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Homogenization and memory effect of a three by three system

Dedicated to Professor Chiu-Chun Chang on his sixtieth birthday

By

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Abstract

The homogenization of 3×3 system of differential equations related to the Coriolis and Lorentz forces are studied. It generates memory effects. The memory (or nonlocal) kernel is described by the Volterra integral equation. When the coefficient is independent of time, the memory kernel can be characterized explicitly in terms of Young's measure. The kinetic formulation of the homogenized equation is also obtained.

1. Introduction

This paper is devoted to the study of the memory (or nonlocal) effects induced by homogenization of the system of differential equations

$$\mathcal{L}^\epsilon U^\epsilon := \frac{\partial}{\partial t} U^\epsilon(x,t) - A^\epsilon(x,t) U^\epsilon(x,t) = g(x,t), \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

with initial data complemented by

$$U^\epsilon(x,0) = U_0(x) = (u_1(x,0), u_2(x,0), u_3(x,0))^t, \qquad x \in \Omega, \tag{1.2}$$

where Ω is an open set in $\mathbf{R}^3$ and ϵ denotes the small parameter, $0 < \epsilon \ll 1$. Here

$$\begin{aligned} U^\epsilon(x,t) &= (u_1^\epsilon(x,t), u_2^\epsilon(x,t), u_3^\epsilon(x,t))^t \\ g(x,t) &= (g_1(x,t), g_2(x,t), g_3(x,t))^t \end{aligned}$$

are real-valued vector functions and

$$A^\epsilon(x,t) = -2 \begin{pmatrix} 0 & w_3^\epsilon(x,t) & -w_2^\epsilon(x,t) \\ -w_3^\epsilon(x,t) & 0 & w_1^\epsilon(x,t) \\ w_2^\epsilon(x,t) & -w_1^\epsilon(x,t) & 0 \end{pmatrix} \tag{1.3}$$

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is the skew-symmetric matrix. In particular, we will consider the case when

$$w_j^\epsilon(x,t) = -\frac{1}{2}w^\epsilon(x,t)b_j(x), \qquad j = 1,2,3 \tag{1.4}$$

then the matrix A^ϵ is rewritten as

$$A^\epsilon(x,t) = w^\epsilon(x,t)\begin{pmatrix} 0 & b_3(x) & -b_2(x) \\ -b_3(x) & 0 & b_1(x) \\ b_2(x) & -b_1(x) & 0 \end{pmatrix} \equiv w^\epsilon(x,t)J(x). \tag{1.5}$$

We also assume that the sequence of scalar measurable functions $\{w^\epsilon\}_\epsilon$ satisfies the bounds

$$0 < a_- \leq w^\epsilon(x,t) \leq a_+, \qquad \text{a.e. in } \Omega \times (0,T) \tag{1.6}$$

and are equicontinuous in t, i.e., there are functions ϕ such that $\phi(\tau) \to 0$ as $\tau \to 0$; and

$$|w^\epsilon(x,t) - w^\epsilon(x,s)| \leq \phi(|t-s|). \tag{1.7}$$

In mathematical models of microscopically non-homogeneous media, various local characteristics are usually described the functions w^ϵ. In other words, homogenization extracts homogeneous effective parameters from disorder or heterogeneous media. Therefore we will deal with sequences $\{w^\epsilon\}_\epsilon$ which describe microscopic quantities and macroscopic quantities are limits of sequences for a suitable weak topology. The homogenization theory studies the behavior of the solution sequence $\{U^\epsilon\}_\epsilon$ as $\epsilon \to 0$ and asks whether average behavior can be discerned from differential equations that are subject to high-frequency fluctuations when those fluctuations result from a dependence on two widely separated spatial scales.

Homogenization problems which induce memory or nonlocal effects are difficult, and despite three decades of research, the available results are still restricted to particular types of equations. The nonlocal effects may appear by homogenization had first been noticed by Enrique Sanchez-Palencia [19] (using asymptotic expansions in a periodic setting), for questions like Visco-Elasticity or for some memory effects in Electricity corresponding to the fact that some coefficients depend upon frequency. J-L. Lions had invented examples where one needed to introduce pseudo-differential operators (with an interpretation as memory effects) [22].

In order to understand this kind of problem, Luc Tartar started thinking about this problem in 1980 with a simplified model where such memory or nonlocal effect appears [21], [22], [23]. The basic fact is that if the microscopic constitutive law has highly oscillating coefficients, the macroscopic constitutive law will present an integral term, or memory term, having a kernel depending on the way those oscillations are produced. This result explains that the mathematical meaning for the absorption and spontaneous emission rules in quantum mechanics is that effective equations often have extra nonlocal terms in space and time. In [22] there is another example in which transport with fluctuating velocity induces some kind of a diffusion effect with memory. Indeed, the nonlocal theory provides a convenient setting in which to study transport in macroscopically heterogeneous system [13], [14]. In connection with the homogenization to the other sciences, we refer to the lecture notes by G. Allaire [2] in material science and A. Mikelić [18] in porous media.

Following Tartar's approach [22], the boundary value problem of the general second order differential equation with time-independent coefficients is discussed thoroughly by N. Antonić [6]. The memory kernel is described by using the eigenfunction expansion and a representation theorem for Nevanlinna function. Using a factorization of the second-order operator and the Dunford–Taylor integral representation theorem Y. Amirat, K. Hamdache and A. Ziani [3], [4] derive a nonlocal limiting equation with source terms. For the Γ-convergence approach to the memory effect we will refer to M. Mascarenhas [17] (see also De Giorgi [10] for the original motivation). The recent important works on the memory effect problems, their applications and the survey will be referred to Y. Amirat, K. Hamdache and A. Ziani [3], [4], [5] and R. Alexandre [1].

The memory term is a convolution in time due to the time translation invariance. For the time-dependent coefficient, the invariance principle by translation in times fails, and the Laplace and/or Fourier transform and the standard homogenization techniques are no longer valid. Our approach will be essentially based on the result due to Tartar [23], as we shall show in section 2, the same Volterra (or Volterra-Green) integral equation also occurs in this situation. In fact, the Volterra equation is the generalization of the moments relation after taking the Laplace or Fourier transform [5]. The Dirac-like system is studied in [11].

The organization of the paper is as follows: In section 2, we prove the main result concerning the homogenization of the 3×3 matrix system with time-dependent coefficient. We show that the limiting system (homogenized equation) is a system of integro-differential equations. The memory kernel is described by the Volterra equation. Section 3 is devoted to the characterization of the memory kernel. We consider the special structures of w^ϵ and represent the weak limits and the memory kernels explicitly in terms of the Young's measure and the associated parameterized measure. Moreover, we give another proof by Dunford-Taylor integral and reformulate the homogenized equation obtained as the kinetic formulation by introducing the kinetic variable. In section 4, we derive the modified moment relation by proving a representation theorem first. In section 5, We give two applications of the homogenization results to the Coriolis and Lorentz forces.

