & \longrightarrow & P^{0} \oplus X^{-1} & \longrightarrow & X^0 & \longrightarrow & 0.
\end{array}$$

It follows from a successive use of Theorem 2.1.(1) that $\mathrm{Coker}(P^{-m-1} \to P^{-m})$ is a C-reflexive R-module for $m \geq n$.

(2) $\Rightarrow$ (3) : Let $m \geq n$, and set $X = \mathrm{Coker}(X^{-m-1} \to X^{-m})$ and $P = \mathrm{Coker}(P^{-m-1} \to P^{-m})$. Since $P^\bullet$ is a projective resolution of M, there is a chain map $\sigma^\bullet : P^\bullet \to X^\bullet$ of complexes over R:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, "\sigma^{-m}"'] & P^{-m+1} \arrow[r] \arrow[d, "\sigma^{-m+1}"'] & P^{-m+2} \arrow[r] \arrow[d, "\sigma^{-m+2}"'] & \cdots \arrow[r] & P^0 \arrow[d, "\sigma^0"'] \\
0 \arrow[r] & X \arrow[r] & X^{-m+1} \arrow[r] & X^{-m+2} \arrow[r] & \cdots \arrow[r] & X^0
\end{tikzcd}$$

Taking the mapping cone of $\sigma^\bullet$, we see that there is an exact sequence

$$0 \to P \to P^{-m+1} \oplus X \to P^{-m+2} \oplus X^{-m+1} \to \cdots \to P^0 \oplus X^{-1} \to X^0 \to 0.$$

It then follows from Theorem 2.1.(1) and 2.1.(2) that X is a C-reflexive R-module.

(3) $\Rightarrow$ (1): Trivial. $\square$

Imitating the way of defining the G-dimension in [1, Theorem 3.13], we make the following definition.

Definition 2.4. For $M \in R$-mod, we define the G_C-dimension of M by

$$\mathrm{G}_C\text{-}\dim_R M = \inf \left\{ n \;\middle|\; \begin{array}{l} \text{there exists an exact sequence of finite length} \\ 0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^0 \to M \to 0, \\ \text{where each } \; X^i \text{ is a } C\text{-reflexive } R\text{-module.} \end{array} \right\}$$

Here we should note that we adopt the ordinary convention that $\inf \emptyset = +\infty$.

Remark 2.1. First of all we should notice that in the case $R = S = C$, the G_C-dimension is the same as the G-dimension.

Furthermore, comparing with Theorem 5.1 below, we are able to see that the G_C-dimension extends the Cohen-Macaulay dimension over a commutative ring R. More precisely, suppose that R and S are commutative local rings. If there is a semi-dualizing (R, S)-bimodule, then R must be isomorphic to S as we will show later in Lemma 5.1. Thus semi-dualizing bimodules are nothing but semi-dualizing R-modules in this case. One can define the Cohen-Macaulay dimension of an R-module M as

$$\text{CM-dim}M = \inf\{\mathrm{G}_C\text{-}\dim_R M \mid C \text{ is a semi-dualizing } R\text{-module}\}.$$

Let C_1 and C_2 be semi-dualizing R-modules. And suppose that an R-module M satisfies $G_{C_1}\text{-}\dim M < \infty$ and $G_{C_2}\text{-}\dim M < \infty$. Then we can show that $G_{C_1}\text{-}\dim M = G_{C_2}\text{-}\dim M$ ($= \operatorname{depth} R - \operatorname{depth} M$) (c.f. Lemma 5.2). In other words, if the rings R and S are commutative, then the value of the G_C-dimension is constant for any choice of semi-dualizing modules C whenever it is finite. But it follows the next example, if R is non-commutative, this is no longer true.

Example 2.3. Let Q be a quiver $e_1 \longrightarrow e_2$, and let $R = kQ$ be the path algebra over an algebraic closed field k. Put $P_1 = Re_1$, $P_2 = Re_2$, $I_1 = \mathrm{Hom}_k(e_1R, k)$, and $I_2 = \mathrm{Hom}_k(e_2R, k)$. Then, it is easy to see that the only indecomposable left R-modules are P_1, P_2 ($\cong I_1$), and I_2, up to isomorphism. Putting $C_1 = P_1 \oplus P_2 = R$ and $C_2 = I_1 \oplus I_2$, we note that $\mathrm{End}_R(C_1) = \mathrm{End}_R(C_2) = R^{op}$ (here, R^{op} is the opposite ring of R), and that C_1 and C_2 are semi-dualizing (R,R)-bimodules. In this case we have that $G_{C_1}\text{-}\dim I_2 (= \text{G-dim } I_2) = 1$ and $G_{C_2}\text{-}\dim I_2 = 0$, which take different finite values.

Theorem 2.2. *If* $\mathrm{G}_C\text{-}\dim_R M < \infty$ *for a module* $M \in R\text{-mod}$*, then*

$$\mathrm{G}_C\text{-}\dim_R M = \sup\{\ n \mid \mathrm{Ext}^n_R(M, C) \neq 0\ \}.$$

Proof. We prove the theorem by induction on $\mathrm{G}_C\text{-}\dim_R M$.

Assume first that $\mathrm{G}_C\text{-}\dim_R M = 0$. Then M is a C-reflexive module, and hence we have $\sup\{\ n \mid \mathrm{Ext}^n_R(M, C) \neq 0\ \} = 0$ from the definition.

Assume next that $\mathrm{G}_C\text{-}\dim_R M = 1$. Then there exists an exact sequence $0 \to X^{-1} \to X^0 \to M \to 0$ where X^0 and X^{-1} are C-reflexive R-modules. Then it is clear that $\mathrm{Ext}^i_R(M, C) = 0$ for $i > 1$. We must show that $\mathrm{Ext}^1_R(M, C) \neq 0$. To do this, suppose $\mathrm{Ext}^1_R(M, C) = 0$. Then we would have an exact sequence

$$0 \to \mathrm{Hom}_R(M, C) \to \mathrm{Hom}_R(X^0, C) \to \mathrm{Hom}_R(X^{-1}, C) \to 0. \tag{1}$$

Then, writing the functor $\mathrm{Hom}_S(\mathrm{Hom}_R(-, C), C)$ as F, we get from this the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \cong\big\downarrow & & \cong\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & F(X^{-1}) & \longrightarrow & F(X^0) & \longrightarrow & F(M) & \longrightarrow & 0,
\end{array}$$

hence the natural map $M \to F(M)$ is also an isomorphism. Furthermore, it also follows from (1) that $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for $i > 0$. Therefore we would have $\mathrm{G}_C\text{-}\dim_R M = 0$, a contradiction. Hence $\mathrm{Ext}^1_R(M, C) \neq 0$ as desired.

