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The stability of the family of A_2-type arrangements

By

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Abstract

We give a necessary and sufficient condition for the stability and the freeness of the family of A_2-type arrangements. Moreover, we determine explicitly when the normalization of the sheafification of its module of reduced logarithmic vector fields is isomorphic to $T_{\mathbf{P}^2}(-2)$.

0. Introduction

A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of codimension one affine hyperplanes of a fixed vector space V. We say $\mathcal{A}$ is *central* if each hyperplane is a vector subspace of V. For each central hyperplane arrangement $\mathcal{A}$, we can define the $S(:= \mathrm{Sym}(V^*))$-module $D(\mathcal{A}) \simeq D_0(\mathcal{A}) \oplus S(-1)$ (see Definition 1.1 and 1.7), and call $D(\mathcal{A})$ (resp:$D_0(\mathcal{A})$) *the module of logarithmic vector fields* (resp:*the module of reduced logarithmic vector fields*). Roughly speaking, these modules consist of derivations of S tangent to hyperplanes in $\mathcal{A}$. In the study of hyperplane arrangements, the structure of $D(\mathcal{A})$ is intensively studied. For example, we say an arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module, and the combinatorial characterization of free arrangements is one of the most important problems in the arrangement theory. On the other hand, recently, instead of the module $D_0(\mathcal{A})$ the sheafification $\widetilde{D_0(\mathcal{A})}$ is also studied. It is known that $\widetilde{D_0(\mathcal{A})}$ is a reflexive sheaf, and the study gives new insight into the theory of arrangements. These results are often obtained by using algebraic geometry. In algebraic geometry, the stability (or the semistability) of torsion free sheaves on the projective space $\mathbf{P}^n$ is an important concept and plays a key role in the vector bundle theory and moduli problem (for the definition of the stability, see Definition 1.9). We say an arrangement $\mathcal{A}$ in a vector space V is *stable* (resp:*semistable*) if the reflexive (and hence torsion free) sheaf $\widetilde{D_0(\mathcal{A})}$ is a stable (resp:semistable) torsion free sheaf on $\mathbf{P}(V)$. The stability of normal crossing arrangements was studied in [1], and some criterions for the stability of arrangements were given in [8]. In this article we study the stability of the family of arrangements which are not normal crossing in codimension two. To

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apply algebraic geometry to arrangement theory, it is important to study the stability of arrangements. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We say $\{\mathcal{A}(k)\}_{k\geq 1}^{\infty}$ is a *family of arrangements in a* $\mathbb{K}$*-vector space* V if it consists of the set of arrangements $\mathcal{A}(k)$ $(k \in \mathbb{Z}_{k>0})$ in a fixed vector space V satisfying $\mathcal{A}(k) \subsetneqq \mathcal{A}(k+1)$ for all k. To state the main theorem, we introduce the following definition of the family of arrangements.

Definition 0.1. Let $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ be a family of arrangements.

(a) The family $\{\mathcal{A}(k)\}$ is called *free* if there exists an integer k_0 such that $\mathcal{A}(k)$ is a free arrangement for $k \geq k_0$.

(b) The family $\{\mathcal{A}(k)\}$ in a vector space V is called *stable* if there exists an integer k_0 such that $\widetilde{D_0(\mathcal{A}(k))}$ is a stable reflexive sheaf on $\mathbf{P}(V)$ for $k \geq k_0$.

For the rest of this section, let V be a three-dimensional vector space over $\mathbb{K}$ unless otherwise specified, and $\{X, Y, Z\}$ be a basis for V^*. In this article, we consider the stability of the following arrangements.

Definition 0.2. An arrangement $\mathcal{A}$ in V is called an A_2*-type arrangement* if $\mathcal{A}$ is defined by

$$\begin{aligned} X &= a_1 Z,\ (a_1+1)Z, \ldots, a_2 Z, \\ Y &= b_1 Z,\ (b_1+1)Z, \ldots, b_2 Z, \\ X+Y &= c_1 Z,\ (c_1+1)Z, \ldots, c_2 Z, \\ Z &= 0, \end{aligned}$$

where $a_1 \leq a_2,\ b_1 \leq b_2$, and $c_1 \leq c_2$ are integers.

Definition 0.3. A family of arrangements $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ in V is called a *family of* A_2*-type arrangements* if $\mathcal{A}(k)$ is an A_2-type arrangement defined by

$$\begin{aligned} X &= (-k+a_1+1)Z, \ldots, (k+a_2-1)Z, \\ Y &= (-k+b_1+1)Z, \ldots, (k+b_2-1)Z, \\ X+Y &= (-k+c_1+1)Z, \ldots, (k+c_2-1)Z, \\ Z &= 0, \end{aligned}$$

where $a_1 \leq a_2,\ b_1 \leq b_2$, and $c_1 \leq c_2$ are integers.

For example, the pictures of $\{\mathcal{A}(k)\}|_{Z=1}$ $(k = 1, 2, 3)$ when $(a_1, a_2, b_1, b_2, c_1, c_2) = (0, 0, 0, 0, 0, 0)$ are drawn in Figure 1.

Note that the initial arrangement $\mathcal{A}(1)$ determines the family of A_2-type arrangements. Since the arrangement in Definition 0.2 is just the cone over affine arrangements associated with the root system of type A_2, we call it the A_2-type arrangement. Note that the A_2-type arrangement is a special case of the *deformation of the Coxeter arrangement of type* A_2 defined in [6]. It is obvious that these arrangements are not normal crossing in codimension

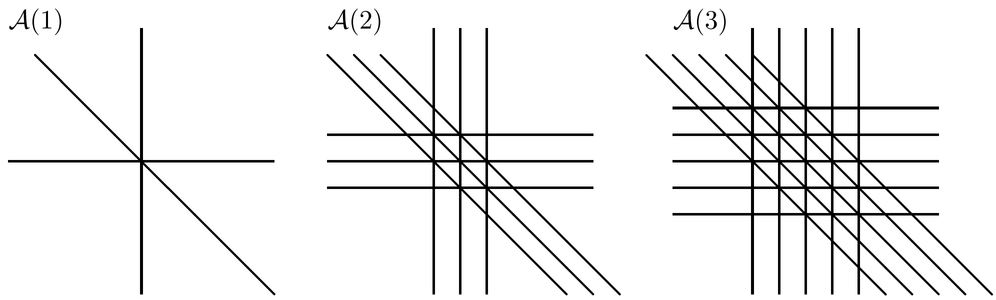

Figure 1. Example of A_2-type arrangements

two. By induction it is easy to see that any family of A_2-type arrangements is contained in the family such that the initial arrangement $\mathcal{A}(1)$ is (1) or (2) below.

