J. Math. Kyoto Univ. (JMKYAZ)
46-3 (2006), 553–582

Non commutative algebraic spaces of finite type over Dedekind domains

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2000 *Mathematics Subject Classification*(*s*). 14A22(16S32)
Received October 11, 2005
Revised December 28, 2005

1. Introduction

In this paper we consider non commutative algebraic spaces of finite type over Dedekind domains. We mainly deal with cases where the Dedekind domains are finitely generated over $\mathbb{Z}$ (that is, mixed characteristic case).

Non commutative "coordinate algebras" are then reduced modulo primes.

In many important cases they are then finite over their centers. We may analyze them using usual techniques of algebraic geometry.

We define "NAS" 's , candidates for non commutative algebraic spaces. We do this in two steps. The first step is to define "LNAS" 's (lesser NAS) as objects which are sheaves of algebras over usual (commutative) algebraic spaces. Then the second step is to define NAS's as objects which are LNAS when reduced to the positive characteristic case. (Our definition of non commutative algebraic spaces is not yet commonly accepted. Therefore we refrain from using the general term "non commutative algebraic spaces" and will keep on using the rather odd abbreviational term "NAS" instead to express our objects instead throughout this paper.)

Our theory is a three floor building.

The first floor is the view of Rosenberg (which was originated by Grothendieck) that non commutative spaces are realized by abelian categories.

The second floor is a theory of sheaf of algebras over a usual commutative scheme. We specially focus on non zero characteristic cases.

The third floor is the theory of ultra filter. We use it to go back from non zero characteristic cases to characteristic zero cases.

In our view of NAS, there always exist commutative counterpart (shadows) of them, and a bunch of matrix algebras (phantoms). They reflect the properties of the NAS. We may also say that the relationship of NAS's and their shadows are something like the one of quantum mechanics and classical counter parts.

We first summarize preliminaries concerning abelian categories in Section 2.

Then we define NAS (Definition 3.2). It is essentially an abelian category with "arithmetic charts." Charts are LNAS (Definition 3.1) which may be dealt with as an object of usual commutative algebraic geometry. We also introduce a notion of a shadow and phantoms of a NAS.(Definition 3.2). One of a NAS of the easiest kind is affine NAS (Definition3.3).

We also define "mild localizations" (Definitions 3.8) and "mild rational maps" (Definition 3.10).

In dealing with NAS, we note that there is an interesting phenomenon special to non commutative objects. A finitely generated algebra can have infinite holes (Lemma 4.1). We call them "albert holes" (Definition 4.1). Almost commutative algebras have no albert hole (Corollary 4.2). We also prove a Proposition which guarantees us flatness of algebra homomorphism when the algebras have no albert hole (Proposition 4.1).

For any affine LNAS, its shadow has skew symmetric bilinear form on cotangent bundle. In other words, the affine coordinate algebra has a structure of a Poisson algebra (Proposition 5.1).

Using this we deal with formal completion of Weyl algebras over fields of non zero characteristics. Differential equations and techniques developed in [13] are generalized to these "formal Weyl algebras".

Then we show that any K-algebra endomorphism of a Weyl algebra $A_n(K)$ over a field K of characteristic 0 is flat (Theorem 5.1).

The last section is devoted to showing some examples of our theory. They are essentially known as facts, but we would like to present a new view.

As the first example, we treat D-modules on algebraic spaces. (Proposition 6.1). On each non singular algebraic space X, the category (D_X-mod) of D_X-modules gives a standard example of NAS. It has the cotangent bundle T^*X as a shadow, as one might expect.

In the last two subsection we study universal enveloping algebras of semi simple Lie algebras from our point of view. First we treat the algebra itself (Proposition 6.2) And then we consider quotient of $\mathfrak{U}(\mathfrak{g})$ by its two-sided ideals (Proposition 6.3.) It is closely related to the Steinberg map.

Remark 1. After this paper was written, the author has noticed the existence of a preprint "The Jacobian conjecture is stably equivalent to the Dixmier conjecture" (math.RA/0512171) by Belov-Kanel and Maxim Kontsevich. We would like to point out the similarity of their method and ours. The method of introducing a Poisson product in a algebra in the preprint is a important special case of the one introduced in this paper.

Acknowledgements. Deep appreciation goes to Professor Jun'ichi Matsuzawa for his having guided the author to the theory of Steiberg maps. Deep appreciation also goes to Professor Akira Ishii for his good advice. The author is grateful to the colleagues in Kochi University. The author also expresses his gratitude to Professor Takuro Mochizuki for constant encouragement. Last but not least, the author expresses his gratitude to the referee of the paper for the careful proof-reading and comments to improve the paper.

2. Preliminaries

2.1. Linear abelian categories

All the algebras and rings in this paper are assumed to be unital and associative unless otherwise stated.

We review the following definition.

Definition 2.1. Let R be a (unital associative) commutative ring. An R-linear abelian category is an abelian category $\mathcal{C}$ equipped with the following structures.

1. For each objects $M_1, M_2 \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(M_1, M_2)$ admits a structure of R-module.

2. Compositions of morphisms respect this structure. That means,

$$(a.f) \circ g = a.(f \circ g) = f \circ (a.g)$$

holds for all $a \in R$ and for all (composable) arrows f, g in $\mathcal{C}$.
An R-functor from an R-linear abelian category to another is an additive functor which respects R-module structure above.

The category (R-mod) of R-modules is an R-linear abelian category.

We say an R-linear abelian category is **augmented** if there is given a covariant R-functor from to (R-mod) to $\mathcal{C}$. The augmentation is said to be **flat** if it is exact.

A functor between two augmented R-linear abelian category is an isomorphism if it is exact, is fully faithful and commutes with augmentation.

2.1.1. A linear abelian category modulo an ideal

Definition 2.2. Let R be a commutative ring. Let $\mathcal{C}$ be an R-linear abelian category. For each ideal I of R, we may define an R/I-linear abelian category $\mathcal{C}/I$ as a full sub category of $\mathcal{C}$ whose object is given by

$$\mathrm{Ob}(\mathcal{C}/I) = \{M \in \mathrm{Ob}(\mathcal{C}); IM = 0\}.$$

(Note: $IM = 0$ has only symbolical meaning. It should be interpreted as

$$M \overset{\times a}{\to} M$$

is identical to 0 for any $a \in I$.)

Example 2.1. Let $\mathcal{A}$ be a sheaf of algebra over an algebraic space X. Let $\mathcal{C}_{\mathrm{qcoh}}$ (respectively $\mathcal{C}_{\mathrm{coh}}$) be the category of quasicoherent (respectively coherent) $\mathcal{A}$-modules. $\mathcal{C}_{\mathrm{qcoh}}/p$ (respectively $\mathcal{C}_{\mathrm{coh}}/p$ is the category of quasicoherent respectively coherent) $\mathcal{A}/p\mathcal{A}$-modules.

Note that if I is finitely generated then there exists a morphism f of noncommutative space from $\mathcal{C}/I$ to $\mathcal{C}$.

$$f^*(M) = M/IM$$
$$f_*(M) = M$$

(IM is well-defined since we assumed I to be finitely generated. Indeed, if $I = a_1R + a_2R + \cdots + a_sR$, then

$$IM = \sum_i a_i M = \mathrm{Image}\left(\bigoplus_i M \overset{(a_1,a_2,a_3,\ldots,a_s)}{\longrightarrow} M\right),$$

the right hand side being independent of the choise of $\{a_i\}$.)

2.1.2. Tensor products

Lemma 2.1. *Assume R is a noetherian commutative ring. Let $\mathcal{C}$ be an R-linear abelian category. Then for any finitely generated R-module N, R-tensor products*

$$X \otimes_R N$$

exists in $\mathcal{C}$ with the following property.

There exists a homomorphism

$$N \ni n \mapsto \varphi_n \in \mathrm{Hom}_{\mathcal{C}}(X, X \otimes_R N)$$

such that for each object $Y \in \mathrm{Ob}(\mathcal{C})$, a map

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes_R N, Y) \to \mathrm{Hom}_{R-\mathrm{module}}(N, \mathrm{Hom}_{\mathcal{C}}(X, Y)).$$

given by

$$\psi \mapsto [n \mapsto \psi \circ \varphi_n]$$

is an isomorphism of R-modules. Furthermore, any R-linear functor commutes with these tensor products.

Proof. It is easy to show that $X \otimes_R N$ is uniquely determined by the universal property above.

To show the existence, we consider a free resolution

$$R^{m_1} \overset{(a_{ij})}{\to} R^{m_2} \overset{(n_j)}{\to} N \to 0$$

of the R-module N. Then we define $X \otimes_R N$ by the following exact sequence.

$$X^{\oplus m_1} \overset{(a_{ij})}{\to} X^{\oplus m_2} \to X \otimes_R N \to 0 \tag{2.1}$$

(That means, the cokernel of the morphism (a_{ij}) above.)

Now for any given element $n \in N$, we choose a lift (b_j) of N to R^{m_2}. That means, (b_j) is an element of R^{m_2} such that $\sum_j b_j n_j = n$ holds. Then we define ϕ_n by

$$X \overset{b_j}{\to} X^{\oplus m_2} \to X^{\oplus m_2} / \mathrm{Image}((a_{ij})) = X \otimes_R N$$

We may easily see that ϕ_n is independent of the choice of the lift (b_j) and that $X \otimes_R N$ with the above $\{\varphi_n\}_{n \in N}$ satisfies the required properties. □

2.2. Limit using ultra filters

The theory of ultra filters and limits using them play an important role in our theory. We briefly record certain definitions here. See for example [13] for details.

Definition 2.3. Let R be a commutative Dedekind domain. Let $\mathcal{U}$ be a free (that means, non principal) ultra filter on $\mathrm{Spm}(R)$. Then for any R-module M, we define its $\mathcal{U}$-limit as

$$M_{\mathcal{U}} = \prod^{*}(M/\mathfrak{p}M) = \left(\prod_{\mathfrak{p}\in \mathrm{Spm}(R)} (M/\mathfrak{p}M) \right) / M_{0,\mathcal{U}}$$

where $M_{0,\mathcal{U}}$ is a submodule of $\prod_{\mathfrak{p}\in\mathrm{Spm}(R)}(M/\mathfrak{p}M)$ defined as

$$M_{0,\mathcal{U}} = \{(a_{\mathfrak{p}}); \exists U \in \mathcal{U} \text{ such that } (a_{\mathfrak{p}} = 0 \quad (\forall \mathfrak{p} \in U))\}$$

We note that when the module M is the R itself, then the resulting module $R_{\mathcal{U}}$ carries a natural structure of an R-algebra.