2. Homogenization of time-dependent case

In the following, without loss of generality, we assume $U^\epsilon(x,0) = U_0(x) = 0$. The solution sequences of (1.1) can be represented as

$$U^\epsilon(x,t) = (\mathcal{L}^\epsilon)^{-1} g \equiv \int_0^t G^\epsilon(x,s,t) g(x,s)\, ds, \tag{2.1}$$

where

$$G^\epsilon(x,s,t) = \Phi^\epsilon(x,t)\Phi^\epsilon(x,s)^{-1}, \tag{2.2}$$

is the Green's function for the initial value problem of the first order linear differential operator $\mathcal{L}^\epsilon$ defined by (1.1). Here Φ^ϵ is the unique solution of the matrix differential equation

$$\frac{\partial}{\partial t}\Phi^\epsilon(x,t) - J(x)w^\epsilon(x,t)\Phi^\epsilon(x,t) = 0, \qquad \Phi^\epsilon(x,0) = I \tag{2.3}$$

where I is the 3×3 identity matrix. The matrix function Φ^ϵ so defined is called the *matrizant* of the system (1.1). For simplicity we introduce the matrix function

$$B^\epsilon(x,s,t) \equiv w^\epsilon(x,t) G^\epsilon(x,s,t) \tag{2.4}$$

in $\Omega \times (0,T) \times (0,T)$. As in Tartar [23], the sequence of measurable functions $\{w^\epsilon\}_\epsilon$ is uniformly bounded in $L^\infty(\Omega \times (0,\infty))$, so that according to the Banach-Alaoglu-Bourbaki theorem a norm bounded set is relatively compact in weak-$*$ topology, we may extract a subsequence still denoted by $\{w^\epsilon\}_\epsilon$ with

$$w^\epsilon \overset{w}{\rightharpoonup} w^0 \qquad \text{weak } * \quad \text{in} \quad L^\infty\big(\Omega \times (0,T)\big).$$

Compactness requires more than just boundedness here because of the strong topology in t. For this reason we appeal to Arzela-Ascloi theorem which asserts that $\{f_n\}$ is a relatively compact set in $C([0,\infty); w\text{-}L^\infty(\Omega))$ if and only if

(i) $\{f_n(t)\}$ is a relatively compact set in $w\text{-}L^\infty(\Omega)$ for all $t \geq 0$;
(ii) $\{f_n\}$ is a equicontinuous in $C([0,\infty);\, L^\infty(\Omega))$.

Since w^ϵ is equicontinuous in t, this implies that the functions $G^\epsilon(x,s,t)$ are bounded with bounded derivatives in s and t and that B^ϵ are bounded with bounded derivatives in s and are equicontinuous in t. According to Arzela-Ascoli theorem, we may extract a subsequence such that

$$\left\{\begin{array}{rcllll} w^\epsilon(\cdot,t) & \overset{w}{\rightharpoonup} & w^0(\cdot,t) & \text{in} & L^\infty(\Omega) \ \text{weak } * & \forall t \in (0,T), \\ G^\epsilon(\cdot,s,t) & \overset{w}{\rightharpoonup} & G^0(\cdot,s,t) & \text{in} & L^\infty(\Omega) \ \text{weak } * & \forall t \in (0,T), \\ B^\epsilon(\cdot,s,t) & \overset{w}{\rightharpoonup} & B^0(\cdot,s,t) & \text{in} & L^\infty(\Omega) \ \text{weak } * & \forall t \in (0,T). \end{array}\right. \tag{2.5}$$

Because of the quadratic nature of the second term of the left hand side of (1.1) the weak convergence in (2.5) does not imply

$$w^\epsilon(\cdot,t)G^\epsilon(\cdot,s,t) \overset{w}{\rightharpoonup} w^0(\cdot,t)G^0(\cdot,s,t) \qquad \text{in} \quad L^\infty(\Omega) \ \text{ weak } * \tag{2.6}$$

$\forall t \in (0,T)$. Consequently, the problem of passage to the limit involves further investigation. Following as the same procedure as Tartar [23] (see also [1], [5], [11]), for $(x,s,t) \in \Omega \times (0,T) \times (0,T)$ we define the matrix function C by the formula

$$C(x,s,t) = B^0(x,s,t) - w^0(x,t)G^0(x,s,t), \tag{2.7}$$

which describes the corrector of the weak limit. Plugging (2.1), the explicit expression of U^ϵ, into (1.1) we find that U^ϵ satisfies the differential integral equation;

$$\frac{\partial}{\partial t}U^\epsilon(x,t) = J(x)\int_0^t w^\epsilon(x,t)G^\epsilon(x,s,t)g(x,s)\,ds + g(x,t). \tag{2.8}$$

On the other hand, it follows from (2.1) and (2.5) that the weak limit of U^ϵ is given by

$$U^0(x,t) = \int_0^t G^0(x,s,t)g(x,s)\,ds, \tag{2.9}$$

while the weak limit of $w^\epsilon U^\epsilon$ converges weakly to Z^0 given by

$$Z^0(x,t) \equiv \int_0^t B^0(x,s,t)g(x,s)ds. \tag{2.10}$$

Taking the limit in (2.8) and using (2.7) we have

$$\begin{aligned}\frac{\partial}{\partial t}U^0(x,t) &= J(x)w^0(x,t)U^0(x,t)\\ &\quad + J(x)\int_0^t C(x,s,t)g(x,s)\,ds + g(x,t).\end{aligned} \tag{2.11}$$

Next, we introduce the kernel $D(x,s,t)$ solution of the resolvent (or Volterra-Green) equation

$$D(x,s,t) = J(x)C(x,s,t) - J(x)\int_s^t C(x,s,\sigma)D(x,\sigma,t)\,d\sigma. \tag{2.12}$$

Integrating by part and using the condition $D(x,s,s) = 0$, we obtain from (2.11) that

$$\begin{aligned}\frac{\partial}{\partial t}U^0(x,t) &= w^0(x,t)J(x)U^0(x,t)\\ &\quad - \int_0^t K(x,s,t)U^0(x,s)\,ds + g(x,t)\end{aligned} \tag{2.13}$$

where the kernel K is given by

$$K(x,s,t) = \frac{\partial}{\partial s}D(x,s,t) + D(x,s,t)J(x)w^0(x,s) \tag{2.14}$$

with $(x,s,t) \in \Omega \times (0,T) \times (0,T)$. We thus have proved, as ϵ goes to zero, the microscopic equation gives place to the macroscopic one, in the following sense.

