J. Math. Kyoto Univ. (JMKYAZ)
45-4 (2005), 711–741

A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra

By

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(with an Appendix written by Catharina Stroppel)

Abstract

We show that there exists a natural non-degenerate pairing of the homomorphism space between two neighbor standard modules over a quasi-hereditary algebra with the first extension space between the corresponding costandard modules. This pairing happens to be a special representative in a general family of pairings involving standard, costandard and tilting modules. In the graded case, under some "Koszul-like" assumptions (which we prove are satisfied for example for the blocks of the category $\mathcal{O}$), we obtain a non-degenerate pairing between certain graded homomorphism and graded extension spaces. This motivates the study of the category of linear complexes of tilting modules for graded quasi-hereditary algebras. We show that this category realizes the module category for the quadratic dual of the Ringel dual of the original algebra. As a corollary we obtain that in many cases Ringel and Koszul dualities commute.

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Received December 6, 2004
Revised July 6, 2005

8. Appendix (written by Catharina Stroppel)

1. Introduction and description of the results

Quasi-hereditary algebras are important objects of study in modern representation theory. One of the central motivations for this study is the fact that quasi-hereditary algebras have exceptionally nice homological properties, for example, they have finite global dimension; they give rise to derived equivalences via Ringel's tilting modules; often, they are Koszul etc. This paper was originated by the following surprising fact, which can be considered as one more contribution to the list of nice homological properties of quasi-hereditary algebras:

Theorem 1. *Let A be a quasi-hereditary algebra over an algebraically closed field $\Bbbk$. Let further $\Delta(i-1), \Delta(i)$ be two neighbor standard modules for A and $\nabla(i-1), \nabla(i)$ be the corresponding costandard modules. Then there exists a non-degenerate bilinear pairing,*

$$\mathrm{Hom}_A\left(\Delta(i-1), \Delta(i)\right) \times \mathrm{Ext}^1_A\left(\nabla(i), \nabla(i-1)\right) \to \Bbbk.$$

Theorem 1 will be proved in Section 3. Actually, a bilinear pairing described above can be defined for an arbitrary pair of (co)standard modules, however, the non-degeneracy can be guaranteed only for neighbors. The next natural step is to try to generalize this to higher extensions. We do this in Section 4 using tilting resolutions of costandard modules. We prove the following non-degeneracy result in Section 5:

Theorem 2. *Let A be a positively graded quasi-hereditary algebra over an algebraically closed field $\Bbbk$ with standard modules $\Delta(i)$ and costandard modules $\nabla(i)$, $i = 1, \dots, n$. Assume that the grading on A induces a positive grading on the Ringel dual $R(A)$ of A. Assume further that for every $j = 0, 1, \dots$ the j-th term of the projective resolution of every standard $R(A)$-module is generated in degree j. Then for every $i, j \in \{1, \dots, n\}$ and every $l \in \{0, 1, \dots\}$ there is a non-degenerate bilinear pairing between the space $\mathrm{ext}^l_A(\nabla(i), \nabla(j)\langle -l\rangle)$ of homogeneous extensions, where $\langle -l\rangle$ denotes the shift of grading, and the space $\mathrm{hom}^l_A(\Delta(j)\langle -l\rangle, \mathcal{T}(\Delta(i))^{-l})$ of homogeneous morphisms, where $\mathcal{T}(\Delta(i))^{-l}$ denotes the l-th term of the tilting resolution of $\Delta(i)$.*

If A is a positively graded quasi-hereditary algebra, then the grading on A induces a natural grading on all standard, costandard, and tilting modules. In particular, one can define a linear complex of projective or tilting modules (for projective modules this means that the j-th component of the complex is generated in degree j, for tilting modules the definition is analogous but more subtle). The conditions of Theorem 2 can be reformulated in terms of linear complexes of tilting modules. This motivates the study of the category $\mathfrak{T}$ of linear complexes of tilting A-modules. It turns out that this category is closely connected with Koszul duality. Namely, using $\mathfrak{T}$ in Section 6 we prove the following statement:

Theorem 3. *Let A be a positively graded quasi-hereditary algebra over an algebraically closed field $\Bbbk$ and $R(A)$ is the Ringel dual of A. Assume that*

(i) *standard A-modules admit linear tilting coresolutions,*

(ii) *costandard A-modules admit linear tilting resolutions.*

Then the Koszul dual $E(R(A))$ of $R(A)$ is quasi-hereditary and $\mathfrak{T}$ is equivalent to the category of locally finite-dimensional graded $E(R(A))^{opp}$-modules.

If we further assume that

(iii) *the grading on $R(E(R(A)))$, induced from $\mathfrak{T}$, is positive,*

then the algebras A, $R(A)$, $E(A)$, $R(E(A))$ and $E(R(A))$ are standard Koszul quasi-hereditary algebras (in the sense of [ADL]*), moreover, we have the isomorphism $E(R(A)) \cong R(E(A))$ as quasi-hereditary algebras. In other words, Koszul and Ringel dualities commute on A.*

We finish the paper by proving that all conditions of Theorem 3 are satisfied for the associative algebras, associated with the blocks of the BGG category $\mathcal{O}$. This is done in Section 7, where we also derive some consequences about the structure of tilting modules in $\mathcal{O}$. In the Appendix, written by Catharina Stroppel, it is shown that all conditions of Theorem 3 are satisfied for the associative algebras, associated with the blocks of the parabolic category $\mathcal{O}$ in the sense of [RC]. As the main tool in the proof of the last result, it is shown that Arkhipov's twisting functor on $\mathcal{O}$ (see [AS, KM]) is gradable.

Acknowledgements. The research was done during the visit of the second author to Uppsala University, which was partially supported by the Royal Swedish Academy of Sciences, and by The Swedish Foundation for International Cooperation in Research and Higher Education (STINT). This support and the hospitality of Uppsala University are gratefully acknowledged. The first author was also partially supported by the Swedish Research Council. We thank Catharina Stroppel for many useful remarks and comments on the preliminary version of the paper and for writing the Appendix.

2. Notation

Let $\Bbbk$ be an algebraically closed field. If the opposite is not emphasized, in this paper by a module we mean a *left* module. For a module, M, we denote by $\mathrm{Rad}(M)$ its radical. For a $\Bbbk$-vector space, V, we denote the dual space by V^*.

Let A be a basic $\Bbbk$-algebra, which is quasi-hereditary with respect to the natural order on the indexing set $\{1, 2, \dots, n\}$ of pairwise-orthogonal primitive idempotents e_i (see [CPS, DR1, DR2] for details). Let $P(i)$, $\Delta(i)$, $\nabla(i)$, $L(i)$, and $T(i)$ denote the projective, standard, costandard, simple and tilting A-modules, associated to e_i, $i = 1, \dots, n$, respectively. Set $P = \oplus_{i=1}^n P(i)$, $\Delta = \oplus_{i=1}^n \Delta(i)$, $\nabla = \oplus_{i=1}^n \nabla(i)$, $L = \oplus_{i=1}^n L(i)$, $T = \oplus_{i=1}^n T(i)$. We remark that, even if the standard A-modules are fixed, the linear order on the indexing set of primitive idempotents, with respect to which the algebra A is quasi-hereditary, is not unique in general. We denote by $R(A)$ and $E(A)$ the Ringel and Koszul duals of A respectively. A graded algebra, $B = \oplus_{i\in\mathbb{Z}} B_i$, will be called *positively graded* provided that $B_i = 0$ for all $i < 0$ and $\mathrm{Rad}(B) = \oplus_{i>0} B_i$.

