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On extensions of projective indecomposable modules

By

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Introduction

Let G be a finite group and p a prime. Let (K, R, k) be a p-modular system. We assume that K contains the $|G|$-th roots of unity and that k is algebraically closed. Suppose we are given a normal subgroup N of G such that G/N is a p-group and a G-invariant block b of N such that $N = QC_N(Q)$ for a defect group Q of b. Then, as is well-known, b has (up to isomorphism) a unique projective indecomposable RN-module V. It seems natural to ask whether there exists an extension U to G of V such that a vertex of U intersects N trivially. Let B be a unique block of G covering b. In Section 3, we obtain two necessary conditions such a module U must satisfy. Let P be a vertex of U and W a P-source of U. Then

(1) PQ is a defect group of B;

(2) W is an endo-permutation module, which is identified with a lift of a source of a unique simple kG-module in B.

(cf. Proposition 3.3, Corollary 3.17.)

In Section 4 we study the case where G/N is cyclic (and (1) holds for a p-subgroup P with $P \cap Q = 1$) and show that any indecomposable RG-module in B with vertex P and a P-source W as in (2) is actually an extension of V.

(Although we have mentioned only RG-modules, we also obtain similar results for kG-modules.)

In Section 1 we define an action of the group of capped endo-permutation modules over p-groups P (Dade [1, 2]) on the set of indecomposable P-modules. In Section 2 we determine vertices and sources of certain indecomposable modules.

Notation and convention

Let o denote R or k. For oG-modules V_i $(i = 1, 2)$, $V_1 \otimes V_2$ stands for $V_1 \otimes_o V_2$. Also for a direct product $G = G_1 \times G_2$ and oG_i-modules V_i $(i = 1, 2)$, $V_1 \times V_2$ stands for the external tensor product $V_1 \otimes_o V_2$. We denote by 1_G the trivial oG-module of rank one. For an RG-module U, let $U^* = U/\pi U$, where πR is the maximal ideal of R. For a kG-module X, an RG-module L such that

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$L^* \cong X$ is said to be a lift of X. For an oG-module U, let $U^\wedge$ be the dual module of U. For a subgroup Q ($\neq 1$) of G, let $\mathcal{H}(Q)$ be the set of all proper subgroups of Q. Let $I(oG)$ be the augmentation ideal of oG. Throughout this paper all oG-modules are assumed to be o-free of finite rank. Since we often use such expressions as "a unique module (up to isomorphism)" we suppress for brevity the words "(up to isomorphism)" in most cases.

1. Groups of capped endo-permutation modules

Let P ($\neq 1$) be a p-group. For a set $\mathcal{X}$ of subgroups of P and oP-modules U, V, we write $U \equiv V \oplus O(\mathcal{X})$, if there exists an $\mathcal{X}$-projective oP-module W (or 0) such that $U \cong V \oplus W$. (In particular, $U \equiv O(\mathcal{X})$ means that U is $\mathcal{X}$-projective.) An oP-module V is called (Dade [1]) an endo-permutation module if $V \otimes V^\wedge$ is a permutation module, where $V^\wedge$ is the dual module of V. An endo-permutation oP-module V is said to be capped, if V has an indecomposable summand with vertex P. In that case, such a summand is determined up to isomorphism and is denoted by $\operatorname{cap}(V)$ ([1, p. 470]). Let $\operatorname{Ep}(oP)$ be the set of (isomorhism classes of) indecomposable endo-permutation oP-modules with vertex P. (In [1], $\operatorname{Ep}(oP)$ is denoted by $\operatorname{Ind}_P(oP)$.) As in [1, Corollary 3.13 and Proposition 6.5], $\operatorname{Ep}(oP)$ forms an abelian group:

For $U, V \in \operatorname{Ep}(oP)$, the product $U \cdot V$ is a unique indecomposable summand with vertex P of $U \otimes V$. So $U \otimes V \equiv U \cdot V \oplus O(\mathcal{H}(P))$. (In Dade's notation [1], $U \cdot V \cong \operatorname{cap}(U \otimes V)$.) In $\operatorname{Ep}(oP)$ the identity is 1_P (the trivial oP-module of rank one) and the inverse of V is $V^\wedge$. So $V \otimes V^\wedge \equiv 1_P \oplus O(\mathcal{H}(P))$.

Let $\operatorname{Ind}(oP)$ be the set of (isomorhism classes of) non-projective indecomposable oP-modules. In this section we define a vertex-preserving action of the group $\operatorname{Ep}(oP)$ on the set $\operatorname{Ind}(oP)$. Let $\mathcal{Q}$ be a set of representatives of P-conjugacy classes of all subgroups ($\neq 1$) of P. For any $Q \in \mathcal{Q}$, let $\operatorname{Ind}(oP|Q)$ be the set of (isomorphism classes of) indecomposable oP-modules with vertex Q. Then we have

$$\operatorname{Ind}(oP) = \bigcup_{Q \in \mathcal{Q}} \operatorname{Ind}(oP|Q) \quad \text{(disjoint)}.$$

Thus it suffices to define an action of $\operatorname{Ep}(oP)$ on $\operatorname{Ind}(oP|Q)$ for each $Q \in \mathcal{Q}$. We begin with the case where $Q = P$.

Lemma 1.1. *For $W \in \operatorname{Ep}(oP)$ and $V \in \operatorname{Ind}(oP|P)$, let $W \otimes V \cong \bigoplus_i X_i$ be a decomposition of $W \otimes V$ into indecomposable summands X_i. Then there is a unique X_i with vertex P.*

Proof. Tensoring with $W^\wedge$, we get $\bigoplus_i W^\wedge \otimes X_i \equiv V \oplus O(\mathcal{H}(P))$, since $W^\wedge \otimes W \equiv 1_P \oplus O(\mathcal{H}(P))$. Thus, for some i, $W^\wedge \otimes X_i \equiv V \oplus O(\mathcal{H}(P))$ and then P is a vertex of X_i. On the other hand, if $j \neq i$, $W^\wedge \otimes X_j \equiv O(\mathcal{H}(P))$. Tensoring with W, we get that $X_j \equiv O(\mathcal{H}(P))$, as required. $\square$

Let us denote the summand X_i in the above lemma by $W \cdot V$. (If $V \in \operatorname{Ep}(oP) (\subseteq \operatorname{Ind}(oP|P))$, two definitions of $W \cdot V$ are at hand, but they coincide with each other, of course.) So we have $W \otimes V \equiv W \cdot V \oplus O(\mathcal{H}(P))$ with $W \cdot V \in \mathrm{Ind}(oP|P)$. This defines an action of $\mathrm{Ep}(oP)$ on $\mathrm{Ind}(oP|P)$. Namely we have:

Proposition 1.2. *Let $W, W' \in \mathrm{Ep}(oP)$ and $V \in \mathrm{Ind}(oP|P)$. Then*
(i) $W \cdot (W' \cdot V) \cong (W \cdot W') \cdot V$, *and*
(ii) $1_P \cdot V \cong V$.

Proof. (i) Since $W \otimes (W' \otimes V) \cong (W \otimes W') \otimes V$, the result follows.
(ii) This is obvious. □

To define an action of $\mathrm{Ep}(oP)$ on $\mathrm{Ind}(oP|Q), Q \in \mathcal{Q}$, we need the following proposition. We note that for any $W \in \mathrm{Ep}(oP)$, W_Q is capped and $\mathrm{cap}(W_Q)$ is well-defined ([1, Proposition 3.10]).

Proposition 1.3. *Let $Q \in \mathcal{Q}$. For $W \in \mathrm{Ep}(oP)$ and $V \in \mathrm{Ind}(oP|Q)$, let $W \otimes V \cong \bigoplus_i X_i$ be a decomposition of $W \otimes V$ into indecomposable summands X_i. Then there is an X_i with vertex Q and the isomorphism class of such X_i is uniquely determined. In fact, X_i is then isomorphic to $(\mathrm{cap}(W_Q) \cdot X)^P$ for a Q-source X of V.*

Proof. Let X be a Q-source of V. Since $V \cong X^P$ by Green's theorem, we get $W \otimes V \cong (W_Q \otimes X)^P$. Since

$$W_Q \equiv m \times \mathrm{cap}(W_Q) \oplus O(\mathcal{H}(Q))$$

for a positive integer m, we have

$$W \otimes V \equiv m \times (\mathrm{cap}(W_Q) \cdot X)^P \oplus O(\mathcal{H}(Q)),$$

where $\mathrm{cap}(W_Q) \cdot X$ is defined by the action of $\mathrm{Ep}(oQ)$ on $\mathrm{Ind}(oQ|Q)$. Since $(\mathrm{cap}(W_Q) \cdot X)^P$ is indecomposable with vertex Q by Green's theorem, the result follows. □

Definition 1.4. Let $Q \in \mathcal{Q}$. For $W \in \mathrm{Ep}(oP)$ and $V \in \mathrm{Ind}(oP|Q)$, put

$$W \cdot V = (\mathrm{cap}(W_Q) \cdot X)^P,$$

where X is a Q-source of V.

This defines an action of $\mathrm{Ep}(oP)$ on $\mathrm{Ind}(oP|Q)$. Namely we have:

Theorem 1.5. *Let $W, W' \in \mathrm{Ep}(oP)$ and $V \in \mathrm{Ind}(oP|Q)$, where $Q \in \mathcal{Q}$. Then*
(i) $W \cdot (W' \cdot V) \cong (W \cdot W') \cdot V$, *and*
(ii) $1_P \cdot V \cong V$.