We also note that when M is an increasing union of finite R-submodules M_j, then we have

$$M \otimes_R R_{\mathcal{U}} = \varinjlim_j (M_j)_{\mathcal{U}} \subset M_{\mathcal{U}}.$$

2.2.1. Frobenius maps Let R be a commutative Dedekind domain which is finitely generated over $\mathbb{Z}$. Let $\mathcal{U}$ be an ultra filter on $\mathrm{Spm}(R)$. We put

$$\mathbb{Z}^{*\mathcal{U}} = \left(\prod_{\mathfrak{p}\in \mathrm{Spm}(R)} \mathbb{Z} \right) / \sim$$

where "$\sim$" is an equivalence relation defined by

$$(a_{\mathfrak{p}}) \sim (b_{\mathfrak{p}}) \iff \exists U \in \mathcal{U} \text{ such that } (a_{\mathfrak{p}} = b_{\mathfrak{p}} \quad (\forall \mathfrak{p} \in U)).$$

Definition 2.4. For any $n = (n_{\mathfrak{p}}) \in \mathbb{Z}^{*\mathcal{U}}$, we define **a Frobenius automorphism of $R_{\mathcal{U}}$ of type** n as follows

$$R_{\mathcal{U}} \ni (x_{\mathfrak{p}}) \mapsto (F_{\mathfrak{p}}^{n_{\mathfrak{p}}}(x)) \in R_{\mathcal{U}}$$

where $F_{\mathfrak{p}}$ is the usual Frobenius map

$$F_{\mathfrak{p}}(x) = x^p \quad (p = \mathrm{char}(R/\mathfrak{p}))$$

3. Definitions

Just as manifold is defined as a combination of

$$\text{(topological space)} + \text{(charts on it)}$$

we define NAS as an R-linear abelian category with p-charts on it.

3.1. LNAS: a lesser model of our main object

We first define LNAS (L for "lesser" or "little").

Definition 3.1. Let k be a commutative ring. We say $X = (\mathcal{C}, |X|, \mathcal{A})$ is an **LNAS** over k with a shadow $|X|$ if it satisfies the following conditions.

1. $\mathcal{C}$ is a k-linear abelian category.
2. $|X|$ is an algebraic space over k with structure sheaf $\mathcal{O}_{|X|}$.
3. $\mathcal{A}$ is a sheaf of algebra over $|X|$ which contains $\mathcal{O}_{|X|}$ as a central subalgebra.
4. $\mathcal{A}$ is $\mathcal{O}_{|X|}$-coherent as an $\mathcal{O}_{|X|}$-module.
5. $\mathcal{C}$ is isomorphic to the k-linear abelian category ($\mathcal{A}$-mod) of $\mathcal{A}$-modules.

If the structure sheaf $\mathcal{O}_{|X|}$ of $|X|$ coincides with the center of $\mathcal{A}$, we call $|X|$ the principal shadow of X.

We often denote the category $\mathcal{C}$ above by $\mathrm{Qcoh}(X)$.

3.2. NAS: our candidate for non commutative algebraic space

Definition 3.2. Let R be a commutative Dedekind domain. A NAS X over R consists of the following data

1. An R-linear abelian category $\mathcal{C}$ which is flat over R. (See Definition 2.1).
2. An LNAS $X_{(\mathfrak{p})}$ over $R/\mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spm}(R)$.
3. An isomorphism (a "$\mathfrak{p}$-chart") $\mathcal{C}/\mathfrak{p} \cong X_{(\mathfrak{p})}$ of $R/\mathfrak{p}$-linear abelian categories for each $\mathfrak{p} \in \mathrm{Spm}(R)$.

If furthermore we have an algebraic space Y over R such that $Y \times_{\mathrm{Spec}\, R} \mathrm{Spec}(R/\mathfrak{p}) \cong |X_{(\mathfrak{p})}|$ for all $\mathfrak{p} \in \mathrm{Spm}(R)$, we call $\mathcal{C}$ a NAS **with a shadow** Y. If furthermore again each space $|X_\mathfrak{p}|$ coincides with the principal shadow of $X_\mathfrak{p}$, Y is called **the principal shadow** of X.

We define the **phantom** of X as a collection $\{F_\mathfrak{p}\}_{\mathfrak{p}\in\mathrm{Spm}(R)}$ of the typical fiber $F_\mathfrak{p}$ for each $\mathcal{A}_\mathfrak{p}$.

We often denote the category $\mathcal{C}$ above as $\mathrm{Qcoh}(X)$.

Definition 3.3. Let A be an algebra over a commutative Dedekind domain R. The category (A-mod) of A-modules is called **affine NAS** if it is a NAS with some affine shadow. If this is the case, we also say "A yields an affine NAS".

It is easy to see that an algebra A over a commutative Dedekind domain R yields a NAS over R if and only if for any $\mathfrak{p} \in \mathrm{Spm}(R)$, $A/\mathfrak{p}A$ is finite over its center.

Definition 3.4. Let R be a commutative Dedekind domain. Let A be an R-algebra. We assume that A yields an affine NAS with an affine shadow $X = \mathrm{Spec}(B)$ for some commutative R-algebra B. By definition we have an injection $\psi_\mathfrak{p} : B/\mathfrak{p}B \hookrightarrow Z(A/\mathfrak{p}A)$ for each $\mathfrak{p} \in \mathrm{Spm}(R)$.

We say A is of **moderate growth** over B if there exist sets S, T which satisfy the following conditions.

1. S is a finite generator of A. We denote by $\deg_A(f)$ the degree of f when we write f as a polynomial of S with coefficients in R.

2. T is a finite generator of B. We denote by $\deg_B(f)$ the degree of f when we write f as a polynomial of T with coefficients in R.

3. There exist positive constants $c_1, c_2, \{a_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathrm{Spm}(R)}$ such that

$$c_1 \deg_{B/\mathfrak{p}B} f \leq a_p \deg_{A/\mathfrak{p}A}(\psi_{\mathfrak{p}}(f)) \leq c_2 \deg_{B/\mathfrak{p}B} f$$

holds for all $f \in B/\mathfrak{p}B$ and for all $\mathfrak{p} \in \mathrm{Spm}(R)$.

3.3. Morphisms of NAS

Definition 3.5. A morphism f from a LNAS X_1 to a LNAS X_2 is a pair $(|f|, f^*)$ of the following data.

1. A morphism $|f| : |X_1| \to |X_2|$ of usual algebraic spaces.
2. An $\mathcal{O}_{X_1}$-algebra sheaf homomorphism $f^* : |f|^*(\mathcal{A}_{X_2}) \to \mathcal{A}_{X_1}$

A morphism f is said to be **étale** if $|f|$ is étale and $|f^*|$ is an étale local isomorphism of $\mathcal{A}$.

Definition 3.6. A morphism f from X_1 to X_2 is a morphism of non commutative space $\mathrm{Qcoh}(X_1) \to \mathrm{Qcoh}(X_2)$ which is compatible with $\mathfrak{p}$-charts. That means, f consists of the following data.

1. A right exact functor $f^* : \mathrm{Qcoh}(X_2) \to \mathrm{Qcoh}(X_1)$.
2. A morphism $f_{\mathfrak{p}} : X_{1,\mathfrak{p}} \to X_{2,\mathfrak{p}}$ of LNAS for each $\mathfrak{p} \in \mathrm{Spm}(R)$.
3. f^* modulo $\mathfrak{p}$ is equal to $f_{\mathfrak{p}}^*$ for each $\mathfrak{p} \in \mathrm{Spm}(R)$.

Definition 3.7. Let f be a morphism of NAS. then

1. f is said to be **flat** if f^* is exact.
2. f is said to be **étale** if it is flat and $f_{\mathfrak{p}}$ is étale (in the sense of Definition 3.5) for all $\mathfrak{p}$.

3.4. Localizations of NAS

A localization of an algebra A is obtained by adding inverses of elements of A. It is known that such localization may not flat. (See for example [11, Counter Example 6.1]). Therefore we employ the following definition.

Definition 3.8. Let R be a commutative Dedekind domain. Let A be an R-algebra which yields NAS. Then an étale localization of A is an R-algebra B which satisfies the following properties.

1. There exists multiplicative subset S of A.
2. B is obtained by adding inverses of S to A.
3. B is étale over A. (Definition 3.7)

Definition 3.9. Let R be a commutative Dedekind domain. Let A be an R-algebra which yields a NAS with an affine shadow $Y = \mathrm{Spec}(A_0)$. Then a **mild localization** of (A, A_0) is a pair (B, B_0) of an R-algebra B and a commutative R-algebra B_0 which satisfies the following properties.

1. There exists an element $f \in A_0 \setminus \cup_{\mathfrak{p}\in\mathrm{Spm}(R)} \mathfrak{p}A_0$ such that $B_0 = A_0[f^{-1}]$.
2. There exists a element $g \in A$ such that $B = \langle A, g^{-1}\rangle$.
3. B yields a shadow Y_f. To be more precise, for any $\mathfrak{p} \in \mathrm{Spm}(R)$, the algebra $B/\mathfrak{p}B$ is finite over $(B_0/\mathfrak{p}B_0)$.

4. f is flat.

The flatness of f is automatically implied by other conditions if f belongs to a certain class of morphisms. We discuss this in the next section.

