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Herz and Herz type Hardy spaces estimates of multilinear integral operators for the extreme cases

By

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Abstract

In this paper, the endpoint estimates for some multilinear operators related to certain fractional singular integral operators on Herz and Herz type Hardy spaces are obtained.

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1. Introduction and notations

Let T be the Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$, a classical result of Coifman, Rochberg and Weiss (see [6]) states that the commutator $[b,T]f = T(bf) - bTf$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when T is replaced by the fractional integral operator. In [10], the boundedness properties of the commutators for the extreme values of p are obtained. In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, have been developed (see [8], [9], [12], [13]). The main purpose of this paper is to establish the endpoint continuity properties of some multilinear operators related to certain non-convolution type fractional singular integral operators on Herz and Herz type Hardy spaces.

First, let us introduce some notations(see [8], [9], [12], [13], [14]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q} |Q|^{-1}\int_Q |f(y)-f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^{\#} \in L^\infty$ and define $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$; We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such

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that

$$\|f\|_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q| dy < \infty.$$

It is well-known that (see [9], [14])

$$\|f\|_{CMO} \approx \sup_{r>1} \inf_{c \in C} |Q(0,r)|^{-1} \int_Q |f(x) - c| dx.$$

Definition 1. Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that

$$\|f\|_{B_p^\delta} = \sup_{r>1} r^{-n(1/p - \delta/n)} \|f \chi_{Q(0,r)}\|_{L^p} < \infty.$$

For $k \in Z$, define $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and $\tilde{\chi}_k$ the characteristic function of C_k for $k \ge 1$ and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 2. Let $0 < p < \infty$ and $\alpha \in R$.

(1) The homogeneous Herz space $\dot{K}_p^\alpha(R^n)$ is defined by

$$\dot{K}_p^\alpha(R^n) = \{f \in L_{loc}^p(R^n \setminus \{0\}) : \|f\|_{\dot{K}_p^\alpha} < \infty\},$$

where

$$\|f\|_{\dot{K}_p^\alpha} = \sum_{k=-\infty}^{\infty} 2^{k\alpha} \|f \chi_k\|_{L^p};$$

(2) The nonhomogeneous Herz space $K_p^\alpha(R^n)$ is defined by

$$K_p^\alpha(R^n) = \{f \in L_{loc}^p(R^n) : \|f\|_{K_p^\alpha} < \infty\},$$

where

$$\|f\|_{K_p^\alpha} = \sum_{k=0}^{\infty} 2^{k\alpha} \|f \tilde{\chi}_k\|_{L^p};$$

If $\alpha = n(1 - 1/p)$, we denote that $\dot{K}_p^\alpha(R^n) = \dot{K}_p(R^n)$, $K_p^\alpha(R^n) = K_p(R^n)$.

Definition 3. Let $1 < p < \infty$.

(1) The homogeneous Herz type Hardy space $H\dot{K}_p(R^n)$ is defined by

$$H\dot{K}_p(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_p(R^n)\},$$

and

$$\|f\|_{H\dot{K}_p} = \|G(f)\|_{\dot{K}_p};$$

(2) The nonhomogeneous Herz type Hardy space $HK_p(R^n)$ is defined by

$$HK_p(R^n) = \{f \in S'(R^n) : G(f) \in K_p(R^n)\},$$

and

$$\|f\|_{HK_p} = \|G(f)\|_{K_p};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 4. Let $1 < p < \infty$. A function $a(x)$ on R^n is called a central $(n(1-1/p), p)$-atom (or a central $(n(1-1/p), p)$-atom of restrict type), if

1) $\text{Supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$);
2) $\|a\|_{L^p} \leq |B(0, r)|^{1/p-1}$,
3) $\int_{R^n} a(x)dx = 0$.

Lemma 1 (see [9], [13])**.** *Let $1 < p < \infty$. A temperate distribution f belongs to $H\dot{K}_p(R^n)$(or $HK_p(R^n)$) if and only if there exist central $(n(1-1/p), p)$-atoms (or central $(n(1-1/p), p)$-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$\|f\|_{H\dot{K}_p}(\text{ or } \ \|f\|_{HK_p}) \approx \sum_j |\lambda_j|.$$

2. Theorems

In this paper, we will consider a class of multilinear operators related to some integral operators, whose definition are following.

Let $\lambda > 1$, $\delta > 0$ and m be a fixed positive integer and A be a function on R^n. We denote that

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\beta| \leq m} \frac{1}{\beta!} D^\beta A(y)(x-y)^\beta$$

and

$$Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\beta| = m} \frac{1}{\beta!} D^\beta A(x)(x-y)^\beta.$$

Definition 5. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,
(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1-\delta)}$,
(3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

The multilinear Littlewood-Paley operator is defined by

$$g_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_{t,1}^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_{t,1}^A(f)(x,y) = \int_{R^n} \frac{f(z)\psi_t(y-z)}{|x-z|^m} R_{m+1}(A;x,z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_{t,1}(f)(y) = f * \psi_t(y)$. We also define that

$$g_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_{t,1}(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [15]).