Finally assume that $\mathrm{G}_C\text{-}\dim_R M = m > 1$. Then there exists an exact sequence $0 \to X^{-m} \to X^{-m+1} \to \cdots \to X^0 \to M \to 0$ such that each X^i is a C-reflexive R-module. Putting $M' = \mathrm{Coker}(X^{-2} \to X^{-1})$, we note that the sequence $0 \to M' \to X^0 \to M \to 0$ is exact and $\mathrm{G}_C\text{-}\dim_R M' = m - 1 > 0$. Therefore, the induction hypothesis implies that $\sup\{\ n \mid \mathrm{Ext}^n_R(M', C) \neq 0\ \} = m-1$. Since $\mathrm{Ext}^n_R(X^0, C) = 0$ for $n > 0$, it follows that $\sup\{\ n \mid \mathrm{Ext}^n_R(M, C) \neq 0\ \} = m$ as desired. □

If R is a left and right noetherian ring and if $R = S = C$, then the equality $\mathrm{G}_R\text{-}\dim_R M = \text{G-dim } M$ holds by definition. We should remark that if R is a Gorenstein commutative ring, then any R-module M has finite G-dimension and it can be embedded in a short exact sequence of the form $0 \to F \to X \to M \to 0$ with $\mathrm{pd}\, F < \infty$ and G-dim $X = 0$. Such a short exact sequence is called a Cohen-Macaulay approximation of M. For the details, see [2].

We can prove an analogue of this result. To state our theorem, we need several notations from [2]. Now let C be a semi-dualizing (R, S)-bimodule as before. We denote by $\mathcal{G}_C$ the full subcategory of R-mod consisting of all C-reflexive R-modules, and by $\mathcal{R}_R(C)$ the full subcategory consisting of R-modules of finite G_C-dimension. And $\mathrm{add}(C)$ denotes the subcategory of all direct summands of direct sums of copies of C. It is obvious that $\mathrm{add}(C) \subseteq \mathcal{G}_C$ and that the objects of $\mathrm{add}(C)$ are injective objects in $\mathcal{G}_C$, indeed $\mathrm{Ext}^i_R(X, C) = 0$ for $X \in \mathcal{G}_C$ and $i > 0$. The following lemma says that C is an injective cogenerator of $\mathcal{G}_C$.

Lemma 2.2. *Suppose an R-module X is C-reflexive, hence $X \in \mathcal{G}_C$. Then there exists an exact sequence $0 \to X \to C^0 \to X^1 \to 0$ where $C^0 \in \mathrm{add}(C)$ and $X^1 \in \mathcal{G}_C$. In particular, we can resolve X by objects in $\mathrm{add}(C)$ as*

$$0 \to X \to C^0 \to C^1 \to C^2 \to \cdots, \ (C^i \in \mathrm{add}(C)).$$

Proof. It follows from Theorem 2.1.(3) that $Y = \mathrm{Hom}_R(X, C)$ is a C-reflexive S-module. Take an exact sequence $0 \to Y' \to S^{\oplus n} \to Y \to 0$ to get the the first syzygy S-module Y' of Y. Applying the functor $\mathrm{Hom}_S(-, C)$, we obtain an exact sequence $0 \to X \to C^{\oplus n} \to \mathrm{Hom}_S(Y', C) \to 0$. Since Y' is a C-reflexive S-module, we see that $\mathrm{Hom}_S(Y', C)$ is a C-reflexive R-module again. $\square$

To state the theorem, let us denote

$$\widehat{\mathrm{add}(C)} = \left\{ F \in R\text{-mod} \;\middle|\; \begin{array}{l} \text{there is an exact sequence of finite length} \\ 0 \to C^{-n} \to C^{-n+1} \to \cdots \to C^0 \to F \to 0 \\ \text{where each } \ C^i \in \mathrm{add}(C) \end{array} \right\}.$$

Then it is easy to prove the following result in a completely similar way to the proof of [2, Theorem 1.1].

Theorem 2.3. *Let $M \in R$-mod, and suppose $\mathrm{G}_C\text{-}\dim_R M < \infty$, hence $M \in \mathcal{R}_R(C)$. Then there exist short exact sequences*

$$0 \to F_M \to X_M \to M \to 0 \tag{2}$$

$$0 \to M \to F^M \to X^M \to 0 \tag{3}$$

where X_M and X^M are in G_C, and F_M and F^M are in $\widehat{\mathrm{add}(C)}$.

Remark 2.2. Let X be a C-reflexive R-module. Since $\mathrm{Ext}^i(X, C) = 0$ for $i > 0$, it follows that $\mathrm{Ext}^i(X, F) = 0$ for $F \in \widehat{\mathrm{add}(C)}$ and $i > 0$. Hence, from (2), we have an exact sequence

$$0 \to \mathrm{Hom}_R(X, F_M) \to \mathrm{Hom}_R(X, X_M) \to \mathrm{Hom}_R(X, M) \to 0.$$

This means that any homomorphism from any C-reflexive R-module X to M factors through the map $X_M \to M$. In this sense, the exact sequence (2) gives an approximation of M by the subcategory G_C.

Remark 2.3. We can of course define $\mathrm{G}_C\text{-}\dim_S N$ for an S-module N as in the same manner as we define G_C-dimension. And it is clear by symmetry that it satisfies that $\mathrm{G}_C\text{-}\dim_S N = \sup\{n | \ \mathrm{Ext}_S^n(N, C) \neq 0\}$ if the former is finite etc.

3. G_C-dimensions for complexes

Again in this section, we assume that R (resp. S) is a left (resp. right) noetherian ring. We denote by $\mathfrak{D}^b(R\text{-mod})$ (resp. $\mathfrak{D}^b(\text{mod-}S)$) the derived category of R-mod (resp. mod-S) consisting of complexes with bounded finite homologies.

For a complex $M^\bullet$ we always write it as

$$\cdots \to M^{n-1} \xrightarrow{\partial_M^n} M^n \xrightarrow{\partial_M^{n+1}} M^{n+1} \xrightarrow{\partial_M^{n+2}} M^{n+2} \to \cdots ,$$

and the shifted complex $M^\bullet[m]$ is the complex with $M^\bullet[m]^n = M^{m+n}$ and $\partial_{M[m]}^n = (-1)^m \partial_M^{m+n}$.

According to Foxby [9], we define the *supremum*, the *infimum* and the *amplitude* of a complex $M^\bullet$ as follows;

$$\begin{cases} s(M^\bullet) = \sup\{\ n \mid \mathrm{H}^n(M^\bullet) \neq 0\ \}, \\ i(M^\bullet) = \inf\{\ n \mid \mathrm{H}^n(M^\bullet) \neq 0\ \}, \\ a(M^\bullet) = s(M^\bullet) - i(M^\bullet). \end{cases}$$

Note that $\mathrm{H}(M^\bullet) = 0 \iff s(M^\bullet) = -\infty \iff i(M^\bullet) = +\infty \iff a(M^\bullet) = -\infty$.