Definition 3.10. Let R be a commutative Dedekind domain. Let A be an R-algebra which yields a NAS with an affine shadow $Y = \mathrm{Spec}(A_0)$. Let C be an R-algebra which yields a NAS with an affine shadow $Y = \mathrm{Spec}(C_0)$. A **mild rational map** from (C, C_0) to (A, A_0) is defined to be a pair of

1. an R-algebra homomorphism ϕ from (C, C_0) to a mild localization (B, B_0) of (A, A_0)
2. an R-algebra homomorphism ϕ_0 from C_0 to A_0

such that

$$\phi|_{C_0/\mathfrak{p}C_0} = \phi_0 \text{ modulo } \mathfrak{p}$$

holds for any $\mathfrak{p} \in \mathrm{Spm}(R)$. (Note that $C_0/\mathfrak{p}C_0$ (respectively, $A_0/\mathfrak{p}A_0$) may be regarded as a subalgebra of $C/\mathfrak{p}C$ (respectively, $A/\mathfrak{p}A$).

Lemma 3.1. *In the Definition above, let*

$$\iota_{\mathcal{U}} : C \hookrightarrow \lim_{\mathfrak{p}\to\mathcal{U}} (C/\mathfrak{p}C)$$

be the canonical injection. Let

$$j_{\mathfrak{p}} : C_0/\mathfrak{p}C_0 \to C/\mathfrak{p}C$$

be the inclusion given in the definition of affine NAS.

Then ϕ_0 is uniquely determined by ϕ if we use the following relation.

$$\iota_{\mathcal{U}}(\phi_0(x)) = \lim_{\mathfrak{p}\to\mathcal{U}} \phi(j_{\mathfrak{p}}(x \textit{ modulo } \mathfrak{p})) \qquad (x \in C_0)$$

4. Flatness

4.1. An example

Lemma 4.1. *Let $R_0 = \mathbb{Z}[1/2]$. For any commutative R_0-algebra R, we define an R-algebra $A(R)$ as follows.*

$$A(R) = R\langle \xi, \eta, \xi^{-1}, \eta^{-1}\rangle/(\eta\xi - \xi\eta - 1).$$

Let $\theta = \xi\eta \in A(R)$. We define $B(R)$ as follows.

$$B(R) = A(R)\langle (2\theta - 1)^{-1}\rangle$$

Then the following statements hold.

1. $A(R) = A(R_0) \otimes_{R_0} R, \quad B(R) = B(R_0) \otimes_{R_0} R.$
2. $A(\mathbb{F}_p) \cong B(\mathbb{F}_p)$ *for any odd prime p.*
3. $Z(A(\mathbb{F}_p)) \cong Z(B(\mathbb{F}_p)) \cong R_0[\xi^p, \eta^p, (\xi^p)^{-1}, (\eta^p)^{-1}]$
4. $A(R_0), B(R_0)$ *yields NAS over R_0.*

5. *$\theta - j$ is invertible in $A(R_0)$ for all $j \in \mathbb{Z}$.*
6. *θ is transcendent over R_0.*
7. *$A(R_0) \not\cong B(R_0)$.*

Proof. (1) : obvious.
(2) Let l be an integer.

$$\xi^{-l}\theta\xi^{l} = \theta + l$$

θ is invertible. In $A(R_0)$ for any l, $\xi\eta + l$ is invertible.

$$\xi^{-l}\theta^{-1}\xi^{l} = (\theta + l)^{-1}$$

Now, let us take an odd prime p. Then in $\mathbb{F}_p$, $1/2$ is equal to a integer $(p+1)/2$. Thus in $A(\mathbb{F}_p)$, $2\theta-1$ is invertible. Thus we know that $A(\mathbb{F}_p) \cong B(\mathbb{F}_p)$ for any odd prime p.
(3) Easy. (See [12], [13] for details.)
(4) is a direct consequence of (3).
(5), (6) Easy.
(7) Consider a representation ρ of $A(\mathbb{Z}[1/2])$ on $M = \mathbb{Q}[x, x^{-1}]$ given by

$$\rho(\xi) = x, \quad \rho(\eta) = \frac{d}{dx} + \frac{1}{2}x^{-1}$$

Then $\rho(2\theta - 1).1 = 0$. So $2\theta - 1$ is not invertible in $A(R_0)$. That means, $A(R_0) \not\cong B(R_0)$. □

Note that (5) and (6) in the above Lemma are special to non commutative algebras. The above example shows that we may have "infinite holes" in our algebras. This strange behavior also appears as the properties (2), (7).

Corollary 4.1. *In the above example, we put $M = A(R_0)/A(R_0).(2\theta - 1)$. Then we have*

1. *M is a non-zero $A(R_0)$-module.*
2. *$M/\mathfrak{p}M = 0$ for any $\mathfrak{p} \in \mathrm{Spm}(R_0)$.*

This motivates the definition in the next subsection.

4.2. Albert holes

In this subsection we extract a property of the example in the previous subsection.

Definition 4.1. Let R be a commutative Dedekind domain with infinite number of primes ($\#\,\mathrm{Spm}(R) = \infty$). Let A be an algebra over R. An albert hole M of A over R is an A-module such that the following conditions hold.

1. M is a noetherian non zero A-module.
2. For any $f \in R \setminus \{0\}$, the multiplication map $M \overset{f\times}{\to} M$ is bijective.

Remark 2. Let M be an albert hole of A over R. Let M_1 be a maximal A-submodule of M. Then M/M_1 is also an albert hole of A over R. Thus we see that when A has an albert hole over R, A always have an irreducible one.

Remark 3. Let M be an irreducible A-module. We take a non zero element $f \in R$. Then both $f.M$ and f-torsion of M is a submodule of M. Thus M satisfies the condition (2) of Definition above if $f.M \neq 0$.

We may generalize the notion of albert hole to categories.

Definition 4.2. Let R be a commutative Dedekind domain with infinite number of primes. Let $\mathcal{C}$ an R-linear abelian category. An albert hole M of $\mathcal{C}$ over R is an object of $\mathcal{C}$ such that

1. $M \neq 0$.
2. M is noetherian. (That means, any increasing sequence of subobjects of M stabilizes.)
3. For any $f \in R \setminus \{0\}$, $M \xrightarrow{\times f} M$ is an isomorphism.

The following Lemma is fundamental.

Lemma 4.2. *If A is commutative finitely generated algebra over R, then A has no albert hole over R.*

Proof. Suppose on the contrary that A has an albert hole M. We may assume M is irreducible. Since A is commutative, this implies that M is isomorphic to a field. By Nullstellensatz, we see that M is a finite extension of $Q(R)$. There exists at least one homomorphism from M to $\overline{Q(R)}$. Let S be the integral closure of R in M. Then M is generated by S and an inverse h^{-1} of an element ("common denominator of generators") h. This means $\mathrm{Spm}(S)$ is finite set (included in $V(h)$). On the other hand, $\mathrm{Spm}(S) \to \mathrm{Spm}(R)$ is surjective since S is finite integral extension over R. This is contrary to the assumption made on R. □

Lemma 4.3. *Let A be a finitely generated R-algebra. Then the following conditions are equivalent.*

1. *A has no albert hole over R.*
2. *There exists no nonzero finitely generated A-module M which satisfies*

$$M \otimes_R (R/\mathfrak{p}) = 0 \quad (\forall \mathfrak{p} \in \mathrm{Spm}(R)). \tag{4.1}$$

Proof. (2) $\implies$ (1): Assume A has an albert hole M. Then by condition (2) we have $M = 0$. A contradiction.
(1) $\implies$ (2): Let M be a finitely generated A-module with $M/\mathfrak{p}M = 0$ for all $\mathfrak{p} \in \mathrm{Spm}(R)$. Let M_1 be a maximal A-submodule of M (which exists because M is finitely generated. (Zorn's lemma)). Then the quotient $N = M/M_1$ is also a non-zero finitely generated A-module which satisfy the condition (4.1). Since A has no albert hole, we deduce that there exists $f \in R \setminus \{0\}$, $m \in N$ such that $f.m = 0$.

$$\{n \in N; f.n = 0\}$$

is a non zero submodule of N. Since N is irreducible, we have $f.N = 0$. Let

$$(f) = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\dots\mathfrak{p}_n^{e_n}$$

be the prime decomposition of ideal (f) in R. Again by the irreducibility of N, we see immediately that there exists a maximal ideal $\mathfrak{p} \in \mathrm{Spm}(R)$ among $\mathfrak{p}_i$'s such that $\mathfrak{p}.N = 0$. Thus $N/\mathfrak{p}.N = 0$, a contradiction. □

The following corollary is a generalization of Lemma 4.2

Corollary 4.2. *Let A be a finitely generated R-algebra. Assume A is almost commutative. That means, A has a filtration Γ such that $\mathrm{gr}^\Gamma(A)$ is commutative. Then A has no albert hole over R. A is noetherian.*

Proof. Let M be a finitely generated A-module with generators $\{m_1, m_2, \dots, m_l\}$. Then we have a filtration M defined by

$$\Gamma_i(M) = \Gamma_i(A).m_1 + \Gamma_i(A).m_2 + \cdots + \Gamma_i(A).m_l$$

$\mathrm{gr}^\Gamma(M)$ is a $\mathrm{gr}^\Gamma(A)$-module generated by (the class of) $\{m_1, m_2, \dots, m_l\}$.

We have

$$M = 0 \iff \mathrm{gr}^\Gamma(M) = 0 \iff \mathrm{gr}^\Gamma(M) \otimes_R R/\mathfrak{p} = 0 \quad (\forall \mathfrak{p})$$

where the last equivalence is guaranteed by Lemma 4.2.

On the other hand, since $\otimes_R R/\mathfrak{p}$ is right-exact, we have

$$\mathrm{gr}^\Gamma(M) \otimes_R R/\mathfrak{p} \cong \mathrm{gr}^\Gamma(M \otimes_R R/\mathfrak{p})$$

Combining the above two, we see

$$M = 0 \iff \mathrm{gr}^\Gamma(M \otimes_R R/\mathfrak{p}) = 0 \quad (\forall \mathfrak{p}) \iff M \otimes_R R/\mathfrak{p} = 0 \quad (\forall \mathfrak{p})$$

Thus by the Lemma above we see that A has no albert hole.

It is also easy to see that A is noetherian. □

The treatment here essentially follows [8].

