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Compact radial operators on the harmonic Bergman space*

By

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Abstract

We study the characterizing problem of the compactness of radial operators on the harmonic Bergman space. We show that under an oscillation condition, the compactness is equivalent to the boundary vanishing conditions of the certain Berezin transforms. As an application, we characterize compact Toeplitz operators with radial symbol on the harmonic Bergman space.

1. Introduction

For $p \geq 1$, we let $L^p = L^p(D, A)$ denote the usual Lebesgue space of the open unit disk D in the complex plane. Here, the letter A denotes the normalized area measure on D. The harmonic Bergman space b^2 is the subspace of L^2 consisting of all complex-valued harmonic functions on D. As is well known, the harmonic Bergman space b^2 is a closed subspace of L^2 and hence is a Hilbert space. We will write Q for the Hilbert space orthogonal projection from L^2 onto b^2. Each point evaluation is easily verified to be a bounded linear functional on b^2. Hence, for each $z \in D$, there exists a unique function R_z in b^2 which has the following reproducing property:

$$u(z) = \langle u, R_z \rangle$$

for every $u \in b^2$. Here and elsewhere, the notation $\langle\ ,\ \rangle$ denotes the usual inner product in L^2. We let K_z be the well-known holomorphic Bergman kernel given by

$$K_z(w) = \frac{1}{(1 - w\bar{z})^2} \qquad (w \in D)$$

and k_z be the L^2-normalized kernels defined by $k_z = (1 - |z|^2)K_z$. It turns out that there is a simple relation between the harmonic Bergman kernel R_z and

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the holomorphic Bergman kernel K_z: $R_z = K_z + \overline{K_z} - 1$. Thus, the explicit formula of R_z is given by

$$R_z(w) = \frac{1}{(1 - w\bar{z})^2} + \frac{1}{(1 - \bar{w}z)^2} - 1 \qquad (w \in D).$$

Let r_z be the L^2-normalized kernels defined by $r_z = R_z/||R_z||_2$.

One can see that the projection Q has the following integral representation:

$$Q\varphi(z) = \int_D R_z\varphi\, dA \qquad (z \in D) \tag{1.1}$$

for functions $\varphi \in L^2$. The integral representation of Q above shows that Q naturally extends to an integral operator via (1.1) from L^1 into the space of all harmonic functions on D. See Chapter 8 of [1] for more information and related facts.

Given a function $u \in L^1$, the *Toeplitz operator* T_u *with symbol* u is defined by

$$T_u f = Q(uf)$$

for functions $f \in b^2 \cap L^\infty$. The operator T_u is densely defined and not bounded in general. If u is a bounded symbol, then clearly T_u is bounded on b^2. We also note that there are lots of unbounded symbols to induce bounded Toeplitz operators on b^2. For examples, it turns out that every positive integrable functions with a certain Carleson condition induces bounded Toeplitz operators on b^2. See [3] for details.

Also, the compactness of Toeplitz operators has been characterized in terms of the boundary vanishing property of the Berezin transform of the symbol. Given a function $u \in L^1$, the Berezin transform $\tilde{u}$ of u is defined by

$$\tilde{u}(z) = \int_D u|r_z|^2\, dA \qquad (z \in D).$$

It was proved in [5] and [8] independently that for a bounded radial symbol u, T_u is compact on b^2 if and only if $\tilde{u}(z) \to 0$ as $|z| \to 1$. Recently, it was proved in [3] that for a positive symbol $u \in L^1$, T_u is compact on b^2 if and only if $\tilde{u}(z) \to 0$ as $|z| \to 1$.

In this paper, we consider radial operators (to be defined below) on the harmonic Bergman space b^2 and study the same characterizing problem of compact radial operator. As we will see in Section 3, Toeplitz operators with radial symbol are examples of radial operators on b^2.

The corresponding problem, as well as its essential version, for Toeplitz operators on the holomorphic Bergman space has been studied by several authors as in [2], [4], [7] and [9].

Given a bounded operator T on b^2, we define $Rad(T)$ to be the operator

$$Rad(T) = \frac{1}{2\pi}\int_0^{2\pi} U_t^* T U_t\, dt$$

where U_t is the unitary operator given by $(U_t f)(z) = f(e^{-it}z)$ for $f \in b^2$ and $z \in D$. The integral definition above means that

$$\langle Rad(T)f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle U_t^* T U_t f, g\rangle dt$$

for $f, g \in b^2$. We say that a bounded operator T on b^2 is radial if $T = Rad(T)$.