Proof. (i) Let X be a Q-source of V. We have

$$\begin{aligned} W \cdot (W' \cdot V) &\cong W \cdot (\mathrm{cap}(W'_Q) \cdot X)^P \\ &\cong \left\{\mathrm{cap}(W_Q) \cdot (\mathrm{cap}(W'_Q) \cdot X)\right\}^P \\ &\cong \left\{\left(\mathrm{cap}(W_Q) \cdot \mathrm{cap}(W'_Q)\right) \cdot X\right\}^P \\ &\cong \{\mathrm{cap}((W \cdot W')_Q) \cdot X\}^P, \end{aligned}$$

since the map sending W to $\mathrm{cap}(W_Q)$ is a group homomorphism from $\mathrm{Ep}(oP)$ to $\mathrm{Ep}(oQ)$ ([1, Proposition 3.15]). Hence $W \cdot (W' \cdot V) \cong (W \cdot W') \cdot V$.

(ii) This is obvious. □

2. Extensions of indecomposable modules and the Green correspondence

In this section, G is a group and N is a normal subgroup of G such that G/N is a p-group. Suppose we are given an indecomposable oG-module U such that U_N is indecomposable. Let $\mathrm{Ind}(o[G/N], U)$ be the set of (isomorphism classes of) all indecomposable $o[G/N]$-modules W such that $\mathrm{Inf}(W) \otimes U$ is indecomposable, where Inf denotes the inflation via the natural homomorphism $G \to G/N$. We have:

Lemma 2.1. *Every indecomposable $k[G/N]$-module belongs to $\mathrm{Ind}(k[G/N], U)$ and every indecomposable endo-permutation $R[G/N]$-module belongs to $\mathrm{Ind}(R[G/N], U)$.*

Proof. The first assertion is proved in [5, Theorem VII 9.12]. If W is an indecomposable endo-permutation $R[G/N]$-module, then W^* is indecomposable ([1, Corollary 6.3]). So the second follows from [8, Lemma 1.1(i)]. □

In the following we assume that our U satisfies:

For a vertex P of U, it holds that $G = PN, P \cap N = 1$ and a P-source of U is an endo-permutation oP-module.

In this situation we shall determine the vertices and sources of $\mathrm{Inf}(W) \otimes U$ for $W \in \mathrm{Ind}(oP, U)$. (Here P is naturally identified with G/N.) Let W_0 be a P-source of U. Since $N_G(P) = P \times C_N(P)$, the Green correspondent of U with respect to $(G, N_G(P), P)$ is of the form $W_0 \times Y$ for a projective indecomposable $oC_N(P)$-module Y. We begin with a special case.

Lemma 2.2. *For every $W \in \mathrm{Ind}(oP, U)$ with vertex P, $\mathrm{Inf}(W) \otimes U$ is the Green correspondent of $(W_0 \cdot W) \times Y$ with respect to $(G, N_G(P), P)$. Here $W_0 \cdot W$ is defined as in Section 1.*

Proof. Clearly $(W_0 \cdot W) \times Y \mid (W \otimes W_0) \times Y$ and $(W \otimes W_0) \times Y \mid (\mathrm{Inf}(W) \otimes U)_{N_G(P)}$. Here $\mathrm{Inf}(W) \otimes U$ is P-projective and $(W_0 \cdot W) \times Y$ has vertex P, so P is a vertex of $\mathrm{Inf}(W) \otimes U$ and the result follows. □

Theorem 2.3. *For every $W \in \mathrm{Ind}(oP, U)$, $\mathrm{Inf}(W) \otimes U$ has a vertex and a source in common with $W_0 \cdot W$.*

Proof. We claim that for any subgroup Q of P, U_{QN} has vertex Q and source $\mathrm{cap}((W_0)_Q)$. Indeed, we have that $\mathrm{cap}((W_0)_Q) \mid (U_{QN})_Q = U_Q$ and that $U_{QN} \mid ((W_0)^G)_{QN} \cong ((W_0)_Q)^{QN}$. So the claim follows.

Now let $W \in \mathrm{Ind}(oP, U)$. Let Q be a vertex of W and let X be a Q-source of W. By the above, the Green correspondent of U_{QN} with respect to $(QN, Q \times C_N(Q), Q)$ is of the form $\mathrm{cap}((W_0)_Q) \times Y'$ for a projective indecomposable $oC_N(Q)$-module Y'. Now $(\mathrm{Inf}(X) \otimes U_{QN})^G \cong \mathrm{Inf}(X^P) \otimes U \cong \mathrm{Inf}(W) \otimes U$. Hence $\mathrm{Inf}(X) \otimes U_{QN}$ is indecomposable and has a vertex and a source in common with $\mathrm{Inf}(W) \otimes U$. By Lemma 2.2, $\mathrm{Inf}(X) \otimes U_{QN}$ has vertex Q and the Green correspondent of it with respect to $(QN, Q \times C_N(Q), Q)$ is $(\mathrm{cap}((W_0)_Q) \cdot X) \times Y' \cong (W_0 \cdot W) \times Y'$. Thus the result follows. □

3. Sources of extensions of projective indecomposable modules

In this section, by (G, N, b), we mean the following data:

($\sharp$) G is a group, N is a normal subgroup of G, b is a G-invariant block of N such that $N = QC_N(Q)$ for a defect group Q of b.

Given such data, clearly Q is normal in G. Let V be the unique projective indecomposable oN-module in b. (In an earlier version of the present paper, the author treated the case when Q was central in N. The possibility of relaxing this condition to the one as above was pointed out by the referee.)

The following extends slightly a result of Dade, cf. [2, Theorem 13.13].

Theorem 3.1. *With the notation above, suppose that there is an extension U to G of V. Let P be a vertex of U and W a P-source of U. Then the following conditions are equivalent.*

(i) $P \cap N = 1$.

(ii) $P \cap Q = 1$.

(iii) $U^\wedge \otimes U$ *is a trivial source oG-module.*

(iv) U_P *is an endo-permutation oP-module.*

(v) W *is an* (*indecomposable*) *endo-permutation oP-module* (*with vertex* P).

(vi) $\mathrm{rank}_o W$ *is prime to p.*

Proof. (i) $\Leftrightarrow$ (ii): Let B be the block of G to which U belongs. Let D be a defect group of B with $P \leqq D$. Since b is a G-invariant block covered by B, we have $Q = D \cap N$. So $P \cap Q = P \cap D \cap N = P \cap N$. Thus the result follows.

(i) $\Rightarrow$ (v): Clearly U_{PN} has vertex P and P-source W, so we may assume $G = PN$. We show that we may assume Q is central in N. We first note that $C_N(Q)$ is a normal subgroup of G with $|G/C_N(Q)|$ a power of p. Let b_0 be the unique block of $C_N(Q)$ covered by b. Then b_0 is G-invariant. Clearly b_0 has defect group $Z(Q)$, which is central in $C_N(Q)$. Let L be an indecomposable $o[PC_N(Q)]$-module such that $L \mid W^{PC_N(Q)}$ and that $U|L^G$. Then, by Green's theorem, $U \cong L^G$, since $PC_N(Q)$ is a subnormal subgroup of G with $|G : PC_N(Q)|$ a power of p. So L has vertex P and W is a P-source of L. By Mackey decomposition, $V \cong U_N \cong (L^G)_N \cong \big(L_{C_N(Q)}\big)^N$, since $PC_N(Q) \cap N = (P \cap N)C_N(Q) = C_N(Q)$. This yields that $L_{C_N(Q)}$ is the unique projective indecomposable $oC_N(Q)$-module in b_0. Thus we may assume $G = PC_N(Q)$ and Q is central in N.

Consider the block ideal b as an oG-module via the conjugation action. We claim $\mathrm{Inv}_Q(U^\wedge \otimes U) \cong b$ as oG-modules. Indeed, let $\rho : b \to \mathrm{End}_o(U) \cong U^\wedge \otimes U$ be the representation of b on U. Clearly ρ induces an oG-homomorphism, say ρ', from b to $\mathrm{Inv}_Q(\mathrm{End}_o(U)) = \mathrm{End}_{oQ}(U)$. It suffices to show that ρ' is an isomorphism. Clearly ρ' is injective, since U_N is a unique projective indecomposable module in b. To prove that ρ' is surjective we first consider the case when $o = k$. Put $U_Q \cong n(kQ)$ for an integer n. Then $\dim_k \mathrm{End}_{kQ}(U) = n^2|Q|$. On the other hand, $\dim_k b = (\dim_k U)^2/|Q|$. So $\dim_k \mathrm{End}_{kQ}(U) = \dim_k b$. Hence ρ' is surjective. This shows that when $o = R$, $\mathrm{End}_{RQ}(U) = \mathrm{Im}\,\rho' + \pi\,\mathrm{End}_{RQ}(U)$, so $\mathrm{End}_{RQ}(U) = \mathrm{Im}\,\rho'$ by Nakayama's lemma. Thus the claim is proved.