Theorem 2.1. *Under the hypotheses* (1.6) − (1.7)*, there exists a subsequence of* $\{w^\epsilon\}_\epsilon$ *and a kernel* K *defined on* $\Omega \times (0,T) \times (0,T)$*, measurable in* x *and* t*, such that* U^ϵ *converges in* $W^{1,\infty}(0,T;L^\infty(\Omega))$ *weak* $*$ *to* U^0 *solution of* (2.13) *where the kernel* D *defined in* $\Omega \times (0,T) \times (0,T)$ *is a solution of* (2.12) *and the kernel* K *is given by* (2.14)*.*

This theorem also answers the typical question in homogenization theory. If the solutions U^ϵ of the problems $\mathcal{L}^\epsilon U^\epsilon = g$ converge weakly to U^0, can an operator $\mathcal{L}^0$ be found such that U^0 is a solution of the problem $\mathcal{L}^0 U^0 = g$, and is $\mathcal{L}^0$ of the same type as $\mathcal{L}^\epsilon$? The answer is negative. Indeed, it is given by

$$
\begin{aligned}
\mathcal{L}^0 U^0 \equiv {} & \frac{\partial}{\partial t} U^0(x,t) - w^0(x,t) J(x) U^0(x,t) \\
& + \int_0^t K(x,s,t) U^0(x,s)\, ds
\end{aligned}
\tag{2.15}
$$

which is an integro-differential operator, i.e., the homogenization process generates memory or nonlocal effects described by integro-differential equations (Alexandre [1], Amirat-Hamdache-Ziani [3], [4], [5], Antonić [6], Jiang-Lin [11], [12], [15], Tartar [21], [22]).

Eq. (1.1) can be seen as the Newton law. (See section 5, the Coriolis and Lorentz forces for example.) The vector function U^ϵ is the particle velocity, the term $A^\epsilon(x,t)U^\epsilon$ is the friction force and A^ϵ the friction coefficient tensor. Therefore the the memory effect induced by homogenization shows that Eq. (1.1) generates asymptotically the generalized Langevin law, i.e., the the friction force contains a memory or nonlocal term. The memory or nonlocal effect also explains qualitatively something about irreversibility. One may start from an equation which is time reversible and a limiting process may make an irreversible equation appear.

3. Characterization of the memory kernel

In this section we will characterize the memory kernel K by using the Young measure introduced by Tartar in 1980's [8], [20], [23]. When the coefficient w^ϵ is independent of time then Eq. (1.1) is invariant by translation in t. It is known that linear operator commuting with translations have to be given by convolution. We assume that $\{w^\epsilon\}_\epsilon$ is a sequence of measurable functions that satisfies the bounds

$$
0 < a_- \le w^\epsilon(x) \le a_+, \qquad \text{a.e. in } \ \Omega. \tag{3.1}
$$

One should notice that in this section the spatial domain Ω need not be an open set of $\mathbf{R^3}$ and may be any measure space endowed with measure having no atoms. It follows from (3.1) that there exists a family of probability measure (Young measure) $d\nu_x$ with support in the interval (a_-, a_+) such that, after extracting a subsequence, for which we keep the index ϵ,

$$
w^\epsilon \xrightarrow{w} w^0 \qquad \text{weakly} * \quad \text{in} \qquad L^\infty(\Omega) \tag{3.2}
$$

with

$$w^0 = \int_\Lambda \lambda d\nu_x(\lambda) \equiv \langle \lambda, d\nu_x \rangle \quad \text{a.e.} \tag{3.3}$$

Similarly, we also have

$$A^\epsilon(x) = w^\epsilon(x)J \overset{w}{\rightharpoonup} A^0 = w^0(x)J \tag{3.4}$$

weakly $*$ in $L^\infty(\Omega)$. For convenience, we will assume $\mathbf{b} = (b_1, b_2, b_3)^t$ is a unit vector $|\mathbf{b}| = 1$ then direct calculation shows

$$J^2 = \mathbf{b} \otimes \mathbf{b} - I\,, \qquad J^3 + J = 0. \tag{3.5}$$

The minimal polynomial of J has the three simple roots $0, i, -i$. The Lagrange interpolation formula for e^{tJ} has the form $1 + \sin tx + (1 - \cos t)x^2$. Therefore the matrizant of the system (1.1) can be represented as

$$\Phi^\epsilon(x,t) = \exp\big(tw^\epsilon(x)J\big) = I + (\sin tw^\epsilon)J + (1 - \cos tw^\epsilon)J^2 \tag{3.6}$$

hence

$$G^\epsilon(x,s,t) = \Phi^\epsilon(x,t)\Phi^\epsilon(x,s)^{-1} = \Phi^\epsilon(x,t-s) = \exp\big((t-s)w^\epsilon(x)J\big) \tag{3.7}$$

and

$$B^\epsilon(x,s,t) = w^\epsilon(x)\exp\big((t-s)w^\epsilon(x)J\big). \tag{3.8}$$

It follows from (3.3) and the Young's fundamental theorem that the weak limits of $G^\epsilon(x,s,t)$ and $B^\epsilon(x,s,t)$ are given respectively by

$$G^\epsilon(x,s,t) \overset{w}{\rightharpoonup} G^0(x,s,t) = \int_\Lambda e^{\lambda(t-s)J} d\nu_x(\lambda), \tag{3.9}$$

$$B^\epsilon(x,s,t) \overset{w}{\rightharpoonup} B^0(x,s,t) = \int_\Lambda \lambda e^{\lambda(t-s)J} d\nu_x(\lambda)\,. \tag{3.10}$$

The fluctuation part is therefore given by

$$C = B^0 - w^0 G^0 = \int_\Lambda \big(\lambda - w^0(x)\big)e^{\lambda(t-s)J} d\nu_x(\lambda). \tag{3.11}$$

The key step to obtain the explicit form of the memory kernel K is to obtain the resolvent kernel D of (2.12) first. In this case it is a convolution type and the Laplace transform is available to solve the integral equation. We denote by L the Laplace transform with respect to the time variable t and p the corresponding transformed variable then applying the Laplace transform to (2.12) yields

$$\begin{aligned} LD(x,s,p) &= (I + JLC)^{-1}JLC = \big(I + L(JC)\big)^{-1}L(JC) \\ &= \int_\Lambda (\lambda - w^0(x))J(pI - \lambda J)^{-1} d\nu_x(\lambda) \\ &\quad \times \left[I + \int_\Lambda (\lambda - w^0(x))J(pI - \lambda J)^{-1} d\nu_x(\lambda)\right]^{-1}. \end{aligned} \tag{3.12}$$