Let H_1 be the Hilbert space $H_1 = \{h : \|h\| = (\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1})^{1/2} < \infty\}$. Then for each fixed $x \in R^n$, $F_{t,1}^A(f)(x,y)$ and $F_{t,1}(f)(y)$ may be viewed as a mapping from $(0,+\infty)$ to H_1, and it is clear that

$$g_\lambda^A(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_{t,1}^A(f)(x,y) \right\| \text{ and}$$

$$g_\lambda(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_{t,1}(f)(y) \right\|.$$

We also consider the variant of g_λ^A, which is defined by

$$\tilde{g}_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |\tilde{F}_{t,1}^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$\tilde{F}_{t,1}^A(f)(x,y) = \int_{R^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} \psi_t(y-z) f(z)dz.$$

Definition 6. Let $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_{t,2}^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_{t,2}^A(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz.$$

We denote that

$$F_{t,2}(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)dz.$$

We also define that

$$\mu_\lambda(f)(x)=\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|F_{t,2}(f)(y)|^2\frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz integral operator(see [16]).

Let H_2 be the Hilbert space $H_2=\{h:\|h\|=(\int\int_{R_+^{n+1}}|h(y,t)|^2dydt/t^{n+3})^{1/2}<\infty\}$, then for each fixed $x\in R^n$, $F_{t,2}^A(f)(x,y)$ and $F_{t,2}(f)(y)$ may be viewed as a mapping from $(0,+\infty)$ to H_2, and it is clear that

$$\mu_\lambda^A(f)(x)=\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_{t,2}^A(f)(x,y)\right\| \text{ and}$$

$$\mu_\lambda(f)(x)=\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_{t,2}(f)(y)\right\|.$$

The variant of μ_λ^A is defined by

$$\tilde{\mu}_\lambda^A(f)(x)=\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|\tilde{F}_{t,2}^A(f)(x,y)|^2\frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$\tilde{F}_{t,2}^A(f)(x,y)=\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}\frac{Q_{m+1}(A;x,z)}{|x-z|^m}f(z)dz.$$

More generally, we consider the following multilinear operators related to certain convolution operators.

Definition 7. Let $F(x,t)$ define on $R^n\times[0,+\infty)$, we denote that

$$F_t(f)(x)=\int_{R^n}F(x-y,t)f(y)dy$$

and

$$F_t^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}F(x-y,t)f(y)dy.$$

Let H be the normed space $H=\{h:\|h\|<\infty\}$. For each fixed $x\in R^n$, we view $F_t(f)(x)$ and $F_t^A(f)(x)$ as a mapping from $[0,+\infty)$ to H. Then, the multilinear operator related to F_t is defined by

$$T^A(f)(x)=\|F_t^A(f)(x)\|;$$

We also define that $T(f)(x)=\|F_t(f)(x)\|$.

It is clear that Definition 5 and 6 are the particular examples of Definition 7. Note that when $m=0$, T^A is just the commutator of T and A(see [11], [16]).

It is well known that multilinear operator, as a non-trivial extension of the commutator, is of great interest in harmonic analysis and has been widely studied by many authors(see [3]–[5]). In [7], the weighted $L^p(p>1)$-boundedness of the multilinear operator related to some singular integral operator are obtained; In [2], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will establish the endpoint continuity properties of the multilinear operators g_λ^A and $\tilde{g}_\lambda^A$, μ_λ^A and $\tilde{\mu}_\lambda^A$ on Herz and Herz type Hardy spaces.

We shall prove the following theorems in Section 3.

Theorem 1. *Let $0<\delta<n$, $1<p<n/\delta$ and $D^\beta A\in BMO(R^n)$ for all β with $|\beta|=m$. Then g_λ^A and μ_λ^A all map $B_p^\delta(R^n)$ continuously into $CMO(R^n)$.*

Theorem 2. *$0<\delta<n$, $1<p<n/\delta$, $1/q=1/p-\delta/n$ and $D^\beta A\in BMO(R^n)$ for all β with $|\beta|=m$. Then $\tilde{g}_\lambda^A$ and $\tilde{\mu}_\lambda^A$ all map $H\dot{K}_p(R^n)$ (or $HK_p(R^n)$) continuously into $\dot{K}_q^\alpha(R^n)$(or $K_q^\alpha(R^n)$) with $\alpha=n(1-1/p)$.*

Theorem 3. *Let $0<\delta<n$, $1<p<n/\delta$ and $D^\beta A\in BMO(R^n)$ for all β with $|\beta|=m$.*

(i) *If for any cube Q and $u\in 3Q\setminus 2Q$, there is a constant $C>0$ such that*

$$\frac{1}{|Q|}\int_Q\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\sum_{|\beta|=m}\frac{1}{\beta!}|D^\beta A(x)-(D^\beta A)_Q|\times\int_{(4Q)^c}\frac{(u-z)^\beta}{|u-z|^m}\psi_t(y-z)f(z)dz\right\|dx\le C\|f\|_{B_p^\delta},$$

then $\tilde{g}_\lambda^A$ maps $B_p^\delta(R^n)$ continuously into $CMO(R^n)$.

(ii) *If for any cube Q and $u\in 3Q\setminus 2Q$, there is a constant $C>0$ such that*

$$\frac{1}{|Q|}\int_Q\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\sum_{|\beta|=m}\frac{1}{\beta!}|D^\beta A(x)-(D^\beta A)_Q|\times\int_{(4Q)^c}\frac{(u-z)^\beta}{|u-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(z)}(y,t)}{|y-z|^{n-1-\delta}}f(z)dz\right\|dx\le C\|f\|_{B_p^\delta},$$

then $\tilde{\mu}_\lambda^A$ maps $B_p^\delta(R^n)$ continuously into $CMO(R^n)$.