Put $D = PQ$. For the P-source W of U, we claim $W^D | U_D$. Indeed, since $W | U_P$, there is an indecomposable summand X of U_D such that $W | X_P$. Then P is a vertex of X and W is a P-source of X. Hence $X \cong W^D$ by Green's theorem. So the claim follows. We also have $W^D \cong \mathrm{Inf}(W) \otimes (1_P)^D$, where $\mathrm{Inf}(W)$ is defined through the natural isomorphism $D/Q \cong P$. Hence

$$\mathrm{Inf}(W)^\wedge \otimes (1_P)^D \otimes \mathrm{Inf}(W) \otimes (1_P)^D \,|\, (U^\wedge \otimes U)_D.$$

By Mackey decomposition, $(1_P)^D | (1_P)^D \otimes (1_P)^D$, so

$$\mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W) \otimes (1_P)^D \,|\, (U^\wedge \otimes U)_D.$$

Since $\mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W)$ is trivial on Q,

$$\begin{aligned}\mathrm{Inv}_Q\big(\mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W) \otimes (1_P)^D\big) &= \mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W) \otimes \mathrm{Inv}_Q((1_P)^D)\\ &\cong \mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W) \otimes 1_D\\ &\cong \mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W),\end{aligned}$$

as oD-modules. So, by the above,

$$\mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W) \,|\, \mathrm{Inv}_Q((U^\wedge \otimes U)_D) \;\cong\; b_D,$$

as oD-modules. Since b is a direct summand of oN, we get that $\mathrm{Inf}(W)^\wedge \otimes \mathrm{Inf}(W)$ is a permutation oD-module by Green's theorem. Restriction to P shows that $W^\wedge \otimes W$ is a permutation oP-module, as required.

(i) and (v) $\Rightarrow$ (iii): Put $H = PN$. It suffices to show $U_H^\wedge \otimes U_H$ is a trivial source module. We have $U_H | W^H \cong \mathrm{Inf}(W) \otimes (1_P)^H$, where $\mathrm{Inf}(W)$ is defined through the natural isomorphism $H/N \cong P$. Thus

$$U_H^\wedge \otimes U_H \,|\, \mathrm{Inf}(W)^\wedge \otimes (1_P)^H \otimes \mathrm{Inf}(W) \otimes (1_P)^H,$$

which is a permutation module. So (iii) follows.

(iii) $\Rightarrow$ (iv): Since $(U^\wedge \otimes U)_P$ must be a permutation module by Green's theorem, the result follows.

(iv) $\Rightarrow$ (v): This is clear.

(v) $\Rightarrow$ (vi): This follows from [1, Lemma 6.4].

(vi) $\Rightarrow$ (i): As a direct summand of $U_{P\cap N}$, $W_{P\cap N}$ is projective, so $P\cap N = 1$. $\square$

The following follows from Green's theorem ([9, Problem 6(iii) on p.302]).

Lemma 3.2. *Let M be a normal subgroup of a group H. Let X be an oH-module such that X_M is indecomposable. Then $\mathrm{vx}(X)M$ contains a p-Sylow subgroup of H.*

Proposition 3.3. *Let U, P be as in Theorem* 3.1. *Let B be the block of G to which U belongs. Then PQ is a defect group of B.*

Proof. Let D be a defect group of B such that $D \geqq P$. By Lemma 3.2, $|G : PN|$ is prime to p, so $DN = PN$. Hence $D = P(D\cap N) = PQ$, since $D \cap N = Q$. Thus PQ is a defect group of B. $\square$

Hereafter we consider exclusively (G, N, b) satisfying ($\sharp$) for which G/N is a p-group. We are interested in the existence of extensions to G of the unique projective indecomposable oN-module in b which satisfy the condition (i) of Theorem 3.1. So in view of Proposition 3.3, we add an assumption on defect groups of the unique block of G covering b, and consider (G, N, B, b, P) such that:

($\sharp\sharp$) (G, N, b) satisfies the condition ($\sharp$) above, B is a unique block of G covering b, P is a p-subgroup ($\neq 1$) of G with $G = PN$ and $P \cap N = 1$, and PQ is a defect group of B.

Given such data, we let $D = PQ$. Let V be the unique projective indecomposable RN-module in b as before.

We begin by determining all indecomposable oG-modules with vertex P in B and similar modules in a (unique) block of G/Q dominated by B. First we prepare some group-theoretical facts.

Lemma 3.4. *With the notation above, we have*

$$O^p(N_G(D)) \leqq C_N(D) \leqq C_N(P) \leqq N_G(P) \leqq N_G(D).$$

Proof. It suffices to show $O^p(N_G(D)) \leqq C_N(D)$, the rest being obvious. Let x be a p'-element of $N_G(D)$. Then $[D, x] \leqq D\cap N = Q$ and $[Q, x] = 1$, since $x \in C_N(Q)$. As is well-known, this implies $[D, x] = 1$. So $x \in C_N(D)$, as required. $\square$

Let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)$. Let $\{b_i; 1 \leqq i \leqq t\}$ be the set of blocks of $N_G(P)$ such that $b_i^{\,G} = B$. Since $N_G(P) = P \times C_N(P)$, each b_i covers a unique block b'_i of $C_N(P)$. Let $\{\beta_m\}$ be the set of blocks of $C_N(D)$ covered by $\widetilde{B}$ (note that $C_N(D) \lhd N_G(D)$). For a block β, let $\mathrm{l}(\beta)$ be the number of irreducible Brauer characters in β. We have the following.

Lemma 3.5. (i) *Each* $b'_i, 1 \leqq i \leqq t$, *covers some* β_m.
(ii) $\mathrm{l}(\widetilde{B}) = 1$ *and* $\mathrm{l}(b'_i) = 1$ *for all* $i, 1 \leqq i \leqq t$.

Proof. (i) By the First Main Theorem ${b_i}^G = B$ if and only if ${b_i}^{N_G(D)} = \widetilde{B}$. On the other hand, since $N_G(D)/C_N(D)$ is a p-group by Lemma 3.4, $\mathrm{Br}_P(e_{\widetilde{B}}) = e_{\widetilde{B}} \in kC_N(D)$, where $e_{\widetilde{B}}$ is the block idempotent of $kN_G(D)$ corresponding to $\widetilde{B}$ and $\mathrm{Br}_P : ZkN_G(D) \to ZkN_G(P)$ is the Brauer homomorphism. Thus

$$\sum e_{b'_i} = \sum e_{b_i} = \sum e_{\beta_m}$$

and the result follows.

(ii) As is well-known, $D \cap C_N(D)$ is a defect group of β_m. Clearly $D \cap C_N(D)$ is central in $C_N(D)$. Thus $\mathrm{l}(\beta_m) = 1$. Then $\mathrm{l}(\widetilde{B}) = 1$, since $N_G(D)/C_N(D)$ is a p-group. Similarly $\mathrm{l}(b'_i) = 1$ by (i). □

Let $1 \leqq i \leqq t$. By Lemma 3.5, b'_i has a unique projective indecomposable $RC_N(P)$-module. This module is denoted by Y_i. For every $W \in \mathrm{Ind}(RP|P)$, let $U_i(W)$ be the Green correspondent of $W \times Y_i$ with respect to $(G, N_G(P), P)$ (note that $W \times Y_i$ is indecomposable and has vertex P [9, Problem 9 on p.302]). For every $W \in \mathrm{Ind}(kP|P)$, let $U'_i(W)$ be the Green correspondent of $W \times {Y_i}^*$ with respect to $(G, N_G(P), P)$. For these modules, we have the following.

Proposition 3.6. *The set* $\{U_i(W); 1 \leqq i \leqq t, W \in \mathrm{Ind}(RP|P)\}$ *is a set of representatives of the isomorphism classes of all indecomposable* RG*-modules with vertex* P *in* B. *Further, for* $1 \leqq i \leqq t$ *and* $W \in \mathrm{Ind}(RP|P)$, $U_i(W)_N$ *is a multiple of* V.

Proposition 3.6′. *The set* $\{U'_i(W); 1 \leqq i \leqq t, W \in \mathrm{Ind}(kP|P)\}$ *is a set of representatives of the isomorphism classes of all indecomposable* kG*-modules with vertex* P *in* B. *Further, for* $1 \leqq i \leqq t$ *and* $W \in \mathrm{Ind}(kP|P)$, $U'_i(W)_N$ *is a multiple of* V^*.

Proof. We prove only Proposition 3.6; the proof of Proposition 3.6′ is similar. Let L be an indecomposable RG-module with vertex P in B. Let W be a P-source of L. Then, since $N_G(P) = P \times C_N(P)$, we have $W^{N_G(P)} \cong W \times RC_N(P)$, so the Green correspondent of L with respect to $(G, N_G(P), P)$ is of the form $W \times Y$ for a projective indecomposable $RC_N(P)$-module Y. Then $Y \cong Y_i$ for some i by the Nagao-Green theorem [9, Theorem 5.3.12]. So $L \cong U_i(W)$. It also follows from the Nagao-Green theorem that modules of the form $U_i(W)$ lie in B. Thus the first assertion follows.

