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The compressible Euler equations for an isothermal gas with spherical symmetry

By

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Abstract

We shall study isothermal gas dynamics with spherical symmetry. In this case, existence theorems have been obtained outside a solid ball. However, little is known for the case including the origin, because the equation has a singularity there. In this paper, we will present discontinuous solutions for this case, by introducing a certain non-homogeneous conservation laws and using a modified Glimm's scheme.

1. Introduction

The compressible Euler equations for an isentropic gas in three dimensional space are given by

$$\begin{aligned} \rho_t + \nabla\cdot(\rho\vec{u}) &= 0, \\ (\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u} + pI) &= 0 \end{aligned} \tag{1.1}$$

with the equation of state

$$p = a^2\rho^\gamma, \tag{1.2}$$

where density ρ, velocity $\vec{u}$ and pressure p are functions of $x \in \mathbf{R}^3$ and $t \geq 0$, while $a > 0$ and $\gamma \geq 1$ are given constants and I is a 3 dimensional unit matrix.

In this paper, we will prove the local existence of solution for the case of spherical symmetry with $\gamma = 1$; i.e., the isothermal gas case. In this case global weak solutions are known to exist outside a solid ball at the origin in [5] and [6]. We consider this problem including the origin.

As we will be seen below, our proof does not work without these restrictions. Thus, we look for solutions of the form

$$\rho = \rho(t, |x|), \quad \vec{u} = \frac{x}{|x|}u(t, |x|). \tag{1.3}$$

Then, denoting $r = |x|$, (1.1) becomes

$$\begin{aligned} \rho_t + \frac{1}{r^2}(r^2\rho u)_r &= 0, \\ \rho(u_t + uu_r) + p_r &= 0. \end{aligned} \tag{1.4}$$

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Set $\tilde{\rho} = r^2\rho$. Then we have from (1.4)

$$\begin{aligned} &\tilde{\rho}_t + (\tilde{\rho}u)_r = 0, \\ &u_t + uu_r + \frac{a^2\tilde{\rho}_r}{\tilde{\rho}} = \frac{2a^2}{r}. \end{aligned} \tag{1.5}$$

Now, we suppose $u(t,0) = 0$ and introduce the Lagrangian mass coordinates

$$\tau = t, \qquad \xi = \int_0^r \tilde{\rho}(t,r)dr. \tag{1.6}$$

Then $\xi > 0$ as long as $\tilde{\rho} > 0$ for $r > 0$, and (1.5) is reformulated as

$$\begin{aligned} &\tilde{\rho}_t + \tilde{\rho}^2 u_\xi = 0, \\ &u_t + a^2\tilde{\rho}_\xi = \frac{2a^2}{r}. \end{aligned} \tag{1.7}$$

Set $v = 1/\tilde{\rho}$ and note that the inverse transformation to (1.6) is given by

$$t = \tau, \qquad r = \int_0^\xi v(t,\zeta)d\zeta. \tag{1.8}$$

Then after changing τ to t and ξ to x respectively, (1.7) is written as

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + \left(\frac{a^2}{v}\right)_x = \frac{2a^2}{\int_0^x v(t,\xi)d\xi}. \end{aligned} \tag{1.9}$$

Remark 1.1. If w is constant and $v(t,x) = wx^{-2/3}$, the above transformation implies that $\rho(t,x)$ becomes constant.

We consider the initial boundary value problem for (1.9) in $t \geq 0$, $x \geq 0$ with following boundary and initial conditions

$$\begin{aligned} U(0,x) &\doteq (\bar{v}(x),\bar{u}(x)) = (\bar{w}(x)x^{-\frac{2}{3}},\bar{u}(x)) \\ &= \begin{cases} U^- = (w^- x^{-\frac{2}{3}},0), & 0 < x < \bar{x}, \\ U^+ = (w^+(x)x^{-\frac{2}{3}},u(x)), & \bar{x} < x \end{cases} \end{aligned} \tag{1.10}$$

and

$$u(t,0) = 0 \qquad \text{for} \quad t > 0. \tag{1.11}$$

Our main result is as follows.

Theorem 1.2. *There exist δ_0, δ_1 and $T > 0$ with the following property. For every initial data of the form* (1.10) *with*

$$\textit{Tot.Var.}\ (\bar{v},\bar{u}) < \delta_1,\ \max\{\sup_x w^+(x), w^-\} > \delta_0, \tag{1.12}$$

the initial boundary value problem (1.9) *through* (1.11) *has a weak solution defined in $0 \leq t \leq T$, where w^- is a constant.*

For simplicity, we consider the following initial conditions from now on.

$$U(0,x) \doteq (\bar{v}(x), \bar{u}) = (\bar{w}x^{-\frac{2}{3}}, \bar{u}) = \begin{cases} U^- = (w^- x^{-\frac{2}{3}}, 0), & 0 < x < \bar{x}, \\ U^+ = (w^+ x^{-\frac{2}{3}}, u), & \bar{x} < x, \end{cases} \tag{1.13}$$

where w^-, w^+ and u are constants.

First, we consider the auxiliary equation

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + \left(\frac{a^2}{v}\right)_x &= \frac{2a^2}{3xv}. \end{aligned} \tag{1.14}$$

We will use the idea of [5] and Riemann solutions of (1.14) to construct approximate solutions. Then, notice that both (1.9) and (1.14) have a steady-state solution of the form $v(t,x) = w_0 x^{-2/3}$, $u(t,x) = u_0$, where w_0 and u_0 are constants. This is the key to guarantee the existence of solution.