In what follows, we will use the notation w to denote the function $w \mapsto w$. Also, for a bounded operator T on b^2, we let $a_n(T) = (n+1)\langle T(w^n), w^n\rangle$ and $\tilde{a}_n(T) = (n+1)\langle T(\bar{w}^n), \bar{w}^n\rangle$ for $n = 0, 1, 2, \dots$.

The next theorem is our main result.

Theorem 1.1. *Let T be a bounded radial operator on b^2. Suppose $n(a_n(T) - a_{n-1}(T))$ and $n(\tilde{a}_n(T) - \tilde{a}_{n-1}(T))$ are bounded sequences. Then T is compact on b^2 if and only if $\langle Tk_z, k_z\rangle \to 0$ and $\langle T\bar{k}_z, \bar{k}_z\rangle \to 0$ as $|z| \to 1$.*

Note that a Toeplitz operator on b^2 is a radial operator if and only if the symbol is a radial function (see Proposition 3.2). So, Toeplitz operators with radial symbol are examples of radial operators on b^2. As an application of Theorem 1, we characterize compact Toeplitz operators with radial symbol satisfying a certain oscillation condition in terms of the boundary vanishing property of the Berezin transform of the symbol.

Theorem 1.2. *Let $u \in L^1$ be a radial function for which T_u is bounded on b^2. Suppose*

$$M = \sup_{0 \le r < 1} \left| u(r) - \frac{1}{1-r^2}\int_r^1 u(t)t\,dt \right| < \infty. \tag{1.2}$$

Then T_u is compact on b^2 if and only $\tilde{u}(z) \to 0$ as $|z| \to 1$.

One can easily check that every bounded functions satisfies condition (1.2). Hence we have the following corollary of Theorem 1.2. The following was originally proved in [5] and [8] independently on the ball using the completely different methods.

Corollary 1.1. *Let u be a bounded radial function on D. Then T_u is compact on b^2 if and only if $\tilde{u}(z) \to 0$ as $|z| \to 1$.*

In Section 2, we prove Theorem 1. In Section 3, we study radial Toeplitz operators and prove Theorem 1.2.

2. Proof of Theorem 1

Let $e_0 = 1$ and $e_{2n-1}(z) = \sqrt{n+1}z^n$, $e_{2n}(z) = \sqrt{n+1}\bar{z}^n$ for $n = 1, 2, \dots$ and $z \in D$. Then the sequence $\{e_n\}$ forms an orthonormal basis for b^2.

Proposition 2.1. *Let T be a bounded radial operator on b^2. Then T is a diagonal operator with respect to the orthonormal basis $\{e_n\}$.*

Proof. We first note that $U_te_0 = 1$, $U_t(e_{2n-1}) = e^{-int}e_{2n-1}$ and $U_t(e_{2n}) = e^{int}e_{2n}$ for $n = 1, 2, \ldots$. Using this, we can easily see

$$\frac{1}{2\pi}\int_0^{2\pi} \langle TU_te_n, U_te_m\rangle\, dt = 0 \qquad (n \neq m).$$

Since T is radial, it follows that

$$\langle Te_n, e_m\rangle = \langle Rad(T)(e_n), e_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle TU_te_n, U_te_m\rangle dt = 0$$

whenever $n \neq m$. Hence T is a diagonal operator with respect to the orthonormal basis $\{e_n\}$ and the diagonal elements of T under the basis $\{e_n\}$ are given by $\langle Te_n, e_n\rangle$. The proof is complete. □

In the proof of Theorem 1, we need the following Tauberian theorem.

Lemma 2.1. *Let $\{c_k\}$ be a bounded sequence of complex numbers. If*

$$\lim_{t\to 1}(1-t)\sum_{n=0}^{\infty} c_nt^n \to 0,$$

then we have

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} c_k = 0.$$

Proof. See Theorem 2 of [6]. □

Note that the normalized kernel k_z can be expressed as

$$k_z(w) = (1-|z|^2)\sum_{n=0}^{\infty}(n+1)w^n\bar{z}^n \qquad (w \in D).$$

So, for a given bounded operator T on b^2, we have

$$\langle Tk_z, k_z\rangle = (1-|z|^2)^2 \sum_{n,m=0}^{\infty} (n+1)(m+1)\langle Tw^n, w^m\rangle \bar{z}^n z^m \tag{2.1}$$

for every $z \in D$. Similarly, we also have

$$\langle T\bar{k}_z, \bar{k}_z\rangle = (1-|z|^2)^2 \sum_{n,m=0}^{\infty} (n+1)(m+1)\langle T\bar{w}^n, \bar{w}^m\rangle z^n \bar{z}^m$$

for every $z \in D$.