Lemma 4.4. *Let R be a commutative Dedekind domain with infinite number of primes. Let B be noetherian, finitely generated R-algebra (which is not necessarily commutative). Let U be a finitely generated B-module. Consider the following three submodules of U.*

1.

$$S_1 = \bigcap_{\mathfrak{p} \in \mathrm{Spm}(R), m \geq 0} \{x \in U; 1 \otimes x \textit{ in } R_\mathfrak{p} \otimes U \textit{ is an element of } \in \mathfrak{p}^m (R_\mathfrak{p} \otimes U)\}$$

2.

$$S_2 = \bigcap_{\mathfrak{p} \in \mathrm{Spm}(R), m \geq 0} (\mathfrak{p}^m).U$$

3.

$$S_3 = \bigcap_{f \in R \setminus \{0\}} f.U$$

Then we have $S_1 = S_2 = S_3$ and it is either zero or an albert hole of B over R.

Proof. That $S_2 \subset S_1$ holds is immediate.

Since every non-zero principal ideal (f) in the commutative Dedekind domain R is written as a product of prime ideals of R, we see that $S_3 \subset S_2$

It remains to prove $S_1 \subset S_3$.

Now let x be an element of S_1 and $f \in R \setminus \{0\}$. From the definition of S_1 and the independence of valuation, we have

$$gx \in fU.$$

with f, g : relatively prime. That means, there exist $h_1, h_2 \in R$ such that

$$gh_1 + fh_2 = 1$$

holds. Then we have

$$x = (gh_1 + fh_2)x \in fU.$$

Thus we see that $x \in S_3$.

Since U is finitely generated B-module, B is a noetherian B-module. Being a submodule of U, S_3 also is a finitely generated B-module.

Let T be the torsion of U with respect to R, namely,

$$T = \{t \in U; \exists f \in R \text{ such that } f.t = 0\}$$

Since U is noetherian, we see that there exists an element $f_0 \in R$ such that $f_0.T = 0$. We notice that

$$f_0.U \cap T = 0$$

holds. Indeed, if $s = f_0 s' \in f_0 U \cap T$, then s' itself is an element of T and hence $s = f_0 s' = 0$. It is now easy to see that for any element x of S_3 and for any element f in $R \setminus \{0\}$, there exists unique $y \in f_0 U$ such that $x = f.y$ holds. From the uniqueness we easily see that y is actually an element of S_3. □

4.3. Flatness criterion

Proposition 4.1 (flatness criterion). *Let R be a Dedekind domain with infinite number of primes. Let A, B be noetherian, finitely generated, flat R-algebras. Assume B has no albert hole. Let $\phi : A \to B$ be an R-algebra homomorphism such that $\phi \mod \mathfrak{p} : A/\mathfrak{p}A \to B/\mathfrak{p}B$ is left flat. Then ϕ is left flat.*

Proof. Let $\mathfrak{p} \in \mathrm{Spm}(A)$. In the sequel, we use the following notation and essentially follow the argument which appears in [8]

$$A_n = A/\mathfrak{p}^{n+1}A, \quad B_n = B/\mathfrak{p}^{n+1}B$$

Then we have

$$\begin{aligned} B \otimes_A \mathfrak{p}A &\cong B \otimes_A A \otimes_R \mathfrak{p} && \text{(flatness of } A \text{ over } R) \\ &\cong B \otimes_R \mathfrak{p} \\ &\cong \mathfrak{p}B && \text{(flatness of } B \text{ over } R) \\ &= B.(\mathfrak{p}A) \end{aligned}$$

Similarly we have the following isomorphism.

$$(B/\mathfrak{p}^{n+1}B)\otimes_{(A/\mathfrak{p}^{n+1})}(\mathfrak{p}A/\mathfrak{p}^{n+1}A)\cong \mathfrak{p}B/\mathfrak{p}^{n+1}B \tag{4.2}$$

Now consider an exact sequence

$$0\to \mathfrak{p}A/\mathfrak{p}^{n+1}A\to A/\mathfrak{p}^{n+1}A\to A/\mathfrak{p}A\to 0$$

and the following long exact sequence associated with it.

$$\begin{aligned}0=\operatorname{Tor}_1^{A/\mathfrak{p}^{n+1}A}(B/\mathfrak{p}^{n+1}B, A/\mathfrak{p}^{n+1}A)&\to \operatorname{Tor}_1^{A/\mathfrak{p}^{n+1}A}(B/\mathfrak{p}^{n+1}B, A/\mathfrak{p}A)\\ \to (B/\mathfrak{p}^{n+1}B)\otimes_{(A/\mathfrak{p}^{n+1}A)}(\mathfrak{p}A/\mathfrak{p}^{n+1}A)&\to B/\mathfrak{p}^{n+1}B\end{aligned}$$

In view of isomorphism (4.2), we see that the last arrow in the sequence above is injective. Hence we obtain

$$\operatorname{Tor}_1^{A/\mathfrak{p}^{n+1}A}(B/\mathfrak{p}^{n+1}B, A/\mathfrak{p}A)=0. \tag{4.3}$$

For any $A/\mathfrak{p}A$ module N, choose a free $A/\mathfrak{p}A$-module F_0 such that

$$0\to K\to F_0\to N\to 0$$

is exact. Then we have the following exact sequence.

$$\begin{aligned}\operatorname{Tor}_1^{A_n}(B_n,F_0)&\to \operatorname{Tor}_1^{A_n}(B_n,N)\\ &\to B_n\otimes_{A_n}K\to B_n\otimes_{A_n}F_0\end{aligned}$$

Note that for N, F_0, tensor products with B_n over A_n are same as those with B_0 over A_0. Since B_0 is flat over A_0 by hypothesis, we therefore see that the arrow in the bottom is injective. On the other hand, from equation (4.3) we obtain $\operatorname{Tor}_1^{A_n}(B_n,F_0)=0$ to conclude

$$\operatorname{Tor}_1^{A_n}(B_n,N)=0.$$

If N is an A_n-module which is not necessarily an A_0 -module, we pay our attention to an exact sequence

$$0\to \mathfrak{p}N\to N\to N/\mathfrak{p}N\to 0$$

and see by induction that

$$\operatorname{Tor}_1^{A_n}(B_n,N)=0 \qquad (\forall N : A_n\text{-module})$$

holds. That means, B_n is left-flat over A_n

Assume

$$0\to \mathfrak{a}\to A$$

exact. That means, $\mathfrak{a}$ is an left ideal of A.

Let K be the kernel of $B \otimes_A \mathfrak{a} \to B$.

$$0 \to K \to B \otimes_A \mathfrak{a} \to B$$

Now consider the following exact sequence

$$0 \to \mathfrak{a}/(\mathfrak{p}^n A \cap \mathfrak{a}) \to A/\mathfrak{p}^n A.$$

Since B_n is flat over A_n, we see that

$$0 \to (B \otimes_A \mathfrak{a})/(B \otimes_A (\mathfrak{p}^n A \cap \mathfrak{a})) \to B/\mathfrak{p}^n B$$

is exact. Therefore we have

$$K \subset \mathrm{Image}(B \otimes_A (\mathfrak{p}^n A \cap \mathfrak{a}) \to B \otimes_A \mathfrak{a}).$$

From an argument similar to the one in a proof of the Artin-Rees theorem [8, Theorem 8.5], we see that there exists a positive integer n_0 such that

$$\mathfrak{a} \cap \mathfrak{p}^m A \subset \mathfrak{p}^{m-n_0} \mathfrak{a}$$

holds for all $m > n_0$. Thus we have

$$K \subset \mathrm{Image}(\mathfrak{p}^{m-n_0} B \otimes_A \mathfrak{a} \to B \otimes_A \mathfrak{a}).$$

Since this holds for any $\mathfrak{p} \in \mathrm{Spm}(R)$, if we put

$$S = \cap_{\mathfrak{p},m} \mathfrak{p}^m (B \otimes_A \mathfrak{a}).$$

then we have $K \subset S$. We only need to show that $S = 0$. We put

$$U = B \otimes_A \mathfrak{a}.$$

Since S is a finitely generated $C = B \otimes_R A$-module, it is noetherian. We deduce from Lemma 4.4 that S is equal to 0. □

5. Main results

5.1. Poisson bracket defined by a lifting

For any field k of characteristic $p \neq 0$, we denote by $W_n(k)$ the ring of Witt vectors of length n.

Proposition 5.1. *Let k be a perfect field of characteristic $p \neq 0$. Let A be an algebra over k. We assume A is finite over its center $Z(A)$. Assume there is given an algebra A_2 which is free as a $W_2(k)$-module such that $A_2/pA_2 \cong A$. Then*

1. *We can introduce a "bracket product" on $Z(A)$*

$$\{\bullet, \bullet\} : Z(A) \times Z(A) \to Z(A)$$

defined by

$$p \cdot \{f_1, f_2\} = [\hat{f}_1, \hat{f}_2] (= \hat{f}_1 \hat{f}_2 - \hat{f}_2 \hat{f}_1) \text{ modulo } p^2 \qquad (\forall f_1, f_2 \in Z(A))$$

where $\hat{f}_1, \hat{f}_2$ are the lift of f_1, f_2 to A_2.

2. *The bracket product satisfies the following properties.*
(a) $\{f, g\} = -\{g, f\} \qquad (\forall f, \forall g \in Z(A))$.
(b) $\{f_1 f_2, g\} = \{f_1, g\} f_2 + f_1 \{f_2, g\} \qquad (\forall f_1, \forall f_2, \forall g \in Z(A))$.

3. *Assume furthermore that there exists an $W_3(k)$-algebra A_3 which is free as a $W_3(k)$-module such that*

$$A_3 / p^2 A_3 \cong A_2.$$

Then the bracket product satisfies the Jacobian identity. In other words, $Z(A)$ is a Poisson algebra with the bracket product.