Suppose in the following that $\mathrm{H}(M^\bullet) = 0$. A complex $M^\bullet$ is called bounded if $s(M^\bullet) < \infty$ and $i(M^\bullet) > -\infty$ (hence $a(M^\bullet) < \infty$). And $\mathfrak{D}^b(R\text{-mod})$ is, by definition, consisting of bounded complexes with finitely generated homology modules. (We remark that for each component M^n of $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ is not necessary finitely generated.) Thus, whenever $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, we have

$$-\infty < i(M^\bullet) \leq s(M^\bullet) < +\infty.$$

and $a(M^\bullet)$ is a non-negative integer.

We remark that the category R-mod can be identified with the full subcategory of $\mathfrak{D}^b(R\text{-mod})$ consisting of all the complexes $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ with $s(M^\bullet) = i(M^\bullet) = a(M^\bullet) = 0$ or otherwise $M^\bullet \cong 0$. Through this identification we always think of R-mod as the full subcategory of $\mathfrak{D}^b(R\text{-mod})$.

For a complex $P^\bullet$, if each component P^i is a finitely generated projective module, then we say that $P^\bullet$ is a projective complex. For any complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, we can construct a projective complex $P^\bullet$ and a chain map $P^\bullet \to$

$M^\bullet$ that yields an isomorphism in $\mathfrak{D}^b(R\text{-mod})$. We call such $P^\bullet \to M^\bullet$ a semi-projective resolution of $M^\bullet$. If $\mathrm{H}(M^\bullet) \neq 0$ and $s = s(M^\bullet)$ is finite, then we can take a semi-projective resolution $P^\bullet$ of $M^\bullet$ in the form;

$$\cdots \to P^{s-2} \xrightarrow{\partial_P^{s-1}} P^{s-1} \xrightarrow{\partial_P^{s}} P^s \to 0 \to 0 \to \cdots, \ \text{(i.e. } P^i = 0 \text{ for } i > s).$$

We call such a semi-projective resolution with this additional property a *standard* projective resolution of $M^\bullet$.

For a projective complex $P^\bullet (\not\cong 0)$ and an integer n, we can consider two kinds of truncated complexes:

$$\begin{cases} \tau^{\leq n} P^\bullet = (\cdots \to P^{n-2} \xrightarrow{\partial_P^{n-1}} P^{n-1} \xrightarrow{\partial_P^{n}} P^n \to 0 \to 0 \to \cdots) \\ \tau^{\geq n} P^\bullet = (\cdots \to 0 \to 0 \to P^{n} \xrightarrow{\partial_P^{n+1}} P^{n+1} \xrightarrow{\partial_P^{n+2}} P^{n+2} \to \cdots) \end{cases}$$

Definition 3.1 (ω-operation)**.** Let $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, $\mathrm{H}(M^\bullet) \neq 0$ and $s = s(M^\bullet)$. Taking a standard projective resolution $P^\bullet$ of $M^\bullet$, we define the projective complex $\omega P^\bullet$ by

$$\omega P^\bullet = (\tau^{\leq s-1} P^\bullet)[-1]. \tag{4}$$

Note from this definition that $\omega P^\bullet$ and $P^\bullet[-1]$ share the same components in degree $\leq s$. We can also see from the definition that there is a triangle of the form

$$\omega P^\bullet \to P^s[-s] \to M^\bullet \to \omega P^\bullet[1]. \tag{5}$$

Therefore, if $M^\bullet$ is a module $M \in R$-mod, then $\omega P^\bullet$ is isomorphic to a first syzygy module of M. Note that $\omega P^\bullet$ is not uniquely determined by $M^\bullet$. Actually it depends on the choice of a standard projective resolution $P^\bullet$, but is unique up to a projective summand in degree s. It is easy to prove the following lemma.

Lemma 3.1. *Let $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, $\mathrm{H}(M^\bullet) \neq 0$ and let $P^\bullet$ be a standard projective resolution of $M^\bullet$. Now suppose that $a(M^\bullet) > 0$. Then,*
(1) $i(\omega P^\bullet) = i(M^\bullet) + 1$,
(2) $0 \leq a(\omega P^\bullet) < a(M^\bullet)$.

Proof. Let $s = s(M^\bullet)$. Since the complexes $P^\bullet$ and $\omega P^\bullet[1]$ share the same components in degree $\leq s-1$, we have that $\mathrm{H}^i(M^\bullet) = \mathrm{H}^i(P^\bullet) = \mathrm{H}^{i+1}(\omega P^\bullet)$ for $i \leq s-2$ and that $\mathrm{H}^{s-1}(M^\bullet) = \mathrm{H}^{s-1}(P^\bullet)$ is embedded into $\mathrm{H}^s(\omega P^\bullet)$. The lemma follows from this observation. □

It follows from this lemma that applying the ω-operation several times to a given projective complex $P^\bullet$, we will have a complex with amplitude 0, i.e. a shifted module.

Definition 3.2. Let $M^\bullet$ and $P^\bullet$ be as in the lemma. Then there is the least integer b with $\omega^b P^\bullet$ having amplitude 0. Thus there is a module $T \in R$-mod such that $\omega^b P^\bullet \cong T[-c]$ for some $c \in \mathbb{Z}$. We call such a module T the *trunk module* of the complex $M^\bullet$.

Remark 3.4. Let $M^\bullet$ and $P^\bullet$ be as in the lemma. Set $i = i(M^\bullet)$, and we see that the trunk module T is isomorphic to $\tau^{\leq i} P^\bullet[i]$ in $\mathfrak{D}^b(R\text{-mod})$, hence $T \cong \mathrm{Coker}(P^{i-1} \to P^i)$. Note that the trunk module T is unique only in the stable category $\underline{R\text{-mod}}$.

Note that the integer b in Definition 3.2 is not necessarily equal to $a(M^\bullet)$. For instance, consider the complex $M^\bullet = P^\bullet = R[2] \oplus R$. Then $a(M^\bullet) = 2$ and $T = \omega^1 P^\bullet[-1] = R$.

Now we fix a semi-dualizing (R, S)-bimodule C. Associated to it, we can consider the following subcategory of $\mathfrak{D}^b(R\text{-mod})$.

Definition 3.3. For a semi-dualizing (R, S)-bimodule C, we denote by $\mathcal{R}_R(C)$ the full subcategory of $\mathfrak{D}^b(R\text{-mod})$ consisting of all complexes $M^\bullet$ that satisfy the following two conditions.

(1) $\mathbf{R}\,\mathrm{Hom}_R(M^\bullet, C) \in \mathfrak{D}^b(\text{mod-}S)$.

(2) The natural morphism $M^\bullet \to \mathbf{R}\,\mathrm{Hom}_S(\mathbf{R}\,\mathrm{Hom}_R(M^\bullet, C), C)$ is an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.

If R is a left and right noetherian ring and if $R = S = C$, then we should note from the papers of Avramov-Foxby [3, (4.1.7)] and Yassemi [15, (2.7)] that $\mathcal{R}_R(R) = \{\ M^\bullet \in \mathfrak{D}^b(R\text{-mod}) \mid \text{G-dim}\ M^\bullet < \infty\}$.