Now, we prove Theorem 1.

Proof of Theorem 1. First assume T is compact on b^2. Since the normalized kernels k_z and $\bar{k}_z$ converge weakly to 0 in b^2 as $|z| \to 1$, we have $\langle Tk_z, k_z\rangle \to 0$ and $\langle T\bar{k}_z, \bar{k}_z\rangle \to 0$ as $|z| \to 1$.

Now, assume $\langle Tk_z, k_z\rangle \to 0$ and $\langle T\bar{k}_z, \bar{k}_z\rangle \to 0$ as $|z| \to 1$. Since T is a radial operator on b^2, the proof of Proposition 2.1 shows that $\langle Tw^n, w^m\rangle = 0$ whenever $n \neq m$. It follows from (2.1) that

$$\begin{aligned}\langle Tk_z, k_z\rangle &= (1-|z|^2)^2 \sum_{n,m=0}^{\infty} (n+1)(m+1)\langle Tw^n, w^m\rangle \bar{z}^n z^m \\ &= (1-|z|^2)^2 \sum_{n=0}^{\infty}(n+1)a_n(T)|z|^{2n} \\ &= (1-|z|^2)\left\{a_0(T) + \sum_{n=1}^{\infty}[(n+1)a_n(T) - na_{n-1}(T)]|z|^{2n}\right\}\end{aligned}$$

for every $z \in D$. Since $\langle Tk_z, k_z\rangle \to 0$ as $|z| \to 1$ by assumption, we have

$$\lim_{t\to 1}(1-t)\left\{a_0(T) + \sum_{n=1}^{\infty}[(n+1)a_n(T) - na_{n-1}(T)]t^n\right\} = 0.$$

Note that

$$|a_n(T)| = (n+1)|\langle T(w^n), w^n\rangle| \leq ||T||$$

for all n and the sequence $n(a_n(T)-a_{n-1}(T))$ is bounded by assumption. Here, $||T||$ is the operator norm of T. It follows that the sequence $(n+1)a_n(T) - na_{n-1}(T)$ is bounded because $(n+1)a_n(T)-na_{n-1}(T) = n(a_n(T)-a_{n-1}(T))+ a_n(T)$. It follows from Lemma 2.1 that

$$\lim_{n\to\infty} \frac{1}{n+1}\left\{a_0(T) + \sum_{k=1}^{n}[(k+1)a_k(T) - ka_{k-1}(T)]\right\} = 0.$$

On the other hand, we note

$$\frac{1}{n+1}\left\{a_0(T) + \sum_{k=1}^{n}[(k+1)a_k(T) - ka_{k-1}(T)]\right\} = a_n(T)$$

for each n. So, $a_n(T) \to 0$ as $n \to \infty$ and hence

$$\lim_{n\to\infty}\langle Te_{2n-1}, e_{2n-1}\rangle = \lim_{n\to\infty}(n+1)\langle T(w^n), w^n\rangle = 0. \tag{2.2}$$

Similarly, we also have

$$\langle T\bar{k}_z, \bar{k}_z\rangle = (1-|z|^2)\left\{\tilde{a}_0(T) + \sum_{n=1}^{\infty}[(n+1)\tilde{a}_n(T) - n\tilde{a}_{n-1}(T)]\right\}|z|^{2n}$$

for every $z \in D$. Since $\langle T\bar{k}_z, \bar{k}_z\rangle \to 0$ as $|z| \to 1$ by assumption, we also have by the similar argument above

$$\lim_{n\to\infty}\langle Te_{2n}, e_{2n}\rangle = 0. \tag{2.3}$$

It follows from (2.2) and (2.3) that the diagonal elements $\langle Te_n, e_n\rangle$ of T under the basis $\{e_n\}$ goes to 0 as $n \to \infty$. Hence T is compact on b^2. The proof is complete. □

