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Inverse theorems for Bernstein-Chlodowsky type polynomials

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Abstract

In this paper we establish two inverse theorems for Bernstein-Chlodowsky type polynomials of two variables in a rectangular and a triangular domain.

1. Introduction

In this paper, we study the inverse theorems for two of generalizations of Bernstein-Chlodowsky type polynomials of two variables on a triangular domain and a rectangular domain.

There are many investigations devoted to the inverse theorems for the classical Bernstein polynomials, as well as by the two-dimensional Bernstein polynomials and their generalizations by means of different techniques being the main tools the classical modulus of continuity [1], [5], [6], [7].

In the multivariate setting the paper of Ditzian [6] yields several characterizations of the classes of functions for the classical Bernstein polynomials on the simplex or on the cube.

On the other hand, since they have been defined on the unbounded region, Bernstein-Chlodowsky polynomials [4], [10] have not been studied so well and we don't know of papers, devoted to the two dimensional case.

Inverse results are devoted to the task of determining the class of functions f for which

$$|B_n^*(f;x,y) - f(x,y)| = O\left(\left(\frac{b_n}{n}\right)^{\alpha}\right), \qquad 0 < \alpha < 1$$

and

$$|B_{n,m}(f;x,y) - f(x,y)| = O\left(\left\{\min\left(\frac{b_n}{n}, \frac{c_m}{m}\right)\right\}^{\alpha}\right), \qquad (0 < \alpha < 1).$$

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The paper is set out as follows:
In Section 2, we give the inverse theorem for the following Bernstein-Chlodowsky type polynomials of two variables on a triangular domain:

$$B_n^*(f;x,y) = \sum_{k=0}^{n} C_n^k \left(1 - \frac{x+y}{b_n}\right)^{n-k} \times \sum_{j=0}^{k} f\left(\frac{k-j}{n}b_n, \frac{j}{n}b_n\right) C_k^j \left(\frac{x}{b_n}\right)^{k-j} \left(\frac{y}{b_n}\right)^j .$$

In Section 3, we give the inverse theorems for the following Bernstein-Chlodowsky type polynomials of two variables on a rectangular domain:

$$B_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k}{n}b_n, \frac{j}{m}c_m\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k \times \left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j} .$$

2. Bernstein-Chlodowsky type polynomials on a triangular domain

In this section we prove the inverse theorem for the operator $B_n^*(f;x,y)$, Bernstein-Chlodowsky type polynomials on a triangular domain.

Let (b_n) be an increasing sequence of positive real numbers and satisfy the properties: $\lim\limits_{n\to\infty} b_n = \infty$ and the sequence $(\frac{b_n}{n})$ decreases to zero as $n \to \infty$.

Let Δ_{b_n} be the triangular domain:

$$\Delta_{b_n} = \{(x,y) \colon x \geq 0, y \geq 0, x+y \leq b_n\}.$$

On the triangular domain we can define the following Bernstein-Chlodowsky type polynomials:

$$B_n^*(f;x,y) = \sum_{k=0}^{n} C_n^k \left(1 - \frac{x+y}{b_n}\right)^{n-k} \times \sum_{j=0}^{k} f\left(\frac{k-j}{n}b_n, \frac{j}{n}b_n\right) C_k^j \left(\frac{x}{b_n}\right)^{k-j} \left(\frac{y}{b_n}\right)^j , \tag{2.1}$$

where $(x,y) \in \Delta_{b_n}$.

Theorem 2.1. *For any sufficiently large fixed positive real number A, the relation*

$$\lim_{n\to\infty} \max_{(x,y)\in\Delta_A} |B_n^*(f;x,y) - f(x,y)| = 0$$

holds for all functions f which are continuous in $x \geq 0$, $y \geq 0$ and satisfy the condition

$$|f(x,y)| \leq M_f(1+x^2+y^2),$$

where M_f is a positive constant depending on the function f only.