2. The Cauchy problem of the auxiliary equation

The homogeneous equation corresponding to (1.9) is

$$\begin{aligned} w_t - u_x &= 0, \\ u_t + \left(\frac{a^2}{w}\right)_x &= 0. \end{aligned} \tag{2.1}$$

Its Jacobian matrix has the two real distinct eigenvalues

$$\lambda_1 = -\frac{a}{w}, \qquad \lambda_2 = \frac{a}{w} \tag{2.2}$$

with corresponding eigenvectors

$$r_1 = \left(1, \frac{a}{w}\right), \qquad r_2 = \left(-1, \frac{a}{w}\right). \tag{2.3}$$

Let

$$U^- \doteq (w^-, u^-), \tag{2.4}$$

where w^- and u^- are constants and $w^- > 0$. The 1-rarefaction curve through U^- is

$$\mathbf{R}_1 = \left\{(w,u) : u - u^- = a\log w - a\log w^-\right\}. \tag{2.5}$$

Similarly, the 2-rarefaction curve through U^- is

$$\mathbf{R}_2 = \left\{(w,u) : u - u^- = -a\log w + a\log w^-\right\}. \tag{2.6}$$

These shock curve are computed as

$$\mathbf{S}_1 = \left\{(w,u) : u - u^- = \frac{a}{\sqrt{ww^-}}(w - w^-), \quad w^- > w\right\} \tag{2.7}$$

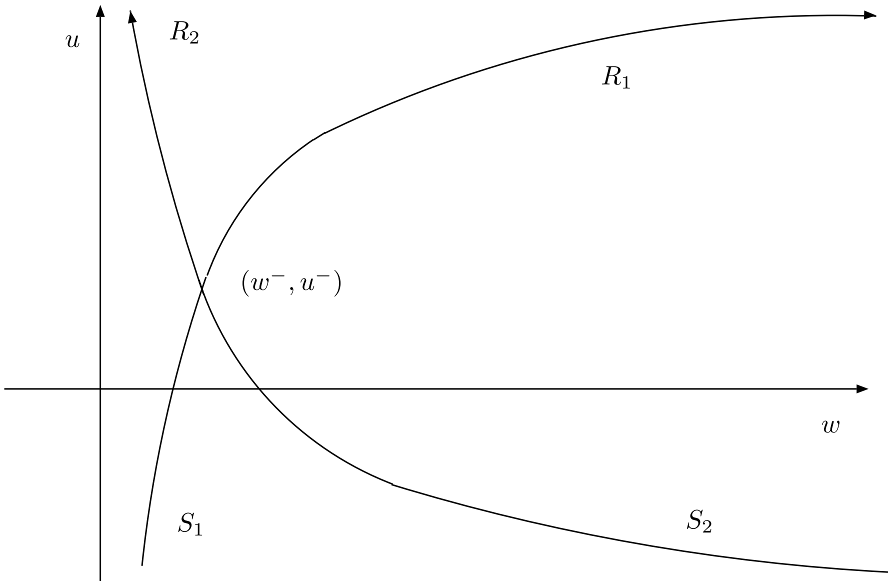

Figure 2.1. Shock curves and rarefaction curves in (w, u)-plane.

and

(2.8) $$\mathbf{S}_2 = \left\{(w,u) : u - u^- = -\frac{a}{\sqrt{ww^-}}(w - w^-), \quad w^- < w\right\}.$$

Now we consider Cauchy problem for (1.14) with initial data

(2.9) $$v(0,x) = \bar{w}(x)x^{-\frac{2}{3}}, \quad u(0,x) = \bar{u}(x),$$

provided $\bar{w}$ and $\bar{u}$ are BV functions. By BV we denote the space of functions of bounded variation on $\mathbf{R}_+ = (0, \infty)$. This problem is essentially the same as that of (2.1). In fact, let $v(t,x) = w(t,x)x^{-2/3}$, then (1.14) becomes

$$\begin{aligned} w_t - x^{\frac{2}{3}}u_x &= 0, \\ u_t + x^{\frac{2}{3}}\left(\frac{a^2}{w}\right)_x &= 0. \end{aligned}$$

Moreover, let

(2.10) $$\xi = 3x^{\frac{1}{3}}.$$

Then we have

$$\begin{aligned} w_t - u_\xi &= 0, \\ u_t + \left(\frac{a^2}{w}\right)_\xi &= 0. \end{aligned}$$

The equation is solved for the case of large data in [7].

Finally, we observe Riemann problem for (1.14) with initial data

$$U(0,x) = \begin{cases} U^- = (w^- x^{-\frac{2}{3}}, u^-), & 0 < x < \bar{x}, \\ U^+ = (w^+ x^{-\frac{2}{3}}, u^+), & \bar{x} < x, \end{cases} \tag{2.11}$$

where w^+, w^-, u^- and u^+ are constants. In view of above transformation, 1- and 2-rarefaction waves are

$$v(t,x) \doteq \begin{cases} w^- x^{-\frac{2}{3}}, & t < -\dfrac{3w^-}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ w^+ x^{-\frac{2}{3}}, & t > -\dfrac{3w^+}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ w x^{-\frac{2}{3}}, & t = -\dfrac{3w}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \quad w \in [w^-, w^+], \end{cases}$$

$$u(t,x) \doteq \begin{cases} u^-, & t < -\dfrac{3w^-}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ u^+, & t > -\dfrac{3w^+}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ u^- + a \log w - a \log w^-, & t = -\dfrac{3w}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \quad w \in [w^-, w^+], \end{cases}$$

where $(w^+, u^+) \in \mathbf{R}_1$, and

$$v(t,x) \doteq \begin{cases} w^- x^{-\frac{2}{3}}, & t > \dfrac{3w^-}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ w^+ x^{-\frac{2}{3}}, & t < \dfrac{3w^+}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ w x^{-\frac{2}{3}}, & t = \dfrac{3w}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \quad w \in [w^+, w^-], \end{cases}$$

$$u(t,x) \doteq \begin{cases} u^-, & t > \dfrac{3w^-}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ u^+, & t < \dfrac{3w^+}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ u^- - a \log w + a \log w^-, & t = -\dfrac{3w}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \quad w \in [w^+, w^-], \end{cases}$$

where $(w^+, u^+) \in \mathbf{R}_2$, respectively.