We remark in passing that the boundedness of $n(a_n(T) - a_{n-1}(T))$ and $n(\tilde{a}_n(T) - \tilde{a}_{n-1}(T))$ is essential in Theorem 1. For example, let's consider a composition operator $C_\varphi : b^2 \to b^2$ defined by $C_\varphi f = f \circ \varphi$ for $f \in b^2$ where $\varphi(z) = -z$. Then one can check that C_φ is a bounded radial operator on b^2 and

$$\langle C_\varphi k_z, k_z\rangle = \langle C_\varphi \bar{k}_z, \bar{k}_z\rangle = \frac{(1-|z|^2)^2}{(1+|z|^2)^2} \qquad (z \in D).$$

Hence $\langle C_\varphi k_z, k_z\rangle \to 0$ and $\langle C_\varphi \bar{k}_z, \bar{k}_z\rangle \to 0$ as $|z| \to 1$. On the other hand, C_φ is a unitary operator on b^2 and hence it is not compact on b^2. In fact, we see $a_n(C_\varphi) = \tilde{a}_n(C_\varphi) = (-1)^n$ and hence $n(a_n(C_\varphi) - a_{n-1}(C_\varphi)) = n(\tilde{a}_n(C_\varphi) - \tilde{a}_{n-1}(C_\varphi)) = 2n(-1)^n$ is not bounded.

3. Radial Toeplitz operators

Given a function $f \in L^1$, the radialization $\mathcal{R}f$ of f is the function on D defined by

$$\mathcal{R}f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)\,dt \qquad (z \in D).$$

Using the rotation invariance of the measure dt, we see that $\mathcal{R}f \in L^1$. Moreover, if T_f is bounded on b^2, then one can easily prove that $T_{\mathcal{R}f}$ is also bounded on b^2.

Recall that a function f is called a radial function if $f = \mathcal{R}f$. Hence, f is a radial function if and only if $f(z) = f(|z|)$ for all $z \in D$.

The next proposition shows that the radial operator of a Toeplitz operator with symbol u is another Toeplitz operator with symbol $\mathcal{R}u$.

Proposition 3.1. *Let $u \in L^1$ for which T_u is bounded on b^2. Then we have $Rad(T_u) = T_{\mathcal{R}u}$ on b^2.*

Proof. Let f, g be two harmonic polynomials in b^2. By Fubini's theorem, we see

$$\begin{aligned}
\langle Rad(T_u)f, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle U_t^* T_u U_t f, g\rangle\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle Q(uU_t f), U_t g\rangle\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle uU_t f, U_t g\rangle\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_D u(w) f(e^{-it}w)\bar{g}(e^{-it}w)\,dA(w)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\int_D u(e^{it}w)f(w)\bar{g}(w)\,dA(w)\,dt \\
&= \int_D f(w)\bar{g}(w)\frac{1}{2\pi}\int_0^{2\pi} u(e^{it}w)\,dt\,dA(w) \\
&= \int_D f\bar{g}\mathcal{R}u\,dA \\
&= \langle T_{\mathcal{R}u}f, g\rangle.
\end{aligned}
$$

It follows that $\langle Rad(T_u)f, g\rangle = \langle T_{\mathcal{R}u}f, g\rangle$ for every harmonic polynomials f, g. Since the set of all harmonic polynomials is dense in b^2, we have the desired result. The proof is complete. □

Proposition 3.2. *Let $u \in L^1$ for which T_u is bounded on b^2. Then T_u is a radial operator on b^2 if and only if u is a radial function on D.*

Proof. First suppose u is radial and hence $u = \mathcal{R}u$. By Proposition 3.1, we have

$$T_u = T_{\mathcal{R}u} = Rad(T_u).$$

Hence T_u is a radial operator on b^2.

To prove the converse implication, we first note that for a given $f \in L^1$, $T_f = 0$ on b^2 if and only if $f = 0$. Indeed, if $T_f = 0$, then $Q(fz^n) = 0$ for every n. So, using (1.1), we have

$$\int_D \left(\frac{1}{(1-z\bar{w})^2} + \frac{1}{(1-\bar{z}w)^2} - 1\right) f(w)w^n\,dA(w) = 0$$

for every $z \in D$ and n. Now, differentiate m times under the integral sign with respect to the variable z and insert $z = 0$. The result is

$$\int_D f(w)w^n\bar{w}^m\,dA(w) = 0$$

for every $n, m = 0, 1, \ldots$. Since the set of all polynomials in z and $\bar{z}$ is dense in $C(\bar{D})$ and $C(\bar{D})$ is dense in L^1, we have $f = 0$.