For an abelian category, $\mathcal{A}$, we denote by $D^b(\mathcal{A})$ the corresponding bounded derived category and by $K(\mathcal{A})$ the corresponding homotopy category. In particular, for an associative algebra, B, we denote by $D^b(B)$ the bounded derived category of B−mod and by $K(B)$ the homotopy category of B−mod. For $M \in B$−mod we denote by $M^\bullet$ the complex defined via $M^0 = M$ and $M^i = 0$, $i \neq 0$.

We will say that an A-module, M, is *Ext-injective* (resp. *Ext-projective*) *with respect to a module*, N, provided that $\mathrm{Ext}_A^k(X, M) = 0$, $k > 0$, (resp. $\mathrm{Ext}_A^k(M, X) = 0$, $k > 0$) for any subquotient X of N.

When we say that a graded algebra is Koszul, we mean that it is Koszul with respect to this grading.

3. A bilinear pairing between Hom_A and Ext_A^1

3.1. Homomorphisms and extensions between standard modules

The following observation is the starting point of this paper. Fix $1 < i \leq n$. According to the classical BGG-reciprocity for quasi-hereditary algebras (see for example [DR2, Lemma 2.5]), we have that $[I(i-1) : \nabla(i)] = [\Delta(i) : L(i-1)]$, where the first number is the multiplicity of $\nabla(i)$ in a costandard filtration of $I(i-1)$, and the second number is the usual composition multiplicity. The quasi-heredity of A, in particular, implies that $\Delta(i-1)$ is Ext-projective with respect to $\mathrm{Rad}(\Delta(i))$ and hence $[\Delta(i) : L(i-1)] = \dim \mathrm{Hom}_A(\Delta(i-1), \Delta(i))$.

The number $[I(i-1) : \nabla(i)]$ can also be reinterpreted. Again, the quasi-heredity of A implies that any non-zero element from $\mathrm{Ext}_A^1(\nabla(i), \nabla(i-1))$ is in fact lifted from a non-zero element of $\mathrm{Ext}_A^1(L(i), \nabla(i-1))$ via the map, induced by the projection $\Delta(i) \twoheadrightarrow L(i)$. Since the module $\nabla(i-1)$ has simple socle, it follows that any non-zero element of $\mathrm{Ext}_A^1(L(i), \nabla(i-1))$ corresponds to an occurrence of $L(i)$ in the socle of the cokernel of the canonical inclusion $\nabla(i-1) \hookrightarrow I(i-1)$. From this one easily derives the equality $[I(i-1) : \nabla(i)] = \dim \mathrm{Ext}_A^1(\nabla(i), \nabla(i-1))$. Altogether, we obtain the following equality: $\dim \mathrm{Hom}_A(\Delta(i-1), \Delta(i)) = \dim \mathrm{Ext}_A^1(\nabla(i), \nabla(i-1))$. In the present section we give a more conceptual explanation for it. In fact, we show that the spaces $\mathrm{Hom}_A(\Delta(i-1), \Delta(i))$ and $\mathrm{Ext}_A^1(\nabla(i), \nabla(i-1))$ are connected via a non-degenerate bilinear pairing.

For every $i = 1, \dots, n$ we fix a non-zero homomorphisms, $\alpha_i : \Delta(i) \to \nabla(i)$. Note that α_i is unique up to a non-zero scalar. For $j < i$ let $f \in \mathrm{Hom}_A(\Delta(j), \Delta(i))$ and

$$\xi : \nabla(j) \overset{\beta}{\hookrightarrow} X \overset{\gamma}{\twoheadrightarrow} \nabla(i) \tag{3.1}$$

be a short exact sequence. Applying $\mathrm{Hom}_A(\Delta(i), {}_-)$ and $\mathrm{Hom}_A(\Delta(j), {}_-)$ to ξ

gives us the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(\Delta(i), X) \arrow[r, "\sim"] \arrow[d, "\circ f"] & \mathrm{Hom}_A(\Delta(i), \nabla(i)) \\ \mathrm{Hom}_A(\Delta(j), \nabla(j)) \arrow[r, "\sim"] & \mathrm{Hom}_A(\Delta(j), X) \arrow[r] & 0. \end{tikzcd}$$

Via this diagram the pair (f, ξ) determines a linear map,

$$\Phi_{(f,\xi)} : \mathrm{Hom}_A(\Delta(i), \nabla(i)) \to \mathrm{Hom}_A(\Delta(j), \nabla(j)).$$

Since $\{\alpha_i\}$ is a basis of $\mathrm{Hom}_A(\Delta(i), \nabla(i))$ and $\{\alpha_j\}$ is a basis of the space $\mathrm{Hom}_A(\Delta(j), \nabla(j))$, we can define $\langle f, \xi\rangle$ to be the unique element from $\Bbbk$ such that $\Phi_{(f,\xi)}(\alpha_i) = \langle f, \xi\rangle\alpha_j$.

Lemma 1.

(1) *Let $\xi' : \nabla(j) \hookrightarrow Y \twoheadrightarrow \nabla(i)$ be a short exact sequence, which is congruent to ξ. Then $\langle f, \xi\rangle = \langle f, \xi'\rangle$ for any f as above. In particular, $\langle\cdot\,,\cdot\rangle$ induces a map from $\mathrm{Hom}_A(\Delta(j), \Delta(i)) \times \mathrm{Ext}^1_A(\nabla(i), \nabla(j))$ to $\Bbbk$ (we will denote the induced map by the same symbol $\langle\cdot\,,\cdot\rangle$ abusing notation).*

(2) *The map $\langle\cdot\,,\cdot\rangle : \mathrm{Hom}_A(\Delta(j), \Delta(i)) \times \mathrm{Ext}^1_A(\nabla(i), \nabla(j)) \to \Bbbk$ is bilinear.*

Proof. This is a standard direct calculation. □

Note that the form $\langle\cdot\,,\cdot\rangle$ is independent, up to a non-zero scalar, of the choice of α_i and α_j. Since the algebras A and A^{opp} are quasi-hereditary simultaneously, using the dual arguments one constructs a form,

$$\langle\cdot\,,\cdot\rangle' : \mathrm{Hom}_A(\nabla(i), \nabla(j)) \times \mathrm{Ext}^1_A(\Delta(j), \Delta(i)) \to \Bbbk,$$

and proves a dual version of Lemma 1.

Theorem 4. *Let $j = i - 1$. Then the bilinear from $\langle\cdot\,,\cdot\rangle$ constructed above is non-degenerate.*

We remark that in the case $j < i - 1$ the analogous statement is not true in general, see the example at the beginning of Subsection 3.2.