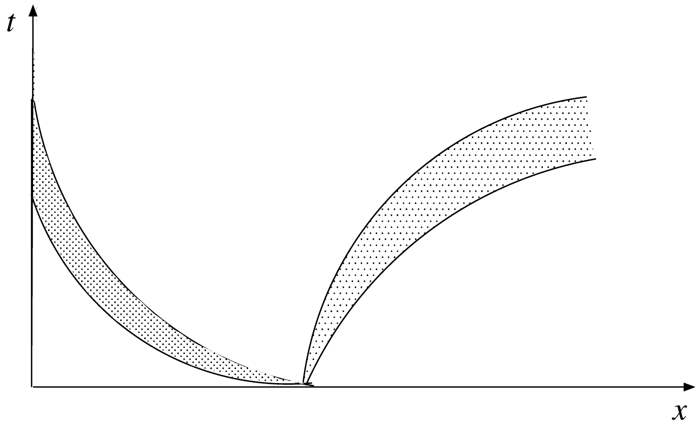

Figure 2.2. 1-rarefaction and 2-rarefaction waves.

Similarly, 1- and 2-shocks are

$$v(t,x) \doteq \begin{cases} w^- x^{-\frac{2}{3}}, & t < -\dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ w^+ x^{-\frac{2}{3}}, & t > -\dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \end{cases}$$

$$u(t,x) \doteq \begin{cases} u^-, & t < -\dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ u^+, & t > -\dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \end{cases}$$

where $(w^+, u^+) \in \mathbf{S}_1$, and

$$v(t,x) \doteq \begin{cases} w^- x^{-\frac{2}{3}}, & t > \dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \\ w^+ x^{-\frac{2}{3}}, & t < \dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - \bar{x}^{\frac{1}{3}}), \end{cases}$$

$$u(t,x) \doteq \begin{cases} u^-, & t > \dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - x_0^{\frac{1}{3}}), \\ u^+, & t < \dfrac{3\sqrt{w^- w^+}}{a}(x^{\frac{1}{3}} - x_0^{\frac{1}{3}}), \end{cases}$$

where $(w^+, u^+) \in \mathbf{S}_2$, respectively.

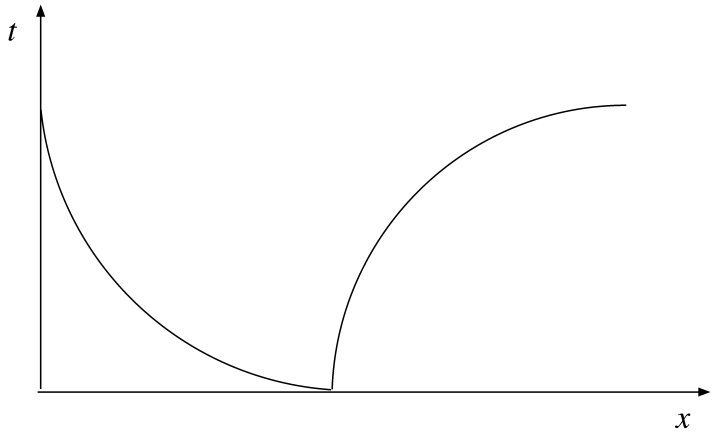

Figure 2.3. 1-shock and 2-shock.

3. Construction of approximate solutions

To construct the approximate solutions, we shall use the difference scheme developed in [7]. For $l, h > 0$, define

$$Y = \{(n,m);\ n = 1,2,3,\dots,\ m = 1,3,5,\dots\},$$
$$A = \prod_{(m,n)\in Y} [\{nh\} \times ((m-1)l, (m+1)l)], \tag{3.1}$$

where l/h will be determined later. Choose a point $\{a_{nm}\} \in A$ randomly, and write $a_{nm} = (nh, c_{nm})$. For $n = 0$, we set $c_{0m} = ml$. We denote approximate solutions by $v^l = w^l x^{-2/3}$ and u^l. Mesh lengths l and h are chosen so that $l/h > a/(\inf w^l)$, for some T. Here T will also be determined later. We shall show later that there exists a w_* such that $\inf w^l \geq w_* > 0$. Considering (2.10), let $\varphi(x) = (1/27)x^3$.

For $0 \leq t < h$, $\varphi(ml) \leq x < \varphi((m+2)l)$, m:odd, we define

$$\begin{aligned} v^l(t,x) &= v_0^l(t,x), \\ u^l(t,x) &= u_0^l(t,x) + E^l(t,x)t, \end{aligned} \tag{3.2}$$

where $v_0^l(t,x)$ and $u_0^l(t,x)$ are the solutions of (1.14) with initial data

$$U_0^l(0,x) = \begin{cases} (\bar{w}(\varphi(ml))x^{-\frac{2}{3}}, \bar{u}(\varphi(ml))), & x < \varphi((m+1)l), \\ (\bar{w}(\varphi((m+2)l))x^{-\frac{2}{3}}, \bar{u}(\varphi((m+2)l))), & \varphi((m+1)l) < x, \end{cases} \tag{3.3}$$

where $\bar{w}, \bar{v}$ and $\bar{u}$ are in (1.13), and

$$E^l(t,x) = \frac{2a^2}{\int_0^{\varphi(ml)} \bar{v}(\xi)d\xi} - \frac{2a^2}{3\bar{v}(\varphi(ml)) \cdot \varphi(ml)}. \tag{3.4}$$