Proof. Simple calculations show that

$$\begin{aligned} B_n^*(1;x,y) &= 1, \\ B_n^*(t_1;x,y) &= x, \\ B_n^*(t_2;x,y) &= y, \\ B_n^*(t_1^2;x,y) &= x^2 + \frac{x(b_n - x)}{n}, \\ B_n^*(t_2^2;x,y) &= y^2 + \frac{y(b_n - y)}{n}. \end{aligned}$$

Now we can apply Korovkin's type approximation theorem, proved in [8], and this gives the proof. □

Given a triangular region Δ_A, no matter how large for some n, Δ_{b_n} will contain Δ_A and therefore the theorem gives a solution of the approximation problem.

Note also that by the properties, $\lim\limits_{n\to\infty} b_n = \infty$ and the sequence $(\frac{b_n}{n})$ decreasing to zero as $n \to \infty$, the triangular domain Δ_{b_n} extends to the infinite quadrant $x > 0$, $y > 0$ as $n \to \infty$ and therefore in effect we established a theorem on the approximation of continuous functions by polynomials (2.1) on an unbounded set.

Now we investigate the inverse theorem for $B_n^*(f;x,y)$.

Theorem 2.2. *If f satisfies the following condition*

$$|B_n^*(f;x,y) - f(x,y)| \le M \left(\frac{b_n}{n}\right)^\alpha, \qquad (0 < \alpha < 1)$$

for some positive constant M, then $f \in Lip\,(\alpha; C(\Delta_{b_n}))$.

Proof. we can show easily,

$$\begin{aligned} \frac{\partial B_n^*(f;x,y)}{\partial x} &= \frac{n}{b_n} \sum_{k=0}^{n-1} C_{n-1}^k \left(1 - \frac{x+y}{b_n}\right)^{n-k-1} \\ &\times \sum_{j=0}^{k} \left[f\left(\frac{k+1-j}{n} b_n, \frac{j}{n} b_n\right) - f\left(\frac{k-j}{n} b_n, \frac{j}{n} b_n\right)\right] \\ &\times C_k^j \left(\frac{x}{b_n}\right)^{k-j} \left(\frac{y}{b_n}\right)^j, \\ \frac{\partial B_n^*(f;x,y)}{\partial y} &= \frac{n}{b_n} \sum_{k=0}^{n-1} C_{n-1}^k \left(1 - \frac{x+y}{b_n}\right)^{n-k-1} \\ &\times \sum_{j=0}^{k} \left[f\left(\frac{k-j}{n} b_n, \frac{j+1}{n} b_n\right) - f\left(\frac{k-j}{n} b_n, \frac{j}{n} b_n\right)\right] \\ &\times C_k^j \left(\frac{x}{b_n}\right)^{k-j} \left(\frac{y}{b_n}\right)^j. \end{aligned}$$

Using the properties of the modulus of continuity [2], [3], we can obtain

$$\left|\frac{\partial B_n^*(f;x,y)}{\partial x}\right| \le \left(\frac{n}{b_n}+\frac{1}{\delta}\right)\omega^{(1)}(f;\delta),$$

$$\left|\frac{\partial B_n^*(f;x,y)}{\partial y}\right| \le \left(\frac{n}{b_n}+\frac{1}{\delta}\right)\omega^{(2)}(f;\delta),$$

$$\left|\frac{\partial B_n^*(f;x,y)}{\partial x}\right| + \left|\frac{\partial B_n^*(f;x,y)}{\partial y}\right| \le 2\left(\frac{n}{b_n}+\frac{1}{\delta}\right)\omega(f;\delta).$$

For any $(x_1\ ,\ \ y_1)$, $(x_2\ ,\ y_2)$of points in Δ_{b_n}, we have

$$\left|\int_{x_1}^{x_2}\left|\frac{\partial B_n^*(f;x,y)}{\partial x}\right|dx\right| + \left|\int_{y_1}^{y_2}\left|\frac{\partial B_n^*(f;x,y)}{\partial y}\right|dy\right|$$
$$\le \sqrt{2}\left(\frac{n}{b_n}+\frac{1}{\delta}\right)\omega(f;\delta)\left[(x_1-x_2)^2+(y_1-y_2)^2\right]^{1/2}.$$