Proof. First let us fix a non-zero $f \in \mathrm{Hom}_A(\Delta(i-1), \Delta(i))$. Since $\Delta(i-1)$ has simple top, there exists a submodule, $M \subset \Delta(i)$, which is maximal with respect to the condition $p \circ f \neq 0$, where $p : \Delta(i) \to \Delta(i)/M$ is the natural projection. Denote $N = \Delta(i)/M$. The module N has simple socle, which is isomorphic to $L(i-1)$, and $p \circ f : \Delta(i-1) \to N$ is a non-zero projection onto the socle of N. Now we claim that $\mathrm{Rad}(N) \hookrightarrow \nabla(i-1)$. Indeed, $\mathrm{Rad}(\Delta(i)) \twoheadrightarrow \mathrm{Rad}(N)$ by the definition of N and hence $\mathrm{Rad}(N)$ can have only composition subquotients of the form $L(t)$, $t < i$, since A is quasi-hereditary. But since $\mathrm{Rad}(N)$ has the simple socle $L(i-1)$, the quasi-heredity of A implies $\mathrm{Rad}(N) \hookrightarrow \nabla(i-1)$. Let C denote the cokernel of this inclusion. The module

N is an extension of $\mathrm{Rad}(N)$ by $L(i)$ and is indecomposable. This implies that the short exact sequence $\xi : \mathrm{Rad}(N) \hookrightarrow N \twoheadrightarrow L(i)$ represents a non-zero element in $\mathrm{Ext}^1_A(L(i), \mathrm{Rad}(N))$. Let us apply $\mathrm{Hom}_A(L(i), \_)$ to the short exact sequence $\mathrm{Rad}(N) \hookrightarrow \nabla(i-1) \twoheadrightarrow C$ and remark that $\mathrm{Hom}_A(L(i), C) = 0$ as $[C : L(s)] \neq 0$ implies $s < i - 1$ by above. This gives us an inclusion, $\mathrm{Ext}^1_A(L(i), \mathrm{Rad}(N)) \hookrightarrow \mathrm{Ext}^1_A(L(i), \nabla(i-1))$, and hence there is a short exact sequence, $\xi' : \nabla(i-1) \hookrightarrow N' \twoheadrightarrow L(i)$, induced by ξ.

Lemma 2. $\mathrm{Ext}^1_A(L(i), \nabla(i-1)) \cong \mathrm{Ext}^1_A(\nabla(i), \nabla(i-1))$.

Proof. We apply $\mathrm{Hom}_A(\_, \nabla(i-1))$ to the short exact sequence $L(i) \hookrightarrow \nabla(i) \twoheadrightarrow D$, where D is the cokernel of the inclusion $L(i) \hookrightarrow \nabla(i)$. This gives the following part in the long exact sequence:

$$\mathrm{Ext}^1_A(D, \nabla(i-1)) \to \mathrm{Ext}^1_A(\nabla(i), \nabla(i-1)) \to \\ \to \mathrm{Ext}^1_A(L(i), \nabla(i-1)) \to \mathrm{Ext}^2_A(D, \nabla(i-1)).$$

But D contains only simple subquotients of the form $L(s)$, $s \leq i-1$. This means that $\nabla(i-1)$ is Ext-injective with respect to D because of the quasi-heredity of A. The statement follows. $\square$

Applying Lemma 2 we obtain that the sequence ξ' gives rise to a short exact sequence, $\xi'' : \nabla(i-1) \hookrightarrow N'' \twoheadrightarrow \nabla(i)$. Moreover, by construction it also follows that N is isomorphic to a submodule in N''. Consider ξ'' with $X = N''$ in (3.1). Using the inclusion $N \hookrightarrow N''$ we obtain that the composition $\varphi \circ f$ is non-zero, implying $\langle f, \xi'' \rangle \neq 0$. This proves that the left kernel of the form $\langle \cdot\,, \cdot \rangle$ is zero.

To prove that the right kernel is zero, we, basically, have to reverse the above arguments. Let $\eta : \nabla(i-1) \hookrightarrow X \twoheadrightarrow \nabla(i)$ be a non-split short exact sequence. Quasi-heredity of A implies that $\nabla(i-1)$ is Ext-injective with respect to $\nabla(i)/\operatorname{soc}(\nabla(i))$. Hence η is in fact a lifting of some non-split short exact sequence, $\eta' : \nabla(i-1) \hookrightarrow X' \twoheadrightarrow L(i)$ say. In particular, it follows that X' and thus also X has simple socle, namely $L(i-1)$. Further, applying $\mathrm{Hom}_A(\Delta(i), \_)$ to η, and using the fact that $\Delta(i)$ is Ext-projective with respect to X, one obtains that there is a unique (up to a scalar) non-trivial map from $\Delta(i)$ to X. Let Y be its image. Then Y has simple top, isomorphic to $L(i)$. Furthermore, all other simple subquotients of X are isomorphic to $L(s)$, $s < i$, and hence Y is a quotient of $\Delta(i)$. Since $\Delta(i-1)$ is Ext-projective with respect to $\mathrm{Rad}(\Delta(i))$, we can find a map, $\Delta(i-1) \to \mathrm{Rad}(\Delta(i))$, whose composition with the inclusion $\mathrm{Rad}(\Delta(i)) \hookrightarrow \Delta(i)$ followed by the projection from $\Delta(i)$ onto Y is non-zero. The composition of the first two maps gives us a map, $h : \Delta(i-1) \to \Delta(i)$, such that $\langle h, \eta \rangle \neq 0$. Therefore the right kernel of the form $\langle \cdot\,, \cdot \rangle$ is zero as well, completing the proof. $\square$

Corollary 1.
(1) $\mathrm{Hom}_A(\Delta(i-1), \Delta(i)) \cong \mathrm{Ext}^1_A(\nabla(i), \nabla(i-1))^*$.
(2) $\mathrm{Ext}^1_A(\Delta(i-1), \Delta(i)) \cong \mathrm{Hom}_A(\nabla(i), \nabla(i-1))^*$.

Proof. The first statement is an immediate corollary of Theorem 4 and the second statement follows by duality since A^{opp} is quasi-hereditary as soon as A is, see [CPS, page 93]. □

Corollary 2. *Assume that A has a* simple preserving duality*, that is a contravariant exact equivalence, which preserves the iso-classes of simple modules. Then*

(1) $\mathrm{Hom}_A(\Delta(i-1), \Delta(i)) \cong \mathrm{Ext}^1_A(\Delta(i-1), \Delta(i))^*$.

(2) $\mathrm{Ext}^1_A(\nabla(i), \nabla(i-1)) \cong \mathrm{Hom}_A(\nabla(i), \nabla(i-1))^*$.