Now, suppose T_u is a radial operator. By Proposition 3.1 again, we have $T_u = Rad(T_u) = T_{\mathcal{R}u}$. So, $T_{u-\mathcal{R}u} = 0$ on b^2 and hence $u = \mathcal{R}u$ by the observation above. Hence u is radial. The proof is complete. □

Before proving Theorem 2, we have a simple lemma.

Lemma 3.1. *Let $u \in L^1$ be a radial function. Then*

$$\tilde{u}(z) = \frac{2\langle T_uK_z, K_z\rangle - 1}{||R_z||_2^2} \qquad (z \in D).$$

Proof. Since u is radial, by integration in polar coordinates, we see

$$\langle uK_z, \bar{K}_z\rangle = \langle uK_z, 1\rangle = \langle u\bar{K}_z, K_z\rangle = \langle u\bar{K}_z, 1\rangle$$

for every $z \in D$. Now, using the relation $R_z = K_z + \overline{K_z} - 1$, we have the desired result. This completes the proof. $\square$

For a bounded Toeplitz operator T_u, we note that

$$\langle T_u(w^n), w^n \rangle = \langle T_u(\bar{w}^n), \bar{w}^n \rangle = \int_D u(w)|w|^{2n}\, dA(w)$$

and hence $a_n(T_u) = \tilde{a}_n(T_u)$ for every $n = 0, 1, \ldots$.

Now, we prove Theorem 2.

Proof of Theorem 1.2. Since r_z converges weakly to 0 in b^2 as $|z| \to 1$, the compactness of T implies the boundary vanishing property of $\tilde{u}$.

For the converse, suppose $\tilde{u}(z) \to 0$ as $|z| \to 1$. Since u is radial, we have T_u is a radial operator on b^2 by Proposition 3.2. We also note that

$$\langle T_u K_z, K_z \rangle = \langle T_u \bar{K}_z, \bar{K}_z \rangle$$

and $||R_z||_2$ is equivalent to $||K_z||_2 = \frac{1}{1-|z|^2}$. Hence $\tilde{u}(z) \to 0$ as $|z| \to 1$ if and only if $\langle T_u k_z, k_z \rangle = \langle T_u \bar{k}_z, \bar{k}_z \rangle \to 0$ as $|z| \to 1$ by Lemma 3.1. Note $a_n(T_u) = \tilde{a}_n(T_u)$ for all n. So, to prove the converse implication, it is sufficient to show the boundedness of the sequence $n(a_n - a_{n-1})$ where $a_n = a_n(T_u)$ for simplicity. Since u is radial, by integration in polar coordinates, we have $a_n = 2(n+1)\int_0^1 u(r) r^{2n+1}\, dr$ for each n. We note that

$$2n^2 \int_0^1 r^{2n-1}(1-r^2)\, dr = \frac{n}{n+1}$$

and by an integration by parts

$$2n^2 \int_0^1 r^{2n-1} \int_r^1 u(t)t\, dt\, dr = n \int_0^1 u(r) r^{2n+1}\, dr = \frac{n}{2n+2} a_n$$

for $n = 1, 2, \ldots$. It follows that

$$\begin{aligned}
&\left| 2n^2 \int_0^1 u(r) r^{2n-1}(1-r^2)\, dr \right| \\
&\qquad \le \left| 2n^2 \int_0^1 \left\{ u(r) - \frac{1}{1-r^2}\int_r^1 u(t)t\, dt \right\} r^{2n-1}(1-r^2)\, dr \right. \\
&\qquad\qquad \left. + 2n^2 \int_0^1 \left\{ \frac{1}{1-r^2}\int_r^1 u(t)t\, dt \right\} r^{2n-1}(1-r^2)\, dr \right| \\
&\qquad \le M + |a_n|
\end{aligned}$$

for $n = 1, 2, \cdots$. Note that $|a_n| \le ||T||$ for all n. It follows that

$$\begin{aligned}
|n(a_n - a_{n-1})| &= \left| 2n \int_0^1 u(r) r^{2n+1}\, dr - 2n^2 \int_0^1 u(r) r^{2n-1}(1-r^2)\, dr \right| \\
&\le M + 2|a_n| \\
&\le M + 2||T||
\end{aligned}$$

for $n = 1, 2, \ldots$. Therefore, the sequence $n(a_n - a_{n-1})$ is bounded. This completes the proof. □

Concluding remark. It is easy to see that our result also holds for weighted harmonic Bergman spaces on D with radial weights like $(\alpha + 1)(1 - |z|^2)^\alpha$, $\alpha > -1$ with some obvious adjustments.

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