3. Proofs of theorems

We begin with the following

Main Theorem. *Let $0<\delta<n$, $1<p<n/\delta$ and $D^\beta A\in BMO(R^n)$ for all β with $|\beta|=m$. Suppose that T^A is the same as in Definition 7 such*

that T is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p,q \in (1,+\infty)$ with $1/q = 1/p - \delta/n$. If T^A satisfies the cancellation condition:

$$\|F_t^A(f)(x) - F_t^A(f)(0)\| \le C\|f\|_{B_p^\delta}$$

for any cube $Q = Q(0,d)$ with $d > 1$, supp$f \subset (2Q)^c$ and $x \in Q$. Then T^A is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.

To prove the theorem, we need the following lemma.

Lemma 2 (see [5]). *Let A be a function on R^n and $D^\beta A \in L^q(R^n)$ for $|\beta| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\beta|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Main Theorem. We have to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T^A(f)(x) - C_Q| dx \le C\|f\|_{B_p^\delta}$$

holds for any cube $Q = Q(0,d)$ with $d > 1$. Fix a cube $Q = Q(0,d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_{\tilde{Q}} x^\beta$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\beta \tilde{A} = D^\beta A - (D^\beta A)_{\tilde{Q}}$ for all β with $|\beta| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m} F(x-y,t) f_2(y) dy \\ &+ \int_{R^n} \frac{R_m(\tilde{A};x,y)}{|x-y|^m} F(x-y,t) f_1(y) dy \\ &- \sum_{|\beta|=m} \frac{1}{\beta!} \int_{R^n} \frac{F(x-y,t)(x-y)^\beta}{|x-y|^m} D^\beta \tilde{A}(y) f_1(y) dy, \end{aligned}$$

then

$$\begin{aligned} \left| T^A(f)(x) - T^{\tilde{A}}(f_2)(0) \right| &= \left| \|F_t^A(f)(x)\| - \|F_t^{\tilde{A}}(f)(0)\| \right| \\ &\le \left\| F_t \left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1 \right)(x) \right\| \\ &+ \sum_{|\beta|=m} \frac{1}{\beta!} \left\| F_t \left(\frac{(x-\cdot)^\beta}{|x-\cdot|^m} D^\beta \tilde{A} f_1 \right)(x) \right\| \\ &+ \|F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(0)\| = I(x) + II(x) + III(x), \end{aligned}$$

thus

$$\frac{1}{|Q|}\int_Q \left|T^A(f)(x) - T^{\tilde{A}}(f_2)(0)\right| dx$$
$$\leq \frac{1}{|Q|}\int_Q I(x)dx + \frac{1}{|Q|}\int_Q II(x)dx + \frac{1}{|Q|}\int_Q III(x)dx = I + II + III.$$

Now, let us estimate I, II and III, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 2, we get

$$R_m(\tilde{A}; x, y) \leq C|x-y|^m \sum_{|\beta|=m} \|D^\beta A\|_{BMO},$$

thus, by the $L^p(R^n)$ to $L^q(R^n)$ boundedness of T for $1 < p, q < \infty$ with $1/q = 1/p - \delta/n$, we get

$$\begin{aligned}
I &\leq \frac{C}{|Q|}\int_Q |T\left(\sum_{|\beta|=m} \|D^\beta A\|_{BMO} f_1\right)(x)|dx \\
&\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO}\left(\frac{1}{|Q|}\int_Q |T(f_1)(x)|^q dx\right)^{1/q} \\
&\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO}|Q|^{-1/q}\|f_1\|_{L^p} \\
&\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO} d^{-n(1/p-\delta/n)}\|f\chi_{2Q}\|_{L^p} \\
&\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.
\end{aligned}$$

Secondly, taking $q, r, s > 1$ such that $1/r = 1/s - \delta/n$, $qs < p$, then by the L^s to L^r-boundedness of T and Hölder's inequality, denoting that $1/q + 1/q' = 1$, we gain

$$\begin{aligned}
II &\leq \frac{C}{|Q|}\int_Q |T(\sum_{|\beta|=m}(D^\beta A - (D^\beta A)_{\tilde{Q}})f_1)(x)|dx \\
&\leq C\sum_{|\beta|=m}\left(\frac{1}{|Q|}\int_Q |T((D^\beta A - (D^\beta A)_{\tilde{Q}})f_1)(x)|^r dx\right)^{1/r} \\
&\leq C\sum_{|\beta|=m}|Q|^{-1/r}\left(\int |(D^\beta A(x) - (D^\beta A)_{\tilde{Q}})f_1(x)|^s dx\right)^{1/s} \\
&\leq C\sum_{|\beta|=m}|Q|^{-1/r}\left(\int_{\tilde{Q}} |D^\beta A(x) - (D^\beta A)_{\tilde{Q}}|^{sq'} dx\right)^{1/(sq')}\left(\int_{\tilde{Q}} |f_1(x)|^{qs} dx\right)^{1/(qs)} \\
&\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}|Q|^{1/(sq')}|Q|^{-1/r}\left(\int_{\tilde{Q}} |f_1(x)|^p dx\right)^{1/p}|Q|^{(p-qs)/(pqs)}
\end{aligned}$$

$$\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}|\tilde{Q}|^{\delta/n-1/p}\|f\chi_{\tilde{Q}}\|_{L^p}$$
$$\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.$$

For III, using the cancellation condition of T, we know $III \leq C\|f\|_{B_p^\delta}$. This completes the proof of Main Theorem. □

To prove Theorem 1, 2 and 3, we need the following lemma.