$$\begin{aligned} X &= 0,\ Z, \dots,\ cZ\ (c \geq 0), \\ Y &= 0, \\ X + Y &= (a-1)Z, \dots, (a+b)Z\ (b \geq -1), \\ Z &= 0. \end{aligned} \tag{1}$$

$$\begin{aligned} X &= 0,\ Z, \dots,\ cZ\ (c \geq 0), \\ Y &= 0,\ Z, \\ X + Y &= (a-1)Z, \dots, (a+b)Z\ (b \geq -1), \\ Z &= 0. \end{aligned} \tag{2}$$

Here $a,\ b,\ c$ are integers. Now we can state the main results in this article. See Definition 1.11 for the definition of the normalization of vector bundles.

Theorem 0.4. *Let $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ be the family of A_2-type arrangements in V such that $\mathcal{A}(1)$ is defined by*

$$\begin{aligned} X &= 0,\ Z, \dots, cZ\ (c \geq 0), \\ Y &= 0, \\ X + Y &= (a-1)Z, \dots,\ (a+b)Z\ (b \geq -1), \\ Z &= 0. \end{aligned}$$

Then the following holds:

(a) *The family $\{\mathcal{A}(k)\}$ is free if and only if*

$$2a + b - c = 0,\ 1,\ 2.$$

(b) *There exists an integer k_0 such that the normalization of $D_0(\widetilde{\mathcal{A}(k)})$ is isomorphic to $T_{\mathbf{P}^2} \otimes \mathcal{O}(-2)$ for $k \geq k_0$ if and only if*

$$2a + b - c = -1,\ 3.$$

(*c*) *The family* $\{\mathcal{A}(k)\}$ *is stable if*

$$2a+b-c>3,$$

or

$$2a+b-c<-1.$$

Theorem 0.5. *Let* $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ *be the family of* A_2*-type arrangements in* V *such that* $\mathcal{A}(1)$ *is defined by*

$$\begin{aligned} X &= 0,\ Z, \ldots, cZ\ (c \geq 0), \\ Y &= 0,\ Z, \\ X+Y &= (a-1)Z, \ldots,\ (a+b)Z\ (b \geq 1), \\ Z &= 0. \end{aligned}$$

Then the following holds:

(*a*) *The family* $\{\mathcal{A}(k)\}$ *is free if and only if*

$$2a+b-c=1,\ 2,\ 3.$$

(*b*) *There exists an integer* k_0 *such that the normalization of* $\widetilde{D_0(\mathcal{A}(k))}$ *is isomorphic to* $T_{\mathbf{P}^2} \otimes \mathcal{O}(-2)$ *for* $k \geq k_0$ *if and only if*

$$2a+b-c=0,\ 4.$$

(*c*) *The family* $\{\mathcal{A}(k)\}$ *is stable if*

$$2a+b-c>4,$$

or

$$2a+b-c<0.$$

From these theorems, we can see that the stability and the freeness of $\widetilde{D_0(\mathcal{A}(k))}$ are determined by the combinatorics of $\mathcal{A}(k)$. Moreover, we give a partial answer to the 3-shift problem, which is the conjecture on the root system posed by Yoshinaga (see Remark 3.3).

The organization of this article is as follows. In Section 1, we review some definitions and results on hyperplane arrangements and the stability of vector bundles on the projective space $\mathbf{P}^n$. In Section 2, we prove Theorem 0.4 through several steps. In Section 3, we prove Theorem 0.5. Since the proof is parallel to that of Theorem 0.4, we give an outline of the proof. As a corollary of these proofs, we give a partial answer to the 3-shift problem.

Notation. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{K}$ denotes an algebraically closed field of characteristic zero. In this article, let the variables a, b, c denote integers. For a vector space V over $\mathbb{K}$, V^* denotes the dual vector space of

V. Let $S = \oplus_{d \in \mathbb{Z}} S_d$ be a commutative graded ring with a unit, where S_d is a homogeneous part of S with degree d. We assume that S is noetherian, $S_d = 0$ for all $d < 0$, $S_0 = \mathbb{K}$ and S is generated by S_1 as a $\mathbb{K}$-algebra for every graded ring S. $\mathrm{Der}_{\mathbb{K}}(S)$ is the S-module of $\mathbb{K}$-linear derivations of S. For any integer $d \in \mathbb{Z}$ and a graded S-module M which is finitely generated over S, M_d is a homogeneous part of M with degree d. We assume that $M_d = 0$ for all $d < 0$. $\widetilde{M}$ denotes the sheafification of M, so $\widetilde{M}$ is a coherent sheaf on $\mathbf{Proj}(S)$. For a vector bundle E on the projective space $\mathbf{P}^n_{\mathbb{K}}$, $c_i(E)$ denotes the i-th Chern class of E and we put the Chern polynomial $c_t(E)$ of E as

$$c_t(E) := \sum_{i=0}^{n} c_i(E) t^i.$$

For a finite set A, its cardinality is denoted by $|A|$.