Using the fact that $\mathbf{b}$ is a unit vector $|\mathbf{b}| = 1$, the matrix J can be diagonalized by the matrix P;

$$J = PMP^{-1} \tag{3.13}$$

where

$$P = \begin{pmatrix} b_1 & b_1b_3 + ib_2 & b_1b_3 - ib_2 \\ b_2 & b_2b_3 - ib_1 & b_2b_3 + ib_1 \\ b_3 & b_3^2 - 1 & b_3^2 - 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{3.14}$$

It follows that the fluctuation matrix function C given by (3.11) becomes

$$C(x, t-s) = \int_\Lambda (\lambda - \omega^0(x)) P e^{\lambda(t-s)M} P^{-1} d\nu_x(\lambda), \tag{3.15}$$

and from which taking the Laplace transform yields

$$LC = iP \begin{pmatrix} 0 & 0 & 0 \\ 0 & \int_\Lambda \frac{\lambda - \omega^0(x)}{z+\lambda} d\nu_x(\lambda) & 0 \\ 0 & 0 & \int_\Lambda \frac{\lambda-\omega^0(x)}{z-\lambda} d\nu_x(\lambda) \end{pmatrix} P^{-1} \tag{3.16}$$

where $z = pi$. Note that the family of Young measures $d\nu_x$ associated to the sequence $\{w^\epsilon\}_\epsilon$ is linked with the parametrized measure $d\mu_x$ by the *moments relation* [4], [5];

$$\int_\Lambda \frac{1}{z-\lambda} d\mu_x(\lambda) = \int_\Lambda (z-\lambda) d\nu_x(\lambda) - \left(\int_\Lambda \frac{1}{z-\lambda} d\nu_x(\lambda) \right)^{-1} \tag{3.17}$$

which was introduced by Tartar [22] through the Nevalinna-Pick arguments. It plays the central role for characterization of the memory or nonlocal kernel. Employing the moments relation (3.17) we derive the following equalities

$$\int_\Lambda \frac{\lambda - \omega^0(x)}{z+\lambda} d\nu_x(\lambda) = 1 - \left[1 - \frac{1}{z+\omega^0(x)} \int_\Lambda \frac{1}{z+\lambda} d\mu_x(\lambda) \right]^{-1} \tag{3.18}$$

and

$$\int_\Lambda \frac{\lambda - \omega^0(x)}{z-\lambda} d\nu_x(\lambda) = -1 + \left[1 - \frac{1}{z-\omega^0(x)} \int_\Lambda \frac{1}{z-\lambda} d\mu_x(\lambda) \right]^{-1} \tag{3.19}$$

which, after some computation, imply

$$\begin{aligned} LD &= (I + L(JC))^{-1} L(JC) \\ &= P \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{z+\omega^0(x)} \int_\Lambda \frac{1}{z+\lambda} d\mu_x(\lambda) & 0 \\ 0 & 0 & \frac{1}{z-\omega^0(x)} \int_\Lambda \frac{1}{z-\lambda} d\mu_x(\lambda) \end{pmatrix} P^{-1}. \end{aligned} \tag{3.20}$$

Accordingly, after taking inverse Laplace transform, the matrix function D is represented explicitly as

$$D(x,t) = -P \begin{pmatrix} 0 & 0 & 0 \\ 0 & \int_\Lambda \eta_+(\lambda) d\mu_x(\lambda) & 0 \\ 0 & 0 & \int_\Lambda \eta_-(\lambda) d\mu_x(\lambda) \end{pmatrix} P^{-1}$$
$$(3.21) \qquad = -\int_\Lambda \int_0^t \exp\left[(\omega^0(x)(t-\sigma) + \lambda\sigma) PMP^{-1}\right] d\mu_x(\lambda)$$
$$= -\int_\Lambda \int_0^t e^{(\omega^0(x)(t-\sigma)+\lambda\sigma)J} d\sigma d\mu_x(\lambda).$$

where

$$(3.22) \qquad \eta_\pm(\lambda) = \int_0^t e^{\pm i\omega^0(x)(t-\sigma)} e^{i\pm\lambda\sigma} d\sigma.$$

Hence, from equation (2.14), the memory kernel K is deduced by

$$(3.23) \qquad K(x,t-s) = \frac{\partial D(x,t-s)}{ds} + D(x,t-s)\omega^0(x)J = \int_\Lambda e^{\lambda(t-s)J} d\mu_x(\lambda).$$

Theorem 3.1. *Let the sequence of scalar functions* $\{w^\epsilon\}_\epsilon$ *satisfy* (3.1); *then, up to a subsequence of* $\epsilon \to 0$, *there exists a kernel* K *associated with* $\{w^\epsilon\}_\epsilon$ *and defined on* $\Omega \times (0,T)$ *such that for all* $g(x,t) \in L^\infty(\Omega \times (0,T))$ *the sequence* $\{U^\epsilon\}$ *of solutions to* (1.1) *converges in* $W^{1,\infty}(0,T; L^\infty(\Omega))$ *weak* $*$ *to* U^0, *a solution of*

$$(3.24) \qquad \frac{\partial}{\partial t} U^0(x,t) = w^0(x)J(x)U^0(x,t) - \int_0^t K(x,t-s)U^0(x,s)\, ds + g(x,t)$$

$$(3.25) \qquad U^\epsilon(x,0) = U_0(x), \qquad x \in \Omega$$

where w^0 *given by* (3.3) *is the weak* $*$ *limit of* w^ϵ *and the memory kernel* K *is measurable in* $x \in \Omega$ *and admits the integral representation*

$$(3.26) \qquad K(x,t) = \int_\Lambda e^{\lambda t J} d\mu_x(\lambda) \qquad a.e. \quad x \in \Omega,$$

$\{\mu_x\}$ *being a family of positive parametrized measures, with measurable dependence in* x *and having its support in* $[a_-, a_+]$.

Another interesting method to obtain the effective equation is the Dunford-Taylor integral which was used by Y. Amirat, K. Hamdache and A. Ziani [4], [5] (see also R. Alexandre [1]) replacing the Fourier transform to obtain the kinetic formulation of the homogenized equation. Following their idea, we write the solution sequences in the operator form

$$U^\epsilon(x,t) = \left(\mathcal{T} - w^\epsilon(x)I\right)^{-1} g(x,t), \qquad \mathcal{T} = \frac{\partial}{\partial t}. \tag{3.27}$$