Proof. Apply the simple preserving duality to the statement of Corollary 1. □

3.2. Some generalizations

It is very easy to see that the statement of Theorem 4 does not extend to the case $j < i-1$. For example, consider the path algebra A of the following quiver:

$$1 \longleftarrow 2 \longleftarrow 3\ .$$

This algebra is hereditary and thus quasi-hereditary. Moreover, it is directed and thus standard modules are projective and costandard modules are simple. One easily obtains that $\mathrm{Hom}_A(\Delta(1), \Delta(3)) = \Bbbk$ whereas $\mathrm{Ext}^1_A(\nabla(3), \nabla(1)) = 0$. The main reason why this happens is the fact that the non-zero homomorphism $\Delta(1) \to \Delta(3)$ factors through $\Delta(2)$ (note that $1 < 2 < 3$).

Let us define another pairing in homology. For $i \in \{1, \dots, n\}$ denote by $\varrho_i : \Delta(i) \twoheadrightarrow L(i)$ the canonical projection. For $j < i$ let $f \in \mathrm{Hom}_A(\Delta(j), \Delta(i))$ and

$$\xi : \nabla(j) \overset{\beta}{\hookrightarrow} X \overset{\gamma}{\twoheadrightarrow} L(i)$$

be a short exact sequence. Applying $\mathrm{Hom}_A(\Delta(i), {}_-)$ and $\mathrm{Hom}_A(\Delta(j), {}_-)$ to ξ gives us the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{Hom}_A(\Delta(i), X) \arrow[r, "\sim"] \arrow[d, "\circ f"] & \mathrm{Hom}_A(\Delta(i), L(i)) \\
\mathrm{Hom}_A(\Delta(j), \nabla(j)) \arrow[rr, "\sim"] & & \mathrm{Hom}_A(\Delta(j), X) \arrow[r] & 0.
\end{tikzcd}$$

Via this diagram the pair (f, ξ) determines a linear map,

$$\overline{\Phi}_{(f,\xi)} : \mathrm{Hom}_A(\Delta(i), L(i)) \to \mathrm{Hom}_A(\Delta(j), \nabla(j)).$$

Since $\{\varrho_i\}$ is a basis of $\mathrm{Hom}_A(\Delta(i), L(i))$ and $\{\alpha_j\}$ is a basis of the space $\mathrm{Hom}_A(\Delta(j), \nabla(j))$, we can define $\overline{\langle f, \xi \rangle}$ to be the unique element from $\Bbbk$ such that $\overline{\Phi}_{(f,\xi)}(\varrho_i) = \overline{\langle f, \xi \rangle}\alpha_j$. One checks that this defines a bilenear map,

$$\overline{\langle \cdot\,, \cdot \rangle} : \mathrm{Hom}_A(\Delta(j), \Delta(i)) \times \mathrm{Ext}^1_A(L(i), \nabla(j)) \to \Bbbk.$$

Proposition 1. *Let N be the (possibly zero) quotient of $\Delta(i)$, maximal with respect to the following conditions*: $[\mathrm{Rad}(N) : L(s)] \neq 0$ *implies* $s \leq j$; $[\mathrm{Soc}(N) : L(s)] \neq 0$ *implies* $s = j$. *Then*

(1) *the rank of the form* $\overline{\langle\cdot\,,\cdot\rangle}$ *equals the multiplicity* $[\mathrm{Soc}(N) : L(j)]$, *which, in turn, equals* $\dim \mathrm{Ext}^1_A(\underline{L(i)}, \nabla(j))$;

(2) *the left kernel of* $\overline{\langle\cdot\,,\cdot\rangle}$ *is the set of all morphisms* $f : \Delta(j) \to \Delta(i)$ *such that* $\pi \circ f = 0$, *where* $\pi : \Delta(i) \twoheadrightarrow N$ *is the natural projection.*

Proof. The proof is analogous to that of Theorem 4. □

Analyzing the proof of Lemma 2 it is easy to see that there is no chance to hope for any reasonable relation between the spaces $\mathrm{Ext}^1_A(L(i), \nabla(j))$ and $\mathrm{Ext}^1_A(\nabla(i), \nabla(j))$ in general. However, we have the following:

Proposition 2.

(1) *The right kernel of* $\langle\cdot\,,\cdot\rangle$ *coincides with the kernel of the homomorphism* $\tau : \mathrm{Ext}^1_A(\nabla(i), \nabla(j)) \to \mathrm{Ext}^1_A(L(i), \nabla(j))$ *coming from the long exact sequence in homology.*

(2) *Let* $j = i - 2$. *Then* τ *is surjective*; *the rank of* $\langle\cdot\,,\cdot\rangle$ *coincides with the rank of* $\overline{\langle\cdot\,,\cdot\rangle}$; *and the left kernel of* $\langle\cdot\,,\cdot\rangle$ *coincides with the left kernel of* $\overline{\langle\cdot\,,\cdot\rangle}$.

Proof. The first statement follows from the proof of Theorem 4. To prove the second statement we remark that for $j = i-2$ we have $\mathrm{Ext}^k_A(X, \nabla(i-2)) = 0$, $k > 1$, for any simple subquotient X of $\nabla(i)/L(i)$. This gives the surjectivity of τ, which implies all other statements. □

We remark that all results of this subsection have appropriate dual analogues.

4. A generalization of the bilinear pairing to higher Ext's

Let us go back to the example at the beginning of Subsection 3.2, where we had a hereditary algebra with $\mathrm{Ext}^1_A(\nabla(3), \nabla(1)) = 0$, $\mathrm{Hom}_A(\Delta(1), \Delta(3)) = \Bbbk$, and such that any morphism from the last space factors through $\Delta(2)$. One can have the following idea: the vector space $\mathrm{Hom}_A(\Delta(1), \Delta(3))$ decomposes into a product of $\mathrm{Hom}_A(\Delta(1), \Delta(2))$ and $\mathrm{Hom}_A(\Delta(2), \Delta(3))$, by Theorem 4 the space $\mathrm{Hom}_A(\Delta(1), \Delta(2))$ is dual to $\mathrm{Ext}^1_A(\nabla(2), \nabla(1))$ and the space $\mathrm{Hom}_A(\Delta(2), \Delta(3))$ is dual to $\mathrm{Ext}^1_A(\nabla(3), \nabla(2))$, perhaps this means that the product of $\mathrm{Hom}_A(\Delta(1), \Delta(2))$ and $\mathrm{Hom}_A(\Delta(2), \Delta(3))$ should correspond to the product of the spaces $\mathrm{Ext}^1_A(\nabla(2), \nabla(1))$ and $\mathrm{Ext}^1_A(\nabla(3), \nabla(2))$ and thus should be perhaps paired with $\mathrm{Ext}^2_A(\nabla(3), \nabla(1))$ and not $\mathrm{Ext}^1_A(\nabla(3), \nabla(1))$? In our example this argument does not work directly either since the algebra we consider is hereditary and thus Ext^2_A simply vanish. However, one can observe that for $j = i - k$, $k \in \mathbb{N}$, one could define a $\Bbbk$-linear map from $\mathrm{Ext}^k_A(\nabla(i), \nabla(j))^*$

to $\mathrm{Hom}_A(\Delta(j),\Delta(i))$ via

$$\mathrm{Ext}_A^k\left(\nabla(i),\nabla(j)\right)^* \xrightarrow{f} \bigotimes_{l=0}^{k-1}\mathrm{Ext}_A^1\left(\nabla(i-l),\nabla(i-l-1)\right)^* \cong \text{ (by Corollary 1)}$$
$$\cong \bigotimes_{l=0}^{k-1}\mathrm{Hom}_A^1\left(\Delta(i-l-1),\Delta(i-l)\right) \xrightarrow{g} \mathrm{Hom}_A\left(\Delta(j),\Delta(i)\right),$$

where g is the usual composition of k homomorphisms, and f is the dual map to the Yoneda composition of k extensions. This map would give a bilinear pairing between $\mathrm{Ext}_A^k\left(\nabla(i),\nabla(j)\right)^*$ and $\mathrm{Hom}_A\left(\Delta(j),\Delta(i)\right)^*$, which could also be interesting. However, we do not study this approach in the present paper.