1. Preliminaries

In this section, we review some elementary definitions which will be used in this article. First we recall those of hyperplane arrangements, for which we refer the reader to [5]. Let us fix an l-dimensional $\mathbb{K}$-vector space $V \simeq \mathbb{K}^l$. A *hyperplane arrangement* (or a *simple arrangement*) $\mathcal{A}$ is a finite collection of codimension one affine subspaces in V. We often say an "arrangement" instead of a "hyperplane arrangement", and call an arrangement in an l-dimensional vector space an "l-arrangement". We say an arrangement $\mathcal{A}$ is *central* if each hyperplane in $\mathcal{A}$ is a vector subspace of V. In this article, we assume all arrangements are "central" and non-empty. Note we can regard a central l-arrangement as the arrangement in $\mathbf{P}^{l-1} \simeq \mathbf{P}(V)$. Let $\{X_1, \ldots, X_l\}$ be a basis for V^* and put $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[X_1, \ldots, X_l]$. For each hyperplane $H \in \mathcal{A}$, let us fix a nonzero linear form $\alpha_H \in V^*$ such that its kernel is H, and put

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H.$$

Definition 1.1. For an arrangement $\mathcal{A}$, the S-module $D(\mathcal{A})$ is defined by

$$\begin{aligned} D(\mathcal{A}) :&= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H \ (\forall H \in \mathcal{A})\} \\ &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q(\mathcal{A})) \in S \cdot Q(\mathcal{A})\}. \end{aligned}$$

We call $D(\mathcal{A})$ the *module of logarithmic vector fields* (with respect to $\mathcal{A}$). We say a nonzero element $\theta = \sum_{i=1}^{l} f_i \dfrac{\partial}{\partial X_i} \in D(\mathcal{A})$ is *homogeneous of degree* p if $f_i \in S_p$ for $1 \leq i \leq l$. An arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module. When $\mathcal{A}$ is free, there exists a homogeneous basis $\{\theta_1, \ldots, \theta_l\}$ for $D(\mathcal{A})$. Then the *exponents* of a free arrangement $\mathcal{A}$ are defined by

$$\exp(\mathcal{A}) := (\deg(\theta_1), \ldots, \deg(\theta_l)).$$

It is known that $\exp(\mathcal{A})$ do not depend on the choice of a basis.

Next, we define a multiarrangement, which was introduced and studied by Ziegler in [13].

Definition 1.2 ([13]). We say a pair $(\mathcal{A}, m)$ is a *multiarrangement* if $\mathcal{A}$ is a simple arrangement and

$$m : \mathcal{A} \to \mathbb{Z}_{>0}$$

is a map from $\mathcal{A}$ to positive integers. The map m is called a *multiplicity function*.

A simple arrangement $\mathcal{A}$ can be thought of as a multiarrangement with $m \equiv 1$. By the same way as for simple arrangements, we define the module of logarithmic vector fields $D(\mathcal{A}, m)$ for a multiarrangement $(\mathcal{A}, m)$.

Definition 1.3. For a multiarrangement $(\mathcal{A}, m)$, the S-module $D(\mathcal{A}, m)$ is defined by

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \ (\forall H \in \mathcal{A})\}.$$

Let $H_0 \in \mathcal{A}$ be a hyperplane in an arrangement $\mathcal{A}$. The restriction of $\mathcal{A}$ to H_0 is a simple arrangement $\mathcal{A} \cap H_0 := \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}$. This restriction has a natural structure of the multiarrangement $(\mathcal{A} \cap H_0,\, m)$, i.e., the multiplicity function $m : \mathcal{A} \cap H_0 \to \mathbb{Z}_{>0}$ is defined by

$$m : \mathcal{A} \cap H_0 \ni H' \mapsto |\{H \in \mathcal{A} \mid H \cap H_0 = H'\}|.$$

For details, see [13] or [12]. It is known that $D(\mathcal{A}, m)$ is a reflexive module (e.g., see Theorem 4.75 in [5] and Theorem 5 in [13]). We can define the freeness and exponents of the multiarrangements by the same way as for simple arrangements. The exponents of a free multiarrangement are sometimes called *multi-exponents*. The following theorem due to K. Saito is useful to see the freeness of an arrangement and determine its basis.

Theorem 1.4 (Saito's criterion). *Let $(\mathcal{A}, m)$ be an l-multiarrangement, $D(\mathcal{A}, m)$ be its module of logarithmic vector fields, and $\theta_1, \ldots, \theta_l \in D(\mathcal{A}, m)$ be homogeneous elements. Then the following two conditions are equivalent*:

(1) $\{\theta_1, \ldots, \theta_l\}$ *forms a basis for $D(\mathcal{A}, m)$ over S.*

(2) $\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_l = u \prod_{H \in \mathcal{A}} \alpha_H^{m(H)} (\frac{\partial}{\partial X_1} \wedge \ldots \wedge \frac{\partial}{\partial X_l})$ *for some $u \in \mathbb{K} \backslash \{0\}$.*

For the proof, see Theorem 4.19 in [5] and Theorem 8 in [13]. We often consider the sheafification of the S-module $D(\mathcal{A})$, and its Chern polynomial can be calculated from the combinatorics of $\mathcal{A}$. To see this, let us introduce some notations. The *characteristic polynomial* of an arrangement $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) := \sum_{X \in L_{\mathcal{A}}} \mu(X) t^{\dim X},$$

where $L_{\mathcal{A}}$ is a lattice which consists of the intersections of elements of $\mathcal{A}$, ordered by reverse inclusion, $\hat{0} := V$ is the unique minimal element of $L_{\mathcal{A}}$ and $\mu : L_{\mathcal{A}} \longrightarrow \mathbb{Z}$ is the Möbius function defined as follows:

$$\begin{aligned} &\mu(\hat{0}) = 1, \\ &\mu(X) = -\sum_{Y<X} \mu(Y), \text{ if } \hat{0} < X. \end{aligned}$$

It is known that for a central arrangement $\mathcal{A}$, its characteristic polynomial $\chi(\mathcal{A}, t)$ can be divided by $(t-1)$. Moreover, the *reduced characteristic polynomial* $\chi_0(\mathcal{A}, t)$ is defined by

$$\chi_0(\mathcal{A}, t) := \chi(\mathcal{A}, t)/(t-1)$$

and the *Poincaré polynomial* $\pi(\mathcal{A}, t)$ by

$$\pi(\mathcal{A}, t) := \sum_{X \in L_{\mathcal{A}}} \mu(X)(-t)^{\operatorname{codim} X}.$$

The polynomials $\chi(\mathcal{A},\, t)$ and $\pi(\mathcal{A},\, t)$ are related as follows:

$$\chi(\mathcal{A},\, t) = t^l \pi(\mathcal{A},\, -1/t),$$

and these polynomials are important concepts in the theory of hyperplane arrangements. Actually there are a lot of combinatorial or geometric interpretations of the characteristic polynomial. For details, see [5]. We can use $\pi(\mathcal{A}, t)$ to calculate the Chern polynomial.