The Dunford-Taylor integral representation enables to write (3.27) as

$$U^\epsilon(x,t) = \left(\frac{1}{2\pi i}\int_\Gamma (zI - w^\epsilon(x)J)^{-1}(z-\mathcal{T})^{-1}dz\right) g(x,t). \tag{3.28}$$

where Γ is the closed curve which contains the spectrum of $\mathcal{T}$. At this point, as mentioned in (3.17) – (3.19), we know from Tartar [22] (see also [1], [4], [5]) after modification that there exists a family of parametrized measures $d\mu_x(\lambda)$ associated with the Young measure $d\nu_x(\lambda)$ such that

$$\begin{aligned}(zI - w^\epsilon(x)J)^{-1} &\overset{w}{\rightharpoonup} \int_\Lambda (zI-\lambda J)^{-1} d\nu_x(\lambda)\\ &= \left[zI - \omega^0(x)J + \int_\Lambda (zI-\lambda J)^{-1} d\mu_x(\lambda)\right]^{-1}\end{aligned} \tag{3.29}$$

in $L^\infty(\Omega)$ weak-$*$. Relation (3.29) combined with expression (3.28) yields

$$\begin{aligned}U^0(x,t) = \frac{1}{2\pi i}\int_\Gamma &\left[zI - w^0(x)J + \int_\Lambda (zI-\lambda J)^{-1} d\mu_x(\lambda)\right]^{-1}\\ &\times (z-\mathcal{T})^{-1} g(x,t)dz.\end{aligned} \tag{3.30}$$

Applying the Dunford-Taylor integral again

$$\mathcal{T}U^0(x,t) - w^0(x)JU^0(x,t) + \int_\Lambda (\mathcal{T}-\lambda J)^{-1} U^0(x,t)d\mu_x(\lambda) = g(x,t), \tag{3.31}$$

which is equivalent that the limit U^0 of $\{U^\epsilon\}$ solves

$$\frac{\partial U^0}{\partial t} - w^0(x)JU^0 + \int_\Lambda W(x,t,\lambda)d\mu_x(\lambda) = g(x,t), \tag{3.32}$$

where the integrant, i.e., the auxiliary function $W(x,t,\lambda)$ satisfies

$$\frac{\partial W}{\partial t} - \lambda JW - U^0 = 0, \qquad W(x,0,\lambda) = 0. \tag{3.33}$$

Thus, as ϵ goes to zero, the microscopic equation gives place to the macroscopic one, in the following sense:

Theorem 3.2. *The effective equation* (3.24) – (3.26) *admits a kinetic formulation as the following well-posed system*:

$$\begin{aligned}\frac{\partial U^0}{\partial t} &= w^0(x)J(x)U^0(x,t) - \int_\Lambda W(x,t,\lambda)d\mu_x(\lambda) + g(x,t),\\ \frac{\partial W}{\partial t} &= \lambda JW + U^0, \qquad U^0(x,0) = U_0(x), \quad W(x,0,\lambda) = 0,\end{aligned} \tag{3.34}$$

for $t \in (0,T)$, $x \in \Omega$ *and* $\lambda \in \Lambda$.

Remark. The kinetic formulation (3.34) is equivalent to (3.24) − (3.26). Since the auxiliary function $W(x,t,\lambda)$ is given explicitly by

$$W(x,t,\lambda) = \int_0^t e^{\lambda(t-s)J} U^0(x,s)ds\,. \tag{3.35}$$

and the kernel K is the same as (3.26). Therefore (3.34) is the same as (3.24).

4. Yet another approach to the memory kernel

As mentioned in section 2, the key step in deriving the kernel K is the moment relation which is equivalent to the resolvent equation after taking the Laplace or Fourier transform. It is also noticed that the family of parametrized measures μ_x may be entirely described in terms of the Young measures ν_x through the moment relation. This relationship is very important because it tells how the memory effect, produced in the macroscopic equation, depends on the way the sequence $\{w^\epsilon\}$ oscillates. In this section we first prove a representation lemma directly related to the resolvent equation (2.12).

Lemma 4.1. *There exists a Radon measure $\tilde{\mu}_x$, associated with Young measures ν_x, on $\Lambda = [a_-, a_+]$ which are measurable for x such that the solution $D(x,s,t)$ of the resolvent equation* (2.12) *is given explicitly by*

$$D(x,s,t) = \int_\Lambda (\lambda - w^0(x))Je^{\lambda(t-s)J} d\tilde{\mu}_x(\lambda). \tag{4.1}$$

Proof. For fixed $s,t \in [0,T]$ and $x \in \Omega$, we denote by $\mathcal{H}$ as the set

$$\mathcal{H} \equiv \{< \phi_{\bar{x},\bar{s},\bar{t}}u, v >: \Lambda \to \mathbf{R} \mid \bar{s},\bar{t} \in [0,T]; \bar{x} \in \Omega; u,v \in \mathbf{R^3}\} \equiv \{\phi_{\bar{x},\bar{s},\bar{t}}\}.$$

where $\phi \in C(\Omega \times \Lambda \times [0,T] \times [0,T];\mathbf{R})$. Let M be the vector space generated by $\mathcal{H}$, then it is obvious that M is the subspace of the space $C(\Lambda)$. We define a family of the linear operators, associated with Young measures ν_x, $T_x : M \to \mathbf{R}$ by

$$\begin{aligned}\langle T_x, < \phi_{\bar{x},\bar{s},\bar{t}}u, v >\rangle &= \int_\Lambda < \phi_{\bar{x},\bar{s},\bar{t}}(\lambda)u, v > d\nu_x(\lambda)\\ &+ \int_0^T \chi_{[\bar{s},\bar{t}]}(\bar{\sigma}) < \int_\Lambda \phi_{\bar{x},\bar{\sigma},\bar{t}}(\lambda)d\nu_x(\lambda)D(\bar{x},\bar{s},\bar{\sigma})u, v > d\bar{\sigma},\end{aligned} \tag{4.2}$$

Due to the Young measures ν_x are measurable for x, we note that the family of the operators $\{T_x\}$ are measurable for the variable x from the definition (4.2); it is then easy to see that

$$|\langle T_x, \phi_{\bar{x},\bar{s},\bar{t}}u, v\rangle| \le C\| < \phi_{\bar{x},\bar{s},\bar{t}}u, v > \|_{C(\Lambda)}$$

where C is a constant. This shows that $\{T_x\}$ are bounded functionals on M. It follows from Hahn-Banach Theorem, there exists a family of bounded functionals $\{\mathcal{T}_x\}$ on $C(\Lambda)$ such that $\mathcal{T}_x|_M = T_x$; therefore applying the Riesz representation theorem we deduce that there exists a family of Radon measures $\{\tilde{\mu}_x\}$ on Λ such that

$$\langle \mathcal{T}_x, \psi\rangle = \int_\Lambda \psi(\lambda) d\tilde{\mu}_x(\lambda), \qquad \forall \psi \in C(\Lambda). \tag{4.3}$$