Instead, we are going to try to extend the pairing we discussed in the previous section to higher extensions using some resolutions. This leads us to the following definition. Choose a minimal tilting resolution,

$$\mathcal{C}^\bullet : 0 \longrightarrow T_k \xrightarrow{\varphi_k} \dots \xrightarrow{\varphi_2} T_1 \xrightarrow{\varphi_1} T_0 \xrightarrow{\varphi_0} \nabla \longrightarrow 0,$$

of ∇ (see [Ri, Section 5] for the existence of such resolution). Denote by $\mathcal{T}(\nabla)^\bullet$ the corresponding complex of tilting modules. Fix $l \in \{0,\dots,k\}$ and consider the following part of the resolution above:

$$\begin{tikzcd} & \Delta \arrow[d, dotted, "f"] & \\ T_{l+1} \arrow[r, "\varphi_{l+1}"] & T_l \arrow[r, "\varphi_l"] \arrow[d, dotted, "g"] & T_{l-1}. \\ & \nabla & \end{tikzcd}$$

For every $f \in \mathrm{Hom}_A(\Delta, T_l)$ and every $g \in \mathrm{Hom}_A(T_l,\nabla)$ the composition gives

$$g \circ f \in \mathrm{Hom}_A(\Delta,\nabla) = \oplus_{i=1}^n \mathrm{Hom}_A(\Delta(i),\nabla(i)) = \oplus_{i=1}^n \Bbbk\alpha_i.$$

Hence $g\circ f = \sum_{i=1}^n a_i\alpha_i$ for some $a_i \in \Bbbk$ and we can denote $\widetilde{\langle f,g\rangle}^{(l)} = \sum_{i=1}^n a_i \in \Bbbk$. Obviously $\widetilde{\langle \cdot\,,\cdot\rangle}^{(l)}$ defines a bilinear map from $\mathrm{Hom}_A(\Delta,T_l)\times\mathrm{Hom}_A(T_l,\nabla)$ to $\Bbbk$. This map induces the bilinear map

$$\langle f,g\rangle^{(l)} : \mathrm{Hom}_A(\Delta,T_l)\times\mathrm{Hom}_{Com}\left(\mathcal{T}(\nabla)^\bullet,\nabla^\bullet[l]\right)\to\Bbbk$$

(where Hom_{Com} means the homomorphisms of complexes). We remark that we have an obvious inclusion $\mathrm{Hom}_{Com}\left(\mathcal{T}(\nabla)^\bullet,\nabla^\bullet[l]\right)\subset\mathrm{Hom}_A(T_l,\nabla)$ since the complex $\nabla^\bullet[l]$ is concentrated in one degree.

Theorem 5. *Let $f \in \mathrm{Hom}_A(\Delta,T_l)$ and $g \in \mathrm{Hom}_{Com}\left(\mathcal{T}(\nabla)^\bullet,\nabla^\bullet[l]\right)$. Assume that g is homotopic to zero. Then $\langle f,g\rangle^{(l)} = 0$. In particular, $\langle\cdot\,,\cdot\rangle^{(l)}$ induces a bilinear map, $\mathrm{Hom}_A(\Delta,T_l)\times\mathrm{Ext}_A^l(\nabla,\nabla)\to\Bbbk$.*

The form, constructed in Theorem 5 will be denoted also by $\langle\cdot\,,\cdot\rangle^{(l)}$ abusing notation. We remark that both the construction above and Theorem 5 admit appropriate dual analogues.

Proof. Since $\mathcal{T}(\nabla)^\bullet$ is a complex of tilting modules, the second statement of the theorem follows from the first one and [Ha, Chapter III(2), Lemma 2.1]. To prove the first statement we will need the following auxiliary statement.

Lemma 3. *Let $\beta : \Delta(i) \to T(j)$ and $\gamma : T(j) \to \nabla(k)$. Then $\gamma \circ \beta \neq 0$ if and only if $i = j = k$, $\beta \neq 0$ and $\gamma \neq 0$.*

Proof. Using the standard properties of tilting modules, see for example [Ri], we have $[T(i) : L(i)] = 1$, $\dim \mathrm{Hom}_A(\Delta(i), T(i)) = 1$ and any non-zero element in this space is injective, $\dim \mathrm{Hom}_A(T(i), \nabla(i)) = 1$ and any non-zero element in this space is surjective. Hence in the case $i = j = k$ the composition of non-zero γ and β is a non-zero projection of the top of $\Delta(i)$ to the socle of $\nabla(i)$. This proves the "if" statement.

To prove the "only if" statement we note that $\gamma \circ \beta \neq 0$ obviously implies $i = k$. Assume that $j \neq i$ and $\gamma \circ \beta \neq 0$. The module $T(j)$ has a costandard filtration, which we fix, and $\Delta(i)$ is a standard module. Hence, by [Ri, Theorem 4], β is a linear combination of some maps, each of which comes from a homomorphism, which maps the top of $\Delta(i)$ to the socle of some $\nabla(i)$ in the costandard filtration of $T(j)$ (we remark that this $\nabla(i)$ is a subquotient of $T(j)$ but not a submodule in general). Since the composition $\gamma \circ \beta$ is non-zero and $\nabla(i)$ has simple socle, we have that at least one whole copy of $\nabla(i)$ in the costandard filtration of $T(j)$ survives under γ. But, by [Ri, Theorem 1], any costandard filtration of $T(j)$ ends with the subquotient $\nabla(j) \neq \nabla(i)$. This implies that the dimension of the image of γ must be strictly bigger than $\dim \nabla(i)$, which is impossible. The obtained contradiction shows that $i = j = k$. The rest follows from the standard facts, used in the proof of the "if" part. □

We can certainly assume that $f \in \mathrm{Hom}_A(\Delta(i), T_l)$ and also that $g \in \mathrm{Hom}_{Com}(\mathcal{T}(\nabla)^\bullet, \nabla(i)^\bullet[l])$ for some i. Consider now any homomorphism $h : T_{l-1} \to \nabla(i)$. Our aim is to show that the composition $h \circ \varphi_l \circ f = 0$. Assume that this is not the case and apply Lemma 3 to the components of the following two pairs:

(a) $f : \Delta(i) \to T_l$ and $h \circ \varphi_l : T_l \to \nabla(i)$

(b) $\varphi_l \circ f : \Delta(i) \to T_{l-1}$ and $h : T_{l-1} \to \nabla(i)$.