First of all we should notice the following fact.

Lemma 3.2. *Let C be a semi-dualizing (R, S)-bimodule as above.*

(1) *The subcategory $\mathcal{R}_R(C)$ of $\mathfrak{D}^b(R\text{-mod})$ is a triangulated subcategory which contains R, and is closed under direct summands. In particular, $\mathcal{R}_R(C)$ contains all projective R-modules.*

(2) *Let $P^\bullet$ be a projective complex in $\mathfrak{D}^b(R\text{-mod})$. Then, $P^\bullet \in \mathcal{R}_R(C)$ if and only if $\omega P^\bullet \in \mathcal{R}_R(C)$.*

(3) *Let $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ and let T be a trunk module of $M^\bullet$. Then $M^\bullet \in \mathcal{R}_R(C)$ if and only if $T \in \mathcal{R}_R(C)$.*

Proof. The proof of (1) is standard, and we omit it. For (2) and (3), in the triangle (5), noting that $P[-s] \in \mathcal{R}_R(C)$ and that $\mathcal{R}_R(C)$ is a triangulated category, we see that $P^\bullet \in \mathcal{R}_R(C)$ is equivalent to that $\omega P^\bullet \in \mathcal{R}_R(C)$. Since $T \cong \omega^b P^\bullet[c]$ as in Definition 3.2, this is also equivalent to that $T \in \mathcal{R}_R(C)$. $\square$

The following lemma says that R-modules in $\mathcal{R}_R(C)$ form the subcategory of modules of finite G_C-dimension.

Lemma 3.3. *Let M be an R-module. Then the following two conditions are equivalent.*

(1) $\mathrm{G}_C\text{-}\dim_R M < \infty$,

(2) $M \in \mathcal{R}_R(C)$.

Proof. (1) $\Rightarrow$ (2): Note from the definition that every C-reflexive module belongs to $\mathcal{R}_R(C)$. Since $\mathrm{G}_C\text{-}\dim_R M < \infty$, there is a finite exact sequence $0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^0 \to M \to 0$ where each X^i is C-reflexive.

Since each X^i belongs to $\mathcal{R}_R(C)$ and since $\mathcal{R}_R(C)$ is closed under making triangles, we see that $M \in \mathcal{R}_R(C)$.

(2) $\Rightarrow$ (1): Suppose that $M \in \mathcal{R}_R(C)$ and let $P^\bullet \in \mathfrak{D}^b(R\text{-mod})$ be a (standard) projective resolution of M. Since $\mathbf{R}\operatorname{Hom}_R(M, C)$ is a bounded complex, it follows that $s = s(\mathbf{R}\operatorname{Hom}_R(M, C))$ is a (finite) non-negative integer. Since the complexes $\operatorname{Hom}_R(\omega^s P^\bullet, C)$ and $\operatorname{Hom}_R(P^\bullet[-s], C)$ share the same component in non-negative degree, we see that $\mathrm{H}^i(\mathbf{R}\operatorname{Hom}_R(\omega^s P^\bullet, C)) = \mathrm{H}^{i+s}(\mathbf{R}\operatorname{Hom}_R(P^\bullet, C)) = 0$ for $i \geq 1$. Noting that $\omega^s P^\bullet$ is isomorphic to the s-th syzygy module $\Omega^s M$ of M, we see from this that $\operatorname{Ext}^i(\Omega^s M, C) = 0$ for $i > 0$. It follows from above lemma, we have $\omega^s P^\bullet \in \mathcal{R}_R(C)$ and the natural map $\Omega^s M \to \mathbf{R}\operatorname{Hom}_S(\operatorname{Hom}_R(\Omega^s M, C), C)$ is an isomorphism, equivalently $\Omega^s M \cong \operatorname{Hom}_S(\operatorname{Hom}_R(\Omega^s M, C), C)$ and $\operatorname{Ext}^i(\operatorname{Hom}_R(\Omega^s M, C), C) = 0$ for $i > 0$. Consequently, we see that $\Omega^s M$ is a C-reflexive R-module, hence $\mathrm{G}_C\text{-}\dim_R M \leq s < \infty$. $\square$

Recall from Theorem 2.2 that if an R-module M has finite G_C-dimension, then we have $\mathrm{G}_C\text{-}\dim_R M = s(\mathbf{R}\operatorname{Hom}_R(M, C))$. Therefore it will be reasonable to make the following definition.

Definition 3.4. Let C be a semi-dualizing (R, S)-bimodule and let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$. We define the G_C-dimension of $M^\bullet$ to be

$$\begin{cases} \mathrm{G}_C\text{-}\dim_R M^\bullet = s(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C)) & \text{if } \quad M^\bullet \in \mathcal{R}_R(C), \\ \mathrm{G}_C\text{-}\dim_R M^\bullet = +\infty & \text{if } \quad M^\bullet \notin \mathcal{R}_R(C). \end{cases}$$

Note that this definition is compatible with that of G_C-dimension for R-modules in Section 2. Just noting an obvious equality

$$s(\mathbf{R}\operatorname{Hom}_R(M^\bullet[m], C)) = s(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C)) + m$$

for $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ and $m \in \mathbb{Z}$, we have the following lemma.

Lemma 3.4. *Let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$ and let m be an integer. Then we have*

$$\mathrm{G}_C\text{-}\dim_R M^\bullet[m] = \mathrm{G}_C\text{-}\dim_R M^\bullet + m.$$

Lemma 3.5. *Let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$. Then the following inequality holds*:

$$\mathrm{G}_C\text{-}\dim_R M^\bullet + i(M^\bullet) \geq 0.$$

Proof. If $M^\bullet \cong 0$, then since $i(M^\bullet) = +\infty$, the inequality holds obviously. We may thus assume that $\mathrm{H}(M^\bullet) = 0$. If $M^\bullet \notin \mathcal{R}_R(C)$, then $\mathrm{G}_C\text{-}\dim_R M^\bullet = +\infty$ by definition, and there is nothing to prove. Hence we assume $M^\bullet \in \mathcal{R}_R(C)$. In particular, we have $M^\bullet \cong \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C), C)$. Therefore we have that

$$\begin{aligned} i(M^\bullet) &= i(\mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C), C)) \\ &\geq i(C) - s(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C)) \\ &= -s(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C)) \\ &= -\mathrm{G}_C\text{-}\dim_R M^\bullet. \end{aligned}$$

(For the inequality see Foxby [8, Lemma 2.1].) □

Proposition 3.1. *For a given complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, suppose that $a(M^\bullet) > 0$. Taking a standard projective resolution $P^\bullet$ of $M^\bullet$, we have an equality*

$$\mathrm{G}_C\text{-}\dim_R M^\bullet = \mathrm{G}_C\text{-}\dim_R \omega P^\bullet + 1.$$