Theorem 1.5 ([3, Theorem 4.1])**.** *For a polynomial $F(t) \in \mathbb{Z}[t]$, let $\overline{F(t)}$ denote the class of $F(t)$ in $\mathbb{Z}[t]/(t^l)$. Let $\mathcal{A}$ be a central l-arrangement and assume $\widetilde{D(\mathcal{A})}$ is a vector bundle on $\mathbf{P}(V)$. Then it holds that*

$$c_t(\widetilde{D(\mathcal{A})}) = \overline{\pi(\mathcal{A}, -t)}.$$

In particular, if $l = 3$ and

$$\chi_0(\mathcal{A}, t) = t^2 - c_1 t + c_2,$$

then for any central 3-arrangement $\mathcal{A}$ it holds that

$$c_t(\widetilde{D(\mathcal{A})}) = (1 - c_1 t + c_2 t^2)(1 - t).$$

Recently, Wakamiko gave exponents and an explicit basis for the module of logarithmic vector fields of any 2-multiarrangement consisting of three lines in [11]. We use the exponents in this article.

Theorem 1.6 ([11, Theorem 3.8]). *Let V be a two-dimensional vector space, $\{X, Y\}$ be a basis for V^*, $S := Sym(V^*)$, and $(\mathcal{A}, m)$ be a multiarrangement in V such that*

$$\mathcal{A} = \{X = 0,\ Y = 0,\ X + Y = 0\}$$

and that

$$\begin{aligned} m(\{X = 0\}) &= k_1, \\ m(\{Y = 0\}) &= k_2, \\ m(\{X + Y = 0\}) &= k_3. \end{aligned}$$

Here, $k_1,\ k_2,\ k_3 \in \mathbb{Z}_{>0}$ and assume that

$$k_3 \geq \max\{k_1, k_2\}.$$

Let us put $|k| := \sum_{i=1}^{3} k_i$.

(a) If

$$k_3 < k_1 + k_2 - 1,$$

then it holds that

$$|d_1 - d_2| = \begin{cases} 0 & \text{if } |k| \text{ is even,} \\ 1 & \text{if } |k| \text{ is odd,} \end{cases}$$

where (d_1, d_2) are the multi-exponents of the free S-module $D(\mathcal{A}, m)$.

(b) If

$$k_3 \geq k_1 + k_2 - 1,$$

then the multi-exponents of $(\mathcal{A}, m)$ are $(k_1 + k_2,\ k_3)$.

By Theorem 1.4, it holds that $\sum_{i=1}^{3} k_i = d_1 + d_2$. Hence the exponents of the 2-multiarrangement consisting of three lines can be completely determined by Theorem 1.6.

Next, let us consider the theory of 3-arrangements. Let $\mathcal{A}$ be an arrangement in a three-dimensional vector space V. Then the sheaf $\widetilde{D(\mathcal{A})}$ is a rank three vector bundle on $\mathbf{P}^2$ since $\widetilde{D(\mathcal{A})}$ is reflexive (e.g., see [2]). Fix a basis $\{X, Y, Z\}$ for V^* in such a way that the hyperplane $\{Z = 0\}$ is an element of $\mathcal{A}$. Define $S := S(V^*) = \mathrm{Sym}(V^*) \simeq \mathbb{K}[X, Y, Z]$. In this situation, we define the *module of reduced logarithmic vector fields* $D_0(\mathcal{A})$ as follows:

Definition 1.7. The S-module $D_0(\mathcal{A})$ is defined by

$$D_0(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(Z) = 0\}.$$

Note that for any (central) arrangement $\mathcal{A}$, there exists an derivation

$$\theta_E := X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z} \in D(\mathcal{A}).$$

We call this derivation θ_E the *Euler derivation*. It is obvious that

$$D_0(\mathcal{A}) \simeq D(\mathcal{A})/(S \cdot \theta_E).$$

Hence the structure of $D_0(\mathcal{A})$ does not depend on the choice of the coordinates of V. Moreover, in the notation of Theorem 1.5, it holds that

$$c_t(\widetilde{D_0(\mathcal{A})}) = 1 - c_1 t + c_2 t^2.$$

As we saw above, we can restrict a given arrangement $\mathcal{A}$ on the plane $H_0 := \{Z = 0\} \in \mathcal{A}$. Moreover, we can obtain a multiarrangement $(\mathcal{A} \cap H_0,\ m)$ and the restriction homomorphism

$$\varphi : D_0(\mathcal{A}) \to D(\mathcal{A} \cap H_0,\ m),$$

defined as follows:

$$D_0(\mathcal{A}) \ni \theta \mapsto \theta|_{Z=0} \in D(\mathcal{A} \cap H_0,\ m).$$

For the details of this homomorphism, see [13]. We can compute the codimension (as $\mathbb{K}$-vector spaces) of the image of φ from the characteristic polynomial of $\mathcal{A}$ and the exponents of $D(\mathcal{A} \cap H_0,\ m)$ by the following theorem, which is a variant of Theorem 3.2 in [12].

Theorem 1.8 (Yoshinaga)**.** *With the above notation, let $\{\theta_1, \theta_2\}$ be a basis for a free $S/(S\cdot Z)$-module $D(\mathcal{A}\cap H_0,\ m)$ such that* $\deg(\theta_i) = d_i$ $(i = 1, 2)$. *Then the dimension of* $\operatorname{coker}(\varphi)$ *(as a $\mathbb{K}$-vector space) is finite and is given by*

$$\chi_0(\mathcal{A}, 0) - d_1 d_2.$$

In particular, $\mathcal{A}$ is free if and only if

$$\chi_0(\mathcal{A}, 0) = d_1 d_2.$$

In [12], Yoshinaga showed the same statement as in Theorem 1.8 for the logarithmic differential module $\Omega(\mathcal{A})$, and we can prove Theorem 1.8 by the same way as in [12].

Next, we review some definitions and results on the stability of vector bundles on projective spaces. The reference for the stability of vector bundles is Chapter II of [4]. First, we define the stability and semistability of torsion free sheaves.

Definition 1.9. A torsion free sheaf E on the projective space $\mathbf{P}^n_{\mathbb{K}}$ is said to be *stable* if for any coherent subsheaf $F \subset E$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$ we have

$$\frac{c_1(F)}{\mathrm{rank}(F)} < \frac{c_1(E)}{\mathrm{rank}(E)},$$

and *semistable* if

$$\frac{c_1(F)}{\mathrm{rank}(F)} \leq \frac{c_1(E)}{\mathrm{rank}(E)}.$$

Moreover, we say E is *unstable* if E is not stable. In this article, we will use the following definitions and results.