If $h \circ \varphi_l \circ f \neq 0$, we obtain that both T_l and T_{l-1} contain a direct summand isomorphic to $T(i)$, such that the map φ_l induces a map, $\overline{\varphi}_l : T(i) \to T(i)$, which does not annihilate the unique copy of $L(i)$ inside $T(i)$. Since $T(i)$ is indecomposable, we have that $\mathrm{End}_A(T(i))$ is local and thus the non-nilpotent element $\overline{\varphi}_l \in \mathrm{End}_A(T(i))$ must be an isomorphism. This contradicts the minimality of the resolution $\mathcal{T}(\nabla)^\bullet$. □

We remark that the sequence

$$0 \to \mathrm{Hom}_A(\Delta, T_k) \to \cdots \to \mathrm{Hom}_A(\Delta, T_1) \to \mathrm{Hom}_A(\Delta, T_0) \to \mathrm{Hom}_A(\Delta, \nabla) \to 0,$$

obtained from $\mathcal{C}^\bullet$ using $\mathrm{Hom}_A(\Delta, \_)$, is exact, and that Theorem 5 defines a bilinear pairing between $\mathrm{Ext}^l_A(\nabla, \nabla)$ and the l-th element of this exact sequence. It is also easy to see that the pairing, given by Theorem 5, does not depend (up to an isomorphism of bilinear forms) on the choice of a minimal tilting resolution of ∇. In particular, for every l the rank of $\langle \cdot\,, \cdot \rangle^{(l)}$ is an invariant of the algebra A. By linearity we have that

$$\langle \cdot\,, \cdot \rangle^{(l)} = \oplus_{i,j=1}^n \langle \cdot\,, \cdot \rangle^{(l)}_{i,j},$$

where $\langle \cdot\,, \cdot \rangle^{(l)}_{i,j}$ is obtained by restricting the definition of $\langle \cdot\,, \cdot \rangle^{(l)}$ to the homomorphisms from $\Delta(j)$ (instead of Δ) to the tilting resolution of $\nabla(i)$ (instead of ∇). The relation between $\langle \cdot\,, \cdot \rangle^{(l)}_{(i,j)}$ and the forms we have studied in the previous section can be described as follows:

Proposition 3. $\mathrm{rank}\langle \cdot\,, \cdot \rangle^{(1)}_{i,i-1} = \dim \mathrm{Ext}^1_A(\nabla(i), \nabla(i-1)) = \mathrm{rank}\langle \cdot\,, \cdot \rangle.$

Proof. Straightforward. □

In the general case we have the following:

Corollary 3. $\mathrm{rank}\langle \cdot\,, \cdot \rangle^{(l)}_{i,j}$ *equals the multiplicity of* $T(j)$ *as a direct summand in the* l*-th term of the minimal tilting resolution of* $\nabla(i)$*.*

Proof. Let $T_l = \oplus_{k=1}^n T(k)^{l_k}$ and $\varrho : T_l \twoheadrightarrow \oplus_{k=1}^n \nabla(k)^{l_k}$ be a projection. Since the complex $\mathcal{C}^\bullet$ is exact and consists of elements having a costandard filtration, the cokernel of any map in this complex has a costandard filtration itself since the category of modules with costandard filtration is closed with respect to taking cokernels of monomorphisms, see for example [DR2, Theorem 1]. This implies that φ_l induces a surjection from T_l onto a module having a costandard filtration. Moreover, the minimality of the resolution means that this surjection does not annihilate any of the direct summands. In other words, the kernel of φ_l is contained in the kernel of ϱ. This implies that for the cokernel N of φ_{l+1} we have $\dim \mathrm{Hom}(N, \nabla(j)) = l_j$. Using Lemma 3 it is easy to see that $\dim \mathrm{Hom}(N, \nabla(j))$, in fact, equals $\mathrm{rank}\langle \cdot\,, \cdot \rangle^{(l)}_{i,j}$. This completes the proof. □

We remark that, using Corollary 3 and the Ringel duality (see [Ri, Chapter 6]), we can also interpret $\mathrm{rank}\langle \cdot\,, \cdot \rangle^{(l)}_{i,j}$ as the dimension of $\mathrm{Ext}^l_{R(A)}$ from the i-th standard $R(A)$-module to the j-th simple $R(A)$-module. For the BGG category $\mathcal{O}$ the dimensions of these spaces are given by the Kazhdan-Lusztig combinatorics.

5. Graded non-degeneracy in a graded case

The form $\langle \cdot\,, \cdot \rangle^{(l)}$ is degenerate in the general case. However, in this section we will show that it induces a non-degenerate pairing between the graded homomorphism and extension spaces for graded algebras under some assumptions in the spirit of Koszulity conditions.

Throughout this section we assume that A is positively graded (recall that this means that $A = \oplus_{i\geq 0} A_i$ and $\mathrm{Rad}(A) = \oplus_{i>0} A_i$). We remark that this automatically guarantees that the simple A-modules can be considered as graded modules. We denote by $A-\mathrm{gmod}$ the category of all graded (with respect to the grading fixed above) finitely generated A-modules. The morphisms in $A-\mathrm{gmod}$ are morphisms of A-modules, which *preserve* the grading, that is these morphisms are homogeneous morphisms of degree 0. We denote by $\langle 1\rangle : A-\mathrm{gmod} \to A-\mathrm{gmod}$ the functor, which shifts the grading as follows: $(M\langle 1\rangle)_i = M_{i+1}$.

Forgetting the grading defines a faithful functor from $A-\mathrm{gmod}$ to $A-\mathrm{mod}$. We say that $M \in A-\mathrm{mod}$ admits the *graded lift* $\tilde{M} \in A-\mathrm{gmod}$ (or, simply, is *gradable*) provided that, after forgetting the grading, the module $\tilde{M}$ becomes isomorphic to M. If M is indecomposable and admits a graded lift, then this lift is unique up to an isomorphism in $A-\mathrm{gmod}$ and a shift of grading, see for example [BGS, Lemma 2.5.3].

For $M, N \in A-\mathrm{gmod}$ we set $\mathrm{ext}^i_A(M,N) = \mathrm{Ext}^i_{A-\mathrm{gmod}}(M,N)$, $i \geq 0$. It is clear that, forgetting the grading, we have

$$\mathrm{Ext}^i_A(M,N) = \oplus_{j\in\mathbb{Z}}\, \mathrm{ext}^i_A(M, N\langle j\rangle), \qquad i \geq 0 \tag{5.1}$$

(see for example [BGS, Lemma 3.9.2]).