Proof. Note from Lemma 3.2(2) that $\mathrm{G}_C\text{-}\dim_R M^\bullet < \infty$ if and only if $\mathrm{G}_C\text{-}\dim_R \omega P^\bullet < \infty$. Assume that $n = \mathrm{G}_C\text{-}\dim_R M^\bullet = s(\mathbf{R}\operatorname{Hom}_R(P^\bullet, C)) < \infty$ and let $s = s(M^\bullet)$. We should note from Lemma 3.5 that

$$\begin{aligned} n + s &= \mathrm{G}_C\text{-}\dim_R M^\bullet + s(M^\bullet) \\ &> \mathrm{G}_C\text{-}\dim_R M^\bullet + i(M^\bullet) \\ &\geq 0. \end{aligned}$$

Since the complex $\operatorname{Hom}_R(\omega P^\bullet, C)$ shares the components in degree $\geq -s$ with $\operatorname{Hom}_R(P^\bullet, C)[1]$, we see that $\mathrm{H}^i(\operatorname{Hom}_R(\omega P^\bullet, C)) = \mathrm{H}^{i+1}(\operatorname{Hom}_R(P^\bullet, C))$ for $i > -s$. Since $n > -s$ as above, it follows that $s(\operatorname{Hom}_R(\omega P^\bullet, C)) = s(\operatorname{Hom}_R(P^\bullet, C)) - 1$. □

As we show in the next theorem, the G_C-dimension of a complex is essentially the same as that of its trunk module. In that sense, every argument concerning G_C-dimension of complexes will be reduced to that of modules.

Theorem 3.1. *Let T be the trunk module of a complex $M^\bullet$ as in Definition 3.2. Then there is an equality*

$$\mathrm{G}_C\text{-}\dim_R M^\bullet = \mathrm{G}_C\text{-}\dim_R T - i(M^\bullet).$$

Proof. If $M^\bullet \notin \mathcal{R}_R(C)$, then the both sides take infinity and the equality holds. We assume that $M^\bullet \in \mathcal{R}_R(C)$ hence $\mathrm{G}_C\text{-}\dim_R M^\bullet < \infty$.

We prove the equality by induction on $a(M^\bullet)$. If $a(M^\bullet) = 0$ then $M^\bullet \cong T[-i]$ for the trunk module T and for $i = i(M^\bullet)$. Therefore it follows from Lemma 3.4 $\mathrm{G}_C\text{-}\dim_R M^\bullet = \mathrm{G}_C\text{-}\dim_R T - i$.

Now assume that $a(M^\bullet) > 0$, and let $P^\bullet$ be a standard projective resolution of $M^\bullet$. Noting from Lemma 3.1 that we can apply the induction hypothesis on $\omega P^\bullet$, we get the following equalities from the previous proposition.

$$\begin{aligned} \mathrm{G}_C\text{-}\dim_R M^\bullet &= \mathrm{G}_C\text{-}\dim_R \omega P^\bullet + 1 \\ &= \mathrm{G}_C\text{-}\dim_R T - i(\omega P^\bullet) + 1 \\ &= \mathrm{G}_C\text{-}\dim_R T - i(P^\bullet) \\ &= \mathrm{G}_C\text{-}\dim_R T - i(M^\bullet). \end{aligned}$$

□

As one of the applications of this theorem, we can show the following theorem that generalizes Lemma 2.1 to the category of complexes.

Theorem 3.2. *Let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$. Then the following conditions are equivalent.*

(1) $\mathrm{G}_C\text{-}\dim_R M^\bullet < \infty$,

(2) *There is a bounded complex $X^\bullet$ consisting of C-reflexive modules and there is a chain map $X^\bullet \to M^\bullet$ that is an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.*

Proof. (2) $\Rightarrow$ (1): Note that every C-reflexive R-module belongs to $\mathcal{R}_R(C)$ and that $\mathcal{R}_R(C)$ is closed under making triangles. Therefore any complexes $X^\bullet$ of finite length consisting of C-reflexive modules are also in $\mathcal{R}_R(C)$, hence $\mathrm{G}_C\text{-}\dim_R X^\bullet < \infty$.

(1) $\Rightarrow$ (2): Assume that $\mathrm{G}_C\text{-}\dim_R M^\bullet < \infty$ hence $M^\bullet \in \mathcal{R}_R(C)$. We shall prove by induction on $a(M^\bullet)$ that the second assertion holds. If $a(M^\bullet) = 0$, then there is an R-module T such that $M^\bullet \cong T[-i]$ where $i = i(M^\bullet)$. Since $\mathrm{G}_C\text{-}\dim_R T < \infty$, there is a complex

$$X^\bullet = \left[\, 0 \to X^{-m} \to \cdots \to X^{-2} \to X^{-1} \to X^0 \to 0 \,\right]$$

with each X^i being C-reflexive and a quasi-isomorphism $X^\bullet \to T$. Thus the complex $X^\bullet[-s]$ is the desired complex for $M^\bullet$.

Now suppose $a = a(M^\bullet) > 0$ and take a standard projective resolution $P^\bullet$ of $M^\bullet$. As in (5), we have chain maps $\varphi : P^s[-s] \to M^\bullet$ and $\psi : \omega P^\bullet \to P^s[-s]$ that make the triangle

$$\omega P^\bullet \xrightarrow{\psi} P^s[-s] \xrightarrow{\varphi} M^\bullet \to \omega P^\bullet[1].$$

Since $a(\omega P^\bullet) < a(M^\bullet)$, it follows from the induction hypothesis that there is a chain map $\rho : X^\bullet \to \omega P^\bullet$ that gives an isomorphism in $\mathfrak{D}^b(R\text{-mod})$, where $X^\bullet$ is a complex of finite length with each X^i being C-reflexive. Thus we also have a triangle

$$X^\bullet \xrightarrow{\psi\cdot\rho} P^s[-s] \xrightarrow{\varphi} M^\bullet \to X^\bullet[1].$$

Now take a mapping cone $Y^\bullet$ of $\psi \cdot \rho$. Then it is obvious that $Y^\bullet$ has finite length and each modules in $Y^\bullet$ is C-reflexive, since Y^i is a module X^i with at most directly summing P^s. Furthermore it follows from the above triangle that there is a chain map $Y^\bullet \to M^\bullet$ that yields an isomorphism in $\mathfrak{D}^b(R\text{-mod})$. □

Also in the category $\mathfrak{D}^b(\text{mod-}S)$, we can construct the notion similar to that in $\mathfrak{D}^b(R\text{-mod})$.

Definition 3.5. Let C be a semi-dualizing (R, S)-bimodule. We denote by $\mathcal{R}_S(C)$ the full subcategory of $\mathfrak{D}^b(\text{mod-}S)$ consisting of all complexes $N^\bullet$ that satisfy the following two conditions.