Taking $\delta_n = \frac{b_n}{n}$, we have

$$\left|\int_{x_1}^{x_2}\left|\frac{\partial B_n^*(f;x,y)}{\partial x}\right|dx\right| + \left|\int_{y_1}^{y_2}\left|\frac{\partial B_n^*(f;x,y)}{\partial y}\right|dy\right| \tag{2.2}$$
$$\le \sqrt{2}\left(\frac{1}{\delta_n}+\frac{1}{\delta}\right)\omega(f;\delta)\left[(x_1-x_2)^2+(y_1-y_2)^2\right]^{1/2}.$$

For all fixed natural numbers n, we write

(2.3)
$$|f(x_1,y_1)-f(x_2,y_2)| \le |f(x_1,y_1)-B_n^*(f;x_1,y_1)| + |f(x_2,y_1)-B_n^*(f;x_2,y_1)|$$
$$+ |f(x_2,y_1)-B_n^*(f;x_2,y_1)| + |f(x_2,y_2)-B_n^*(f;x_2,y_2)|$$
$$+ \left|\int_{x_1}^{x_2}\left|\frac{\partial}{\partial x}B_n^*(f;x,y_1)\right|dx\right| + \left|\int_{y_1}^{y_2}\left|\frac{\partial}{\partial y}B_n^*(f;x_2,y)\right|dy\right|.$$

Also $\delta_{n-1} \le 2\delta_n$ for $n = 2,3,\dots$ and for a given $0 < \delta \le 1$ there exists a natural number n such that

$$\delta_n \le \delta \le \delta_{n-1} \le 2\delta_n. \tag{2.4}$$

Using inequalities (2.2) and (2.4) in (2.3), we obtain

$$|f(x_1,y_1)-f(x_2,y_2)| \le M_1\left[\delta^\alpha + \frac{[(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}}{\delta}\omega(f;\delta)\right],$$

where $M_1 = \max\left\{4M, 3\sqrt{2}\right\}$. For $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \le h$ where $0 < h \le 1$ and by the definition of ω, we can write

$$\omega(f;h) \le M_1\left(\delta^\alpha + \frac{h}{\delta}\omega(f;\delta)\right), \qquad 0 < h,\delta \le 1. \tag{2.5}$$

On the other hand, from the inequality (2.5) we get the inequality $\omega(f;h) \le M_2 h^\alpha$ (proof in [1]). Consequently, we have

$$\omega(f;h) \le M_2 h^\alpha, \qquad (0 < \alpha < 1).$$

This completes the proof. □

3. Bernstein-Chlodowsky type polynomials on a rectangular domain

In this section we prove the inverse theorem for the operator $B_{n,m}(f;x,y)$, Bernstein-Chlodowsky type polynomials on a rectangular domain.

Let (b_n) and (c_m) be increasing sequences of positive real numbers and satisfy the properties: $\lim\limits_{n\to\infty} b_n = \lim\limits_{m\to\infty} c_m = \infty$ and the sequences $(\frac{b_n}{n})$ and $(\frac{c_m}{m})$ decrease to zero as $n, m \to \infty$.

For any $b_n > 0, c_m > 0$ we denote by $D_{b_n c_m}$:

$$D_{b_n c_m} = \{(x,y) \colon \ 0 \le x \le b_n, \ 0 \le y \le c_m\}.$$

We can introduce the Bernstein-Chlodowsky type polynomials for a function f of two variables as follows:

$$B_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k}{n}b_n, \frac{j}{m}c_m\right)\varphi_n^k\left(\frac{x}{b_n}\right)\varphi_m^j\left(\frac{y}{c_m}\right), \tag{3.1}$$

where $(x,y) \in D_{b_n c_m}$, $\varphi_n^k(t) = \binom{n}{k} t^k (1-t)^{n-k}$.

Theorem 3.1. *For any sufficiently large fixed positive real numbers A and B, if $f \in C(D_{AB})$ then the equality*

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \max_{(x,y)\in D_{AB}} |B_{n,m}(f;\, x,\, y) - f(x,\, y)| = 0$$

holds.