Lemma 4. *Let $M, N \in A-\mathrm{gmod}$. Then the non-graded trace $\mathrm{Tr}_M(N)$ of M in N, that is the sum of the images of all (non-graded) homomorphism $f : M \to N$, belongs to $A-\mathrm{gmod}$.*

Proof. Any $f : M \to N$ can be written as a sum of homogeneous components $f_i : M \to N\langle i\rangle$, $i \in \mathbb{Z}$, in particular, the image of f is contained in the sum of the images of all f_i. Since the image of a homogeneous map is a graded submodule of N, the statement follows. □

Corollary 4. *All standard and costandard A-modules are gradable.*

Proof. By duality it is enough to prove the statement for standard modules. The module $\Delta(i)$ is defined as a quotient of $P(i)$ modulo the trace of $P(i+1)\oplus\cdots\oplus P(n)$ in $P(0)$. For positively graded algebras all projective modules are obviously graded and hence the statement follows from Lemma 4. □

Proposition 4. *Let $M, N \in A-\mathrm{gmod}$. Then the universal extension of M by N (in the category $A-\mathrm{mod}$) is gradable.*

Proof. As we have mentioned before, we have the equality $\mathrm{Ext}^1_A(M,N) = \oplus_{j\in\mathbb{Z}}\, \mathrm{ext}^1_A(M, N\langle j\rangle)$. Every homogeneous extension obviously produces a gradable module. Since we can construct the universal extension of N by M choosing a homogeneous basis in $\mathrm{Ext}^1_A(M,N)$, the previous argument shows that the obtained module will be gradable. This completes the proof. □

We would like to fix a grading on all modules, related to the quasi-hereditary structure. We concentrate L in degree 0 and fix the gradings on P, Δ, ∇

and I such that the canonical maps $P \twoheadrightarrow L$, $\Delta \twoheadrightarrow L$, $L \hookrightarrow \nabla$ and $L \hookrightarrow I$ are all morphism in $A-\mathrm{gmod}$. The only structural modules, which are left, are tilting modules. However, to proceed, we have to show first that tilting modules are gradable.

Corollary 5. *All tilting A-modules admit graded lifts. Moreover, for T this lift can be chosen such that both the inclusion $\Delta \hookrightarrow T$ and the projection $T \twoheadrightarrow \nabla$ are morphisms in $A-\mathrm{gmod}$.*

Proof. By [Ri, Proof of Lemma 3], the tilting A-module $T(i)$ is produced by a sequence of universal extensions as follows: we start from the (gradable) module $\Delta(i)$, and on each step we extend some (gradable) module $\Delta(j)$, $j < i$, with the module, obtained on the previous step. Using Proposition 4 and induction we see that all modules, obtained during this process, are gradable. The statement about the choice of the lift is obvious. □

We fix the grading on T, given by Corollary 5. This automatically induces a grading on the Ringel dual $R(A) = \mathrm{End}_A(T)^{opp}$. In what follows we always will consider $R(A)$ as a graded algebra with respect to this induced grading.

Note that the same ungraded A-module can occur as a part of different structures, for example, a module can be projective, injective and tilting at the same time. In this case it is possible that the lifts of this module, which we fix for different structures, are different. For example, if we have a non-simple projective-injective module, then, considered as a projective module, it is graded in non-negative degrees with top being in degree 0; considered as an injective module, it is graded in non-positive degrees with socle being in degree 0; and, considered as a tilting module, it has non-trivial components both in negative and positive degrees.

A complex, $\mathcal{X}^\bullet$, of graded projective (resp. injective, resp. tilting) modules will be called *linear* provided that $\mathcal{X}^i \in \mathrm{add}(P\langle i\rangle)$ (resp. $\mathcal{X}^i \in \mathrm{add}(I\langle i\rangle)$, resp. $\mathcal{X}^i \in \mathrm{add}(T\langle i\rangle)$) for all $i \in \mathbb{Z}$.

To avoid confusions between the degree of a graded component of a module and the degree of a component in some complex, to indicate the place of a component in a complex we will use the word *position* instead of the word degree.

We say that $M \in A-\mathrm{gmod}$ admits an *LT-resolution*, $\mathcal{T}^\bullet \twoheadrightarrow M$, (here LT stands for linear-tilting) if $\mathcal{T}^\bullet$ is a linear complex of tilting modules from $A-\mathrm{gmod}$, such that $\mathcal{T}^i = 0$, $i > 0$, and the homology of $\mathcal{T}^\bullet$ is concentrated in position 0 and equals M in this position. One also defines *LT-coresolution* in the dual way. The main result of this section is the following:

Theorem 6. *Let A be a positively graded quasi-hereditary algebra and $1 \leq i, j \leq n$. Assume that*

(i) *$\nabla(i)$ admits an LT-resolution, $\mathcal{T}(\nabla(i))^\bullet \twoheadrightarrow \nabla(i)$;*

(ii) *the induced grading on $R(A)$ is positive.*

Then the form $\langle \cdot\,, \cdot \rangle_{i,j}^{(l)}$ induces a non-degenerate bilinear pairing between

$$\mathrm{hom}_A(\Delta(j)\langle -l\rangle, \mathcal{T}(\nabla(i))^{-l}) \quad \textit{and} \quad \mathrm{ext}_A^l(\nabla(i), \nabla(j)\langle -l\rangle).$$

We remark that Theorem 6 has a dual analogue.

Proof. The assumption (ii) means that

$$\hom_A(\Delta\langle s\rangle, T) \neq 0 \qquad \text{implies} \qquad s \leq 0 \tag{5.2}$$

$$\hom_A(\Delta(k)\langle s\rangle, T(m)) \neq 0 \quad \text{and} \quad k \neq m \qquad \text{implies} \qquad s < 0. \tag{5.3}$$

Hence, it follows that

$$\dim \hom_A\left(\Delta(j)\langle -l\rangle, \mathcal{T}(\nabla(i))^{-l}\right)$$

equals the multiplicity of $T(j)\langle -l\rangle$ as a direct summand of $\mathcal{T}(\nabla(i))^{-l}$, which, using the dual arguments, in turn, equals

$$\dim \hom_A\left(\mathcal{T}(\nabla(i))^{-l}, \nabla(j)\langle -l\rangle\right).$$

From the definition of an LT-resolution and (5.2)–(5.3) we also obtain

$$\hom_A\left(\mathcal{T}(\nabla(i))^{-l+1}, \nabla(j)\langle -l\rangle\right) = 0,$$

which means that there is no homotopy from $\mathcal{T}(\nabla(i))^{\bullet}$ to $\nabla(j)\langle -l\rangle^{\bullet}$. The arguments, analogous to those, used in Corollary 3, imply that any map from $\mathcal{T}(\nabla(i))^{-l}$ to $\nabla(j)\langle -l\rangle$ induces a morphism of complexes from $\mathcal{T}(\nabla(i))^{\bullet}$ to $\nabla(j)\langle -l\rangle^{\bullet}$. Hence

$$\dim \mathrm{ext}_A^l\left(\nabla(i), \nabla(j)\langle -l\rangle\right) = \dim \hom_A\left(\mathcal{T}(\nabla(i))^{-l}, \nabla(j)\langle -l\rangle\right).$$

We can now interpret every $f \in \hom_A\left(\Delta(j)\langle -l\rangle, \mathcal{T}(\nabla(i))^{-l}\right)$ as a fixation of a direct summand of $\mathcal{T}(\nabla)^{-l}$, which is isomorphic to $T(i)\langle -l\rangle$. Projecting it further onto $\nabla(j)\langle -l\rangle$ shows that the left kernel of the form $\langle \cdot, \cdot\rangle_{i,j}^{(l)}$ is zero. Since the dimensions of the left and the right spaces coincide by the arguments above, we obtain that the form is non-degenerate. This completes the proof. $\square$

It is easy to see that the condition (ii) of Theorem 6 does not imply the condition (i) in general. Further, it is also easy to see, for example for the path algebra of the following quiver:

$$\begin{tikzcd} 1 & & 2 \arrow[r] & 3 \arrow[lll, bend right] & 4 \arrow[l] \end{tikzcd},$$

that the condition (i) (even if we assume it to be satisfied for all i) does not imply the condition (ii) in general, However, we do not know if the assumptions of the existence of an LT-resolution for ∇ and, simultaneously, an LT-coresolution for Δ, would imply the condition (ii).