For $0 \le t < h$, $0 \le x < \varphi(l)$, we define v^l and u^l by (3.2) where v^l and u^l are the solution of (1.14) with boundary data

$$v_0^l(0,x) = v^l(\varphi(l)), \quad u_0^l(0,x) = u^l(\varphi(l)), \quad x > 0, \tag{3.5}$$

$$u(t,0) = 0, \quad t > 0, \tag{3.6}$$

and

$$E^l(t,x) = 0. \tag{3.7}$$

Suppose that v^l and u^l are defined for $0 \le t < nh$. For $nh \le t < (n+1)h$, $\varphi(ml) \le x < \varphi((m+2)l)$, m:odd, we define

$$\begin{aligned} v^l(t,x) &= v_0^l(t,x), \\ u^l(t,x) &= u_0^l(t,x) + E^l(t,x) \cdot (t - nh), \end{aligned} \tag{3.8}$$

where v_0^l and u_0^l are the solutions of (1.14) with initial data ($t = nh$)

$$U_0^l(nh,x) = \begin{cases} (w^l(nh-0,\varphi(c_{nm}))x^{-\frac{2}{3}}, u^l(nh-0,\varphi(c_{nm}))), \\ \hfill x < \varphi((m+1)l), \\ (w^l(nh-0,\varphi(c_{n\,m+2}))x^{-\frac{2}{3}}, u^l(nh-0,\varphi(c_{n\,m+2}))), \\ \hfill x > \varphi((m+1)l), \end{cases} \tag{3.9}$$

and

$$E^l(t,x) = \frac{2a^2}{\int_0^{\varphi(ml)} v^l(nh-0,\xi)d\xi} - \frac{2a^2}{3v^l(nh-0,\varphi(ml)) \cdot \varphi(ml)}. \tag{3.10}$$

For $nh \le t < (n+1)h$, $0 \le x < \varphi(l)$, we define v^l and u^l as (3.8), where v_0^l and u_0^l are the solutions of (1.14) with initial ($t = nh$) boundary data

$$v_0^l(nh,x) = w^l(nh-0,\varphi(c_{n1}))x^{-\frac{2}{3}}, \quad u_0^l(nh,x) = u^l(nh-0,\varphi(c_{n1})), \quad x > 0, \tag{3.11}$$

$$u(t,0) = 0, \qquad t > nh, \tag{3.12}$$

and $E^l(t,x)$ is as (3.7).

Remark 3.1. If $\bar{x}^{1/3} - (a/3w_*)T > 0$,

$$\begin{aligned} E^l(t,x) = \frac{2a^2}{\int_0^x v^l(t,\xi)d\xi} - \frac{2a^2}{3xv^l(t,x)} = 0 \\ \text{for} \quad 0 \le t \le T \quad \text{and} \quad 0 \le x \le \left(\bar{x}^{\frac{1}{3}} - \frac{a}{3w_*}t\right)^3. \end{aligned} \tag{3.13}$$

Therefore, no jump exists in the area (see a shaded area in Fig. 3.4).

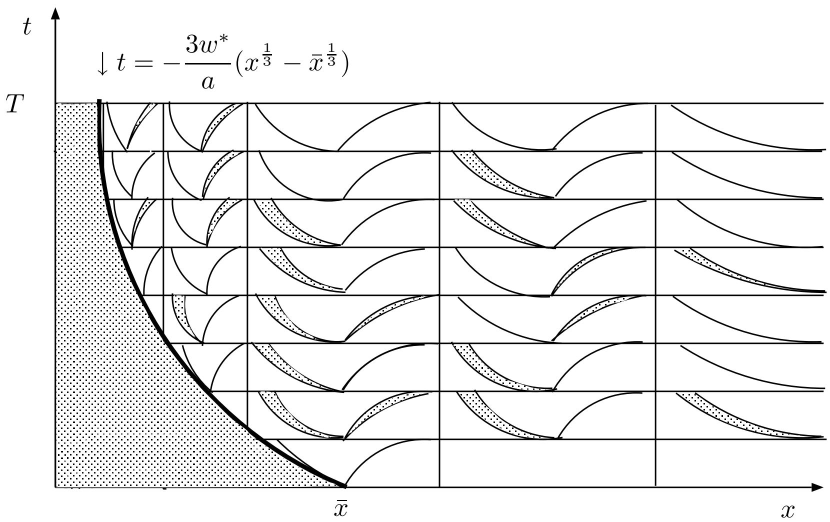

Figure 3.4. Approximate solution.

4. Bounds on the total variation

In this section, we shall prove bounds on total variation of approximate solutions defined in the previous section. So we must prepare lemma.

Lemma 4.1. *Suppose that there exist some positive constants $\delta_2 < \delta_0$ and T such that*

$$\textit{Tot. Var.}\ \{(w^l(t,\cdot), u^l(t,\cdot))\} < \delta_2 \quad \textit{and} \quad \bar{x}^{\frac{1}{3}} - \frac{a}{3w_*}T > 0 \quad \textit{for} \quad 0 \le t \le T.$$

Then,

$$\text{Tot. Var.}\ \{E^l(t,\cdot)\} \le \frac{4a^2}{3w_*(\bar{x}^{\frac{1}{3}} - \frac{a}{3w_*}T)} + \frac{2a^2\delta_2}{3w_*^2(\bar{x}^{\frac{1}{3}} - \frac{a}{3w_*}T)} \quad \textit{for} \quad 0 \le t \le T.$$