Lemma 3. *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta| = m$. Then g_λ^A and μ_λ^A are all bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Proof. For g_λ^A, by Minkowski's inequality and the conditions of ψ, we get

$$g_\lambda^A(f)(x)$$
$$\leq \int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}\left(\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|\psi_t(y-z)|^2\frac{dydt}{t^{1+n}}\right)^{1/2}dz$$
$$\leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}$$
$$\times\left(\int_0^\infty\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}}\frac{dydt}{t^{1+n}}\right)^{1/2}dz$$
$$\leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}$$
$$\times\left[\int_0^\infty\left(t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}\right)tdt\right]^{1/2}dz,$$

noting that

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}} \leq CM\left(\frac{1}{(t+|x-z|)^{2n+2-2\delta}}\right)$$
$$\leq \frac{C}{(t+|x-z|)^{2n+2-2\delta}}$$

(where Mg denotes the Hardy-Littlwood maximal function of g) and

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$g_\lambda^A(f)(x) \leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2}dz$$
$$= C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}}dz.$$

For μ_λ^A, notice that $|x-z| \le 2t$, $|y-z| \ge |x-z|-t \ge |x-z|-3t$ when $|x-y| \le t$, $|y-z| \le t$, and $|x-z| \le t(1+2^{k+1}) \le 2^{k+2}t$, $|y-z| \ge |x-z| - 2^{k+3}t$ when $|x-y| \le 2^{k+1}t$, $|y-z| \le t$, we obtain

$$
\begin{aligned}
&\mu_\lambda^A(f)(x)\\
&\le \int_{R^n}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\right.\\
&\quad \times \left.\left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-\delta-1}|x-z|^m}\right)^2 \chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{1/2} dz\\
&\le C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\\
&\quad \times \left[\int_0^\infty\int_{|x-y|\le t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-3t)^{2n-2\delta-2}}\frac{dydt}{t^{n+3}}\right]^{1/2} dz\\
&\quad + C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\\
&\quad \times \left[\int_0^\infty\sum_{k=0}^\infty\int_{2^kt<|x-y|\le 2^{k+1}t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)t^{-n-3}dydt}{(|x-z|-2^{k+3}t)^{2n-2\delta-2}}\right]^{1/2} dz\\
&\le C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\left[\int_{|x-z|/2}^\infty\frac{dt}{(|x-z|-3t)^{2n-2\delta}}\right]^{1/2} dz\\
&\quad + C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\\
&\quad \times \left[\sum_{k=0}^\infty\int_{2^{-2-k}|x-z|}^\infty 2^{-kn\lambda}(2^kt)^nt^{-n}\frac{2^kdt}{(|x-z|-2^{k+3}t)^{2n-2\delta}}\right]^{1/2} dz\\
&\le C\int_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n-\delta}}dz\\
&\quad + C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n-\delta}}dz\left[\sum_{k=0}^\infty 2^{kn(1-\lambda)}\right]^{1/2}\\
&= C\int_{R^n}\frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}}|f(z)|dz.
\end{aligned}
$$

Thus, the lemma follows from [7]. □

Proof of Theorem 1. From Lemma 3, we know that g_λ and μ_λ are bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Now, it suffices to verify that g_λ^A and μ_λ^A satisfy the cancellation condition in Main Theorem, that is

$$\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_{t,j}^{A}(f)(x,y)-\left(\frac{t}{t+|y|}\right)^{n\lambda/2}F_{t,j}^{A}(f)(0,y)\right\|$$
$$\leq C\|f\|_{B_p^\delta} \text{ for } j=1,2.$$

Let supp$f\subset(2Q(0,d))^c$ and $\tilde{A}(x)=A(x)-\sum\limits_{|\beta|=m}\frac{1}{\beta!}(D^\beta A)_Q x^\beta$.

For g_λ^A, we write, for $x\in Q$,

$$\begin{aligned}
&\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_{t,1}^{\tilde{A}}(f)(x,y)-\left(\frac{t}{t+|y|}\right)^{n\lambda/2}F_{t,1}^{\tilde{A}}(f)(0,y)\\
&=\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\int_{R^n}\left[\frac{1}{|x-z|^m}-\frac{1}{|z|^m}\right]\psi_t(y-z)R_m(\tilde{A};x,z)f(z)dz\\
&\quad+\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\int_{R^n}\frac{\psi_t(y-z)f(z)}{|z|^m}[R_m(\tilde{A};x,z)-R_m(\tilde{A};0,z)]dz\\
&\quad+\int_{R^n}\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}-\left(\frac{t}{t+|y|}\right)^{n\lambda/2}\right]\frac{\psi_t(y-z)R_m(\tilde{A};0,z)f(z)}{|z|^m}dz\\
&\quad-\sum_{|\beta|=m}\frac{1}{\beta!}\int_{R^n}\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\frac{(x-z)^\beta}{|x-z|^m}-\left(\frac{t}{t+|y|}\right)^{n\lambda/2}\frac{(-z)^\beta}{|z|^m}\right]\\
&\quad\times\psi_t(y-z)D^\beta\tilde{A}(z)f(z)dz\\
&:=I_1^t(x)+I_2^t(x)+I_3^t(x)+I_4^t(x).
\end{aligned}$$