Since $P \cap N = 1$, $U_i(W)_N$ is projective. So the second follows. □

We also need to consider certain $o[G/Q]$-modules. Let us introduce some notation. Put $\overline{G} = G/Q$ and for every $H \leqq G$ put $\overline{H} = HQ/Q$. As is well-known, B contains a unique simple kG-module, say S, so there is a unique block $\overline{B}$ of $\overline{G}$ which is dominated by B. Let $\overline{S}$ be the unique simple $k\overline{G}$-module in $\overline{B}$. Clearly $\overline{B}$ has defect group $\overline{D} = \overline{P}$. Let $\bar{b}$ be the unique block of $\overline{N}$ dominated by b. Of course $\bar{b}$ has defect 0. Clearly $(\overline{G}, \overline{N}, \overline{B}, \bar{b}, \overline{P})$ satisfies ($\sharp\sharp$). Let B' be the Brauer correspondent of $\overline{B}$ in $N_{\overline{G}}(\overline{P})$. We have $N_{\overline{G}}(\overline{P}) = \overline{P} \times C_{\overline{N}}(\overline{P})$. So if b' is the block of $C_{\overline{N}}(\overline{P})$ covered by B', b' has defect 0 and b' contains a unique projective indecomposable $RC_{\overline{N}}(\overline{P})$-module Z (with Z^* simple). Via the natural isomorphism $\overline{P} \cong P$, $\mathrm{Ind}(o\overline{P})$ may be identified with $\mathrm{Ind}(oP)$ and we denote by $\overline{W} \in \mathrm{Ind}(o\overline{P})$ the module corresponding to $W \in \mathrm{Ind}(oP)$. Clearly $\overline{P}$ is a vertex of $\overline{S}$. Let $\overline{W}_0$ be a $\overline{P}$-source of $\overline{S}$. So $\overline{W}_0 = \overline{W_0}$ for $W_0 \in \mathrm{Ind}(kP|P)$. Since $\overline{S}_{\overline{N}}$ is the unique simple module in $\bar{b}$, W_0 is an endo-permutation module by Theorem 3.1. The following lemma characterizes W_0 inside G.

Lemma 3.7. *$\overline{W}_0$, or W_0, is unique up to isomorphism and W_0 is (up to isomorphism) a unique indecomposable summand of S_P with vertex P.*

Proof. Since $N_{\overline{G}}(\overline{P}) = \overline{P} \times C_{\overline{N}}(\overline{P})$, we see $\overline{W}_0$ is uniquely determined ([9, Theorem 3.3.6]). Of course, P is a vertex of W_0. If $W = \mathrm{Inf}(\overline{W}_0)$, where Inf is taken via the natural homomorphism $D \to D/Q = \overline{P}$, then W is a D-source of S and $W_P \cong W_0$. So $W_0 | S_P$. Conversely, let L be an indecomposable summand of S_P with vertex P. Then $\overline{L} \,|\, \overline{S}_{\overline{P}}$ and $\overline{P}$ is a vertex of $\overline{L}$. So $\overline{L}$ is a $\overline{P}$-source of $\overline{S}$ and we get $\overline{L} \cong \overline{W}_0$ by the above. Thus $L \cong W_0$. This completes the proof. □

For every $W \in \mathrm{Ind}(RP|P)$, let $\overline{U}(\overline{W})$ be the Green correspondent of $\overline{W} \times Z$ with respect to $(\overline{G}, N_{\overline{G}}(\overline{P}), \overline{P})$. For every $W \in \mathrm{Ind}(kP|P)$, let $\overline{U}'(\overline{W})$ be the Green correspondent of $\overline{W} \times Z^*$ with respect to $(\overline{G}, N_{\overline{G}}(\overline{P}), \overline{P})$.

Since $(\overline{G}, \overline{N}, \overline{B}, \bar{b}, \overline{P})$ satisfies ($\sharp\sharp$), applying Propositions 3.6, 3.6′, and the First Main Theorem, we get the following.

Proposition 3.8. *The set $\{\overline{U}(\overline{W}); W \in \mathrm{Ind}(RP|P)\}$ is a set of representatives of the isomorphism classes of all indecomposable $R\overline{G}$-modules with vertex $\overline{P}$ in $\overline{B}$.*

Proposition 3.8′. *The set $\{\overline{U}'(\overline{W}); W \in \mathrm{Ind}(kP|P)\}$ is a set of representatives of the isomorphism classes of all indecomposable $k\overline{G}$-modules with vertex $\overline{P}$ in $\overline{B}$.*

The indecomposable modules under investigation are closely related to each other. To see this, we need some general facts. In the following Lemmas 3.9, 3.10 and 3.11, let M be a normal subgroup of G. Let $\theta : G \to G/M$ be the natural homomorphism. We define a functor θ^* as follows: For a subgroup H of G with $H \geqq M$ and an oH-module U, we set $\theta^*(U) = U/UI(oM)$. So $\theta^*(U)$ is an $o[H/M]$-module. (We note that $\theta^*(U)$ may be 0 or may not be o-free in general.)

Lemma 3.9. *We have the following isomorphisms.*

(i) $\theta^*(U \oplus V) \cong \theta^*(U) \oplus \theta^*(V)$ *for* oG*-modules* U *and* V.

(ii) $\theta^*(U \otimes \mathrm{Inf}(W)) \cong \theta^*(U) \otimes W$ *for an* oG*-module* U *and an* $o[G/M]$*-module* W.

(iii) $\theta^*(U/\pi U) \cong \theta^*(U)/\pi\theta^*(U)$ *for an* RG*-module* U.

(iv) $\{\theta^*(U)\}_{H/M} \cong \theta^*(U_H)$ *for an* oG*-module* U*, where* $G \geqq H \geqq M$.

(v) $\{\theta^*(U)\}^{G/M} \cong \theta^*(U^G)$ *for an* oH*-module* U*, where* $G \geqq H \geqq M$.

In particular, if U *is an* H*-projective* oG*-module for a subgroup* H *of* G*, then* $\theta^*(U)$ *is* HM/M*-projective.*

Proof. For the proof, we use a well-known isomorphism: $\theta^*(U) \cong U \otimes_{oH} o[H/M]$ for an oH-module U, cf. [9, Theorem 1.9.17(i)]. We extend θ to an algebra homomorphism from oG onto $o[G/M]$ and denote the image of $\alpha \in oG$ by $\overline{\alpha}$.

(i) This is obvious.

(ii) Define $f : (U \otimes \mathrm{Inf}(W)) \otimes_{oG} o[G/M] \longrightarrow (U \otimes_{oG} o[G/M]) \otimes W$ by

$$f((u \otimes w) \otimes \overline{\alpha}) = ((u \otimes \overline{1} \otimes w)\overline{\alpha}, \quad u \in U, w \in W, \alpha \in oG.$$

Then f is an isomorphism; the inverse of f is given by

$$(u \otimes \overline{\alpha}) \otimes w \longrightarrow (u\alpha \otimes w) \otimes \overline{1}, \quad u \in U, w \in W, \alpha \in oG.$$

(iii) Define $f : (U \otimes_R k) \otimes_{kG} k[G/M] \to (U \otimes_{RG} R[G/M]) \otimes_R k$ by

$$f((u \otimes \lambda) \otimes \overline{\varphi(\alpha)}) = u \otimes \overline{\alpha} \otimes \lambda, \quad u \in U, \lambda \in k, \alpha \in RG,$$

where $\varphi : RG \to kG$ is the natural map. Then f is an isomorphism.

(iv) Define $f : U \otimes_{oG} o[G/M] \to U \otimes_{oH} o[H/M]$ by

$$f(u \otimes \overline{\alpha}) = u\alpha \otimes \overline{1}, \qquad u \in U, \alpha \in oG.$$

Then f is an isomorphism; the inverse of f is given by

$$u \otimes \overline{\alpha} \longrightarrow u \otimes \overline{\alpha}, \qquad u \in U, \alpha \in oH.$$

(v) Define $f : (U \otimes_{oH} o[H/M]) \otimes_{o[H/M]} o[G/M] \to (U \otimes_{oH} oG) \otimes_{oG} o[G/M]$ by

$$f((u \otimes \overline{\alpha}) \otimes \overline{\beta}) = (u \otimes 1) \otimes \overline{\alpha}\overline{\beta}, \quad u \in U, \alpha \in oH, \beta \in oG.$$

Then f is an isomorphism; the inverse of f is given by

$$(u \otimes \alpha) \otimes \overline{\beta} \longrightarrow (u \otimes \overline{1}) \otimes \overline{\alpha}\overline{\beta}, \quad u \in U, \alpha, \beta \in oG.$$

The last assertion follows from (i) and (v). This completes the proof. $\square$

Lemma 3.10. *Let* U *be a projective indecomposable* oG*-module and let* T *be a simple* kG*-module corresponding to* U*. Then* $\theta^*(U)$ *is isomorphic to the projective indecomposable* $o[G/M]$*-module corresponding to* T*, if* $M \leqq \mathrm{Ker}\ T$*; a zero module, otherwise.*

Proof. We first consider the case when $o = k$. If $\theta^*(U) \neq 0$, then there is a surjection $\theta^*(U) \to T$, since the head of U is simple and isomorphic to T. Thus $M \leqq \mathrm{Ker}\ T$. Conversely, if $M \leqq \mathrm{Ker}\ T$, then the required conclusion follows by Landrock [6, II 11.15].