From above arguments we note that the family of the Radon measures $\{\tilde{\mu}_x\}$, which are measurable for x, are corresponding to the family of Young measures ν_x. Choosing $\psi(\lambda) =< \phi_{x,s,t}(\lambda)u, v >$, the index x correspoding to the index of $\mathcal{T}_x$, then from (4.2) – (4.3) we obtain

$$\begin{aligned}
&\int_\Lambda < \phi_{x,s,t}(\lambda)u, v > d\tilde{\mu}_x(\lambda) = \langle \mathcal{T}_x, < \phi_{x,s,t}(\lambda)u, v >\rangle \\
&\quad = \int_\Lambda < \phi_{x,s,t}(\lambda)u, v > d\nu(\lambda) + \int_0^T < \int_\Lambda \phi_{x,\sigma,t}(\lambda)d\nu_x(\lambda)D(x, s, \sigma)u, v > d\sigma
\end{aligned}$$

for any $u, v \in \mathbf{R}^3$, thus we have

$$\begin{aligned}
&\int_\Lambda \phi_{x,s,t}(\lambda)d\tilde{\mu}_x(\lambda) = \langle \mathcal{T}_x, \phi_{x,s,t}(\lambda)\rangle \\
&\quad = \int_\Lambda \phi_{x,s,t}(\lambda)d\nu_x(\lambda) + \int_0^T \left[\int_\Lambda \phi_{x,\sigma,t}(\lambda)d\nu_x(\lambda)\right] D(x, s, \sigma)d\sigma.
\end{aligned} \tag{4.4}$$

In particular, let

$$\phi_{x,s,t}(\lambda) = [\lambda - w^0(x)]Je^{\lambda(t-s)J}$$

and use the equations (2.12) and (4.4), we derive the relation

$$\begin{aligned}
D(x, s, t) &= \int_\Lambda \phi_{x,s,t}(\lambda)d\tilde{\mu}_x(\lambda) \\
&= \int_\Lambda \phi_{x,s,t}(\lambda)d\nu_x(\lambda) + \int_0^T \left[\int_\Lambda \phi_{x,\sigma,t}(\lambda)d\nu_x(\lambda)\right] D(x, s, \sigma)d\sigma
\end{aligned}$$

or equivalently

$$D(x, s, t) = \int_\Lambda [\lambda - w^0(x)]Je^{\lambda(t-s)J}d\tilde{\mu}_x(\lambda). \tag{4.5}$$

This completes the proof of the lemma. □

Another proof of the kernel. Applying this representation lemma we can derive the explicit form of the memory kernel K with the help of the Radon measure. Indeed, from (2.14), (4.1) and employing the relation $J^2e^{\lambda(t-s)J} = -e^{\lambda(t-s)J}$ we deduce that

$$K(x, s, t) = -\int_\Lambda (\lambda - w^0(x))^2 J^2 e^{\lambda(t-s)J}d\tilde{\mu}_x(\lambda) = \int_\Lambda e^{\lambda(t-s)J}d\mu_x(\lambda) \tag{4.6}$$

where $d\mu_x(\lambda) = (\lambda - w^0(x))^2 d\tilde{\mu}_x(\lambda)$.

Next we derive the extension of the moment relation (3.17). First we rewrite the resolvent equation (2.12) with the help of the representation formula (3.12)

$$\begin{aligned}\int_\Lambda(\lambda - w^0(x))Je^{\lambda(t-s)J}d\tilde{\mu}_x(\lambda) &= \int_\Lambda(\lambda - w^0(x))Je^{\lambda(t-s)J}d\nu_x(\lambda)\\ -\left(\int_\Lambda(\lambda - w^0(x))Je^{\lambda(t-s)J}d\nu_x(\lambda)\right) &* \left(\int_\Lambda(\lambda - w^0(x))Je^{\lambda(t-s)J}d\tilde{\mu}_x(\lambda)\right).\end{aligned} \tag{4.7}$$

Taking the Laplace transform, we get

$$\begin{aligned}&\int_\Lambda\big(\lambda J - w^0(x)J\big)(zI - \lambda J)^{-1}d\tilde{\mu}_x(\lambda)\\ &\quad= \int_\Lambda(\lambda J - w^0(x)J)(zI - \lambda J)^{-1}d\nu_x(\lambda)\\ &\quad\quad\times\left[I + \int_\Lambda(\lambda J - w^0(x)J)(zI - \lambda J)^{-1}d\nu_x(\lambda)\right]^{-1}\\ &\quad= \left[-I + (zI - w^0(x)J)\int_\Lambda(zI - \lambda J)^{-1}d\nu_x(\lambda)\right]\\ &\quad\quad\times\left[(zI - w^0(x)J)\int_\Lambda(zI - \lambda J)^{-1}d\nu_x(\lambda)\right]^{-1}\\ &\quad= -\left[(zI - w^0(x)J)\int_\Lambda(zI - \lambda J)^{-1}d\nu_x(\lambda)\right]^{-1} + I.\end{aligned} \tag{4.8}$$

Therefore

$$\begin{aligned}\left(\int_\Lambda(zI - \lambda J)^{-1}d\nu_x(\lambda)\right)^{-1} &= zI - w^0(x,t)J\\ &\quad-\left(\int_\Lambda\big(zI - w^0(x)J\big)\big(\lambda J - w^0(x)J\big)\big(zI - \lambda J\big)^{-1}d\tilde{\mu}_x(\lambda)\right)\\ &= zI - w^0(x)J - \left(\int_\Lambda\big(\lambda J - w^0(x)J\big)^2\big(zI - \lambda J\big)^{-1}d\tilde{\mu}_x(\lambda)\right)\end{aligned} \tag{4.9}$$

Here we use the relation

$$\int_\Lambda(\lambda J - w^0(x)J)d\tilde{\mu}_x(\lambda) = 0.$$

So we obtain the following generalization of the moment relation

$$\begin{aligned}&\int_\Lambda\big(\lambda J - w^0(x)J\big)^2(zI - \lambda J)^{-1}d\tilde{\mu}_x(\lambda)\\ &\quad= \int_\Lambda(zI - \lambda J)d\nu_x(\lambda) - \left[\int_\Lambda(zI - \lambda J)^{-1}d\nu_x(\lambda)\right]^{-1}.\end{aligned} \tag{4.10}$$

The extra square term on the left hand side of (4.10) comparing with the standard moment relation indicates the antisymmetric nature of J.