Lemma 1.10 ([4, Lemma 1.2.4, Ch. II])**.** *A torsion free sheaf E on the projective space $\mathbf{P}^n_{\mathbb{K}}$ is stable if and only if $E \otimes \mathcal{O}_{\mathbf{P}^n}(d)$ is stable for some $d \in \mathbb{Z}$.*

Definition 1.11. For a rank two vector bundle E on $\mathbf{P}^n_{\mathbb{K}}$ $(n \geq 2)$, there exists the unique integer $d \in \mathbb{Z}$ such that

$$c_1(E \otimes \mathcal{O}_{\mathbf{P}^n}(d)) \in \{0, -1\}.$$

We call $E \otimes \mathcal{O}_{\mathbf{P}^n}(d)$ the *normalization* of E or the *normalized* E. The *normalized Chern polynomial of* E denotes the Chern polynomial of the normalized E.

Lemma 1.12 ([4, Lemma 1.2.5, Ch. II])**.** *Let E be a rank two bundle on $\mathbf{P}^n$ $(n \geq 2)$ and $E \otimes \mathcal{O}_{\mathbf{P}^n_{\mathbb{K}}}(d)$ be its normalization. Then E is stable if and only if*

$$H^0(\mathbf{P}^n, E \otimes \mathcal{O}_{\mathbf{P}^n_{\mathbb{K}}}(d)) = 0.$$

Moreover, if $c_1(E)$ is even, then E is semistable if and only if $H^0(\mathbf{P}^n, E(d-1)) = 0$.

Lemma 1.13 ([4, Lemma 1.2.7, Ch. II])**.** *For a rank two bundle E on $\mathbf{P}^2_{\mathbb{K}}$, put $c_i := c_i(E)$ $(i = 1, 2)$. If*

$$c_1^2 - 4c_2 \geq 0,$$

then E is unstable.

Theorem 1.14 ([8, Theorem 4.5])**.** *Let $\mathcal{A}$ be an arrangement of d lines in $\mathbf{P}^2$, H_0 be a line in $\mathcal{A}$, and let us put $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$. Then the following holds*:

(i) *If d is odd, then $\mathcal{A}$ is stable if $\mathcal{A}'$ is stable and $|\mathcal{A} \cap H_0| > (d+1)/2$.*

(ii) *If d is odd, then $\mathcal{A}$ is semistable if $\mathcal{A}'$ is semistable and $|\mathcal{A} \cap H_0| > (d-1)/2$.*

(iii) *If d is even, then $\mathcal{A}$ is stable if $\mathcal{A}'$ is semistable and $|\mathcal{A} \cap H_0| > d/2$.*

Theorem 1.15 (Beilinson). *Assume that E is a stable rank two bundle on $\mathbf{P}^2_{\mathbb{K}}$ such that $c_1(E) = -1$ and $c_2(E) = 1$. Then*

$$E \simeq \Omega_{\mathbf{P}^2}(1) \simeq T_{\mathbf{P}^2}(-2).$$

For the proof of Theorem 1.15, see Theorem 3.1.3 and Example 1 of section 3.2 in Chapter II of [4].

2. Proof of Theorem 0.4

In this section, we prove Theorem 0.4. From now on, we fix a three-dimensional $\mathbb{K}$-vector space V and a basis $\{X, Y, Z\}$ for V^*. Let us put $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[X, Y, Z]$. We consider the family of A_2-type arrangements $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ defined in Definition 0.3. As mentioned in the introduction, it suffices to consider the families such that $A(1)$ is (1) or (2). In this section, we consider (1). In this case, the A_2-type arrangement $\mathcal{A}(k)$ is expressed as

$$\begin{aligned} X &= (-k+1)Z, \ldots, (k+c-1)Z \\ Y &= (-k+1)Z, \ldots,\ (k-1)Z, \\ X+Y &= (-k+a)Z, \ldots, (k+a+b-1)Z, \\ Z &= 0. \end{aligned} \tag{3}$$

First, we consider the following six lines to calculate the characteristic polynomial of $\mathcal{A}(k)$:

$$\begin{aligned} A_1 &: = \{X = (-k+1)Z\}, \\ A_2 &: = \{X = (k+c-1)Z\}, \\ A_3 &: = \{Y = (-k+1)Z\}, \\ A_4 &: = \{Y = (k-1)Z\}, \\ A_5 &: = \{X+Y = (-k+a)Z\}, \\ A_6 &: = \{X+Y = (k+a+b-1)Z\}. \end{aligned}$$

Let us call these six lines *exterior lines* in $\mathcal{A}(k)$. For each k, let us fix the exterior line $H_k \in \mathcal{A}(k)$ such that $\alpha_{H_{k+1}} - \alpha_{H_k} = Z$ or $-Z$, and let us consider the multiarrangements $(\mathcal{A}(k) \cap H_k, m_k)$, where m_k is a canonically induced multiplicity function on $\mathcal{A}(k) \cap H_k$.

Definition 2.1. With the above notation, we say an exterior line H_n in $\mathcal{A}(n)$ is *stable* if for all $k \geq n$, it holds that

$$|\{L \in \mathcal{A}(k+1) \cap H_{k+1} |\ m_{k+1}(L) = 2\}| = |\{L \in \mathcal{A}(k) \cap H_k |\ m_k(L) = 2\}| + 1,$$

and

$$|\{L \in \mathcal{A}(k+1) \cap H_{k+1} |\ m_{k+1}(L) = 1\}| = |\{L \in \mathcal{A}(k) \cap H_k |\ m_k(L) = 1\}| + 2.$$