Proof. Before proof, we recall that w_* is defined in Section 3. Observing (3.13) and

$$\left| \frac{2a^2}{3x^{\frac{1}{3}}w(t,x)} - \frac{2a^2}{3y^{\frac{1}{3}}w(t,y)} \right| \le \frac{2a^2}{3x^{\frac{1}{3}}} \left| \frac{w(t,x) - w(t,y)}{w(t,x)w(t,y)} \right| + \frac{2a^2}{3w(t,y)} \left| \frac{1}{x^{\frac{1}{3}}} - \frac{1}{y^{\frac{1}{3}}} \right|,$$

we have

$$\begin{aligned}\text{Tot. Var.}\ \{E^l(t,\cdot)\} \le\ & \frac{2a^2\delta_2}{3w_*^2(\bar{x}^{\frac{1}{3}} - \frac{a}{3w_*}T)} \\ & + \frac{2a^2}{3w_*(\bar{x}^{\frac{1}{3}} - \frac{a}{3w_*}T)} + \frac{2a^2}{3w_*(\bar{x}^{\frac{1}{3}} - \frac{a}{3w_*}T)} \qquad \text{for} \quad 0 \le t \le T.\end{aligned}$$

$\square$

Now system (2.1) is hyperbolic provided $w > 0$, with the characteristic roots and Riemann invariants given by

$$\begin{aligned} \lambda_1 &= -\frac{a}{w}, \qquad & r &= u + a\log w, \\ \lambda_2 &= \frac{a}{w}, \qquad & s &= u - a\log w. \end{aligned} \tag{4.1}$$

It is well-known ([7]) that all shock wave curves in the (r,s)-plane have the same figure (see Fig. 4.5).

The 1-shock wave curve S_1, starting from $(\bar{r},\bar{s})$ can be express in the form

$$s - \bar{s} = f(r - \bar{r}) \qquad \text{for} \quad r \le \bar{r} \tag{4.2}$$

and the 2-shock wave curve S_2, starting from $(\bar{r},\bar{s})$ can also be express in the form

$$r - \bar{r} = f(s - \bar{s}) \qquad \text{for} \quad s \le \bar{s}, \tag{4.3}$$

where

$$0 \le f'(x) < 1, \qquad f''(x) \le 0, \qquad \lim_{x\to-\infty} f'(x) = 1.$$

The 1-rarefaction wave curve R_1, starting from $(\bar{r},\bar{s})$ can be express in the form

$$s - \bar{s} = 0 \qquad \text{for} \quad r \le \bar{r}, \tag{4.4}$$

and the corresponding expression for the 1-rarefaction wave curve R_1, starting from $(\bar{r},\bar{s})$ is

$$r - \bar{r} = 0 \qquad \text{for} \quad s \le \bar{s}. \tag{4.5}$$

Let us consider the Riemann problem (4.6) and (1.14). Denote by Δr (resp. Δs) the absolute value of the variation of the Riemann invariant r (resp. s) in the first (resp. second) shock wave.

Definition 4.2. We denote

$$P(w_l, u_l, w_r, u_r) = \Delta r + \Delta s.$$

Lemma 4.3.

$$P(w_1, u_1, w_3, u_3) \le P(w_1, u_1, w_2, u_2) + P(w_2, u_2, w_3, u_3), \tag{4.6}$$

where u_1, u_2 and u_3 are arbitrary constants and w_1, w_2 and w_3 are arbitrary positive constants.

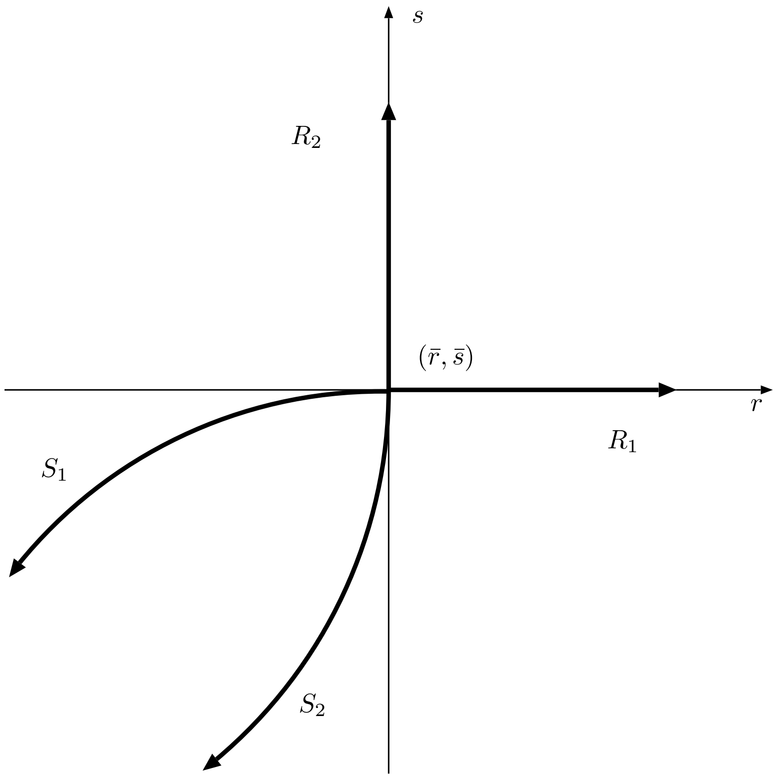

Figure 4.5. Shock wave curves and rarefaction wave curves in (r, s)-plane.