Now we apply the modified moment relation to give another proof of **Theorem 3.2.** From (3.28), the Dunford-Taylor integral representation of the weak limit U^0;

$$U^0(x,t) \equiv \frac{1}{2\pi i}\int_\Gamma \int (zI-\lambda J)^{-1}\, d\nu_x(\lambda)(z-\mathcal{T})^{-1} g(x,t)dz. \tag{4.11}$$

We apply the moment relation (4.10) to rewrite U^0 as

$$\begin{aligned} U^0(x,t) = \frac{1}{2\pi i}\int_\Gamma &\left[zI - w^0(x)J - \int_\Lambda \left(\lambda J - w^0(x)J\right)^2 (zI-\lambda J)^{-1} d\tilde{\mu}_x(\lambda)\right]^{-1} \\ &\times (z-\mathcal{T})^{-1} g(x,t)dz. \end{aligned} \tag{4.12}$$

Applying the Dunford-Taylor integral once more, U^0 satisfies the equation

$$\begin{aligned} &\mathcal{T}U^0(x,t) - w^0(x)JU^0(x,t) \\ &\quad - \int_\Lambda \left(\lambda - w^0(x)\right)^2 J^2 \left(\mathcal{T}-\lambda J\right)^{-1} U^0(x,t) d\tilde{\mu}_x(\lambda) = g(x,t). \end{aligned} \tag{4.13}$$

Thus the limit U^0 of $\{U^\epsilon\}$ satisfies

$$\frac{\partial U^0}{\partial t} - w^0(x)JU^0 - \int_\Lambda \left(\lambda - w^0(x)\right)^2 J^2 W(x,t,\lambda) d\tilde{\mu}_x(\lambda) = g(x,t), \tag{4.14}$$

$$\frac{\partial W}{\partial t} - \lambda JW - U^0 = 0, \qquad W(x,0,\lambda) = 0. \tag{4.15}$$

It follows from (4.15) that the auxiliary function $W(x,t,\lambda)$ is given explicitly by

$$W(x,t,\lambda) = \int_0^t e^{\lambda(t-s)J} U^0(x,s)ds \tag{4.16}$$

and the homogenized equation (4.14) becomes

$$\frac{\partial U^0}{\partial t} - w^0(x)JU^0 + \int_0^t K(x,t-s)U^0(x,s)ds = g(x,t), \tag{4.17}$$

where the kernel K is the same as (4.6)

$$\begin{aligned} K(x,s,t) &= -\int_\Lambda \left(\lambda - w^0(x)\right)^2 J^2 e^{\lambda(t-s)J} d\tilde{\mu}_x(\lambda) \\ &= \int_\Lambda e^{\lambda(t-s)J} d\mu_x(\lambda) \end{aligned} \tag{4.18}$$

and $d\mu_x(\lambda) = (\lambda - w^0(x))^2 d\tilde{\mu}_x(\lambda)$.

5. Application to Coriolis and Lorentz forces

In this section we will apply the homogenization results obtained in the previous sections to analyze the Coriolis and Lorentz forces. The Coriolis force plays a significant role in many oceanographic and meteorological phenomena involving displacements of masses of matter over long distance, such as the circulation pattern of the trade winds and the course of the Gulf stream.

Let $U^\epsilon = (u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)^t \in \mathbf{R}^3$ be the velocity of the particles with respect to the rotating frame, m the mass, ω^ϵ the angular velocity and $\mathbf{g}$ is the gravity acceleration of the earth then the equation of motion is

$$m\frac{dU^\epsilon}{dt} = m\mathbf{g} - 2m\omega^\epsilon \times \mathbf{U}^\epsilon, \qquad \mathbf{U}^\epsilon(\mathbf{x}, \mathbf{0}) = \mathbf{U_0}(\mathbf{x}). \tag{5.1}$$

The Coriolis force on a particle of mass m is $-2m\omega^\epsilon \times \mathbf{U}^\epsilon$. Assume $\omega^\epsilon(\mathbf{x}, \mathbf{t}) = \mathbf{w}^\epsilon(\mathbf{x}, \mathbf{t})\mathbf{b}(\mathbf{x}) = \mathbf{w}^\epsilon(\mathbf{x}, \mathbf{t})(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})^\mathbf{t}$ then

$$\begin{aligned}\omega^\epsilon \times \mathbf{U}^\epsilon &= w^\epsilon(x,t)\mathbf{b} \times U^\epsilon = w^\epsilon(b_1, b_2, b_3)^t \times (u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)^t \\ &= w^\epsilon(x,t)(b_2u_3^\epsilon - b_3u_2^\epsilon, b_3u_1^\epsilon - b_1u_3^\epsilon, b_1u_2^\epsilon - b_2u_1^\epsilon)^t = -w^\epsilon JU^\epsilon\end{aligned}$$

where the skew-symmetric matrix J is the same as (1.5). Therefore we can rewrite (5.1) as

$$\frac{dU^\epsilon}{dt} = \mathbf{g} - 2w^\epsilon\mathbf{b} \times U^\epsilon = \mathbf{g} + 2w^\epsilon JU^\epsilon\,, \qquad U^\epsilon(x,0) = U_0(x). \tag{5.2}$$

Applying (2.1) Theorem we derive the following homogenization result for the Coriolis force (5.2).

Theorem 5.1. *Let w^ϵ satisfy (1.6) – (1.9) then after extraction of a subsequence, there is a kernel K defined on $\mathbf{R}^3 \times (0,T) \times (0,T)$ such that for bounded measurable $\mathbf{g}$ sequence $\{U^\epsilon\}_\epsilon$ of solutions to (5.2) converges weakly $*$ in $W^{1,\infty}(0,T; L^\infty(\mathbf{R}^3))$ to the solution U of*

$$\begin{aligned}\frac{dU}{dt} &= \mathbf{g} - 2w^0(x,t)\mathbf{b} \times U \\ &\quad - \int_0^t K(x,s,t)U(x,s)\,ds\,, \quad U(x,0) = U_0(x).\end{aligned} \tag{5.3}$$

The kernel K is given by

$$K(x,s,t) = \frac{\partial D(x,s,t)}{\partial s} + 2D(x,s,t)J(x)w^0(x,s) \tag{5.4}$$

while the kernel D solves the Volterra-Green equation

$$D(x,s,t) = 2JC(x,s,t) - 2J\int_s^t C(x,s,\sigma)D(x,\sigma,t)d\sigma \tag{5.5}$$

with kernel C defined by (2.5) *and* (2.7).

Furthermore, if the sequence of measurable functions $\{w^\epsilon\}_\epsilon$ is assumed to be independent of time variable t, $w^\epsilon = w^\epsilon(x)$, and satisfies the uniform bound $0 < a_- \le w^\epsilon(x) \le a_+$ then there is a subsequence of $\{w^\epsilon(x)\}$ and a kernel K associated with $w^\epsilon(x)$, such that the sequence U^ϵ of solutions to (5.2) *converges in $L^\infty(0,T;L^\infty(\mathbf{R}^3))$ weak $*$ to the unique solution U of*

$$\begin{aligned} \frac{dU}{dt} &= \mathbf{g} - 2w^0(x)\mathbf{b} \times U \\ &\quad - \int_0^t K(x,t-s)U(x,s)ds\,, \quad U(x,0) = U_0(x). \end{aligned} \tag{5.6}$$

The kernel K is given by

$$K(x,t) = -\int_\Lambda 4(\lambda - w^0(x))^2 J^2 e^{2\lambda t J} d\tilde{\mu}_x(\lambda), \tag{5.7}$$

where $d\tilde{\mu}_x$ is a parametrized family of nonnegative measures having support in Λ associated with the sequence $\{w^\epsilon\}_\epsilon$.