Proof. For any $a, b \in A$ such that at least one of them is in the center $Z(A)$, there exists an element of A, which we denote by $x_{a,b}$, which satisfies the following relation.

$$[\hat{a}, \hat{b}] = p \cdot x_{a,b}$$

Note that $x_{a,b}$ is unique modulo p since we assumed A_2 to be free as a $W_2(k)$-module. We can also easily verify that if both a, b belongs to the center $Z(A)$, the element $x_{a,b}$ modulo p is independent of the choice of the lifts $\hat{a}, \hat{b}$. Thus we have a well-defined A-valued bracket

$$Z(A)^2 \ni (a, b) \mapsto \{a, b\} = (x_{a,b} \text{ modulo } p) \in A$$

on $Z(A)$. Now for any $g \in A$, we have

$$\begin{aligned} p[x_{f_1,f_2}, g] &= [[\hat{f}_1, \hat{f}_2], g] = [[\hat{f}_1, g], \hat{f}_2] + [\hat{f}_1, [\hat{f}_2, g]] = [p x_{f_1,g}, \hat{f}_2] + [\hat{f}_1, p x_{f_2,g}\}] \\ &= p^2 (x_{x_{f_1,g}, f_2} + x_{f_1, x_{f_2,g}}) \\ &\in p^2 A. \end{aligned}$$

Thus $[x_{f_1,f_2}, g] \in pA$. So the bracket is actually $Z(A)$ valued.

It is easy to see that the Poisson bracket has the properties (2) in the Proposition. The statement (3) is also clear. □

Remark 4. We often come to a situation in which we only know a subalgebra Z of $Z(A)$ which happens to be **Poisson closed**, that means, closed under the Poisson bracket.

Combining the above result with the theory of ultra product (Definition 2.3, [13]), we easily deduce the Proposition 5.2 below. Before we state it, let us have the following note to keep the statement shorter.

Note 1. Let us recall that a ring R of integers of an algebraic number field clearly satisfies the following conditions.

1. R has infinite number of primes.
2. The residue fields of R are all perfect.

Proposition 5.2. *Let R be a commutative Dedekind ring R which satisfies the properties* 1. *and* 2. *of the Note above. Let A, B be an R algebra which satisfy the following conditions.*

1. *A is free as an R-module.*
2. *A yields an affine NAS with an affine shadow* $\operatorname{Spec} B$. *(Note that by definition we have an injection $\psi_{\mathfrak{p}} : B/\mathfrak{p}B \hookrightarrow Z(A/\mathfrak{p}A)$ for each $\mathfrak{p} \in \operatorname{Spm}(R)$.)*
3. *Each image* $\operatorname{Image}(\psi_{\mathfrak{p}})$ *is Poisson closed.*

Then for any ultra filter $\mathcal{U}$ on $\operatorname{Spm}(R)$, *the algebra $B_{\mathcal{U}}$ has natural Poisson bracket to make it a Poisson algebra.*

Corollary 5.1. *Let R, A, B as in Proposition above. Assume furthermore that A is of moderate growth over B.* (*Definition* 3.4)*. Then for any ultra filter $\mathcal{U}$, $B \otimes_R R_{\mathcal{U}}$ has natural Poisson bracket to make it a Poisson algebra.*

Proof. Since grading of $A/\mathfrak{p}A$ is defined by the grading of A, we may easily see that

$$\deg_{A/\mathfrak{p}A}\{f, g\} \leq \deg_{A/\mathfrak{p}A} f + \deg_{A/\mathfrak{p}A} g$$

holds for any $f, g \in A/\mathfrak{p}A$. Then for any $f, g \in B/\mathfrak{p}B$, we have the following estimate of degrees.

$$\begin{aligned}
&\deg_{B/\mathfrak{p}B}\{f, g\} \leq a_{\mathfrak{p}}/c_1 \deg_{A/\mathfrak{p}A}\{f, g\} \\
&\leq a_{\mathfrak{p}}/c_1(\deg_{A/\mathfrak{p}A} f + \deg_{A/\mathfrak{p}A} g) \leq c_2/c_1(\deg_{B/\mathfrak{p}B} f + \deg_{B/\mathfrak{p}B} g)
\end{aligned}$$

Thus the bracket product is "continuous". □

Remark 5. There are occasions where the Poisson brackets above already defined over R. That means, $\{f, g\} \in B$ for any $f, g \in B$. In such a case, we may simply say "we have a Poisson bracket on B defined by the lifting".

5.2. Completions of Weyl algebras

The rest of this section is devoted to applying our theory to Weyl algebras. In this subsection we first review the definition of the Weyl algebras and then recall certain results concerning their completions. The method here is almost identical to the one appears in [12], [13].

5.2.1. Definition of Weyl algebras In this subsection, we review the definition of Weyl algebras.

Definition 5.1. Let n be a positive integer. A Weyl algebra $A_n(k)$ over a commutative ring k is an algebra over k generated by $2n$ elements $\{\gamma_1, \gamma_2, \ldots, \gamma_{2n}\}$ with the "canonical commutation relations" (CCR)

$$[\gamma_i, \gamma_j] = h_{ij} \qquad (1 \leq i, j \leq 2n).$$

Where h is a non-degenerate anti-hermitian $2n \times 2n$ matrix of the following form.

$$(h) = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

In the rest of this section, the letter h will always represent the matrix above. We denote by $\bar{h}$ the inverse matrix of h.

5.2.2. The positive characteristic case In this subsection we discuss the case in which k is a field of positive characteristic p. In this case, the Weyl algebra $A_n(k)$ has many both-sided ideals. We may thus consider a completion of $A_n(k)$ with respect to such an ideal.

Definition 5.2. Let k be a ring which satisfies $p.1_k = 0$ for some positive integer $p \neq 0$. We denote by $\hat{A}_n(k)$ the completion of $A_n(k)$ with respect to the ideal I_0 generated by $\{\gamma_i^p\}_{i=1}^{2n}$.

(Note that I_0 above is equal to $\sum_{i=1}^{2n} A_n(k)\gamma_i^p$.)

Many of the arguments developed in the author's previous paper [13] can be generalized to the completion $\hat{A}_n(k)$. We record some of them here.

Lemma 5.1. *Let k be a field of characteristic $p \neq 0$. The following statements hold.*

1. *The center $\hat{Z}_n(k)$ of $\hat{A}_n(k)$ is equal to the formal power series $k[[\gamma_1^p, \gamma_2^p, \ldots, \gamma_{2n+1}^p, \gamma_{2n}^p]]$.*
2. *Let $\hat{\mathfrak{M}}$ be a two sided ideal of $\hat{A}_n(k)$ generated by $\gamma_1^p, \ldots, \gamma_{2n}^p$. Then the residue ring $\hat{A}_n/\hat{\mathfrak{M}}$ is isomorphic to the full matrix ring $M_{p^n}(k)$ and thus the ideal $\hat{\mathfrak{M}}$ is a maximal ideal of $\hat{A}_n$.*
3. *The ideal $\hat{\mathfrak{M}}$ is the unique maximal two sided ideal of $\hat{A}_n(k)$.*

Proof. (1), (2): Easy consequence of Lemma 5.3
(3): Let I be another proper two-sided ideal of $\hat{A}$. Let us assume $I \neq \hat{\mathfrak{M}}$. Then by using the maximality of $\hat{\mathfrak{M}}$ we have $I + \hat{\mathfrak{M}} = \hat{A}_n$. That means, there exists an element $x \in \hat{\mathfrak{M}}$ such that

$$1 + x \in I$$

holds. It is easy to see that formal inverse

$$1 + \sum_{i=1}^{\infty} (-x)^i$$

belongs to $\hat{A}_n$ and is indeed an inverse of $1 + x$. Thus we have $I = \hat{A}_n$. Since $M_{p^n}(k)$ is simple, we see that $I = \hat{A}$ holds. Thus every proper two-sided ideal of $\hat{A}$ is a subset of $\hat{\mathfrak{M}}$. $\square$

Definition 5.3. Let k be a field of characteristic $p \neq 0$.

1. We denote by $\hat{A}_n(k)$ the completion of $A_n(k)$ with respect to the ideal I_0 generated by $\{\gamma_i^p\}_{i=1}^{2n}$. (Although this is a special case of Definition 5.2, we recall it here for the purpose of future references.)
2. $\hat{Z}_n(k) =$ (The center of $A_n(k)$) $= k[[\gamma_1^p, \gamma_2^p, \gamma_3^p, \ldots, \gamma_{2n}^p]]$
3. $\hat{\mathfrak{M}}_n(k) =$ (the unique maximal ideal of $\hat{Z}_n(k)$) $= \sum_{i=1}^{2n} \hat{Z}_n(k)\gamma_i^p$.
4. $T_i =$ (the p-th root of γ_i^p) (in the algebraic closure of the quotient field of $\hat{Z}_n(k)$).

5. $\hat{S}_n(k) = k[[T_1, T_2, T_3, \dots, T_{2n}]]$.

If the field k involved is obvious from contexts, then we shall omit "(k)" and simply denote $\hat{A}_n, \hat{Z}_n, \hat{\mathfrak{M}}_n$, and so on.

The following three lemmas are fundamental in the study of $\hat{A}_n$. The proofs are almost identical with the ones of results on (non-completed) A_n.

Lemma 5.2. *Let $\varphi : \hat{A}_n(k) \to \hat{A}_n(k)$ be a continuous k-algebra homomorphism. Then the following statements hold.*

1. *$\bar{\varphi} : \hat{A}_n(k) \to \hat{A}_n(k)/\hat{\mathfrak{M}}_n(k)$ obtained as a composition of φ with the canonical projection is surjective.*
2. *$\varphi(\hat{Z}_n(k)) \subset \hat{Z}_n(k)$.*

Proof. (1) The kernel I of $\bar{\varphi}$ is a two sided ideal of $\hat{A}_n$ and is therefore a subset of $\hat{\mathfrak{M}}_n$. On the other hand, the associate map

$$\hat{A}_n/I \to \hat{A}_n/\hat{\mathfrak{M}}_n$$

is an injection. By using dimension argument we see that $I = \hat{\mathfrak{M}}_n$ and that the map $\bar{\varphi}$ is a surjection.