Proof. Let $g(x) = -f(-x)$, and set

$$\begin{aligned} P(w_1, u_1, w_2, u_2) &= \Delta r_1 + \Delta s_1, \\ P(w_2, u_2, w_3, u_3) &= \Delta r_2 + \Delta s_2, \\ P(w_1, u_1, w_3, u_3) &= \Delta r_3 + \Delta s_3. \end{aligned}$$

Then it is obvious that

$$\begin{aligned} &\Delta r_3 + g(\Delta s_3) + \Delta s_3 + g(\Delta r_3) \\ &\qquad \leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1) + g(\Delta s_1) + g(\Delta s_2). \end{aligned}$$

We notice that $f'' \leq 0$ and hence

$$\leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1 + \Delta r_2) + g(\Delta s_1 + \Delta s_2).$$

Let $x + g(x) = h(x), \Delta r_3 = p', \Delta s_3 = q', \Delta r_1 + \Delta r_2 = p$ and $\Delta s_1 + \Delta s_2 = q$. Then

$$h(p') + h(q') \leq h(p) + h(q). \tag{4.7}$$

Set $K = h(p') + h(q')$. Under the restriction (4.15) we shall estimate $p+q$ form below. To do this, as h is monotone increasing function, we must estimate $p+q$ from below under the restriction

$$h(p) + h(q) = K. \tag{4.8}$$

We do this by using Lagrange's method of indeterminate coefficients. Set $G(p, q, \lambda) = p + q + \lambda(h(p) + h(q) - K)$. Then

$$G_p = 1 + \lambda h'(p) = 0, \qquad G_p = 1 + \lambda h'(q) = 0.$$

Because $h''(x) > 0$, we have $p = q$. So $p + q$ attains its extremum at $p = q$. We can show that when $p = q$, $p + q$ is minimum under the restriction (4.16). Therefore

$$h(p) = h(q) = \frac{k}{2} = \frac{h(p') + h(q')}{2} \geq h\left(\frac{p' + q'}{2}\right).$$

Hence it follows that

$$p = q \geq \frac{p' + q'}{2}.$$

We thus have

$$p + q \geq p' + q'.$$

□

We denote

$$\begin{aligned}
Z_1 &= \{\varphi(l) - 0, \varphi(l) + 0, \varphi(3l) - 0, \ldots, \varphi(2ml - 1) - 0, \varphi(2ml - 1) + 0, \ldots\},\\
Z_2 &= \{\varphi(2l), \varphi(4l), \varphi(6l), \ldots, \varphi(2ml), \ldots\}.
\end{aligned}$$

Let $Z_{(n)} = Z_1 \cup Z_2 \cup \{\varphi(c_{nm})\}$ and line up the elements $z_{n,i}$ of $Z_{(n)}$ so that $z_{n,i} \leq z_{n,i+1}$. (We regard $\varphi((2m-1)l) - 0 < \varphi((2m-1)l) + 0$ for m:integer.)

Let

$$\begin{aligned}
F(nh - 0, w^l, u^l) &= \sum_{z_{n,i} \in Z_{(n)}} P(w^l(nh - 0, z_{n,i}), u^l(nh - 0, z_{n,i}),\\
&\qquad\qquad w^l(nh - 0, z_{n,i+1}), u^l(nh - 0, z_{n,i+1})),\\
F(nh + 0, w^l, u^l) &= \sum_{m:\mathrm{odd}} P(w^l(\varphi(a_{nm})),\\
&\qquad\qquad u^l(\varphi(a_{nm})), w^l(\varphi(a_{n\,m+2})), u^l(\varphi(a_{n\,m+2}))).
\end{aligned}$$

Using Lemma 4.3, we have

$$F((n+1)h + 0, w^l, u^l) \leq F((n+1)h - 0, w^l, u^l). \tag{4.9}$$

The following equality is obvious from the definition of F, u^l and v^l.

$$F((n+1)h - 0, w_0^l, u_0^l) = F(nh + 0, w^l, u^l). \tag{4.10}$$

We also have

$$
\begin{aligned}
&F((n+1)h-0, w^l, u^l) = F(nh-0, w_0^l, u_0^l) \\
&\quad + \sum_{m:\text{odd}} P(w^l((n+1)h-0, \varphi(ml)-0), u^l((n+1)h-0, \varphi(ml)-0), \\
&\quad w^l((n+1)h-0, \varphi(ml)+0), u^l((n+1)h-0, \varphi(ml)+0)).
\end{aligned} \tag{4.11}
$$

Now choose a positive constant δ_2 such that $\delta_2 < \delta_0$. We observe that if $\text{Tot. Var.}\,\{(w^l(t,\cdot), u^l(t,\cdot))\} < \delta_2$, there exists a constant C_1 depending on δ_2 such that

$$
\text{Tot. Var.}\,\{(w^l(nh+0,\cdot), u^l(nh+0,\cdot))\} \le C_1 \cdot F(nh+0, w^l, u^l). \tag{4.12}
$$

Set

$$
C_2 = \frac{4a^2}{3(\delta_0-\delta_2)\bar{x}^{\frac{1}{3}} - \frac{a}{3(\delta_0-\delta_2)}T)} + \frac{2a^2\delta_2}{3(\delta_0-\delta_2)^2(\bar{x}^{\frac{1}{3}} - \frac{a}{3(\delta_0-\delta_2)}T)}.
$$

Then choose T and δ_1 suitably small such that

$$
C_1 \left\{F(+0, w^l, u^l) + 2C_2T\right\} \le \delta_2 \tag{4.13}
$$

and

$$
\bar{x}^{\frac{1}{3}} - \frac{a}{3(\delta_0-\delta_2)}T > 0.
$$

Let $N = T/h$ and suppose that

$$
F(nh+0, w^l, u^l) \le F(+0, w^l, u^l) + 2C_2nh \qquad (n = 0, 1, \dots, N). \tag{4.14}
$$

Then, from (4.12) and (4.13),

$$
\text{Tot. Var.}\,\{(w^l(nh+0,\cdot), u^l(nh+0,\cdot))\} < \delta_2 \qquad \text{for} \quad n = 0, 1, 2, \dots, N \tag{4.15}
$$

holds. Considering $(\delta_0 - \delta_2)$ to be w_* in Lemma 4.1, we have

$$
\begin{aligned}
&F((n+1)h-0, w^l, u^l) - F((n+1)h-0, w_0^l, u_0^l) \\
&\quad \le 2h \sum_{m:\text{odd}} |E^l(nh, \varphi((m-1)l)) - E^l(nh, \varphi((m+1)l))| \\
&\quad \le 2C_2h.
\end{aligned} \tag{4.16}
$$

From (4.9), (4.10) and (4.16), we have

$$
F((n+1)h+0, w^l, u^l) \le F(nh+0, w^l, u^l) + 2C_2h. \tag{4.17}
$$

By induction, we thus obtain the following lemma.