(1) $\mathbf{R}\operatorname{Hom}_S(N^\bullet, C) \in \mathfrak{D}^b(R\text{-mod})$.

(2) The natural morphism $N^\bullet \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_S(N^\bullet, C), C)$ is an isomorphism in $\mathfrak{D}^b(\text{mod-}S)$.

Definition 3.6. Let C be a semi-dualizing (R, S)-bimodule and let $N^\bullet$ be a complex in $\mathfrak{D}^b(\text{mod-}S)$. We define the G_C-dimension of $N^\bullet$ to be

$$\begin{cases} \mathrm{G}_C\text{-}\dim_S N^\bullet = s(\mathbf{R}\operatorname{Hom}_S(N^\bullet, C)) & \text{if} \quad N^\bullet \in \mathcal{R}_S(C), \\ \mathrm{G}_C\text{-}\dim_S N^\bullet = +\infty & \text{if} \quad N^\bullet \notin \mathcal{R}_S(C). \end{cases}$$

Note that all the properties concerning $\mathcal{R}_R(C)$ and G_C-dimension hold true for $\mathcal{R}_S(C)$ and G_C-dimension by symmetry.

Lemma 3.6. *Let C be a semi-dualizing (R, S)-bimodule as above. Then the functors $\mathbf{R}\operatorname{Hom}_R(-, C)$ and $\mathbf{R}\operatorname{Hom}_S(-, C)$ yield a duality between the categories $\mathcal{R}_R(C)$ and $\mathcal{R}_S(C)$.*

We postpone the proof of this lemma until Theorem 4.1 in the next section, where we prove the duality in more general setting. Using this lemma we are able to prove the following theorem, which generalizes Theorem 2.3. We recall that $\mathrm{add}(C)$ is the additive full subcategory of R-mod consisting of modules that are isomorphic to direct summands of finite direct sums of copies of C.

Theorem 3.3. *Let $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ and suppose that $\mathrm{G}_C\text{-}\dim_R M^\bullet < \infty$. Then there exists a triangle*

$$F^\bullet_M \to X^\bullet_M \to M^\bullet \to F^\bullet_M[1] \tag{6}$$

where $X^\bullet_M$ is a shifted C-reflexive R-module, and $F^\bullet_M$ is a complex that is isomorphic to a complex of finite length consisting of modules in $\mathrm{add}(C)$.

Proof. Let $N^\bullet = \mathbf{R}\operatorname{Hom}_R(M^\bullet, C)$ and let T be a trunk module of $N^\bullet$ in the category $\mathfrak{D}^b(\text{mod-}S)$. We have a triangle of the following type:

$$T[-i] \to P^\bullet \to N^\bullet \to T[-i+1],$$

where $i = i(N^\bullet)$ and $P^\bullet$ is a projective S-complex of length $a(N^\bullet)$. Note that $n = \mathrm{G}_C\text{-}\dim_S T$ is finite as well as $\mathrm{G}_C\text{-}\dim_S N^\bullet < \infty$ by Lemma 3.6. Take the n-th syzygy module of T, and we have a C-reflexive S-module U with the triangle

$$U[-i-n] \to Q^\bullet \to N^\bullet \to U[-i-n+1],$$

where $Q^\bullet$ is again a projective S-complex of finite length. Applying the functor $\mathbf{R}\operatorname{Hom}_S(-, C)$, we have a triangle

$$\mathbf{R}\operatorname{Hom}_S(U, C)[i+n-1] \to M^\bullet \to \mathbf{R}\operatorname{Hom}_S(Q^\bullet, C) \to \mathbf{R}\operatorname{Hom}_S(U, C)[i+n].$$

Note that $\mathbf{R}\operatorname{Hom}_S(U, C)$ is isomorphic to a C-reflexive R-module and that $\mathbf{R}\operatorname{Hom}_S(Q^\bullet, C)$ is a complex of finite length, each component of which is a module in $\mathrm{add}(C)$. $\square$

4. $\mathrm{G}_{C^\bullet}$-dimensions for complexes

The notion of a semi-dualizing bimodule is naturally extended to that of a semi-dualizing complex of bimodules. For this purpose, let $C^\bullet$ be a complex consisting of (R,S)-bimodules and (R,S)-bimodule homomorphisms. Then for a complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, take an R-projective resolution $P^\bullet$ of $M^\bullet$, and we understand $\mathbf{R}\operatorname{Hom}_R(M^\bullet, C^\bullet)$ as the class of complexes of S-modules that are isomorphic in $\mathfrak{D}^b(\text{mod-}S)$ to the complex $\operatorname{Hom}_R(P^\bullet, C^\bullet)$. In this way, $\mathbf{R}\operatorname{Hom}_R(-, C^\bullet)$ yields a functor $\mathfrak{D}^b(R\text{-mod}) \to \mathfrak{D}^b(\text{mod-}S)$. Likewise, $\mathbf{R}\operatorname{Hom}_S(-, C^\bullet)$ yields a functor $\mathfrak{D}^b(\text{mod-}S) \to \mathfrak{D}^b(R\text{-mod})$.

Let $s \in S$. Then we see that the right multiplication $\rho(s) : C^\bullet \to C^\bullet$ is a chain map of R-complexes. Take a projective resolution $P^\bullet$ of $C^\bullet$ as a complex in $\mathfrak{D}^b(R\text{-mod})$ and a chain map $\psi : P^\bullet \to C^\bullet$ of R-complexes. Combining these two, we have a chain map $h(s) = \rho(s) \cdot \psi : P^\bullet \to C^\bullet$, which defines an element of degree 0 in the complex $\operatorname{Hom}_R(P^\bullet, C^\bullet)$. In such a way, we obtain the morphism $h : S \to \mathbf{R}\operatorname{Hom}_R(C^\bullet, C^\bullet)$ in $\mathfrak{D}^b(\text{mod-}S)$, which we call the right homothety morphism. Likewise, we have the left homothety morphism $R \to \mathbf{R}\operatorname{Hom}_S(C^\bullet, C^\bullet)$ in $\mathfrak{D}^b(R\text{-mod})$.

Definition 4.1. Let $C^\bullet$ be a complex consisting of (R,S)-bimodules and (R,S)-bimodule homomorphisms as above. We call $C^\bullet$ a *semi-dualizing complex of bimodules* if the following conditions hold.

(1) The complex $C^\bullet$ is bounded, that is, there are only a finite number of i with $\mathrm{H}^i(C^\bullet) \neq 0$.

(2) The right homothety morphism $S \to \mathbf{R}\operatorname{Hom}_R(C^\bullet, C^\bullet)$ is an isomorphism in $\mathfrak{D}^b(\text{mod-}S)$.

(3) The left homothety morphism $R \to \mathbf{R}\operatorname{Hom}_S(C^\bullet, C^\bullet)$ is an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.

Definition 4.2. We denote by $\mathcal{R}_R(C^\bullet)$ the full subcategory consisting of all complexes $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ that satisfy the following conditions.