Note that $|x-z|\sim|z|$ for $x\in Q$ and $z\in R^n\setminus 2Q$, by Lemma 2 and the following inequality(see[14])

$$|b_{Q_1}-b_{Q_2}|\leq C\log(|Q_2|/|Q_1|)\|b\|_{BMO}, for Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{k+1}Q\setminus 2^kQ$,

$$\begin{aligned}
|R_m(\tilde{A};x,y)|&\leq C|x-y|^m\sum_{|\beta|=m}(\|D^\beta A\|_{BMO}+|(D^\beta A)_{Q(x,y)}-(D^\beta A)_Q|)\\
&\leq Ck|x-y|^m\sum_{|\beta|=m}\|D^\beta A\|_{BMO}.
\end{aligned}$$

Thus, similar to the proof of Lemma 3, we obtain

$$\begin{aligned}
\|I_1^t(x)\| &\le C\int_{R^n\setminus 2Q}\frac{|x||f(z)|}{|z|^{n+m+1-\delta}}|R_m(\tilde{A};x,z)|dz\\
&= C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\frac{|x||f(z)|}{|z|^{n+m+1-\delta}}|R_m(\tilde{A};x,z)|dz\\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k\frac{|x|}{|z|^{n+1-\delta}}|f(z)|dz\\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=1}^{\infty}k2^{-k}(2^kd)^{-n(1/p-\delta/n)}\|f\chi_{2^{k+1}Q}\|_{L^p}\\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=1}^{\infty}k2^{-k}\|f\|_{B_p^\delta}\\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.
\end{aligned}$$

For $I_2^t(x)$, by the formula (see [5]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};0,y)=\sum_{|\gamma|<m}\frac{1}{\gamma!}R_{m-|\gamma|}(D^\gamma\tilde{A};x,0)(x-y)^\gamma$$

and Lemma 2, we get

$$\begin{aligned}
&|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|\\
&\qquad\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\left(|x-x_0|^m+\sum_{0<|\gamma|<m}|x_0-z|^{m-|\gamma|}|x-x_0|^{|\gamma|}\right),
\end{aligned}$$

thus, similar to the estimates of $I_1^t(x)$ and Lemma 3, we get, for $x\in Q$

$$\begin{aligned}
\|I_2^t(x)\| &\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\frac{|x|}{|z|^{n+1-\delta}}|f(z)|dz\\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.
\end{aligned}$$

For $I_3^t(x)$, by the inequality: $a^{1/2}-b^{1/2}\le(a-b)^{1/2}$ for $a\ge b>0$, we obtain, similar to the estimate of Lemma 3 and I_1,

$$\begin{aligned}
&\|I_3^t(x)\| \\
&\le C\int_{R^n\setminus 2Q}\left(\int_{R^{n+1}_+}\left[\frac{t^{n\lambda/2}|x|^{1/2}|\psi_t(y-z)\|f(z)|}{(t+|x-y|)^{(n\lambda+1)/2}|z|^m}|R_m(\tilde{A};0,z)|\right]^2\frac{dydt}{t^{n+1}}\right)^{1/2}dz \\
&\le C\int_{R^n\setminus 2Q}\frac{|f(z)\|x|^{1/2}|R_m(\tilde{A};0,z)|}{|z|^m} \\
&\quad\times\left(\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\lambda+1}\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right)^{1/2}dz \\
&\le C\int_{R^n\setminus 2Q}\frac{|f(z)\|x|^{1/2}|R_m(\tilde{A};0,z)|}{|z|^m}\left(\int_0^\infty\frac{dt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2}dz \\
&\le C\int_{R^n\setminus 2Q}\frac{|f(z)\|x|^{1/2}|R_m(\tilde{A};0,z)|}{|z|^{m+n+1/2-\delta}}dz \\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=1}^\infty 2^{-k/2}(2^kd)^{-n(1/p-\delta/n)}\|f\chi_{2^{k+1}Q}\|_{L^p} \\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.
\end{aligned}$$

For $I_4^t(x)$, similar to the estimates of $I_1^t(x)$ and $I_3^t(x)$, by Hölder's inequality, we get

$$\begin{aligned}
\|I_4^t(x)\| &\le C\sum_{|\beta|=m}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left(\frac{|x|}{|y|^{n+1-\delta}}+\frac{|x|^{1/2}}{|y|^{n+1/2-\delta}}\right)|D^\beta\tilde{A}(y)\|f(y)|dy \\
&\le C\sum_{|\beta|=m}\sum_{k=1}^\infty(2^{-k}+2^{-k/2})(2^kd)^{\delta-n/p} \\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|D^\beta A(y)-(D^\beta A)_{\tilde{Q}}|^{p'}dy\right)^{1/p'}\|f\chi_{2^{k+1}Q}\|_{L^p} \\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=1}^\infty(2^{-k}+2^{-k/2})(2^kd)^{-n(1/p-\delta/n)}\|f\chi_{2^{k+1}Q}\|_{L^p} \\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.
\end{aligned}$$

Thus

$$\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_{t,1}^A(f)(x,y)-\left(\frac{t}{t+|y|}\right)^{n\lambda/2}F_{t,1}^A(f)(0,y)\right\|\le C\|f\|_{B_p^\delta}.$$