Lemma 4.4.

$$
F(nh+0, w^l, u^l) \le F(+0, w^l, u^l) + 2C_2nh \qquad (n = 0, 1, \dots, N). \tag{4.18}
$$

Denote by $F(\tau)$ the sum of the absolute values of variations of r^l and s^l for $t=\tau$. Then for $nh \le \tau < (n+1)h$, we have

$$\begin{aligned} F(\tau) &\le F(nh) + 2h \sum_{m:\text{odd}} |E^l(nh, \varphi((m-1)l)) - E^l(nh, \varphi((m+1)l))| \\ &\le F(nh) + 2C_2 h \\ &\le F(+0) + 2C_2 nh. \end{aligned} \tag{4.19}$$

Therefore, we obtain the following theorem.

Theorem 4.5. *For some $T>0$, the variations of w^l and u^l are bounded uniformly for h and $\{a_{mn}\}$, especially the positive lower bounds of w^l is also uniformly bounded.*

Theorem 4.6. *For any interval $[x_1, x_2] \subset [0, \infty)$, we obtain*

$$\begin{aligned} &\int_{x_1}^{x_2} |w^l(t_2,x) - w^l(t_1,x)| + |u^l(t_2,x) - u^l(t_1,x)| dx \\ &\quad \le M \cdot (|t_2 - t_1| + h), \qquad 0 \le t_1,\ t_2 < T, \end{aligned} \tag{4.20}$$

where M depends on T, x_1 and x_2, but not on l and h.

Proof. Without loss of generality, we assume that

$$nh \le t_1 < (n+1)h < \cdots < (n+k)h \le t_2 < (n+k+1)h.$$

Let

$$\begin{aligned} &\int_{x_1}^{x_2} |u^l(t_2,x) - u^l(t_1,x)| dx \\ &\quad \le I_1 + I_2 \\ &\qquad + \int_{x_1}^{x_2} (|u^l(t_2,x) - u^l((n+k)h+0,x)| \\ &\qquad + |u^l(t_1,x) - u^l((n+1)h-0,x)|) dx, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{x_1}^{x_2} \sum_{i=1}^{k} |u^l((n+i)h+0,x) - u^l((n+i)h-0,x)| dx, \\ I_2 &= \int_{x_1}^{x_2} \sum_{i=1}^{k-1} |u^l((n+i+1)h-0,x) - u^l((n+i)h+0,x)| dx \end{aligned}$$

and

$$k = \left[\frac{t_2 - t_1}{h} \right].$$

Denote by $1_{[\alpha,\beta]}$ the characteristic functions of the interval $[\alpha,\beta]$. We regard Tot. Var.$_{-\varphi(l)<x<\varphi(l)}$ = Tot. Var.$_{0<x<\varphi(l)}$. Then

$$\begin{aligned} I_1 \leq & \sum_{i=0}^{k+1} \sum_{m:\text{integer}} \int_{x_1}^{x_2} (\text{Tot. Var.}_{\varphi(2ml)<x<\varphi((2m+2)l)} u^l((n+i)h-0,x) \\ & \cdot 1_{[\varphi(2ml),\varphi((2m+2)l)]})dx \\ \leq & \tilde{M}\left(\left[\frac{t_2-t_1}{h}\right]+2\right)\cdot\left(\sup_{0\leq t\leq T}\text{Tot. Var.} u^l(t,\cdot)\right)\cdot l. \end{aligned}$$

$$\begin{aligned} I_2 \leq & \sum_{i=0}^{k} \sum_{m:\text{integer}} \int_{x_1}^{x_2} (\text{Tot. Var.}_{\varphi((2m-1)l)<x<\varphi((2m+1)l)} u_0^l((n+i+1)h-0,x) \\ & \cdot 1_{[\varphi((2m-1)l),\varphi((2m+1)l)]} + C_2 h)dx \\ \leq & \sum_{i=0}^{k} \tilde{M}l \cdot \text{Tot. Var.} u_0^l((n+i+1)h-0,\cdot) + C_2(x_2-x_1)h \\ \leq & \left(\left[\frac{t_2-t_1}{h}\right]+1\right)\cdot\left(\tilde{M}l \sup_{0\leq t\leq T}\text{Tot. Var.} u_0^l(t,\cdot) + C_2(x_2-x_1)h\right), \end{aligned}$$

provided that h and l are small enough. Here $\tilde{M}$ depends on T, x_1 and x_2, but not on l and h. The remaining terms can be evaluated similarly. For

$$\int_{x_1}^{x_2} |w^l(t_2,x)-w^l(t_1,x)|dx,$$

we also have a similar estimate. Combining these results gives (4.20). $\square$

5. Convergence of approximate solutions

Let T, δ_1 and δ_2 be the same constants as in the previous section, $h_n = T/n$ and $h_n/l_n = \tilde{\delta} < \delta \doteq a/(\delta_0-\delta_2)$. Consider the sequence (w^{l_n}, u^{l_n}) $(n = 1, 2, \ldots)$. Then from Theorem 4.9, there exists a subsequence which converges in L^1_{loc} to functions (w, u) uniformly for $t \in [0,T]$. Now we shall prove that $w(t,x)$ and $u(t,x)$ are the weak solutions of initial boundary value problem (1.9) through (1.11) provided $\{a_{nm}\}$ is suitably chosen, namely, they satisfy the integral identity