(1) The complex $\mathbf{R}\operatorname{Hom}_R(M^\bullet, C^\bullet)$ of S-modules belongs to $\mathfrak{D}^b(\text{mod-}S)$.

(2) The natural morphism $M^\bullet \to \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C^\bullet), C^\bullet)$ gives an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.

Similarly we can define $\mathcal{R}_S(C^\bullet)$ as the full subcategory consisting of all complexes $N^\bullet$ that satisfy the following conditions.

(1') The complex $\mathbf{R}\operatorname{Hom}_S(N^\bullet, C^\bullet)$ of R-modules belongs to $\mathfrak{D}^b(R\text{-mod})$.

(2') The natural morphism $N^\bullet \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_S(N^\bullet, C^\bullet), C^\bullet)$ gives an isomorphism in $\mathfrak{D}^b(\text{mod-}S)$.

Definition 4.3.

(1) For a complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, we define the $\mathrm{G}_{C^\bullet}$-dimension of $M^\bullet$ as

$$G_{C^\bullet}\text{-}\dim_R M^\bullet = \begin{cases} s(\mathbf{R}\operatorname{Hom}_R(M^\bullet, C^\bullet)) & \text{if } M^\bullet \in \mathcal{R}_R(C^\bullet), \\ +\infty & \text{otherwise.} \end{cases}$$

(2) Similarly we define the $\mathrm{G}_{C^\bullet}$-dimension of a complex $N^\bullet \in \mathfrak{D}^b(\text{mod-}S)$ as

$$G_{C^\bullet}\text{-}\dim_S N^\bullet = \begin{cases} s(\mathbf{R}\operatorname{Hom}_S(N^\bullet, C^\bullet)) & \text{if } N^\bullet \in \mathcal{R}_R(C^\bullet), \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Let $C^\bullet$ be a semi-dualizing complex of (R,S)-bimodules. Then the functors $\mathbf{R}\operatorname{Hom}_R(-, C^\bullet)$ and $\mathbf{R}\operatorname{Hom}_S(-, C^\bullet)$ give rise to a duality between $\mathcal{R}_R(C^\bullet)$ and $\mathcal{R}_S(C^\bullet)$.*

Proof. It is straightforward to see that both functors send complexes with $\mathrm{G}_{C^\bullet}$-dimension finite (over the respective rings), and that the compositions of them are the identity functors (for the respective categories). $\square$

5. $\mathrm{G}_{C^\bullet}$-dimension in the commutative case

In this final section of the paper, we shall observe several properties of G_C-dimension in the case when R and S are commutative local rings. We begin with the following lemma.

Lemma 5.1. *Let R and S be commutative noetherian rings. Suppose that there exists a semi-dualizing (R,S)-bimodules C. Then R is isomorphic to S.*

Proof. Let $\phi : R \to \operatorname{Hom}_R(C, C) = S$ and $\psi : S \to \operatorname{Hom}_S(C, C) = R$ be the homothety morphisms. Since R and S are commutative, we see that they are well-defined ring homomorphism and that $\psi\phi$ (resp. $\phi\psi$) is the identity map on R (resp. S). Hence $R \cong S$ as desired. $\square$

In view of this lemma, we may assume that R coincides with S for our purpose of this section. Thus we may call a semi-dualizing (R,S)-bimodule simply a semi-dualizing module. For a semi-dualizing complex $C^\bullet$, we simply write $\mathcal{R}(C^\bullet)$ for $\mathcal{R}_R(C^\bullet)$. Note that $G_{C^\bullet}\text{-}\dim M^\bullet$ (in this paper) is the same as $\text{G-dim}_{C^\bullet} M^\bullet$ in [5] and $\mathrm{G}_{C^\bullet}\text{-}\dim M^\bullet$ in [10].

From now on, we assume that R is a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. It is known that $G_{C^\bullet}\text{-}\dim M^\bullet$ satisfies the Auslander-Buchsbaum-type equality as well as $\text{G-dim}_R M^\bullet$.

Lemma 5.2. [5, Theorem 3.14] *For $M^\bullet \in \mathcal{R}(C^\bullet)$,*

$$G_{C^\bullet}\text{-}\dim M^\bullet = \operatorname{depth} R - \operatorname{depth} M^\bullet + s(C^\bullet),$$

where the depth $\operatorname{depth} M^\bullet$ *of a complex $M^\bullet$ is defined to be $i(\mathbf{R}\operatorname{Hom}(k, M^\bullet))$.*

We are now able to state the main result of this section.

Theorem 5.1. *The following conditions are equivalent for a local ring $(R, \mathfrak{m}, k)$.*

(1) *R is a Cohen-Macaulay local ring that is a homomorphic image of a Gorenstein local ring.*

(2) *For any finitely generated R-module M, there exists a semi-dualizing module C such that $\mathrm{G}_C\text{-}\dim_R M < \infty$.*

(3) *There exists a semi-dualizing module C such that $\mathrm{G}_C\text{-}\dim_R k < \infty$.*

(4) *For any $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ there exists a semi-dualizing module C such that $\mathrm{G}_C\text{-}\dim_R M^\bullet < \infty$.*

(5) *There exists a semi-dualizing module C such that $\mathcal{R}(C) = \mathfrak{D}^b(R\text{-mod})$.*

(6) *The dualizing complex $D^\bullet$ exists and there exists a semi-dualizing module C such that $\mathrm{G}_C\text{-}\dim_R D^\bullet < \infty$.*

Proof. The implications (5) ⇒ (4) ⇒ (2) ⇒ (3) are trivial.

(3) ⇒ (1): Since $\mathrm{G}_C\text{-}\dim_R k < \infty$, we have $\mathrm{Ext}_R^n(k, C) = 0$ for $n \gg 0$. Hence we see that the injective dimension of C is finite. Therefore R is Cohen-Macaulay. (It is well-known that a commutative local ring which admits a finitely generated module of finite injective dimension is Cohen-Macaulay. For example, see [14].) Note that

$$\begin{aligned}\operatorname{depth} C &= -\mathrm{G}_C\text{-}\dim_R C + \operatorname{depth} R + s(C)\\ &= \operatorname{depth} R\\ &= \dim R.\end{aligned}$$

That is to say, C is a maximal Cohen-Macaulay module. Since the isomorphism $\mathrm{Ext}_R^d(\mathrm{Ext}_R^d(k, C), C) \cong k$, where $d = \dim R$, holds, one can show that C is the dualizing module of R. The existence of the dualizing module of R implies that R is a homomorphic image of a Gorenstein local ring. (See Reiten [12, Theorem (3)] or Foxby [6, Theorem 4.1].)

(1) ⇒ (6): It follows from the condition (1) that R admits the dualizing module K_R. Note that K_R is a semi-dualizing module and isomorphic to the dualizing complex in $\mathfrak{D}^b(R\text{-mod})$. Hence $\mathrm{G}_{K_R}\text{-}\dim K_R = 0 < \infty$.