Now assume $o = R$. Since $\theta^*(U) \mid \theta^*(RG) \cong R[G/M]$ by Lemma 3.9, $\theta^*(U)$ is projective or 0. Then, since $\theta^*(U)/\pi\theta^*(U) \cong \theta^*(U/\pi U)$ by Lemma 3.9, the conclusion follows from the first paragraph. □

Lemma 3.11. *Let H be a subgroup of G with $H \geqq M$. Put $\overline{G} = G/M$ and $\overline{H} = H/M$. Let B (resp. b) be a block of G (resp. H). Assume the following conditions: B (resp. b) dominates a unique block $\overline{B}$ (resp. $\overline{b}$) of $\overline{G}$ (resp. $\overline{H}$); b^G is defined and equals B; $\overline{b}^{\overline{G}}$ is defined. Then $\overline{b}^{\overline{G}} = \overline{B}$.*

Proof ([3, Proposition 1.2.16]). Let $f : ZkG \to Zk\overline{G}$ be the algebra homomorphism induced by the natural homomorphism $G \to \overline{G}$. Define $g : ZkH \to Zk\overline{H}$ similarly. Define $s_H : ZkG \to ZkH$ by $s_H(\widehat{K}) = \sum_{x \in K \cap H} x$, where K are conjugacy classes of G. Define $s_{\overline{H}} : Zk\overline{G} \to Zk\overline{H}$ similarly. Then $s_{\overline{H}} \circ f = g \circ s_H$. From this and our assumption that $b^G = B$, we see that B dominates $\overline{b}^{\overline{G}}$. Thus $\overline{b}^{\overline{G}} = \overline{B}$. □

Now we return to our original situation. Let $\widetilde{b}$ be a block of $N_N(D)$ ($= N_G(D) \cap N$) covered by $\widetilde{B}$. It is easy to see that $N_N(D)$ is the inverse image of $C_{\overline{N}}(\overline{P})$ in G.

Lemma 3.12. (i) *B' is a unique block of $\overline{N_G(D)}$ which is dominated by $\widetilde{B}$.*

(ii) *$\widetilde{b}$ dominates b' and $Y_i^{N_N(D)}$ is a projective indecomposable $RN_N(D)$-module in $\widetilde{b}$.*

Proof. (i) By Lemma 3.5, $\widetilde{B}$ contains a unique simple $kN_G(D)$-module, so $\widetilde{B}$ dominates a unique block B'' of $\overline{N_G(D)} = N_{\overline{G}}(\overline{P})$. Clearly B'' has defect group $\overline{D} = \overline{P}$, and $B''^{\overline{G}} = \overline{B}$ by Lemma 3.11. So $B'' = B'$ by the First Main Theorem. Thus the result follows.

(ii) Since $N_G(D) = DN_N(D)$, we see that $\widetilde{b}$ is a unique block of $N_N(D)$ which is covered by $\widetilde{B}$. This yields the first assertion.

By Lemma 3.4 and Green's theorem $Y_i^{N_N(D)}$ and $Y_i^{N_G(D)}$ are projective indecomposable. By Lemma 3.5, b_i' covers some β_m. Then, by Mackey decomposition, $(Y_i^{N_G(D)})_{C_N(D)}$ has a summand in β_m. Since $N_G(D)/C_N(D)$ is a p-group, it follows that $Y_i^{N_G(D)}$ belongs to $\widetilde{B}$. Then we see that $Y_i^{N_N(D)}$ belongs to $\widetilde{b}$ by the first paragraph. This completes the proof. □

In the rest of this section, let $\theta : G \to G/Q$ be the natural homomorphism. Let θ^* be the functor defined as above. Now we prove the following.

Theorem 3.13. *For every $W \in \mathrm{Ind}(RP|P)$ and $i, 1 \leqq i \leqq t$, we have $\theta^*(U_i(W)) \cong \overline{U}(\overline{W}) \oplus M_i(W)$ for an $\mathcal{H}(\overline{P})$-projective $R\overline{G}$-module $M_i(W)$.*

Theorem 3.13′. *For every* $W \in \mathrm{Ind}(kP|P)$ *and* $i, 1 \leqq i \leqq t$, *we have* $\theta^*(U_i'(W)) \cong \overline{U}'(\overline{W}) \oplus M_i'(W)$ *for an* $\mathcal{H}(\overline{P})$*-projective* $k\overline{G}$*-module* $M_i'(W)$.

Proof. Here we give only the proof of Theorem 3.13; Theorem 3.13′ is proved in a similar way. Put $Y = Y_i$ and $\widetilde{Y} = 1_P \times Y$. First we claim that P acts trivially on $\theta^*(\widetilde{Y}^{N_G(D)})$. Let $u \in P, x \in N_G(D)$ and $y \in \widetilde{Y}$. Put $xux^{-1} = vz$ with $v \in P, z \in Q$. Then

$$(y \otimes x)(u-1) = y \otimes vzx - y \otimes x = (y \otimes x)(x^{-1}zx - 1),$$

since P acts trivially on $\widetilde{Y}$. Thus the claim follows .

Now

$$\begin{aligned} \theta^*\big(\widetilde{Y}^{N_G(D)}\big)_{\overline{N_N(D)}} &\cong \theta^*\big((\widetilde{Y}^{N_G(D)})_{N_N(D)}\big) && \text{(by Lemma 3.9)} \\ &\cong \theta^*\big(Y^{N_N(D)}\big) && \text{(by Mackey decomposition).} \end{aligned}$$

Thus

$$\theta^*\big(\widetilde{Y}^{N_G(D)}\big) \cong 1_{\overline{P}} \times \theta^*\big(Y^{N_N(D)}\big).$$

By Lemmas 3.10 and 3.12(ii), $\theta^*\big(Y^{N_N(D)}\big) \cong Z$. Hence

$$\theta^*\big(\widetilde{Y}^{N_G(D)}\big) \cong 1_{\overline{P}} \times Z. \tag{1}$$

Put $\widetilde{W} = \overline{W} \times 1_{C_{\overline{N}}(\overline{P})}$. Let $\mathrm{Inf}(\widetilde{W})$ be the inflation of $\widetilde{W}$ via the natural homomorphism $N_G(D) \to N_{\overline{G}}(\overline{P})$. Then

$$\begin{aligned} \theta^*\big((W \times Y)^{N_G(D)}\big) &\cong \theta^*\Big(\big(\mathrm{Inf}(\widetilde{W})_{N_G(P)} \otimes \widetilde{Y}\big)^{N_G(D)}\Big) \\ &\cong \theta^*\big(\mathrm{Inf}(\widetilde{W}) \otimes \widetilde{Y}^{N_G(D)}\big) \\ &\cong \widetilde{W} \otimes \theta^*(\widetilde{Y}^{N_G(D)}) && \text{(by Lemma 3.9)} \\ &\cong \overline{W} \times Z && \text{(by (1)).} \end{aligned} \tag{2}$$

Thus

$$\begin{aligned} \theta^*((W \times Y)^G) &\cong \theta^*\big(\{(W \times Y)^{N_G(D)}\}^G\big) \\ &\cong \big\{\theta^*\big((W \times Y)^{N_G(D)}\big)\big\}^{\overline{G}} && \text{(by Lemma 3.9)} \\ &\cong (\overline{W} \times Z)^{\overline{G}} && \text{(by (2))} \\ &\cong \overline{U}(\overline{W}) \oplus M, \end{aligned} \tag{3}$$

where M is an $\mathcal{H}(\overline{P})$-projective module.

On the other hand, $(W \times Y)^G \cong U_i(W) \oplus L$, where L is an $\mathcal{H}(P)$-projective module. So

$$\theta^*((W \times Y)^G) \cong \theta^*(U_i(W)) \oplus \theta^*(L) \tag{4}$$

and, by Lemma 3.9, $\theta^*(L)$ is $\mathcal{H}(\overline{P})$-projective. Comparison of (3) and (4) yields the result. This completes the proof. $\square$

Let $\overline{V}$ be the unique projective indecomposable $R\overline{N}$-module in $\overline{b}$. We determine the extensions to $\overline{G}$ of $\overline{V}$. Let V be the unique projective indecomposable RN-module in b, as before. Let $\overline{T}$ be the unique simple $k\overline{N}$-module in $\overline{b}$. Clearly $\overline{S}$ is a unique extension of $\overline{T}$ to $\overline{G}$.

Lemma 3.14. *We have the following isomorphism.*
(i) $\theta^*(V) \cong \overline{V}$.
(ii) $\theta^*(V/\pi V) \cong \overline{T}$.

Proof. (i) Since the simple kN-module corresponding to V is trivial on Q, the assertion follows from Lemma 3.10.
(ii) By Lemma 3.9 and (i), $\theta^*(V/\pi V) \cong \theta^*(V)/\pi\theta^*(V) \cong \overline{V}/\pi\overline{V} \cong \overline{T}$. $\square$

Lemma 3.15. *An $R\overline{G}$-module L is an extension of $\overline{V}$ if and only if L is a lift of $\overline{S}$.*

Proof. Let L be an extension of $\overline{V}$. Then $L^*_{\overline{N}} \cong \overline{T}$, so $L^* \cong \overline{S}$. Conversely if L is a lift of $\overline{S}$, then $L^*_{\overline{N}} \cong \overline{T}$. Hence $L_{\overline{N}} \cong \overline{V}$. $\square$

Let $\mathrm{Lf}(W_0)$ be a set of representatives of the isomorphism classes of all indecomposable endo-permutation RP-modules W with vertex P such that $W^* \cong W_0$.

Theorem 3.16. *The set $\{\overline{U}(\overline{W}); W \in \mathrm{Lf}(W_0)\}$ is a set of representatives of the isomorphism classes of all extensions of $\overline{V}$ to $\overline{G}$. In particular, the number of isomorphism classes of such extensions equals $|\mathrm{Lf}(W_0)| = |P/P'| > 0$, where P' is the commutator subgroup of P.*

Proof. Let L be an extension of $\overline{V}$ to $\overline{G}$. By Lemma 3.15, L is a lift of $\overline{S}$. So, since L lies in $\overline{B}$, we see that L has vertex $\overline{P}$. If $\overline{W}$ is a $\overline{P}$-source of L, then $W \in \mathrm{Ep}(RP)$ by Theorem 3.1. Clearly we have $\overline{W}^* \mid L^*_{\overline{P}} \cong \overline{S}_{\overline{P}}$. Since $\overline{W}^*$ is indecomposable with vertex $\overline{P}$ (cf. [1, Corollary 6.3] and [2, Proposition 12.1]), we get $\overline{W}^* \cong \overline{W}_0$ by Lemma 3.7. By Proposition 3.8, we get $L \cong \overline{U}(\overline{W})$.