Proof. We can easily prove the following Korovkin type equalities,

$$\begin{aligned} B_{n,m}(1;x,y) &= 1,\\ B_{n,m}(t_1;x,y) &= x,\\ B_{n,m}(t_2;x,y) &= y,\\ B_{n,m}(t_1^2+t_2^2;x,y) &= x^2+y^2+\frac{x(b_n-x)}{n}+\frac{y(c_m-y)}{m}. \end{aligned}$$

If we use the above equalities we can see that

$$\|B_{n,m}(1;x,y)-1\|_{C(D_{AB})}=0,$$
$$\|B_{n,m}(t_1;x,y)-x\|_{C(D_{AB})}=0,$$
$$\|B_{n,m}(t_2;x,y)-y\|_{C(D_{AB})}=0,$$
$$\begin{aligned}\left\|B_{n,m}(t_1^2+t_2^2;x,y)-(x^2+y^2)\right\|_{C(D_{AB})} &= \max_{(x,y)\in D_{AB}}\left|\frac{x(b_n-x)}{n}+\frac{y(c_m-y)}{m}\right| \\ &\le A\frac{b_n}{n}+B\frac{c_m}{m}\to 0 \quad \text{as} \quad n,m\to\infty.\end{aligned}$$

According to Korovkin-type theorem for multivariate functions, the proof of Theorem 3.1 is completed. □

Now we can give an inverse theorem for $B_{n,m}$.

Theorem 3.2. *If f satisfies the following condition*

$$|B_{n,m}(f;x,y)-f(x,y)|\le K\left\{\min\left(\frac{b_n}{n},\frac{c_m}{m}\right)\right\}^{\alpha}, \quad (0<\alpha<1)$$

for some positive constant K, then $f\in Lip\,(\alpha;C(D_{b_nc_m}))$.

Proof. We can write the partial derivative of $B_{n,m}(f;x,y)$ with respect to x, and as for the classical Bernstein polynomials,

$$\begin{aligned}\frac{\partial B_{n,m}(f;x,y)}{\partial x} &= \frac{n}{b_n}\sum_{k=0}^{n-1}\sum_{j=0}^{m}\left[f\left(\frac{k+1}{n}b_n,\frac{j}{m}c_m\right)-f\left(\frac{k}{n}b_n,\frac{j}{m}c_m\right)\right] \\ &\quad\times\varphi_{n-1}^{k}\left(\frac{x}{b_n}\right)\varphi_m^{j}\left(\frac{y}{c_m}\right).\end{aligned}$$

Taking absolute values on both sides and using the properties of the modulus of continuity, we can obtain

$$\begin{aligned}\left|\frac{\partial B_{n,m}(f;x,y)}{\partial x}\right| &\le \frac{n}{b_n}\sum_{k=0}^{n-1}\sum_{j=0}^{m}\omega^{(1)}\left(f;\frac{b_n}{n}\right)\varphi_{n-1}^{k}\left(\frac{x}{b_n}\right)\varphi_m^{j}\left(\frac{y}{c_m}\right) \\ &\le \frac{n}{b_n}\omega^{(1)}(f;\delta)\sum_{k=0}^{n-1}\sum_{j=0}^{m}\left(1+\frac{b_n}{\delta n}\right)\varphi_{n-1}^{k}\left(\frac{x}{b_n}\right)\varphi_m^{j}\left(\frac{y}{c_m}\right) \\ &\le \frac{n}{b_n}\omega^{(1)}(f;\delta)\left[1+\frac{b_n}{\delta n}\right],\end{aligned}$$

and so

$$\left|\frac{\partial B_{n,m}(f;x,y)}{\partial x}\right|\le\omega^{(1)}(f;\delta)\left[\frac{n}{b_n}+\frac{1}{\delta}\right].$$