For μ_λ^A, we write, for $x \in Q$,

$$\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_{t,2}^{\tilde{A}}(f)(x,y) - \left(\frac{t}{t+|y|}\right)^{n\lambda/2} F_{t,2}^{\tilde{A}}(f)(0,y)$$
$$= \int_{|y-z|\le t} \left[\frac{1}{|x-z|^m} - \frac{1}{|z|^m}\right] \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \frac{\Omega(y-z)R_m(\tilde{A};x,z)f(z)}{|y-z|^{n-1-\delta}} dz$$
$$+ \int_{|y-z|\le t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \frac{\Omega(y-z)f(z)}{|y-z|^{n-1-\delta}|z|^m} [R_m(\tilde{A};x,z) - R_m(\tilde{A};0,z)] dz$$
$$+ \int_{|y-z|\le t} \left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} - \left(\frac{t}{t+|y|}\right)^{n\lambda/2}\right] \frac{\Omega(y-z)R_m(\tilde{A};0,z)f(z)}{|y-z|^{n-1-\delta}|z|^m} dz$$
$$- \sum_{|\beta|=m} \frac{1}{\beta!} \int_{|y-z|\le t} \left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \frac{(x-z)^\beta}{|x-z|^m} - \left(\frac{t}{t+|y|}\right)^{n\lambda/2} \frac{(-z)^\beta}{|z|^m}\right]$$
$$\times \frac{\Omega(y-z)D^\beta A(z)f(z)}{|y-z|^{n-1-\delta}} dz$$
$$:= J_1^t(x) + J_2^t(x) + J_3^t(x) + J_4^t(x).$$

Similar to the proof of Lemma 3 and g_λ^A, we obtain

$$\|J_1^t(x)\| \le C \int_{R^n \setminus 2Q} \frac{|x||f(z)|}{|z|^{n+m+1-\delta}} |R_m(\tilde{A};x,z)| dz \le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \|f\|_{B_p^\delta}$$

and

$$\|J_2^t(x)\| \le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^k Q} \frac{|x|}{|z|^{n+1-\delta}} |f(z)| dz$$
$$\le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \|f\|_{B_p^\delta}.$$

For $J_3^t(x)$, similar to the estimates of Lemma 3 and $I_3^t(x)$, we obtain

$$\|J_3^t(x)\| \le C \int_{R^n \setminus 2Q} \left(\int_{R_+^{n+1}} \left[\frac{t^{n\lambda/2}|x|^{1/2}}{(t+|x-y|)^{(n\lambda+1)/2}}\right.\right.$$
$$\left.\left. \times \frac{|f(z)||\Omega(y-z)|\chi_{\Gamma(z)}(y,t)}{|y-z|^{n-1-\delta}|z|^m} |R_m(\tilde{A};0,z)|\right]^2 \frac{dydt}{t^{n+3}}\right)^{1/2} dz$$
$$\le C \int_{R^n \setminus 2Q} \frac{|f(z)||R_m(\tilde{A};0,z)||x|^{1/2}}{|z|^m}$$
$$\times \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda+1} \frac{t^{-n}\chi_{\Gamma(z)}(y,t)}{|y-z|^{2n+2-2\delta}} dydt\right]^{1/2} dz$$

$$\leq C\int_{R^n\setminus 2Q}\frac{|f(z)||R_m(\tilde{A};0,z)||x|^{1/2}}{|z|^{m+n+1/2-\delta}}dz$$
$$\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.$$

For $J_4^t(x)$, similar to the proof of $J_1^t(x)$, $J_3^t(x)$ and $I_4^t(x)$, we obtain

$$\begin{aligned}\|J_4^t(x)\| &\leq C\sum_{|\beta|=m}\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left(\frac{|x|}{|y|^{n+1-\delta}}+\frac{|x|^{1/2}}{|y|^{n+1/2-\delta}}\right)|D^\beta\tilde{A}(y)||f(y)|dy\\ &\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=1}^{\infty}k(2^{-k}+2^{-k/2})(2^kd)^{-n(1/p-\delta/n)}\|f\chi_{2^{k+1}Q}\|_{L^p}\\ &\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{B_p^\delta}.\end{aligned}$$

Thus

$$\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_{t,2}^A(f)(x,y)-\left(\frac{t}{t+|y|}\right)^{n\lambda/2}F_{t,2}^A(f)(0,y)\right\|\leq C\|f\|_{B_p^\delta}.$$

These yield the desired results and complete the proof of Theorem 1. □

Proof of Theorem 2. We only give the proof on homogeneous weighted Herz and Herz Hardy space. To be simply, we denote $\tilde{T}_\lambda^A=\tilde{g}_\lambda^A$ or $\tilde{\mu}_\lambda^A$. Let $f\in H\dot{K}_p(R^n)$, by Lemma 1, $f=\sum_{j=-\infty}^{\infty}\lambda_ja_j$, where $a_j's$ are the central $(n(1-1/p),p)$-atom with supp$a_j\subset B_j=B(0,2^j)$ and $\|f\|_{H\dot{K}_p}\approx\sum_j|\lambda_j|$. We write

$$\begin{aligned}\|\tilde{T}_\lambda^A(f)\|_{\dot{K}_p^\delta} &= \sum_{k=-\infty}^{\infty}2^{k\alpha}\|\chi_k\tilde{T}_\lambda^A(f)\|_{L^q}\\ &\leq \sum_{k=-\infty}^{\infty}2^{k\alpha}\sum_{j=-\infty}^{k-1}|\lambda_j|\|\chi_k\tilde{T}_\lambda^A(a_j)\|_{L^q}+\sum_{k=-\infty}^{\infty}2^{k\alpha}\sum_{j=k}^{\infty}|\lambda_j|\|\chi_k\tilde{T}_\lambda^A(a_j)\|_{L^q}\\ &= L+LL.\end{aligned}$$