(2) Let S be the k-linear span of

$$\{\gamma_1^{i_1}\gamma_2^{i_2}\dots\gamma_{2n}^{i_{2n}}; \qquad i_1, i_2, \dots, i_{2n} \in \{0, 1, \dots, p-1\}\}.$$

Then we note that every element f of $\hat{A}_n$ is written as a "formal power series".

$$\sum_{I \subset \mathbb{N}^{2n}} c_I(f)(\gamma^p)^I \qquad (c_I(f) \in S).$$

Let us assume that there exists an element $\check{g}$ of $\hat{Z}_n$ such that $g = \varphi(\check{g})$ does not belong to the center $\hat{Z}_n$. We may find a smallest (in the graded lexicographical order) index set I_0 such that

$$c_{I_0}(g) \notin k$$

Then there exists an index i such that γ_i does not commute with $c_{I_0}(g)$. By the result of (1) we may find an element $\check{h} \in \hat{Z}_n$ such that its image $h = \varphi(\check{h})$ belongs to $\gamma_i + \hat{\mathfrak{M}}_n$. Observing the leading term of the commutator $[h, g]$, we come to a contradiction. □

Lemma 5.3 ([12, Lemma 1]). *Let k be a field of characteristic p. Then a k-algebra $\mathfrak{M}$ which is generated by $\mu_1, \mu_2, \dots, \mu_{2n}$ with the relations*

$$[\mu_i, \mu_j] = h_{ij}, \quad \mu_i^p = 0 (i, j = 1, 2, \dots, 2n)$$

is isomorphic to the full matrix algebra $M_{p^n}(k)$.

Proof. We have a representation Φ_0 of the algebra on $k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots, x_n^p)$ where

$$\left.\begin{aligned} \Phi_0(\mu_i) &= \text{multiplication by } x_i \\ \Phi_0(\mu_{i+n}) &= \partial/\partial x_i. \end{aligned}\right\} \qquad i = 1, 2, \dots, n$$

□

Using the lemma above we may prove the following lemma.

Lemma 5.4. *Let k be a field of characteristic p. Let us use the abbreviation as mentioned in the end of Definition 5.3. Let $u : \mathrm{Spf}(\hat{S}_n) \to \mathrm{Spf}(\hat{Z}_n)$ be a morphism of affine formal schemes defined by $u^*(\gamma_i^p) = T_i^p$.*

The map u^* extends uniquely to a map $u^*(\hat{A}_n)\check{} \to M_{p^n}(\mathcal{O}_{\hat{S}_n})$ by the formula

$$u^*(\gamma_i) = T_i + \mu_i.$$

Proof. Same as a proof of [12, Lemma 5] □

5.3. The Poisson bracket on the center of a completed Weyl algebra

Lemma 5.5. *Let ξ, η be two elements in an algebra which satisfy $[\eta, \xi] = 1$. Then we have*

$$[\eta^p, \xi^p] \equiv p!(\equiv -p) \quad (\textit{modulo } p^2)$$

Proof. This is a special case of a calculation on differential operators (which appears for example in [9, formula (11,4)]). Indeed, we have

$$\eta^p \xi^p = \sum_{k=1}^{p} \frac{1}{k!} (\partial_\xi)^k.(\xi^p) \cdot (\partial_\eta)^k.(\eta^p) = \sum_{k=1}^{p} k! \binom{p}{k}^2 \xi^{p-k} \eta^{p-k} \equiv \xi^p \eta^p + p!$$
$$(\text{modulo } p^2).$$

□

Corollary 5.2. *Let $\{\gamma_i\}_{i=1}^{2n}$ be standard generators of $A_n(\mathbb{Z})$. Then we have the following relations.*

$$[\gamma_i^p, \gamma_j^p] = -p h_{ij}. \textit{ modulo } p^2.$$

Proposition 5.3. *Let k be a perfect field of characteristic $p \neq 0$. Then the center $\hat{Z}_n(k)$ of the completed Weyl algebra $\hat{A}_n(k)$ has a structure of a Poisson algebra. It is natural in the following sense. Let $k_2 = W_2(k)$ be the ring of Witt vectors of length 2. Then any continuous k_2-algebra homomorphism $\phi : \hat{A}_n(k_2) \to \hat{A}_n(k_2)$ induces a map $\psi : \hat{Z}_n(k) \to \hat{Z}_n(k)$ which preserves the symplectic structure. Therefore it also preserves the symplectic structure on $\hat{S}_n$. The symplectic form on $\hat{S}_n$ is given by*

$$\Omega = \sum_{ij} \bar{h}_{ij} dT_i \wedge dT_j.$$

Proof. The same proof as the one of Proposition 5.1 works. (We need to make use of "formal version" of the Proposition, but the modification needed to make the formal version is trivial. □

5.4. p-curvatures

In this subsection we refine the result obtained for Weyl algebras in [13] to the formal completions. In doing so we would like to make clear that the Cartier operator [1], [5] plays an important role in our theory.

In this subsection, we assume k is a field of characteristic $p \neq 0$ and fix it once and for all. Recall we have topological algebras $\hat{A}_n, \hat{Z}_n, \hat{S}_n$ over k (Definition 5.3).

We may regard $\hat{A}_n$ as a subsheaf of the sheaf $M_{p^n}(\mathcal{O})$ of matrix algebras over $\mathrm{Spf}(\hat{S}_n(k))$.

Lemma 5.6. *We have a canonical connection ∇° on $M_{p^n}(\mathcal{O})$.*

$$\nabla^\circ_{\partial/\partial T_i} = \partial/\partial T_i - \sum_j \bar{h}_{ij} \,\mathrm{Ad}\, \mu_j$$

$$\nabla^\circ_{\partial/\partial T_i} . \gamma_j = \delta_{ij} - \sum_l \bar{h}_{il} h_{lj} = 0$$

It is a connection of an Azumaya algebra, that means, a PGL_{p^n}-bundle. We would like to consider its lift to a GL_{p^n}-bundle. One such is the one defined by

$$\nabla_{\partial/\partial T_i} = \partial/\partial T_i - \sum_j \bar{h}_{ij} \mu_j .$$

The curvature of ∇ is computed as follows.

$$[\nabla_{\partial/\partial T_i}, \nabla_{\partial/\partial T_j}] = \left[\sum_k \bar{h}_{ik}\mu_k, \sum_l \bar{h}_{jl}\mu_l \right] = \sum_{k,l} \bar{h}_{ik}\bar{h}_{jl}h_{kl} = \bar{h}_{ji}$$

For a ("scalar valued") 1-form α consider

$$\nabla^{(\alpha)} = \nabla + \alpha .$$

Any lift of ∇° is of the form $\nabla^{(\alpha)}$ for some α.

The curvature of $\nabla^{(\alpha)}$ is computed as follows.

$$[\nabla^{(\alpha)}_{\partial/\partial T_i}, \nabla^{(\alpha)}_{\partial/\partial T_j}] = g_{ji} + \alpha_{j,i} - \alpha_{i,j}$$

$\nabla^{(\alpha)}$ is integrable if and only if the values above are zero. For example, this is the case if α is equal to

$$\rho = \sum_{i=1}^n T_i dT_{i+n} .$$

If $\nabla^{(\alpha)}$ is integrable, the p-curvature of $\nabla^{(\alpha)}$ is computed in the following way.

$$(\nabla^{(\alpha)}_{\partial/\partial T_i})^p = \left(\partial/\partial T_i - \sum_j g_{ij}\mu_j + \alpha_i \right)^p = \alpha_i^p + (\partial/\partial T_i)^{p-1} . \alpha_i$$

(In the calculation here we use technique which appears often in the theory of p-curvatures. [10, §10.6.3], [13, Lemma 3.4])

In particular, p-curvature of $\nabla^{(\rho)}$ is given by

$$(\nabla^{(\rho)}_{\partial/\partial T_i})^p = \rho_i^p.$$

Suppose now that there is given another set $\{\overline{\gamma_i}\}_{i=1}^{2n}$ of topological generators of $\hat{A}_n$ which satisfies CCR. Then with the same procedure as above we obtain another local parameters $\{\overline{T_i}\}_{i=1}^{2n}$ of $\mathrm{Spm}\,\hat{S}_n$ and another lift $\overline{\nabla}$ of ∇°. Then with the same argument as in [13], we see that there exists a scalar valued 1-form ω and a "gauge transformation" G such that

$$\nabla + \omega = G\overline{\nabla}G^{-1}$$

holds.

$$\nabla + \omega + \overline{\rho} = G(\overline{\nabla} + \overline{\rho})G^{-1}$$

Comparing the p-curvatures of both hands sides, we obtain

$$\langle \omega + \overline{\rho}, \partial_{T_i}\rangle^p + \partial_{T_i}^{p-1}.(\langle \omega + \overline{\rho}, \partial_{T_i}\rangle) = \langle \overline{\rho}, v\rangle^p|_{v=\partial_{T_i}}$$

We simplify it to obtain

$$\omega_i^p + \partial_{T_i}^{p-1}(\omega_i) + (\partial_{T_i})^{p-1}(\langle \overline{\rho}, \partial_{T_i}\rangle) = 0.$$

That means,

$$\omega_i^p + \partial_{T_i}^{p-1}(\omega_i) + (\partial_{T_i})^{p-1}\sum_{j=1}^{n}\left(\overline{T}_j\frac{\partial\overline{T}_{j+n}}{\partial T_i}\right) = 0.$$

This is a differential equation satisfied by ω.

Using the Cartier operator C and its theory, the differential equation above can be rewritten in a form of the following proposition.

Proposition 5.4. *The difference ω of connections ∇ and $G\overline{\nabla}G^{-1}$ satisfies the following equations.*

1. $$d(\omega + \overline{\rho} - \rho) = 0.$$

2. $$\omega = C(\omega + \overline{\rho} - \rho).$$

Corollary 5.3. *When the coordinate change lifts to the ring $W_2(k)$ of Witt vectors of length 2, (that means, $\{\overline{\gamma_i}\}$ lifts to a set of CCR generators of $\hat{A}_n(W_2(k))$) then by Proposition 5.3 we have $d(\overline{\rho} - \rho) = 0$.*

Therefore, in that case we have

1. $d\omega = 0$,

2. $\omega - C(\omega) - C(\overline{\rho} - \rho) = 0$.

As a simple outcome of the result above, we obtain a result on shadows of mild rational automorphisms of affine Weyl algebras $A_n(K)$. To describe this, we first introduce the following notations.

Definition 5.4.