$$\begin{aligned} &\int_0^T\int_0^\infty v\phi_t - u\phi_x dtdx + \int_0^\infty \bar{v}(x)\phi(0,x)dx = 0, \\ &\int_0^T\int_0^\infty u\psi_t + \left(\frac{a^2}{v}\right)\psi_x + \frac{2a^2}{\int_0^x v(t,\xi)d\xi}\psi dtdx + \int_0^\infty \bar{u}(x)\psi(0,x)dx = 0 \end{aligned} \tag{5.1}$$

for any smooth functions ϕ and ψ with compact support in the region $\{(t,x) : 0 \leq t < T, 0 \leq x < \infty\}$ and $\psi(t,0) = 0$. Observing that v_0 and u_0 are weak solutions in each time strip $nh \leq t < (n+1)h$,

$$\begin{aligned}\int_{nh}^{(n+1)h} \int_0^\infty & u^l \psi_t + \left(\frac{a^2}{v^l}\right)\psi_x + \frac{1}{3v^l x}\psi + E^l(t,x)\psi dt dx \\ & + \int_0^\infty u^l(nh+0,x)\psi(nh,x) \\ & - \int_0^\infty u^l((n+1)h-0,x)\psi((n+1)h,x))dx = 0.\end{aligned} \tag{5.2}$$

If we sum this over n, we have

$$\begin{aligned}&\int_0^T \int_0^\infty u^l \psi_t + \left(\frac{a^2}{v^l}\right)\psi_x + \frac{1}{3v^l x}\psi + E^l(t,x)\psi dt dx + \int_0^\infty \bar{u}(x)\psi(0,x)dx \\ &\qquad = -\sum_{k=1}^{N} \int_0^\infty \{u^l(kh+0,x) - u^l(kh-0,x)\} \cdot \psi(kh,x)dx,\end{aligned} \tag{5.3}$$

where $N = T/h$. When $N \to \infty$, the right-hand side of the above equality tends to 0 for almost every $\{a_{nm}\} \in A$ (see [3] or [8]).

Lemma 5.1.

$$E^l(t,x) \to \frac{2a^2}{\int_0^x v(t,\xi)d\xi} - \frac{2a^2}{3xv(t,x)} \qquad (N \to \infty), \tag{5.4}$$

locally uniform for t *and* x.

Proof. Observing (3.13), let $nh \leq t < (n+1)h$, $x \geq (\bar{x}^{1/3} - \frac{a}{3(\delta_0-\delta_2)}t)^3$ and $x \in (\varphi((m-1)l), \varphi((m+1)l))$, m:odd. Then

$$\left| \int_0^x v^l(nh,\xi)d\xi - \int_0^{\varphi(ml)} v^l(nh,\xi)d\xi \right| \leq \|w_*\|_\infty \cdot l. \tag{5.5}$$

On the other hand

$$\int_0^x v^l(t,\xi)d\xi \to \int_0^x v(t,\xi)d\xi \qquad (\mathrm{N} \to \infty), \tag{5.6}$$

locally uniform for t and x.

We have

$$\begin{aligned}&\left| \int_0^x v^l(t,\xi)d\xi - \int_0^x v^l(nh,\xi)d\xi \right| \\ &\quad \leq \int_0^x \left(\sum_{m:\mathrm{odd}} \text{Tot. Var.}_{\varphi((m-1)l)<\xi<\varphi((m+1)l)} w^l(nh,\cdot)\xi^{-\frac{2}{3}} \right. \\ &\qquad \left. \cdot 1_{[\varphi((m-1)l),\varphi((m+1)l)]} \right) d\xi \\ &\quad \leq \sup_{0\leq t\leq T} \text{Tot. Var.} w^l \cdot 2l.\end{aligned} \tag{5.7}$$

From (5.5), (5.6) and (5.7), we have (5.4). □

For each test function ϕ, v_ν and u_ν also satisfy

$$\begin{aligned}&\int_0^T\int_0^\infty (v^l\phi_t - u^l\phi_x)dtdx + \int_0^\infty \bar{v}(x)\phi(0,x)dx \\ &\quad = -\sum_{k=0}^{N}\{v^l(kh+0,x) - v^l(kh-0,x)\}\cdot\phi(kh,x)dx - I_1 - I_2,\end{aligned} \tag{5.8}$$

where

$$I_1 = \sum_{n=0}^{N-1}\int_{nh}^{(n+1)h} E^l(t,0)(t-nh)\phi(t,0)dt$$

and

$$I_2 = \sum_{n=0}^{N-1}\sum_{m:\mathrm{odd}}\int_{nh}^{(n+1)h}\{E^l(t,\varphi(ml)+0) - E^l(t,\varphi(ml)-0)\}(t-nh)\phi(t,\varphi(ml))dt.$$

The first term of the right-hand side of equality (5.9) tends to 0 for almost every $\{a_{nm}\}\in A$ (see [3] or [8]).

Observing Remark 3.1, $I_1 = 0$. Therefore, we shall show that $I_2 \to \infty$ as $N\to\infty$. From Lemma 4.1,

$$\sum_{m:\mathrm{odd}}\int_{nh}^{(n+1)h}\{E^l(t,\varphi(ml)+0) - E^l(t,\varphi(ml)-0)\}(t-nh)\phi(t,\varphi(ml))dt \le C_2h^2.$$

We thus have

$$I_2 \le \|\phi\|_\infty \sum_{n=0}^{N-1} C_2h^2 \le \|\phi\|_\infty C_2hT, \tag{5.9}$$

where C_2 is the same constant as in the previous section. We can conclude that (5.1) holds.

Remark 5.2. From the above arguments, we can replace constants, w^+ and u in (1.13), by BV functions, $w^+(x)$ and $u(x)$, respectively.

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