It is easy to see that all exterior lines in $\mathcal{A}(k)$ are stable for all $k \gg 0$. This fact plays a key role in the stability problem, for we can calculate the characteristic polynomial of $\mathcal{A}(k)$ easily if all exterior lines are stable. Let us explain the reason for it. From the general theory on arrangements, it is obvious that the reduced characteristic polynomial $\chi_0(\mathcal{A}(k), t)$ of the central arrangement $\mathcal{A}(k)$ is equal to the characteristic polynomial of the non-central 2-arrangement $\{\mathcal{A}(k) \setminus \{Z = 0\}\}|_{Z=1} =: d\mathcal{A}(k)$ (the *deconing* of $\mathcal{A}(k)$. See Chapter two of [5]). Let $dH := H|_{Z=1}$ ($H \in \mathcal{A}$) denote the deconing of H. When all exterior lines are stable, we can determine the position of the intersection points $\{dA_i \cap dA_j\}_{1 \le i < j \le 6}$. For example, the intersection point $dA_1 \cap dA_5$ is $(-k+1, a-1)$, hence it is on the line $X = -k+1$ and between the two lines $Y = k-1$ and $Y = -k+1$ for sufficiently large k. So we can draw a picture of an arrangement $d\mathcal{A}(k)$, and we can obtain the characteristic polynomial of $\mathcal{A}(k)$ by using that picture. Then Theorem 1.5 allows us to obtain the Chern polynomial of $\widetilde{D_0(\mathcal{A}(k))}$ as follows:

Lemma 2.2. *With the above notation, for sufficiently large k, it holds that*

$$c_t(\widetilde{D_0(\mathcal{A}(k))}) = 1 - (6k + b + c - 2)t + \left(\left(3k + \frac{1}{2}b + \frac{1}{2}c - 1\right)^2 + \left(a + \frac{1}{2}b - \frac{1}{2}c - \frac{1}{2}\right)^2 - \frac{1}{4}\right)t^2.$$

By the same way, we can prove the following results which will be used later.

Lemma 2.3. *Let $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ be the family of A_2-type arrangements such that $\mathcal{A}(1)$ is defined by*

$$\begin{aligned} X &= 0, Z, \ldots, cZ \ (c \ge 0), \\ Y &= 0, \\ X + Y &= (a-1)Z,\ aZ, \ldots,\ (a+b)Z \ (b \ge -1), \\ Z &= 0. \end{aligned}$$

(*a*) *Let H_k be the exterior line in $\mathcal{A}(k)$ defined by*

$$H_k := \{X + Y = (k + a + b - 1)Z\}.$$

Then for sufficiently large k, it holds that

$$|\mathcal{A}(k) \cap H_k| = 3k + a + b - 1.$$

(*b*) *Let $H'_k \in \mathcal{A}(k)$ be the exterior line defined by*

$$H'_k : = \{X = (k + c - 1)Z\}.$$

Then

$$|\mathcal{A}(k) \cap H'_k| = 3k - a + c.$$

Example 2.1. With the above notation, let us consider the case when $a = b = 1$, and $c = 0$. According to Lemma 2.2, its Chern polynomial is

$$c_t(D_0(\widetilde{\mathcal{A}(k)})) = 1 - (6k - 1)t + (9k^2 - 3k + 1)t^2 \tag{4}$$

for sufficiently large k. However, when $k = 1$, the characteristic polynomial of $\mathcal{A}(1)$ is given by $t^2 - 5t + 6$. Hence it holds that

$$c_t(D_0(\widetilde{\mathcal{A}(1)})) = 1 - 5t + 6t^2,$$

which is not equal to (4) when $k = 1$. Hence the assumption $k \gg 0$ is necessary.

By using Lemma 2.2, we can obtain the following criterion for the freeness of $\{\mathcal{A}(k)\}$.

Proposition 2.4. *The family of A_2-type arrangements $\{\mathcal{A}(k)\}$ is free if and only if*

$$2a + b - c = 0,\ 1,\ 2.$$

Proof. Note that for sufficiently large k,

$$|\mathcal{A}(k)| - 1 = -c_1(D_0(\widetilde{\mathcal{A}(k)})) = 6k + b + c - 2.$$

First, let us assume $b + c$ is even. In this case, we know the multi-exponents of the multiarrangement $(\mathcal{A}(k) \cap \{Z = 0\},\ m)$ due to Theorem 1.6 (since the condition in that theorem is satisfied for sufficiently large k), i.e.,

$$\exp(\mathcal{A}(k) \cap \{Z = 0\},\ m) = \left(3k + \frac{1}{2}b + \frac{1}{2}c - 1,\ 3k + \frac{1}{2}b + \frac{1}{2}c - 1\right).$$

From Theorem 1.8, $\mathcal{A}(k)$ is free if and only if

$$c_2(D_0(\widetilde{\mathcal{A}(k)})) = \left(3k + \frac{1}{2}b + \frac{1}{2}c - 1\right)^2.$$

By Lemma 2.2, this condition is equivalent to

$$(2a + b - c)(2a + b - c - 2) = 0.$$

Secondly, let us assume that $b + c$ is odd. We can use the same argument as above, hence $\mathcal{A}(k)$ is free if and only if

$$\left(3k + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}\right)\left(3k + \frac{1}{2}b + \frac{1}{2}c - \frac{3}{2}\right) = c_2(D_0(\widetilde{\mathcal{A}(k)})).$$

By Lemma 2.2, this condition is equivalent to

$$(2a + b - c - 1)^2 = 0,$$

and these complete the proof. □

Note that when $\mathcal{A}(k)$ is free, its exponents are

$$(5) \qquad \left(1,\ 3k+\frac{1}{2}b+\frac{1}{2}c-1,\ 3k+\frac{1}{2}b+\frac{1}{2}c-1\right)$$

if $b+c$ is even, and

$$(6) \qquad \left(1,\ 3k+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2},\ 3k+\frac{1}{2}b+\frac{1}{2}c-\frac{3}{2}\right).$$

if $b+c$ is odd.

Next, we prove a sufficient and necessary condition for the normalization of $\widetilde{D_0(\mathcal{A})}$ to be isomorphic to $T_{\mathbf{P}^2}(-2)$.