1. Let R be a commutative Dedekind domain. For any R-algebra A which yields an affine NAS with an affine shadow $Y = \mathrm{Spec}(A_0)$, we let $\mathrm{Rat}_0(A; A_0)$ be a set

$$\mathrm{Rat}_0(A; A_0) = \{\text{mild rational map from } (A, A_0) \text{ to itself.}\}$$

(See Definition 3.10 for the definition of mild rational maps)

2. For any scheme X, we let $\mathrm{Rat}(X)$ be a set

$$\mathrm{Rat}(X) = \{\text{rational map from } X \text{ to itself.}\}$$

3. For any scheme X with a symplectic structure ω, we put

$$\mathrm{Rat}_1(X, \omega) = \{f \in \mathrm{Rat}(X); f^*(\omega) = \omega\}.$$

Proposition 5.5. *Let R be a commutative Dedekind domain of characteristic zero which is finitely generated over $\mathbb{Z}$. Let $\mathcal{U}$ be an ultrafilter on* $\mathrm{Spm}(R)$. *Then there exists a map*

$$F : \mathrm{Rat}_0(A_n(R); Z_n(R)) \to \mathrm{Rat}_1(Z_n(R_{\mathcal{U}})), \omega)$$

defined by

$$F((\phi, \phi_0)) = \phi_0 (= \lim_{\mathfrak{p} \to \mathcal{U}} (\phi \textit{ modulo } \mathfrak{p}))$$

Example 5.1. Let c be an algebraic integer. Let $\phi \in \mathrm{Rat}_0(A_1(\mathbb{Z}[c]))$ be given by

$$f(\gamma_1) = \gamma_1 + c(1 + \gamma_2)^{-1},$$
$$f(\gamma_2) = \gamma_2.$$

Then the corresponding map $\phi_0 \in Rat(Z_1(\mathbb{Z}[c]))$ is given by

$$\phi_0(T_1) = T_1 + (c - c^{1/p})(1 + T_2)^{-1}$$
$$\phi_0(T_2).$$

This example shows that the map F defined in the above Proposition somewhat involves Frobenius maps and may not be described as a limit of ordinary polynomial maps.

Before closing this subsection, we cite the following results that are obtained in Author's previous papers [12], [13].

Proposition 5.6.

1. *$A_n(k)$ is an Azumaya algebra over its center $Z_n(k)$.*
2. *The restriction of $\phi_{\mathfrak{p}}$ yields a k-algebra homomorphism $\psi_{\mathfrak{p}} : Z_n(k) \to Z_n(k)$ between the centers.*
3. *$A_n(k) \otimes_{Z_n(k),\psi_{\mathfrak{p}}} Z_n(k) \cong A_n(k)$.*
4. *$\psi_{\mathfrak{p}}$ is flat for almost all (that means, all except finite number of) primes $\mathfrak{p} \in \mathrm{Spm}(R)$.*

5.5. An exact sequence

We employ the following symbols.

Definition 5.5.

$$\mathrm{Aut}(\hat{A}_n) = \{\text{continuous } k\text{-algebra automorphism of } \hat{A}_n\}$$
$$\mathrm{Aut}(\hat{Z}_n) = \{\text{continuous } k\text{-algebra automorphism of } \hat{Z}_n\}$$
$$\mathrm{Int}(\hat{A}_n) = \{\phi \in \mathrm{Aut}(\hat{A}_n); \exists g \in \hat{A}_n^{\times} \text{ such that } \phi(x) = gxg^{-1}\}$$

Proposition 5.7. *We have an exact sequence*

$$1 \to \mathrm{Int}(\hat{A}_n) \to \mathrm{Aut}(\hat{A}_n) \overset{restr.}{\to} \mathrm{Aut}(\hat{Z}_n).$$

Proof. Every continuous k-algebra endomorphism of $\hat{A}_n$ preserves $\hat{Z}_n$. (Lemma 5.2) Assume there is given a continuous k-algebra endomorphism φ of $\hat{A}_n$ which is trivial when restricted to the center $\hat{Z}_n$. Then ϕ is given by "a homomorphism on fibers". Namely, there exists a matrix valued function g on $\mathrm{Spf}(\hat{S}_n)$ such that

$$\phi(x) = gxg^{-1}$$

holds for any $x \in \hat{A}_n$. The differential equations in Proposition 5.4 turns into

$$d\omega = 0, \quad \omega = C(\omega).$$

A result of Illusie [5, Theorem 2.1.17] (originally a result of Cartier) tells us that there exists a function $c \in \hat{S}_n^{\times}$ such that

$$\omega = c^{-1}dc$$

Then we may replace g by cg and assume that $\omega = 0$. This implies that $\nabla^{\circ} g = 0$. This means, $g \in \hat{A}_n$.

Therefore ϕ is internal. □

5.6. Flatness of algebra endomorphisms of Weyl algebras

Theorem 5.1. *Let K be a field of characteristic 0. Then any K-algebra endomorphism $\phi : A_n(K) \to A_n(K)$ of a Weyl algebra is flat.*

Proof. We note that $A_n(R)$ has no albert holes over R for any commutative Dedekind domain R (Corollary 4.2). Let us first assume that K is a number field. Let $\mathfrak{O}$ be the ring of integers in K. Since Weyl algebras are finitely generated over K, we may find a non zero element $f \in \mathfrak{O}$ such that ϕ is defined over $R = \mathfrak{O}_f$. For each maximal ideal $\mathfrak{p} \in \mathrm{Spm}(R)$, ϕ induces a $k = R/\mathfrak{p}$-algebra homomorphism $\phi_{\mathfrak{p}} : A_n(k) \to A_n(k)$. (See Proposition 5.6) Thus we see that $\phi_{\mathfrak{p}}$ is flat for almost all $\mathfrak{p}$. Hence we see that

$$\phi_{R_g} : A_n(R_g) \to A_n(R_g)$$

is flat. This in turn implies that ϕ is flat.

Let us now deal with the case where K is not necessarily a number field. Since the Weyl algebras are finitely generated over K, we may assume K has a finite transcendence degree over a number field $\mathfrak{K}$.

We proceed by induction on transcendence degree d of K over $\mathfrak{K}$. The case $d = 0$ is already proved. For the case $d > 0$, we find a commutative Dedekind domain R over $\mathfrak{K}$ with $K = Q(R)$ (the quotient field of R) such that ϕ is already defined on R. $\phi_R : A_n(R) \to A_n(R)$ Then for each $\mathfrak{p} \in \mathrm{Spm}(R)$, ϕ_R induces a $R/\mathfrak{p}$-homomorphism $\phi_{\mathfrak{p}} : A_n(R/\mathfrak{p}) \to A_n(R/\mathfrak{p})$. By induction hypothesis, $\phi_{\mathfrak{p}}$ is flat. Then by Proposition 4.1, ϕ_R is flat. Thus we see that $A_n(K) \to A_n(K)$ is flat. □

6. Examples

6.1. D-modules over smooth algebraic spaces

Proposition 6.1. *Let R be a commutative Dedekind domain which is finitely generated over $\mathbb{Z}$. Let X be a smooth algebraic space over R. Then the category* (D_X-mod) *of D_X-modules is a NAS. Its principal shadow is* (*the total space of*) *the cotangent bundle T^*X.*

Proof. D_X is a sheaf of $\mathcal{O}_X$-modules which is a sheaf of algebras. Let us denote the algebraic scheme $X \times_R (R/\mathfrak{p})$ by $X_{(\mathfrak{p})}$. (We use the similar notation for all algebraic spaces hereafter in this proof.) Let us take an étale-open affine scheme U of X. Let $B = B(U)$ the affine coordinate ring. We assume that U is sufficiently small so that there exists a coordinate functions $x_1, x_2, \dots, x_n \in B$. Let us write derivations d/dx_i as ∂_i for short.

Then it is easy to see that the center of $(D_U)|_{U_{(\mathfrak{p})}}$ is equal to $Z_{(\mathfrak{p})}(U) = \mathcal{O}^p_{U_{(\mathfrak{p})}}[\partial_1^p, \partial_2^p, \dots, \partial_n^p]$.

The principal shadow of $X_{(\mathfrak{p})}$ is obtained by gluing these centers $Z_{\mathfrak{p}}(U)$ together. The way how to glue is essentially the same as the one described in [13, Example 8.3]. Namely, let U, V be two étale-open coordinate affine scheme of X. Let $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ be coordinate functions of these two respectively. Let a, b be "transition functions" on an appropriate intersection scheme (that means, affine subscheme of $U \otimes_X V$.) Namely,

$$y_i = a_i(x), \quad \frac{\partial}{\partial y_i} = \sum_j b_{ij}(x) \frac{\partial}{\partial x_j}$$

Then one has

$$(y_i)^p = (a_i(x))^p, \quad \left(\frac{\partial}{\partial y_i}\right)^p = \sum_j (b_{ij}(x))^p \left(\frac{\partial}{\partial x_j}\right)^p.$$

The former relation is obvious. The latter is verified using techniques in [13] □

6.2. Universal enveloping algebras of Lie algebras

In this subsection and the next, we fix a complex semisimple Lie algebra $\mathfrak{g}$. It is well known that $\mathfrak{g}$ is actually defined over a commutative Dedekind domain R which is finitely generated over the ring $\mathbb{Z}$ of integers. That means, there exists a $\mathbb{Z}$-Lie algebra $\mathfrak{g}_R$ which is free as a R-module such that $\mathfrak{g}_R \otimes_R \mathbb{C}$ is isomorphic to $\mathfrak{g}$. Furthermore, R may be so chosen that most of the ingredients (such as Cartan subalgebra $\mathfrak{h}$) which appear in the theory of Lie algebras are also defined over it. In this subsection, we fix such R. Let W be the Weyl algebra of $\mathfrak{g}$.

We would like to study the universal enveloping algebra $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ using our technique.

We first summarize some well-known results on semisimple Lie algebras in the following Lemma.

Lemma 6.1. *Let k be an extension field of a residue field $R/\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spm}(R)$. Let p be its characteristic. Let $\mathfrak{U}_k = \mathfrak{U}(\mathfrak{g}_k)$ be the universal enveloping algebra of $\mathfrak{g}_k = \mathfrak{g} \otimes_R k$. Then $\mathfrak{g}_k$ has a unique structure of restricted Lie algebra* ([6, *definition* 4 *of section* V.7].) *That means, $\mathfrak{g}_k$ admits a "p-operator" $X \mapsto X^{[p]}$.*

In terms of a map

$$\phi : \mathfrak{g}_k \ni X \mapsto \phi(X) = X^{[p]} - X^p \in \mathfrak{U}_k,$$

we may put the axioms of the p-operator in the following way.