Now let L_1 be an R-form of an irreducible character of height 0 in $\overline{B}$. Then it is easy to see that L_1 is an extension of $\overline{V}$ to $\overline{G}$. By the above, $L_1 \cong \overline{U}(\overline{W_1})$ with $W_1 \in \mathrm{Lf}(W_0)$. Let $W \in \mathrm{Lf}(W_0)$. By Lemma 2.2, $\overline{U}(\overline{W}) \cong \mathrm{Inf}(\overline{W_1}^{\wedge} \cdot \overline{W}) \otimes L_1$. Since $\overline{W_1}^{\wedge} \cdot \overline{W}$ has R-rank 1 (cf. [2, Proposition 12.1]), $\overline{U}(\overline{W})$ also is an extension of $\overline{V}$. The equality $|\mathrm{Lf}(W_0)| = |P/P'|$ also follows from [2, Proposition 12.1], since K contains the $|G|$-th roots of unity. This completes the proof. $\square$

Now we obtain a necessary condition for an indecomposable RG- (resp. kG-) module in B with vertex P to be an extension of V (resp. V^*).

Corollary 3.17. *Let $W \in \mathrm{Ind}(RP|P)$ and $1 \leqq i \leqq t$. If $U_i(W)$ is an extension of V, then $W \in \mathrm{Lf}(W_0)$. Furthermore, for $W \in \mathrm{Lf}(W_0)$, the following are equivalent.*
(i) *$U_i(W)$ is an extension of V.*
(ii) *$\theta^*(U_i(W)) \cong \overline{U}(\overline{W})$.*

Proof. Assume that $U_i(W)$ is an extension of V. Then we have

$$\theta^*(U_i(W))_{\overline{N}} \cong \theta^*(U_i(W)_N) \cong \theta^*(V) \cong \overline{V}.$$

In particular, $\theta^*(U_i(W))$ is indecomposable. Hence, by Theorem 3.13, $\theta^*(U_i(W)) \cong \overline{U}(\overline{W})$. So $\overline{U}(\overline{W})$ is an extension of $\overline{V}$ and $W \in \mathrm{Lf}(W_0)$ by Theorem 3.16.

Let $W \in \mathrm{Lf}(W_0)$. (i) $\Rightarrow$ (ii): This follows from the first paragraph.

(ii) $\Rightarrow$ (i): Put $U_i(W)_N \cong nV$ for an integer n. Then

$$\theta^*(U_i(W))_{\overline{N}} \cong n\overline{V}.$$

Since $\overline{U}(\overline{W})$ is an extension of $\overline{V}$ by Theorem 3.16, we get $n = 1$. This completes the proof. $\square$

Corollary 3.17$'$. *Let $W \in \mathrm{Ind}(kP|P)$ and $1 \leqq i \leqq t$. If $U'_i(W)$ is an extension of V^*, then $W \cong W_0$. Furthermore, the following are equivalent.*
(i) *$U'_i(W_0)$ is an extension of V^*.*
(ii) *$\theta^*(U'_i(W_0)) \cong \overline{S}$.*

Proof. Assume that $U'_i(W)$ is an extension of V^*. Then we have

$$\theta^*(U'_i(W))_{\overline{N}} \cong \theta^*(U'_i(W)_N) \cong \theta^*(V^*) \cong \overline{T}.$$

Thus $\theta^*(U'_i(W)) \cong \overline{S}$. Hence, by Theorem 3.13$'$, $\theta^*(U'_i(W)) \cong \overline{U}'(\overline{W})$. So $\overline{U}'(\overline{W}) \cong \overline{S}$ and $W \cong W_0$.

(i) $\Rightarrow$ (ii): This follows from the first paragraph.

(ii) $\Rightarrow$ (i): Put $U'_i(W_0)_N \cong nV^*$ for an integer n. Then

$$\theta^*(U'_i(W))_{\overline{N}} \cong n\overline{T}.$$

So $n = 1$ and (i) follows. $\square$

As the following corollaries show, the existence of an extension of V (resp. V^*) to G with vertex P is equivalent to a statement neater than Theorem 3.13 (resp. Theorem 3.13$'$).

Corollary 3.18. *Let $1 \leqq i \leqq t$. The following are equivalent.*
(i) *$U_i(W)$ is an extension of V for every $W \in \mathrm{Lf}(W_0)$.*
(ii) *$U_i(W)$ is an extension of V for some $W \in \mathrm{Lf}(W_0)$.*
(iii) *$\theta^*(U_i(W))$ is indecomposable for some $W \in \mathrm{Lf}(W_0)$.*
(iv) *$\theta^*(U_i(W)) \cong \overline{U}(\overline{W})$ for some $W \in \mathrm{Ep}(RP)$.*

Proof. (i) $\Rightarrow$ (ii): This is trivial.

(ii) $\Rightarrow$ (iii): This follows from Corollary 3.17.

(iii) $\Rightarrow$ (iv): Assume that $\theta^*(U_i(W_1))$ is indecomposable for $W_1 \in \mathrm{Lf}(W_0)$. By Corollary 3.17 (and Theorem 3.13), $U_i(W_1)$ is an extension of V. Let $W \in \mathrm{Ep}(RP)$. By Lemma 2.2,

$$U_i(W) \cong \mathrm{Inf}({W_1}^\wedge \cdot W) \otimes U_i(W_1).$$

So

$$\begin{aligned}\theta^*(U_i(W)) &\cong \mathrm{Inf}({W_1}^\wedge \cdot W) \otimes \theta^*(U_i(W_1)) && \text{(by Lemma 3.9)}\\ &\cong \mathrm{Inf}({W_1}^\wedge \cdot W) \otimes \overline{U}(\overline{W}_1) && \text{(by Theorem 3.13),}\end{aligned}$$

where Inf is taken via the natural isomorphism $\overline{G}/\overline{N} \cong \overline{P} \cong P$. Since $\overline{U}(\overline{W}_1)$ is an extension of $\overline{V}$ by Theorem 3.16, $\theta^*(U_i(W))$ is indecomposable by Lemma 2.1. Thus the result follows from Theorem 3.13.

(iv) $\Rightarrow$ (i): This follows from Corollary 3.17. □

Corollary 3.18′. *Let $1 \leqq i \leqq t$. The following are equivalent.*
(i) $U_i'(W_0)$ *is an extension of* V^*.
(ii) $\theta^*(U_i'(W)) \cong \overline{U}'(\overline{W})$ *for every* $W \in \mathrm{Ind}(kP|P)$.

Proof. (i) $\Rightarrow$ (ii): Let $W \in \mathrm{Ind}(kP|P)$. We have

$$U_i'(W) \cong \mathrm{Inf}({W_0}^\wedge \cdot W) \otimes U_i'(W_0)$$

by Lemma 2.2. So

$$\theta^*(U_i'(W)) \cong \mathrm{Inf}({W_0}^\wedge \cdot W) \otimes \theta^*(U_i'(W_0)) \qquad \text{(by Lemma 3.9).}$$

By Corollary 3.17′, $\theta^*(U_i'(W_0))_{\overline{N}} \cong \overline{T}$, which is simple. Thus we get the result by Lemma 2.1 and Theorem 3.13′.

(ii) $\Rightarrow$ (i): We have $\theta^*(U_i'(W_0)) \cong \overline{U}'(\overline{W}_0) \cong \overline{S}$. So we get the result by Corollary 3.17′. □

4. The case where P is cyclic

In this section, by (G, N, B, b, P) we mean the following data:

($\sharp\sharp\sharp$) (G, N, B, b, P) satisfies the condition ($\sharp\sharp$) in Section 3 and P is cyclic.

We retain the notation introduced in Section 3. We shall prove the following.

Theorem 4.1. *For every i $(1 \leqq i \leqq t)$ and every $W \in \mathrm{Lf}(W_0), U_i(W)$ is an extension of V.*

We postpone the proof for a while and give consequences of Theorem 4.1.

Theorem 4.1′. *For every i $(1 \leqq i \leqq t)$, $U_i'(W_0)$ is an extension of V^*.*

Proof. Let $W \in \mathrm{Lf}(W_0)$. By definition,

$$(W \times Y_i)^G \cong U_i(W) \oplus L,$$

where L is an $\mathcal{H}(P)$-projective module. Reducing modulo p,

$$(W_0 \times Y_i{}^*)^G \cong U_i(W)^* \oplus L^*. \tag{1}$$

On the other hand, by definition,

$$(W_0 \times Y_i{}^*)^G \cong U_i'(W_0) \oplus M, \tag{2}$$

where M is an $\mathcal{H}(P)$-projective module. By Theorem 4.1, $U_i(W)$ is an extension of V. So $U_i(W)^*_N \cong V^*$, and $U_i(W)^*$ is indecomposble. Clearly there exists a vertex A of $U_i(W)^*$ with $A \leqq P$. By Lemma 3.2, $G = AN$. Thus $A = P$. So by (1) and (2), we get $U_i(W)^* \cong U_i'(W_0)$. Therefore $U_i'(W_0)$ is an extension of V^*. □

From Theorem 4.1, Proposition 3.6 and Corollary 3.17, we obtain the following.