Proposition 2.5. *Let $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ be the family of A_2-type arrangements such that $\mathcal{A}(1)$ is defined by*

$$\begin{aligned} X &= 0,\ Z, \dots,\ cZ\ (c \geq 0), \\ Y &= 0, \\ X+Y &= (a-1)Z, \dots,\ (a+b)Z\ (b \geq -1), \\ Z &= 0. \end{aligned}$$

Then there exists an integer k_0 such that the normalization of $\widetilde{D_0(\mathcal{A}(k))}$ is isomorphic to $T_{\mathbf{P}^2} \otimes \mathcal{O}(-2)$ for $k \geq k_0$ if and only if

$$2a+b-c = 3,\ \text{or}\ -1.$$

Proof. The "only if" part follows immediately from Lemma 2.2. We show the "if" part. The strategy of the proof is to show the stability of $\widetilde{D_0(\mathcal{A}(k))}$ and to show that the normalized Chern polynomial is $1-t+t^2$, for they enable us to use Theorem 1.15. Since the proof is the same, we consider only the case when $2a+b-c=3$. Note that

$$|\mathcal{A}(k)| = 6k+b+c-1 = 6k+2a+2b-4.$$

Let H_k be the exterior line defined in Lemma 2.3 (a) and $\{\mathcal{B}(k)\}$ be the family of A_2-type arrangements defined by

$$\mathcal{B}(k) := \mathcal{A}(k) \setminus H_k.$$

From Theorem 1.14, $\mathcal{A}(k)$ is stable if $\mathcal{B}(k)$ is semistable and

$$|\mathcal{A}(k) \cap H_k| > \frac{|\mathcal{A}(k)|}{2}.$$

Since

$$|\mathcal{A}(k) \cap H_k| = 3k + a + b - 1 \text{ (from Lemma 2.3 (a))},$$

and

$$\frac{|\mathcal{A}(k)|}{2} = \frac{6k + 2a + 2b - 4}{2} = 3k + a + b - 2,$$

it suffices to show $\mathcal{B}(k)$ is semistable. First, assume $b > -1$. Then $\mathcal{B}(1)$ is defined by

$$\begin{aligned} X &= 0,\ Z, \dots,\ cZ\ (c \geq 0), \\ Y &= 0, \\ X + Y &= (a-1)Z, \dots,\ (a+b-1)Z, \\ Z &= 0. \end{aligned}$$

Since $2a + (b-1) - c = 2$, Proposition 2.4 shows $\mathcal{B}(k)$ is free with exponents

$$(1,\ 3k + a + b - 3,\ 3k + a + b - 3).$$

Therefore $\widetilde{D_0(\mathcal{B}(k))} \simeq \mathcal{O}(-3k - a - b + 3) \oplus \mathcal{O}(-3k - a - b + 3)$ and this is a semistable vector bundle. When $b = -1$, Lemma 2.2 and Lemma 2.3 (a) shows

$$c_t(\widetilde{D_0(\mathcal{B}(k))}) = 1 - (6k + 2a - 8)t + (9k^2 + 6ak - 24k + a^2 - 8k + 16)t^2.$$

Hence Theorem 1.6 and Theorem 1.8 shows $\widetilde{D_0(\mathcal{B}(k))} \simeq \mathcal{O}(-3k - a + 4) \oplus \mathcal{O}(-3k - a + 4)$, and this is also semistable. Since the normalization of $\widetilde{D_0(\mathcal{A}(k))}$ is $\widetilde{D_0(\mathcal{A}(k))}(3k + a + b - 3)$ and its Chern polynomial is $1 - t + t^2$, Theorem 1.15 completes the proof. □

In particular, $\{\mathcal{A}(k)\}$ is stable when $2a + b - c = -1$ or 3. Summarizing results above, we can show the following classification on the stability and the freeness when $c = 0$.

Proposition 2.6. *Let $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ be the family of A_2-type arrangements such that $\mathcal{A}(1)$ is defined by*

$$\begin{aligned} X &= 0, \\ Y &= 0, \\ X + Y &= (a-1)Z, \dots,\ (a+b)Z\ (b > -1), \\ Z &= 0. \end{aligned}$$

Then the following holds:

(*a*) *The family $\{\mathcal{A}(k)\}$ is free if and only if*

$$2a + b = 0,\ 1,\ 2.$$

(*b*) *There exists an integer* k_0 *such that the normalization of* $\widetilde{D_0(\mathcal{A}(k))}$ *is isomorphic to* $T_{\mathbf{P}^2}(-2)$ *for* $k \geq k_0$ *if and only if*

$$2a + b = -1,\ 3.$$

(*c*) *The family* $\{\mathcal{A}(k)\}$ *is stable if*

$$2a + b > 3$$

or

$$2a + b < -1.$$

Proof. (a) and (b) have been already proved in Proposition 2.4 and Proposition 2.5. So it suffices to show $\widetilde{D_0(\mathcal{A}(k))}$ is stable for sufficiently large k when $2a + b \neq -1,\ 0,\ 1,\ 2,\ 3$. Note

$$-c_1(\widetilde{D_0(\mathcal{A})}) = |\mathcal{A}(k)| - 1 = 6k + b - 2.$$

Let $H_k \in \mathcal{A}(k)$ be the exterior line defined in Lemma 2.3 (a). We show the result by the induction on $2a + b$. First, we show that we may assume

$$2a + b > 3.$$

To show that, let us assume that

$$2a + b < -1.$$

Then replacing X and Y by $-X$ and $-Y$, we obtain a new family of A_2-type arrangements such that the initial arrangement is defined by

$$\begin{aligned} X &= 0, \\ Y &= 0, \\ X + Y &= (-a - b)Z, \ldots,\ (-a + 1)Z, \\ Z &= 0. \end{aligned}$$

Put

$$A := -a - b + 1.$$

Then it holds that

$$2A + b = -2a - b + 2 > 3.$$

Since the structure of the module of logarithmic vector fields $D(\mathcal{A})$ for a central arrangement $\mathcal{A}$ is invariant under every invertible linear transformation of $\mathcal{A}$, we may assume $2a + b > 3$. Now let us assume the statement is true for $2a + b = g - 1$ ($g \geq 4$) (When $2a + b = 3$, the statement is already shown in Proposition 2.5). We show the statement when $2a + b = g$. According to