1. *ϕ is a p-linear map. That means, ϕ is additive and satisfies*

$$\phi(c.X) = c^p \phi(X) \quad (\forall c \in k, \ \forall X \in \mathfrak{g}_k).$$

2. *The range of ϕ lies in the center of $\mathfrak{U}$.*

Proof. See [6, section V.7]. □

Now we have the following Proposition to describe the universal enveloping algebra as an affine NAS.

Proposition 6.2. $\mathfrak{U}(\mathfrak{g})$ *yields an affine NAS with a shadow* $\operatorname{Spec}(\mathfrak{S}(\mathfrak{g}))$ *where $\mathfrak{S}(\mathfrak{g})$ is the symmetric tensor algebra generated by $\mathfrak{g}$.* (*We may, by abuse of language, say that "the shadow is the dual $\mathfrak{g}^*$ of $\mathfrak{g}$".*) *The Poisson bracket on $\mathfrak{S}(\mathfrak{g}_{R_{\mathfrak{U}}})$ introduced as in Corollary* 5.1 (*with Remark after it*) *is given by the following equation*

$$\{X, Y\} = [X, Y] \qquad (\forall X, Y \in \mathfrak{g}_R) \tag{6.1}$$

In particular, the Poisson bracket is actually defined over R (that means, on $\mathfrak{S}(\mathfrak{g})$.)

Proof. For any $\mathfrak{p} \in \mathrm{Spm}(R)$, let us put $k = R/\mathfrak{p}$ and define S_k as a subalgebra of $\mathfrak{U}(\mathfrak{g}_k)$ generated by the range of ϕ. By using the above Lemma and the Poincaré-Birkoff-Witt theorem, we conclude that S_k is isomorphic to a symmetric tensor algebra $\mathfrak{S}(\mathfrak{g}_k)$. It is also clear that $\mathfrak{U}(\mathfrak{g}_k)$ is finite over S_k. The phantom consists of the "reduced enveloping algebra of $\mathfrak{g}_k$" in the sense of [4].

Let us now describe the symplectic structure on the shadow. Let $H \in \mathfrak{g}_R$ be a semisimple element of $\mathfrak{g}_R$. We would like to prove first the formula (6.1) for $X = H$. To this end, we extend R to its finite extension S so that $\mathfrak{g}_S$ is decomposed into $\mathrm{ad}(H)$-eigen spaces. Let X_α be an $\mathrm{ad}(H)$-eigen vector.

$$[H, X_\alpha] = \alpha X_\alpha.$$

In the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, we have

$$H X_\alpha = X_\alpha (H + \alpha).$$

Using this relation we may easily deduce

$$H X_\alpha^p = X_\alpha^p (H + p\alpha).$$

For any polynomial $f \in S[x]$, we have

$$f(H) X_\alpha^p = X_\alpha^p f(H + p\alpha).$$

In particular, we put $f(x) = x - x^p$ and obtain

$$(H - H^p) X_\alpha^p = X_\alpha^p ((H + p\alpha) - (H + p\alpha)^p).$$

$$(H - H^p) X_\alpha^p = X_\alpha^p (H - H^p + p\alpha) \quad (\text{modulo } p^2)$$

$$[\phi(H), \phi(X_\alpha)] = p\alpha\phi(X_\alpha) = p\phi([H, X_\alpha]) \quad (\text{modulo } p^2).$$

Thus by the definition of the Poisson bracket we see

$$\{H, X_\alpha\} = [H, X_\alpha].$$

Since X_α is an arbitrary $\mathrm{ad}(H)$-eigen vector, we see that

$$\{H, X\} = [H, X]$$

holds for any $X \in \mathfrak{g}_S$ (in particular, for any $X \in \mathfrak{g}_R$.)

On the other hand, we know that semisimple elements is dense in $\mathfrak{g}$ (See [3, 23.3 Appendix]).

We thus conclude that the Poisson bracket satisfies the relation described in the statement of the Proposition. $\square$

6.3. Steinberg map

We use the same notation as in the previous subsection. Let G be a Chevalley group of $\mathfrak{g}$. It is known ([3]) that G is defined over a ring of integers.

$\mathfrak{U}(\mathfrak{g})$ has a large center $\mathfrak{Z}$ from the very beginning (that means, without passing to positive characteristic cases).

Lemma 6.2 (See the proof of [3, Theorem 23.3]). *The center $\mathfrak{Z}$ of $\mathfrak{U}$ is a polynomial ring over $\mathbb{C}$ generated by $l = \operatorname{rank}(\mathfrak{h})$-elements $\{c_i\}_{i=1}^{l}$. The principal symbols $\sigma(c_i)$ of c_i's are precisely those who are generators of $\mathfrak{S}(\mathfrak{g})^G \cong \mathfrak{S}(\mathfrak{h})^W$.*

The elements c_i are sometimes referred to as "Casimirs" in the physics literature.

$\mathfrak{U}(\mathfrak{g})$ may be regarded as a sheaf of algebras over an affine space $\operatorname{Spec}(\mathfrak{Z})$. The aim of this subsection is to describe the fiber on a point (maximal ideal) $\mathfrak{M}_a = (c_1 - a_1, c_2 - a_2, \dots, c_l - a_l) \in \operatorname{Spm}(\mathfrak{Z})$ for $(a_1, a_2, \dots, a_l) \in \mathbb{C}^l$.

The inclusion $\mathfrak{S}(\mathfrak{g}) \hookleftarrow \mathfrak{S}(\mathfrak{g})^G \cong \mathfrak{S}(\mathfrak{h})^W$ gives rise to a map on their spectrum. Namely, $\mathfrak{g}^* \to \mathfrak{h}^*/W$. Using the Killing form, we may identify $\mathfrak{g}$ (respectively $\mathfrak{h}$) with its dual $\mathfrak{g}^*$ (respectively, $\mathfrak{h}^*$). Then the map above yields $\mathfrak{g} \to \mathfrak{h}/W$. This map is sometimes called the Steinberg map. We review the following facts.

Theorem 6.1 ([2], [7]). *Let $\chi : \mathfrak{g} \to \mathfrak{h}/W$ be the Steinberg map. The following facts hold.*

1. *χ is flat. The codimension of each fiber is equal to $l = \dim \mathfrak{h}$.*
2. *$\chi^{-1}(\overline{h})$ ($\overline{h} \in \mathfrak{h}/W$) is a finite union of G-orbits (under adjoint action).*
3. *$\chi^{-1}(\overline{h})$ contains a unique G-orbit O_h of a regular element. The orbit O_h is dense open subset in $\chi^{-1}(\overline{h})$. The complement of the orbit O_h is of codimension greater than or equal to 2.*
4. *The set of non-singular point of $\chi^{-1}(\overline{h})$ coincides with the regular element of $\mathfrak{g}$ (in the Lie algebra theoretic sense) in $\chi^{-1}(\overline{h})$.*

Put $A = \mathfrak{U}(\mathfrak{g})/\mathfrak{M}_{\bar{h}}\mathfrak{U}(\mathfrak{g})$ for a maximal ideal $\mathfrak{M}_{\bar{h}}$ of $\mathfrak{Z}$. We would like to show that A yields an affine NAS with shadow X isomorphic to a fiber $\chi^{-1}(\bar{h})$ of the Steinberg map $\mathfrak{g} \to \mathfrak{h}^W$.

Proposition 6.3. *We fix $\bar{h} \in \mathfrak{h}_R/W$. Let us put $a_i = c_i(h)$ and put*

$$\mathfrak{M}_{\bar{h}} = (c_1 - a_1, c_2 - a_2, \dots c_l - a_l)\mathfrak{Z},$$

$$A = \mathfrak{U}(\mathfrak{g})/\mathfrak{M}_{\bar{h}}\mathfrak{U}(\mathfrak{g})$$

1. *For any $\mathfrak{p} \in \operatorname{Spm}(R)$, there exists a central subalgebra $Z_{\mathfrak{p}}$ of $A/\mathfrak{p}A$ defined by*

$$Z_{\mathfrak{p}} = \langle \phi(\mathfrak{S}(\mathfrak{g})), \mathfrak{Z} \rangle / \mathfrak{M}_a \langle \phi(\mathfrak{S}(\mathfrak{g})), \mathfrak{Z} \rangle.$$

2. *A yields an affine NAS with a shadow X which satisfies $X \otimes_R \mathbb{C} \cong \chi^{-1}(\bar{h})$.*

3. *The affine coordinate ring $\mathcal{O}(X)$ on X has a natural structure of Poisson algebra.*

Proof.
(1): obvious
(2): The principal symbol $\sigma(c_i)$ is G-invariant and therefore $\phi(\sigma(c_i))$ is also G-invariant. Therefore, it is a member of $\mathfrak{Z}$. We have b_i such that $\phi(\sigma(c_i)) = b_i$ modulo $\mathfrak{M}_{\hat{h}}$.

$$\phi(\mathfrak{S}(\mathfrak{g}))/(\phi(\sigma(c_i)) - b_i)) \to \langle \phi(\mathfrak{S}(\mathfrak{g})), \mathfrak{Z} \rangle / \mathfrak{M}_{\hat{h}} \langle \phi(\mathfrak{S}(\mathfrak{g})), \mathfrak{Z} \rangle = Z$$

is a surjective map. In other words, we have a closed immersion $\chi^{-1}(\bar{h})_R \hookrightarrow \operatorname{Spec} Z$.

On the other hand, it is easy to see $\operatorname{Spec} Z$ is closed under G-action.

The dimension of $\operatorname{Spec} Z$ is greater or equal to $\dim(\mathfrak{g}) - l$.

Thus we conclude by using the previous theorem that $\chi^{-1}(\bar{h})_R = \operatorname{Spec} Z$.
(3) is a direct consequence of Corollary 5.1. The fact that the Poisson bracket is actually defined over R is proved by the same manner as in Proposition 6.2. □

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