Like this way, it is obvious that

$$\left|\frac{\partial B_{n,m}(f;x,y)}{\partial y}\right| \leq \omega^{(2)}(f;\delta)\left[\frac{m}{c_m}+\frac{1}{\delta}\right].$$

Taking $\delta_{n,m} = \min(\frac{b_n}{n}, \frac{c_m}{m})$, we obtain for any pair x_1, x_2 of points in $[0, b_n]$ and y_1 , y_2 of points in $[0, c_m]$

$$\left|\int_{x_1}^{x_2}\left|\frac{\partial B_{n,m}(f;x,y)}{\partial x}\right| dx\right| \leq \omega^{(1)}(f;\delta)\left[\frac{1}{\delta_{n,m}}+\frac{1}{\delta}\right]|x_2-x_1|,$$

$$\left|\int_{y_1}^{y_2}\left|\frac{\partial B_{n,m}(f;x,y)}{\partial y}\right| dy\right| \leq \omega^{(2)}(f;\delta)\left[\frac{1}{\delta_{n,m}}+\frac{1}{\delta}\right]|y_2-y_1|. \tag{3.2}$$

The sequence $\delta_{n,m}$ decreases to zero as $n, m \to \infty$. Also $\delta_{n-1,m-1} \leq 2\delta_{n,m}$ for $n, m = 2, 3, \ldots$

Hence for a given $0 < \delta \leq 1$ there exist natural numbers n and m such that

$$\delta_{n,m} \leq \delta \leq \delta_{n-1,m-1} \leq 2\delta_{n,m}. \tag{3.3}$$

Using (3.3) in (3.2), we get

$$\left|\int_{x_1}^{x_2}\left|\frac{\partial B_{n,m}(f;x,y)}{\partial x}\right| dx\right| \leq 3\frac{\omega^{(1)}(f;\delta)}{\delta}|x_2-x_1|,$$

$$\left|\int_{y_1}^{y_2}\left|\frac{\partial B_{n,m}(f;x,y)}{\partial y}\right| dy\right| \leq 3\frac{\omega^{(2)}(f;\delta)}{\delta}|y_2-y_1|. \tag{3.4}$$

On the other hand, for all fixed natural numbers n and m, we write

$$\begin{aligned}|f(x_1,y_1)-f(x_2,y_2)| \leq{}& |f(x_1,y_1)-B_{n,m}(f;x_1,y_1)| \\ &+|f(x_2,y_1)-B_{n,m}(f;x_2,y_1)|+|f(x_2,y_1) \\ &-B_{n,m}(f;x_2,y_1)|+|f(x_2,y_2)-B_{n,m}(f;x_2,y_2)| \\ &+\left|\int_{x_1}^{x_2}\left|\frac{\partial}{\partial x}B_{n,m}(f;x,y_1)\right| dx\right| \\ &+\left|\int_{y_1}^{y_2}\left|\frac{\partial}{\partial y}B_{n,m}(f;x_2,y)\right| dy\right|.\end{aligned} \tag{3.5}$$

Using (3.3) and (3.4) in (3.5), we can obtain

$$|f(x_1,y_1)-f(x_2,y_2)| \leq K_1\left[\delta^\alpha+\frac{[(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}}{\delta}\omega(f;\delta)\right], \tag{3.6}$$

where $K_1 = \max\left\{4K, 3\sqrt{2}\right\}$. By the inequality (3.6) we have

$$\omega(f;h) \leq K_2 h^\alpha, \qquad (0<\alpha<1).$$

This completes the proof. □

Corollary 3.1. *If*

$$\begin{aligned} &|B_{n,m}(f;x,\cdot) - f(x,\cdot)| \leq L_1 \left(\frac{b_n}{n}\right)^\alpha, \\ &|B_{n,m}(f;\cdot,y) - f(\cdot,y)| \leq L_2 \left(\frac{c_m}{m}\right)^\beta, \qquad (0<\alpha,\beta<1) \end{aligned}$$

then

$$f \in Lip_y\left(\alpha; C(D_{b_n c_m})\right) \cap Lip_x\left(\beta; C(D_{b_n c_m})\right).$$

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