Our second application is about the Lorentz force. Let $U^\epsilon \in \mathbf{R}^3$ be the velocity of the particles and m the mass. The equation of motion according to the Newton's law satisfies

$$m\frac{\partial U^\epsilon}{\partial t} = \mathcal{F}^\epsilon(x,t), \tag{5.8}$$

where $\mathcal{F}^\epsilon$ is the Lorentz force given by

$$\mathcal{F}^\epsilon(x,t) = q\big(E(x,t) + U^\epsilon(x,t) \times B^\epsilon(x,t)\big)\,. \tag{5.9}$$

Here $E(x,t)$ is the exterior electric field and $B^\epsilon(x,t)$ is the exterior magnetic field which is assumed to satisfy $B^\epsilon(x,t) = w^\epsilon(x,t)\mathbf{b} = w^\epsilon(x,t)\,(b_1,b_2,b_3)^t$. When w^ϵ is independent of t, the homogenization of the Lorentz force was studied by Y. Amirat, K. Hamdache and A. Ziani in [5]. We extend their result to the time dependent case.

Theorem 5.2. *Let w^ϵ satisfy* (1.6) $-$ (1.9). *There exists a subsequence of $\{w^\epsilon\}_\epsilon$ and a kernel K defined on $(0,T)\times(0,T)\times\mathbf{R}^3$, measurable in x and t, such that $\{U^\epsilon\}$ converges in $W^{1,\infty}(0,T;L^\infty(\mathbf{R}^3))$ weak $*$ to U solution of*

$$\begin{aligned} m\frac{\partial U}{\partial t} &= q\big(E(x,t) + w^0(x,t)U(x,t) \times \mathbf{b}\big) \\ &\quad - \int_0^t K(x,s,t)U(x,s)ds, \qquad U(x,0) = U_0(x), \end{aligned} \tag{5.10}$$

where the kernel K is given by

$$K(x,s,t) = m\frac{\partial D(x,s,t)}{\partial s} + D(x,s,t)J(x)w^0(x,s), \tag{5.11}$$

while the kernel D solves the Volterra-Green equation

$$mD(x,s,t) = JC(x,s,t) - J\int_s^t C(x,s,\sigma)D(x,\sigma,t)d\sigma. \tag{5.12}$$

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References

[1] R. Alexandre, *Some results in homogenization tackling memory effects*, Asymptot. Anal. **15** (1997), 229–259.

[2] G. Allaire, Homogenization and Applications to Material Sciences, Lecture Note at Newton Institute, Cambridge, 1999.

[3] Y. Amirat, K. Hamdache and A. Ziani, *Kinetic formulation for a transport equation with memory*, Comm. Partial Differential Equations **16** (1991), 1287–1311.

[4] ———, *Some results on homogenization of convection-diffusion equations*, Arch. Rational Mech. Anal. **114** (1991), 155–178.

[5] ———, *On homogenization of ordinary differential equations and linear transport equations*, In: Calculus of Variations, Homogenization and Continuum Mechanics, (eds. G. Bouchitté, G. Buttazzo and P. Suquet), World Scientific, pp. 29–50, 1994.

[6] N. Antonić, *Memory effects in homogenization linear second-order equations*, Arch. Rational Mech. Anal. **125** (1993), 1–24.

[7] F. Bornemann, *Homogenization in time of singularly perturbed mechanical system*, Lecture Note in Math. **1687**, Springer, 1998.

[8] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, Amer. Math. Soc., Providence, 1990.

[9] P. Gérard, P. Markowich, N. Mauser and F. Poupaud, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math. **L** (1997), 323–379.

[10] E. De Giorgi, *Some remarks on Γ-convergence and least squares method*, In: Composite Media and Homogenization Theory, (ed. Dall Maso and Dell'Antonio), Birkhäuser, Boston, pp. 135–142, 1991.

[11] J. S. Jiang and C. K. Lin, *Homogenization of the Dirac-like system*, Math. Models Methods Appl. Sci. **11**-3 (2001), 433–458.

[12] ______, *Homogenization of second order equation with spatial dependent coefficient*, Discrete Contin. Dynam. Systems **12**-2 (2005), 303–313.

[13] D. L. Koch and J. F. Brady, *A nonlocal description of advection-diffusion with application to dispersion in porous media*, J. Fluid Mech. **180** (1987), 387–403.

[14] D. L. Koch, R. G. Cox, H. Brenner and J. F. Brady, *The effect of order on dispersion in porous media*, J. Fluid Mech. **200** (1989), 173–188.

[15] C. K. Lin and J. S. Jiang, *Nonlocal effects induced by homogenization of a second order differential equation*, In: Proceedings of the IMC'94, (eds. Y. Fong, W.-J. Huang, Y.-J. Lee and N. -C. Wong), World Scientific, pp. 131–148, 1996.

[16] M. L. Mascarenhas, *A linear homogenization problem with time dependent coefficient*, Trans. Amer. Math. Soc. **281** (1984), 179–195.

[17] ______, *Memory effect and Γ-convergence*, Proc. Roy. Soc. Edinburgh **123A** (1993), 311–322.

[18] A. Mikelić, *Homogenization theory and applications to filtration through porous media*, Lecture Note in Math. **1734**, Springer-Verlag, Berlin and New York, 2000.

[19] E. Sanchez-Palencia, *Non-homogeneous media and vibration theory*, Lecture Note in Phys. **127**, Springer-Verlag, Berlin and New York, 1980.

[20] L. Tartar, *Compensated compactness and applications to partial differential equations*, Heriot-Watt Symposium, **IV**, (ed. R. J. Knops), Pitman, New York, pp. 136–211, 1979.

[21] ______, *Remark on homogenization*, In: Homogenization and effective moduli of materials and media, (ed. J. L. Ericksen et al.), Springer-Verlag, New York, pp. 228–246, 1986.

[22] ______, *Nonlocal effects induced by homogenization*, In: Partial Differential Equations and the Calculus of Variations **II** (ed. F. Colombini et al.), pp. 925–937, 1989.

[23] ______, *Memory effects and homogenization*, Arch. Rational Mech. Anal. **111** (1990), 121–133.

[24] ______, *H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh **115A** (1990), 193–230.

[25] V. V. Zhikov, *Estimates for the averaged matrix and the averaged tensor*, Russian Math. Surveys **46** (1991), 65–136.