Corollary 4.2. *The set $\{U_i(W); 1 \leqq i \leqq t, W \in \mathrm{Lf}(W_0)\}$ is a set of representatives of the isomorphism classes of all extensions of V to G with vertex P.*

Also, from Theorem 4.1′, Proposition 3.6′ and Corollary 3.17′, we obtain the following.

Corollary 4.2′. *The set $\{U_i'(W_0); 1 \leqq i \leqq t\}$ is a set of representatives of the isomorphism classes of all extensions of V^* to G with vertex P.*

Proof of Theorem 4.1. First we show that it suffices to consider the case when Q is central in N.

(*Reduction*) Assume that the theorem is true under the assumption that Q is central in N. Then it is true in general.

To see this, put $G_0 = PC_N(Q)$ and $N_0 = C_N(Q)$. Clearly N_0 is a normal subgroup of G with $|G/N_0|$ a power of p. Let b_0 be the unique block of N_0 covered by b. Then, since b is G-invariant, b_0 is G-invariant. Clearly b_0 has defect group $Z(Q)$, which is central in N_0. Let B_0 be the unique block of G_0 covering b_0. Then, as is well-known, (for examaple, cf. [7, Lemma 4.13]), $PZ(Q)$ is a defect group of B_0. Thus (G_0, N_0, B_0, b_0, P) satisfies (♯♯♯).

Let S_0 be the unique simple kG_0-module in B_0. We claim that S_0 is a summand of S_{G_0}. Let U (resp. U_0) be the unique projective indecomposable kG- (resp. kG_0-) module in B (resp. B_0). By Green's theorem, $U_0{}^G$ is projective indecomposable. Now $G = G_0 N$ and $G_0 \cap N = PC_N(Q) \cap N = (P \cap N)C_N(Q) = N_0$. So, by Mackey decomposition, $U_0{}^G$ lies in a block of G covering b, namely B. Thus $U_0{}^G \cong U$. So, by Nakayama relation S_0 is a constituent of S_{G_0}. Then a repeated use of Clifford's theorem proves the claim. Let W_1 be an indecomposable summand of $(S_0)_P$ with vertex P, cf. Lemma 3.7. By the claim, we see $W_1 \cong W_0$ by Lemma 3.7.

Let $1 \leqq i \leqq t$ and $W \in \mathrm{Lf}(W_0)$. Let L be an indecomposable RG_0-module such that $L|W^{G_0}$ and that $U_i(W)|L^G$. Then, by Green's theorem, $U_i(W) \cong L^G$. Thus L has vertex P and W is a P-source of L. Since $L^G \cong U_i(W)$ lies in B, we see that L lies in B_0 by Mackey decomposition. Thus by Proposition 3.6 (applied to (G_0, N_0, B_0, b_0, P)), there exist a block β of $N_{G_0}(P)$ and a projective indecomposable $RC_{N_0}(P)$-module Y in the block of $C_{N_0}(P)$ covered by β such that $\beta^{G_0} = B_0$ and that L is the Green correspondent of $W \times Y$ with respect to $(G_0, N_{G_0}(P), P)$. Hence, by our assumption, $L_{N_0} \cong V_0$, where V_0 is a unique projective indecomposable RN_0-module in b_0. Then $U_i(W)_N \cong (L^G)_N \cong V_0{}^N$ by Mackey decomposition and $V_0{}^N \cong V$ by Green's theorem. Thus $U_i(W)$ is an extension of V, as required.

Hereafter we consider only the case when Q is central in N. We argue by induction on $|G|$. Let $|P| = p^n, n \geqq 1$. Fix i and put $B_0 = b_i, V_0 = Y_i$. Let $W \in \mathrm{Lf}(W_0)$. Put $M = W \times Y_i$ and $U = U_i(W)$.

Let P_1 be the unique subgroup of P of order p. We distinguish two cases:

(Case 1) $N_G(P_1 Q) = G$, (Case 2) $N_G(P_1 Q) \neq G$.

(Case 1) Put $G_1 = N_G(P_1)$ and let $B_1 = B_0{}^{G_1}$. Let D_1 be a defect group of B_1 such that $P \leqq D_1$. Put $N_1 = G_1 \cap N$ and let b_1 be a block of N_1 covered by B_1. We have :

(1.a) (G_1, N_1, B_1, b_1, P) satisfies the same assumption as (G, N, B, b, P).

Clearly $G_1 = PN_1$ with $P \cap N_1 = 1$. We have $D_1 = P(D_1 \cap N_1)$. Since $B_1{}^G = B$, we get $D_1 \cap N_1 \leqq_G D \cap N = Q$. Since Q is normal in G, we get $D_1 \cap N_1 = D_1 \cap Q$. So $D_1 \cap N_1$ is central in N_1 and, since P mormalizes $D_1 \cap N_1$, $D_1 \cap N_1$ is normal in G_1. Thus $D_1 \cap N_1$ is a defect group of b_1. Then, since $|D_1/D_1 \cap N_1| = |G_1/N_1|$, b_1 is G_1-invariant. Thus (1.a) is proved.

We prepare a group-theoretical fact, which enables us to reduce the proof to the case of (G_1, N_1, B_1, b_1, P) by arguments similar to those used in (*Reduction*).

(1.b) $O^p(G) \leqq N_1$.

Let x be a p'-element of G. Since $P_1 Q$ is normal in G by assumption and $x \in N, [x, P_1 Q] \leqq P_1 Q \cap N = Q$. Since Q is central in N, $[x, Q] = 1$. As is well-known, this implies $[x, P_1 Q] = 1$. Thus $x \in C_G(P_1) \cap N \leqq N_1$ and (1.b) follows.

Let X be the Green correspondent of M with respect to $(G_1, N_G(P) = N_{G_1}(P), P)$. So X belongs to B_1 by the Nagao-Green theorem [9, Theorem 5.3.12]. Let V_1 be the unique projective indecomposable RN_1-module in b_1 and let S_1 be the unique simple kG_1-module in B_1. Let W_1 be an indecomposable summand of $(S_1)_P$ with vertex P.

(1.c) (1) $X^G \cong U$.
(2) S_1 is a direct summand of S_{G_1}.
(3) If $X_{N_1} \cong V_1$, then $U_N \cong V$.
(4) $W_1 \cong W_0$.

X^G is indecomposable by (1.b) and Green's theorem. Then P is a common vertex of X and X^G, so (1) follows. By (1)

$$\dim_k \operatorname{Hom}_G(U/\pi U, S) = \dim_k \operatorname{Hom}_{G_1}(X/\pi X, S_{G_1}).$$

On the other hand, by (1.b) and a repeated use of of Clifford's theorem, we get $S_{G_1} \cong mS_1 \oplus L$ for some integer m and a semi-simple module L not involving S_1. Since $\operatorname{Hom}_G(U/\pi U, S) \neq 0$, we get $m \neq 0$ and (2) follows. Then S_{N_1} involves $(S_1)_{N_1}$ and hence $V \mid {V_1}^N$ by Nakayama relation. By Green's theorem, we get ${V_1}^N \cong V$. Then (3) follows from (1) and Mackey decomposition. (4) follows from (2), cf. the proof of (*Reduction*).

By (1.c)(3), it suffices to show that X is an extension of V_1.

Clearly S_1 is trivial on P_1. Hence by (1.c)(4), W_0 is trivial on P_1. So, if $n = 1$, then W_0 is the trivial module and W has R-rank one. Thus $X = M$ is an extension of $V_1 = V_0$. Thus we may assume $n \geqq 2$. In the following put $\overline{H} = HP_1/P_1$ for any subgroup H of G_1. We write $\overline{G}_1$ and $\overline{N}_1$ for $\overline{G_1}$ and $\overline{N_1}$, respectively. Let $\overline{B}_0$ be the unique block of $\overline{N_G(P)}$ dominated by B_0. Let $\overline{B}_1$ be the unique block of $\overline{G}_1$ dominated by B_1. Let $\overline{b}_1$ be the block of $\overline{N}_1$ identified with b_1 via the natural isomorphism $\overline{N}_1 \cong N_1$. Let $\overline{V}_1$ be the module in $\overline{b}_1$ identified with V_1 via the same isomorphism.

(1.d) (1) $(\overline{G}_1, \overline{N}_1, \overline{B}_1, \overline{b}_1, \overline{P})$ satisfies the same assumption as (G, N, B, b, P).
(2) $\overline{B}_0^{\overline{G}_1} = \overline{B}_1$.
(3) $\overline{V}_1$ is a unique projective indecomposable $R\overline{N}_1$-module in $\overline{b}_1$.

Indeed, (1) follows from (1.a). (2) follows from Lemma 3.11. (3) is clear.

As we have shown, W_0 is regarded as a $\overline{P}$-module, which we denote by $\overline{W}_0$. Also, S_1 is regarded as a $\overline{G}_1$-module, which we denote by $\overline{S}_1$. Then the following is clear:

(1.e) $\overline{S}_1$ is a unique simple $k\overline{G}_1$-module in $\overline{B}_1$, $\overline{W}_0 \mid (\overline{S}_1)_{\overline{P}}$, and $\overline{P}$ is a vertex of $\overline{W}_0$.