For LL, by the following equality

$$Q_{m+1}(A;x,y)=R_{m+1}(A;x,y)+\sum_{|\beta|=m}\frac{1}{\beta!}(x-y)^\beta(D^\beta A(x)-D^\beta A(y)),$$

we have, similar to the proof of Lemma 3,

$$\tilde{T}_\lambda^A(f)(x)\leq T_\lambda^A(f)(x)+C\sum_{|\beta|=m}\int_{R^n}\frac{|D^\beta A(x)-D^\beta A(y)|}{|x-y|^{n-\delta}}|f(y)|dy,$$

thus, $\tilde{T}_\lambda^A$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/\delta$ with $1/q = 1/p - \delta/n$ by Lemma 3 and [1]. We see that

$$\begin{aligned} LL &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha} \sum_{j=k}^{\infty} |\lambda_j| \|a_j\|_{L^p} \le C \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} |\lambda_j| 2^{-jn(1-1/p)} \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=-\infty}^{j} 2^{(k-j)n(1-1/p)} \le C \sum_{j=-\infty}^{\infty} |\lambda_j| \le C \|f\|_{H\dot{K}_p}. \end{aligned}$$

To estimate L, we denote that $\tilde{A}(x) = A(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_{2B_j} x^\beta$. Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$ and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\beta|=m} \frac{1}{\beta!}(x-y)^\beta D^\beta A(x)$.

For g_λ^A, we write, by the vanishing moment of a and for $x \in B_k$ with $k \ge j+1$,

$$\begin{aligned} \tilde{F}_{t,1}^A(a_j)(x,y) &= \int_{R^n} \frac{\psi_t(y-z) R_m(\tilde{A};x,z)}{|x-z|^m} a_j(z) dz \\ &\quad - \sum_{|\beta|=m} \frac{1}{\beta!} \int \frac{\psi_t(y-z) D^\beta \tilde{A}(z)(x-z)^\beta}{|x-z|^m} a_j(z) dz \\ &= \int_{R^n} \left[\frac{\psi_t(y-z) R_m(\tilde{A};x,z)}{|x-z|^m} - \frac{\psi_t(y) R_m(\tilde{A};x,0)}{|x|^m} \right] a_j(z) dz \\ &\quad - \sum_{|\beta|=m} \frac{1}{\beta!} \int_{R^n} \left[\frac{\psi_t(y-z)(x-z)^\beta}{|x-z|^m} - \frac{\psi_t(y) x^\beta}{|x|^m} \right] D^\beta \tilde{A}(x) a_j(z) dz, \end{aligned}$$

similar to the proof of Lemma 3 and Theorem 1, we obtain

$$\begin{aligned} &\|\tilde{F}_{t,1}^A(a_j)(x,y)\| \\ &\quad \le C \int_{R^n} \left[\frac{|z|}{|x|^{m+n+1}} + \frac{|z|^{1/2}}{|x|^{m+n+1/2}} \right] |R_m(\tilde{A};x,z)| |a_j(z)| dz \\ &\qquad + C \sum_{|\beta|=m} \int_{R^n} \left[\frac{|z|}{|x|^{n+1-\delta}} + \frac{|z|^{1/2}}{|x|^{n+1/2-\delta}} \right] |D^\beta \tilde{A}(x)| |a_j(z)| dz \\ &\quad \le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left[\frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j/2}}{2^{k(n+1/2-\delta)}} \right] \\ &\qquad + C \sum_{|\beta|=m} \left[\frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j/2}}{2^{k(n+1/2-\delta)}} \right] |D^\beta \tilde{A}(x)|, \end{aligned}$$

thus

$$\begin{aligned}
L &\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=-\infty}^{k-1}|\lambda_j| \\
&\quad\times\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right]2^{kn/q} \\
&\quad + C\sum_{|\beta|=m}\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=-\infty}^{k-1}|\lambda_j| \\
&\quad\times\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right]\left(\int_{B_k}|D^\beta\tilde{A}(x)|^q dx\right)^{1/q} \\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{k=-\infty}^{\infty}2^{kn(1-\delta/n)}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right] \\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=j+1}^{\infty}[2^{j-k}+2^{(j-k)/2}] \\
&\le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=-\infty}^{\infty}|\lambda_j| \le C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\|f\|_{H\dot{K}_p}.
\end{aligned}$$

A same argument as in the proof of Theorem 1 will give the proof of $\tilde{\mu}_\lambda^A$, we omit the details. This completes the proof of Theorem 2. □