Theorem 1.14 and Lemma 2.3 (a), we can show the statement is true if the arrangement $\mathcal{B}(k) := \mathcal{A}(k) \setminus H_k$ is stable and the inequality

$$|\mathcal{A}(k) \cap H_k| = 3k + a + b - 1 > 3k + \frac{b}{2}$$

holds. This inequality is equivalent to

$$2a + b > 2.$$

Since we have assumed $2a + b > 3$, it suffices to show that $\{\mathcal{B}(k)\}$ is a stable family. Assume that $\{\mathcal{B}(k)\}$ is not a stable family. When $b \neq -1$, the induction hypothesis and Proposition 2.4 assert that $\mathcal{B}(k)$ must be free. So it holds that

$$2a + (b - 1) = 0,\ 1,\ 2.$$

Since $\mathcal{A}(k)$ is not free, it must hold that

$$2a + b = 3,$$

but in this case $\{\mathcal{A}(k)\}$ is stable due to Proposition 2.5. When $b = -1$, we can prove the statement by using the same argument as in Proposition 2.5. $\square$

Proof of Theorem 0.4. The statements (a) and (b) in Theorem 0.4 have been already proved in Proposition 2.4 and Proposition 2.5. So it suffices to show (c). By the same way as in the proof of Proposition 2.6, we may assume that $2a + b - c < -1$. Then we can show (c) by the induction on c, using Lemma 2.3 (b) and the same argument as in the proof of Proposition 2.6. $\square$

Example 2.2. We show two examples, each of which satisfies the condition (b) or (c) of Theorem 0.4 but the statement does not hold for small k. These are caused by the fact that not all exterior lines are stable.

(b) In the notation of Theorem 0.4, let us put $a = 1,\ b = 1,\ c = 0$. In this case $2a + b - c = 3$, so for sufficiently large k, the normalization of $\widetilde{D_0(\mathcal{A}(k))}$ is isomorphic to $T_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-2)$. However, the reduced characteristic polynomial of $\mathcal{A}(1)$ is $t^2 - 5t + 6$, hence $\widetilde{D_0(\mathcal{A}(1))}$ cannot be isomorphic to $T_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-4)$.

(c) In the notation of Theorem 0.4, let us put $a = 2,\ b = 6,\ c = 3$. In this case $2a + b - c = 7$, so for sufficiently large k, $\widetilde{D_0(\mathcal{A}(k))}$ is stable. However, the reduced characteristic polynomial of $\mathcal{A}(1)$ is $t^2 - 13t + 41$. Since $(13)^2 - 4 \cdot 41 = 5 > 0$, Lemma 1.13 shows $\widetilde{D_0(\mathcal{A}(1))}$ is not stable (not semistable, in fact).

On the other hand, if the family of A_2-type arrangements satisfies the condition (a) of Theorem 0.4, then $\mathcal{A}(k)$ is free for all $k \geq 1$. This follows from the calculation of characteristic polynomials and Theorem 1.8. However, if we denote $(1,\ d_1^k,\ d_2^k)$ as the exponents of $\mathcal{A}(k)$, then $|d_1^k - d_2^k|$ is not constant for all k. For sufficiently large k, $|d_1^k - d_2^k|$ is equal to 0 if $|\mathcal{A}|$ is odd and 1 if $|\mathcal{A}|$ is even.

3. Proof of Theorem 0.5

In this section, we prove Theorem 0.5. Hence we consider the family of A_2-type arrangements $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ such that the A_2-type arrangement $\mathcal{A}(k)$ is expressed as

$$\begin{aligned} X &= (-k+1)Z, \dots,\ (k+c-1)Z\ (c \geq 0), \\ Y &= (-k+1)Z, \dots, kZ, \\ X+Y &= (-k+a)Z, \dots,\ (k+a+b-1)Z\ (b \geq -1), \\ Z &= 0. \end{aligned} \tag{7}$$

We classify the freeness and the stability of these arrangements. Theorem 0.5 can be obtained by using the same argument as in the previous section, so we give only the outline. We begin with the following lemma, which can be proved by using stable exterior lines.

Lemma 3.1. *With the above notation, for sufficiently large k, it holds that*

$$\begin{aligned} c_t(\widetilde{D_0(\mathcal{A}(k))}) = 1 &- (6k+b+c-1)t \\ &+ \left(\left(3k + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}\right)^2 + \left(a + \frac{1}{2}b - \frac{1}{2}c - 1\right)^2 - \frac{1}{4}\right)t^2. \end{aligned}$$

Proof of Theorem 0.5. We can show (a) and (b) of Theorem 0.5 by the same way as in the proof of Proposition 2.4 and Proposition 2.5, so leave to the reader. Hence it suffices to show (c). Let us put

$$L_k := \{Y = kZ\} \in \mathcal{A}(k)$$

and

$$\mathcal{B}(k) := \mathcal{A}(k) \setminus L_k.$$

We know the stability of $\mathcal{B}(k)$ from Theorem 0.4. Hence by counting the number of points on $\mathcal{A}(k) \cap L_k$ and using Theorem 1.14, we can reduce the proof to Theorem 0.4. □

Remark 3.1. According to the main theorems, we can see that for the family of A_2-type arrangements, the freeness and the stability of the bundle $\widetilde{D_0(\mathcal{A}(k))}$ can be completely determined by the combinatorics of $\mathcal{A}(k)$ for sufficiently large k.

Remark 3.2. In Theorem 0.4 and 0.5, it is easy to find the least integer k_0 such that the statements of the theorems hold for all $k \geq k_0$, i.e., k_0 is just the least integer which satisfies the condition in Theorem 1.6 and makes all the exterior lines in $\mathcal{A}(k)$ stable.

Remark 3.3. Let $\{\mathcal{A}(k)\}_{k=1}^{\infty}$ be the family of A_2-type arrangements such that the initial arrangement is defined as (1) (resp:(2)) in the introduction. Then from Theorem 0.4 and 0.5, it holds that

$$D_0(\widetilde{\mathcal{A}(k)}) \otimes \mathcal{O}_{\mathbf{P}^2}(-3) \simeq D_0(\widetilde{\mathcal{A}(k+1)}). \tag{8}$$

for sufficiently large k if

$$-1 \leq 2a + b - c \leq 3 \text{ (resp} : 0 \leq 2a + b - c \leq 4).$$

Yoshinaga conjectured (8) is true for all the families of A_2-type arrangements. This conjecture is called the 3-*shift problem*. We can see that the Chern polynomials (Lemma 2.2), splitting types on the infinite line $\{Z = 0\}$ (Theorem 1.6), and the stability (Theorem 0.4 and 0.5) support the 3-shift problem.

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