For the group $N_{\overline{G}_1}(\overline{P})$, the following are clear:

(1.f) (1) $\overline{N_G(P)} = N_{\overline{G}_1}(\overline{P}) = \overline{P} \times \overline{C_N(P)}$.
(2) $\overline{C_N(P)} = C_{\overline{N}_1}(\overline{P})$.
(3) There is a natural isomorphism: $\overline{C_N(P)} \cong C_N(P)$.

Let $\overline{V}_0$ be the $\overline{C_N(P)}$-module identified with V_0 via the natural isomorphism in (1.f)(3). The following is clear:

(1.g) $\overline{V}_0$ is a (unique) projective indecomposable $RC_{\overline{N}_1}(\overline{P})$-module in the block of $C_{\overline{N}_1}(\overline{P})$ covered by $\overline{B}_0$.

By (1.d)(1) and (1.e), we can choose $W' \in \mathrm{Lf}(\overline{W}_0)$. Put $M' = W' \times \overline{V}_0$. By (1.g), M' belongs to $\overline{B}_0$. Let X' be the Green correspondent of M' with respect to $(\overline{G}_1, N_{\overline{G}_1}(\overline{P}), \overline{P})$. By applying the induction hypothesis to $(\overline{G}_1, \overline{N}_1, \overline{B}_1, \overline{b}_1, \overline{P})$, we get that X' is an extension of $\overline{V}_1$. Thus:

(1.h) $\mathrm{Inf}(X')$ is an extension of V_1.

On the other hand, we have:

(1.i) $\mathrm{Inf}(X')$ is the Green correspondent of $\mathrm{Inf}(M')$ with respect to $(G_1, N_G(P), P)$.

Now, since $\mathrm{Inf}(W') \in \mathrm{Ep}(RP)$ and $\mathrm{Inf}(W')^* \cong W_0 \cong W^*$, there exists an RP-module L of rank 1 such that $\mathrm{Inf}(W') \cong L \otimes W$, cf. [2, Proposition 12.1]. Let $\mathrm{Inf}(L)$ be the inflation of L to G_1 via the natural homomorphism $G_1 \to G_1/N_1 \cong P$. Then

$$\begin{aligned}
\mathrm{Inf}(M')^{G_1} &\cong \big(\mathrm{Inf}(W') \times V_0\big)^{G_1} \\
&\cong \big(\mathrm{Inf}(L)_{N_G(P)} \otimes M\big)^{G_1} \\
&\cong \mathrm{Inf}(L) \otimes M^{G_1} \\
&\cong (\mathrm{Inf}(L) \otimes X) \oplus A,
\end{aligned}$$

where A is an $\mathcal{H}(P)$-projective module. Thus, by (1.i), $\mathrm{Inf}(X') \cong \mathrm{Inf}(L) \otimes X$. Then, by (1.h), $X_{N_1} \cong V_1$. Thus the proof is complete in (Case 1).

(Case 2) Put $G_2 = N_G(P_1 Q)$. Since $Q \lhd G$ and P is a cyclic p-group, we have the following.

(2.a) $N_G(P) \leqq N_G(D) \leqq G_2$.

Put $B_2 = B_0^{G_2}$ and let b_2 be a block of $G_2 \cap N$ covered by B_2. We have

$G_2 = PN_2$ with $N_2 = G_2 \cap N$.

(2.b) (G_2, N_2, B_2, b_2, P) satisfies the same assumption as (G, N, B, b, P).

It suffices to show that D is a defect group of B_2. (Indeed, if this is the case then b_2 is G_2-invariant and $D \cap N_2 = Q$ is a defect group of b_2.) Since $B_2 = (B_0^{N_G(D)})^{G_2}$, D is contained in a defect group of B_2. On the other hand, since $B_0^G = B$ and $B_0^{G_2} = B_2$, $B_2^G = B$. Thus a defect group of B_2 is contained in a G-conjugate of D. Hence the result follows.

Let X be the Green correspondent of M with respect to $(G_2, N_G(P), P)$.

(2.c) $X^G \cong U \oplus L$ for a projective RG-module L.

Clearly U is the Green correspondent of X with respect to (G, G_2, P) and $X^G \cong U \oplus L$, where L is an $\mathcal{X}$-projective module. Here $\mathcal{X} = \{P \cap P^x; x \in G \setminus G_2\}$. Then it is easy to see $\mathcal{X} = \{1\}$, because P is a cycllc p-group. Thus (2.c) follows.

In the following we put $\overline{H} = HQ/Q$ for any subgroup H of G. By the induction hypothesis applied to $(\overline{G_2}, \overline{N_2}, \overline{B_2}, \overline{b_2}, \overline{P})$ and Corollary 3.18, $X/XI(RQ)$ is indecomposable with vertex $\overline{P}$. So, as in (2.c), we get the following.

(2.d) $(X/XI(RQ))^{\overline{G}} \cong \overline{U} \oplus \overline{L}$ for an indecomposable $R\overline{G}$-module $\overline{U}$ and a projective $R\overline{G}$-module $\overline{L}$.

We now show the following, cf. the proof of Lemma III.5.13 in Feit [4].

(2.e) $\mathrm{Hom}_{G,Q}(U^*, S) = 0$. Here the left hand side denotes the k-vector space of Q-projective kG-homomorphisms from U^* to S.

Let $\phi : U^* \to S$ be a Q-projective kG-homomorphism. Let I be the injective hull of U^* and let $e : U^* \to I$ be the essential homomorphism. Since U^*_Q is projective, $0 \to U^*_Q \to I$ splits. Take a kG-homomorphism $f : I_Q \to U^*_Q$ such that $fe = \mathrm{id}_{U^*}$. Choose a kQ-homomorphlsm $\psi : U^* \to S$ such that $\phi = \mathrm{Tr}_Q^G(\psi)$ and let $g = \mathrm{Tr}_Q^G(\psi f)$. Then $\phi = ge$. Put $I = \bigoplus_s P_s$ with P_s projective indecomposable. If $e(U^*)$ projects onto some P_s, then $P_s | U^*$. This shows U has a projective summand (Feit [4, I.17.11]), a contradiction. Thus $e(U^*) \subseteq IJ(kG)$, where $J(kG)$ is the radical of kG. Hence $\phi(U^*) = 0$, as required.

(2.f) $U/UI(RQ)$ is projective-free.

Indeed,

$$\begin{aligned} 0 &= \mathrm{Hom}_{G,Q}(U^*, S) && \text{(by (2.e))} \\ &= \mathrm{Hom}_{\overline{G},\overline{Q}}(U^*/U^*I(kQ), S) && \text{(since } S \text{ is trivial on } Q) \\ &= \mathrm{Hom}_{\overline{G},\overline{1}}(U^*/U^*I(kQ), S). \end{aligned}$$

Thus $U^*/U^*I(kQ)$ is projective-free. By Lemma 3.9,

$$U^*/U^*I(kQ) \cong (U/UI(RQ))^*.$$

Thus $U/UI(RQ)$ is projective-free.

From (2.c) and Lemma 3.9, we get

(2.g) $(X/XI(RQ))^{\overline{G}} \cong U/UI(RQ) \oplus L/LI(RQ)$.

Thus, by (2.f), comparison of (2.d) and (2.g) shows that

(2.h) $U/UI(RQ) \cong \overline{U}$,

which implies that U is an extension of V. Indeed, (2.h) shows that the condition (iii) of Corollary 3.18 holds, so by Corollary 3.18(i), U is an extension of V. This completes the proof of Theorem 4.1. □

Let (G, N, B, b, P) be as above. Put $|P| = p^n$ and let

$$J(n) = \{j;\ j \text{ is an integer prime to } p \text{ and } 1 \leqq j < p^n\}.$$

For every $j \in J(n)$, let W_j be the unique indecomposable kP-module of dimension j. Let $1 \leqq i \leqq t$. Let W_0 be as above. Define $U_i'(W_j), j \in J(n)$, and $U_i'(W_0)$ as above. We have $U_i'(W_j)_N \cong n_{ij}V^*$ for an integer n_{ij} (cf. Proposition 3.6′). The following shows, in particular, W_0 (and hence a D-source of S) is determined if we know the Green correspondent of $W_1 \times {Y_i}^*$ with respect to $(G, N_G(P), P)$.

Corollary 4.3. *Let* $1 \leqq i \leqq t$ *and* $j \in J(n)$*. Then we have*
(i) $U_i'(W_j) \cong \mathrm{Inf}(W_0 \cdot W_j) \otimes U_i'(W_0)$.
(ii) $n_{ij} \in J(n)$*; in fact, for a fixed* i*, the map* $j \to n_{ij}$ *is a permutation on* $J(n)$ *of order at most* 2.
(iii) $\dim_k W_0 = n_{i1}$.

Proof. We note that ${W_0}^\wedge \cong W_0$.
(i) By Theorem 4.1′, $U_i'(W_0)$ is an extension of V^*. So the result follows from Lemma 2.2.
(ii) By (i), $n_{ij} = \dim_k(W_0 \cdot W_j)$. So $n_{ij} \in J(n)$. If $n_{ij} = m$, then $W_0 \cdot W_j \cong W_m$. Thus $n_{im} = \dim_k(W_0 \cdot W_m) = \dim_k(W_0 \cdot (W_0 \cdot W_j)) = j$.
(iii) We have $n_{i1} = \dim_k(W_0 \cdot W_1) = \dim_k(W_0)$.
This completes the proof. □

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