Proof of Theorem 3. We only give the proof of $\tilde{g}_\lambda^A$. For any cube $Q = Q(0,d)$ with $d>1$, let $f\in B_p(w)$ and $\tilde{A}(x) = A(x) - \sum_{|\beta|=m}\frac{1}{\beta!}(D^\beta A)_{\tilde{Q}}x^\beta$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u\in 3Q\setminus 2Q$,

$$\begin{aligned}
\tilde{F}_{t,1}^A(f)(x,y) &= \tilde{F}_{t,1}^A(f_1)(x,y) + \int_{R^n}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)f_2(z)dz \\
&- \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q)\int_{R^n}\left[\frac{(x-z)^\alpha}{|x-z|^m} - \frac{(u-z)^\alpha}{|u-z|^m}\right]\psi_t(y-z)f_2(z)dz \\
&- \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q)\int_{R^n}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(y-z)f_2(z)dz,
\end{aligned}$$

then

$$\begin{aligned}
&\left|\tilde{g}_\lambda^A(f)(x) - g_\lambda\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m}f_2\right)(0)\right| \\
&= \left\|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}\tilde{F}_{t,1}^A(f)(x,y)\right|\right\|
\end{aligned}$$

$$
\begin{aligned}
&- \left\|\left(\frac{t}{t+|y|}\right)^{n\lambda/2} F_{t,1}\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m} f_2\right)(0)\right\|\Bigg| \\
&\le \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \tilde{F}^A_{t,1}(f)(x,y) - \left(\frac{t}{t+|y|}\right)^{n\lambda/2} F_{t,1}\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m} f_2\right)(0)\right\| \\
&\le \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \tilde{F}^A_{t,1}(f_1)(x,y)\right\| \\
&\quad + \left\|\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \int_{R^n} \frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)\right.\right. \\
&\quad - \left.\left.\left(\frac{t}{t+|y|}\right)^{n\lambda/2} \int_{R^n} \frac{R_m(\tilde{A};0,z)}{|z|^m}\psi_t(-z)\right] f_2(z)dz\right\| \\
&\quad + \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A(x) - (D^\beta A)_Q)\right. \\
&\quad \times \left.\int_{R^n} \left[\frac{(y-z)^\beta}{|y-z|^m} - \frac{(u-z)^\beta}{|u-z|^m}\right]\psi_t(y-z) f_2(z)dz\right\| \\
&\quad + \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \sum_{|\beta|=m} \frac{1}{\beta!}|D^\beta A(x) - (D^\beta A)_Q|\right. \\
&\quad \times \left.\int_{R^n} \frac{(u-z)^\beta}{|u-z|^m}\psi_t(y-z) f_2(z)dz\right\| \\
&= M_1(x) + M_2(x) + M_3(x,u) + M_4(x,u).
\end{aligned}
$$

By the the $L^p(R^n)$ to $L^q(R^n)$-boundedness of $\tilde{g}^A_\lambda$ for $1 < p < n/\delta$ with $1/q = 1/p - \delta/n$, we get

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q M_1(x)dx &\le C\left(\frac{1}{|Q|}\int_Q |\tilde{g}^A_\lambda(f_1)(x)|^q dx\right)^{1/q} \\
&\le C|Q|^{-1/q}\|f_1\|_{L^p} \le C\|f\|_{B^\delta_p};
\end{aligned}
$$

Similar to the proof of Theorem 1, we obtain

$$
\frac{1}{|Q|}\int_Q M_2(x)dx \le C\|f\|_{B^\delta_p}
$$

and

$$
\frac{1}{|Q|}\int_Q M_3(x,u)dx \le C\|f\|_{B^\delta_p}.
$$

Thus, using the estimates of $M_4(x,u)$, we obtain

$$
\frac{1}{|Q|}\int_Q \left|\tilde{g}^A_\lambda(x) - g_\lambda\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m} f_2\right)(0)\right| dx \le C\|f\|_{B^\delta_p}.
$$

This completes the proof of Theorem 3. □

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References

[1] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), 7–16.

[2] W. Chen and G. Hu, *Weak type* (H^1, L^1) *estimate for multilinear singular integral operator*, Adv. Math. (China) **30** (2001), 63–69.

[3] J. Cohen, *A sharp estimate for a multilinear singular integral on* R^n, Indiana Univ. Math. J. **30** (1981), 693–702.

[4] J. Cohen and J. Gosselin, *On multilinear singular integral operators on* R^n, Studia Math. **72** (1982), 199–223.

[5] ———, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math. **30** (1986), 445–465.

[6] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.

[7] Y. Ding and S. Z. Lu, *Weighted boundedness for a class rough multilinear operators*, Acta Math. Sinica **17** (2001), 517–526.

[8] J. Garcia-Cuerva, *Hardy spaces and Beurling algebras*, J. London Math. Soc. **39** (1989), 499–513.

[9] J. Garcia-Cuerva and M. L. Herrero, *A theory of Hardy spaces associated to the Herz spaces*, Proc. London Math. Soc. **69** (1994), 605–628.

[10] E. Harboure, C. Segovia and J. L. Torrea, *Boundedness of commutators of fractional and singular integrals for the extreme values of* p, Illinois J. Math. **41** (1997), 676–700.

[11] L. Z. Liu, *Continuity for commutators of Littlewood-Paley operator on certain Hardy spaces*, J. Korean Math. Soc. **40** (1) (2003), 41–60.

[12] S. Z. Lu and D. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. China Ser. A **38** (1995), 147–158.

[13] ———, *The weighted Herz type Hardy spaces and its applications*, Sci. China Ser. A **38** (1995), 662–673.

[14] E. M. Stein, *Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[15] A. Torchinsky, *The real variable methods in harmonic analysis*, Pure Appl. Math. **123**, Academic Press, New York, 1986.

[16] A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math. **60/61** (1990), 235–243.