(6) ⇒ (5): We may assume that $i(D^\bullet) = 0$. Then note that $\operatorname{depth} D^\bullet = \dim R$. It follows from Lemma 5.2 that

$$\begin{aligned}\mathrm{G}_C\text{-}\dim_R D^\bullet &= \operatorname{depth} R - \operatorname{depth} D^\bullet + s(C)\\ &= \operatorname{depth} R - \dim R\\ &\le 0.\end{aligned}$$

On the other hand, from Lemma 3.5 we have that

$$\mathrm{G}_C\text{-}\dim_R D^\bullet = \mathrm{G}_C\text{-}\dim_R D^\bullet + i(D^\bullet) \ge 0.$$

Consequently, we have $\dim R = \operatorname{depth} R$. Hence R is Cohen-Macaulay. And this implies that $D^\bullet$ is isomorphic to the dualizing module K_R of R. It is obvious that K_R is a semi-dualizing module and every maximal Cohen-Macaulay module is K_R-reflexive. As a result, every R-module has finite G_{K_R}-dimension, hence $\mathcal{R}(K_R)$ contains all R-modules. Then it follows from Theorem 3.1 that $\mathcal{R}(K_R)$ contains all complexes in $\mathfrak{D}^b(R\text{-mod})$, hence $\mathcal{R}(K_R) = \mathfrak{D}^b(R\text{-mod})$. □

Similarly to the above theorem, we can get a result for semi-dualizing complexes.

Theorem 5.2. *The following conditions are equivalent for a local ring* $(R, \mathfrak{m}, k)$.

(1) *R is a homomorphic image of a Gorenstein local ring.*

(2) *For any $M \in R$-mod, there exists a semi-dualizing complex $C^\bullet$ such that $G_{C^\bullet}$-$\dim M < \infty$.*

(3) *There exists a semi-dualizing complex $C^\bullet$ such that $G_{C^\bullet}$-$\dim k < \infty$.*

(4) *For any $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, there exists a semi-dualizing complex $C^\bullet$ such that $G_{C^\bullet}$-$\dim M < \infty$.*

(5) *There exists a semi-dualizing complex $C^\bullet$ such that $\mathcal{R}(C^\bullet) = \mathfrak{D}^b(R\text{-mod})$.*

(6) *The dualizing complex $D^\bullet$ exists.*

Proof. It is easy to prove the implications (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6). The remaining implication (6) $\Rightarrow$ (1) that is the most difficult to prove follows from [11, Theorem 1.2]. □

As final part of the paper we discuss a kind of uniqueness property of semi-dualizing complexes.

Theorem 5.3. *Let $C_1^\bullet$ and $C_2^\bullet$ be semi-dualizing complexes. Suppose that $C_1^\bullet \in \mathcal{R}(C_2^\bullet)$ and $C_2^\bullet \in \mathcal{R}(C_1^\bullet)$. Then $C_1^\bullet \cong C_2^\bullet[a]$ for some $a \in \mathbb{Z}$. In particular, we have $\mathcal{R}(C_1^\bullet) = \mathcal{R}(C_2^\bullet)$.*

For the proof this theorem we need the notion of Poincare and Bass series of a complex.

Remark 5.5. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. For a complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, consider two kinds of formal Laurent series in the variable t;

$$\mathrm{P}_{M^\bullet}(t) = \sum_{n\in\mathbb{Z}} \dim_k \mathrm{H}^{-n}(M^\bullet \overset{\mathbf{L}}{\otimes} k) \cdot t^n,$$
$$\mathrm{I}^{M^\bullet}(t) = \sum_{n\in\mathbb{Z}} \dim_k \mathrm{H}^{n}(\mathbf{R}\operatorname{Hom}(k, M^\bullet)) \cdot t^n.$$

These series are called respectively the Poincare series and the Bass series of $M^\bullet$. As it is shown in Foxby [8, Theorem 4.1(a)], the following equality holds for $M^\bullet, N^\bullet \in \mathfrak{D}^b(R\text{-mod})$.

$$\mathrm{I}^{\mathbf{R}\operatorname{Hom}(M^\bullet, N^\bullet)}(t) = \mathrm{P}_{M^\bullet}(t) \cdot \mathrm{I}^{N^\bullet}(t) \tag{7}$$

Proof of 5.3. Since $C_1^\bullet \in \mathcal{R}(C_2^\bullet)$, we have $C_1^\bullet \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet), C_2^\bullet)$. Hence, we have from (7) that

$$\mathrm{I}^{C_1^\bullet}(t) = \mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t) \cdot \mathrm{I}^{C_2^\bullet}(t).$$

Likewise, it follows from $C_2^\bullet \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet), C_1^\bullet)$ that

$$\mathrm{I}^{C_2^\bullet}(t) = \mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t) \cdot \mathrm{I}^{C_1^\bullet}(t).$$

Since $\mathrm{H}(C_1^\bullet) \neq 0$, one can check that $\mathrm{I}^{C_1^\bullet}(t) \neq 0$. Therefore we have $\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t) \cdot \mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t) = 1$.

Since $\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t)$ and $\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t)$ are formal Laurent series with non-negative coefficients, we have

$$\begin{aligned}
&\operatorname{order}(\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t)) + \operatorname{order}(\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t)) \\
&\qquad = \operatorname{order}(\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t) \cdot \mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t)) \\
&\deg(\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t)) + \deg(\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t)) \\
&\qquad = \deg(\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t) \cdot \mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t))
\end{aligned}$$

Therefore we have $\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet)}(t) = t^a$ and $\mathrm{P}_{\mathbf{R}\operatorname{Hom}(C_2^\bullet, C_1^\bullet)}(t) = t^{-a}$ for some integer a. Thus it follows that

$$\begin{aligned}
C_1^\bullet &\cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(C_1^\bullet, C_2^\bullet), C_2^\bullet) \\
&\cong \mathbf{R}\operatorname{Hom}(R[-a], C_2^\bullet) \\
&\cong C_2^\bullet[a],
\end{aligned}$$

as desired. □

Finally we have an interesting corollary of this theorem.

Corollary 5.1. *Suppose that R admits the dualizing complex $D^\bullet$. Then R is a Gorenstein ring if and only if* G-dim $D^\bullet < \infty$.

Proof. If R is Gorenstein then $D^\bullet \cong R$ thus G-dim $D^\bullet =$ G-dim $R = 0$. Conversely, assume G-dim$D^\bullet < \infty$. Then we have $D^\bullet \in \mathcal{R}(R)$. On the other hand, we have $R \in \mathcal{R}(D^\bullet)$, more generally $\mathcal{R}(D^\bullet)$ contains all R-modules by the definition of dualizing complex. Hence it follows from the theorem that $D^\bullet \cong R[a]$ for some $a \in \mathbb{Z}$, which means R is a Gorenstein ring. □

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