We also would like to remark that the conditions of Theorem 6 are not at all automatic even in very good cases. For example one can check that the path algebra of the following quiver:

$$\begin{tikzcd} 1 & 2 \arrow[l] \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd}$$

is standard Koszul, however, both conditions of Theorem 6 fail.

Let A be a positively graded quasi-hereditary algebra. Following [ADL] we call A *standard Koszul* provided that all standard A-modules admit linear projective resolutions, and all costandard A-modules admit linear injective coresolutions. By [ADL, theorem 1], any standard Koszul algebra is Koszul.

Theorem 7. *Let A be a positively graded quasi-hereditary algebra. Then the following conditions are equivalent*:

(a) *A is standard Koszul and the induced grading on $R(A)$ is positive.*

(b) *All standard A-modules admit LT-resolutions, and all costandard A-modules admit LT-coresolutions.*

The algebras, satisfying the equivalent conditions of Theorem 7 will be called *SCT-algebras* (abbreviating standard-costandard-tilting). Theorem 6 motivates a deeper study of SCT-algebras.

Proof. Our first observation is that for any A, satisfying (b), the induced grading on the $R(A)$ is positive. To prove this it is enough to show that all subquotients in any standard filtration of the cokernel of the map $\Delta(i) \hookrightarrow T(i)$ have the form $\Delta(j)\langle l\rangle$, $l > 0$. This follows by induction in i. For $i = 1$ the statement is obvious, and the induction step follows from the inductive assumption applied to the first term of the linear tilting coresolution of $\Delta(i)$.

Now we claim that the Ringel dual $R(A)$ of A satisfies (a) if A satisfies (b), and vice versa. Assume that A satisfies (b). Applying $\mathrm{Hom}_A(T, \_)$ to the LT-resolution of ∇ we obtain that the k-th component of the projective resolution of the standard $R(A)$-module is generated in degree k. Applying analogous arguments to the LT-coresolution of Δ we obtain that the k-th component of the injective resolution of the costandard $R(A)$-module is generated in degree $-k$. As we have already shown, the induced grading on $R(A)$ is positive. Furthermore, the (graded) Ringel duality maps injective A-modules to tilting $R(A)$-modules, which implies that the grading, induced on A from $R(A)-\mathrm{gmod}$, will coincide with the original grading on A, and hence will be positive as well. This means that $R(A)$ satisfies (a). The arguments in the opposite direction are similar.

To complete the proof it is now enough to show, say, that any algebra, satisfying (b), satisfies (a) as well. The existence of a linear tilting coresolution for Δ and the above proved fact that for any A, satisfying (b), the induced grading on the $R(A)$ is positive, imply $\mathrm{ext}^k(\Delta\langle l\rangle, \Delta) = 0$ unless $l \leq k$. Since A is positively graded, we have that the k-th term of the projective resolution of Δ consists of modules of the form $P(i)\langle -l\rangle$, $l \geq k$. Assume that for some k we have that $P(i)\langle -l\rangle$ with $l > k$ occurs. Since every kernel and cokernel in our resolution has a standard filtration, we obtain that $\mathrm{ext}^k(\Delta, \Delta(i)\langle -l\rangle) \neq 0$ with $l > k$, which contradicts $l \leq k$ above. This implies that Δ has a linear projective resolution. Analogous arguments imply that ∇ has a linear injective coresolution. This completes the proof. □

6. The category of linear complexes of tilting modules

We continue to work under the assumptions of Section 5, moreover, we assume, until the end of this section, that A is such that both A and $R(A)$ are positively graded.

The results of Section 5 motivate the following definition: We say that $M \in A{-}\mathrm{gmod}$ is *T-Koszul* provided that M is isomorphic in $D^b(A{-}\mathrm{gmod})$ to a linear complex of tilting modules. Thus any module, which admits an LT-(co)resolution, is T-Koszul.

We denote by $\mathfrak{T} = \mathfrak{T}(A)$ the category, whose objects are linear complexes of tilting modules and morphisms are all morphisms of graded complexes (which means that all components of these morphisms are homogeneous homomorphisms of A-modules of degree 0). We also denote by $\mathfrak{T}^b$ the full subcategory of $\mathfrak{T}$, which consists of bounded complexes.

Lemma 5.
(1) *$\mathfrak{T}$ is an abelian category.*
(2) *$\langle -1\rangle[1] : \mathfrak{T} \to \mathfrak{T}$ is an auto-equivalence.*
(3) *The complexes $(T(i)^\bullet)\langle -l\rangle[l]$, where $i \in \{1, \dots, n\}$ and $l \in \mathbb{Z}$, constitute an exhaustive list of simple objects in $\mathfrak{T}$.*

Proof. The assumption that the grading on $R(A)$, induced from $A{-}\mathrm{gmod}$, is positive, implies that the algebra $\mathrm{end}_A(T^\bullet)$ is semi-simple. Using this it is easy to check that taking the usual kernels and cokernels of morphisms of complexes defines on $\mathfrak{T}$ the structure of an abelian category. That $\langle -1\rangle[1] : \mathfrak{T} \to \mathfrak{T}$ is an auto-equivalence follows from the definition.

The fact that $\mathrm{end}_A(T^\bullet)$ is semi-simple and the above definition of the abelian structure on $\mathfrak{T}$ imply that any non-zero homomorphism in $\mathfrak{T}$ to the complex $(T(i)^\bullet)\langle -l\rangle[l]$ is surjective. Hence the objects $(T(i)^\bullet)\langle -l\rangle[l]$ are simple. On the other hand, it is easy to see that for any linear complex $\mathcal{T}^\bullet$ and for any $k \in \mathbb{Z}$ the complex $\left(\mathcal{T}^k\right)^\bullet$ is a subquotient of $\mathcal{T}^\bullet$ provided that $\mathcal{T}^k \neq 0$. Hence any simple object in $\mathfrak{T}$ should contain only one non-zero component. In order to be a simple object, this component obviously should be an indecomposable A-module. Therefore any simple object in $\mathfrak{T}$ is isomorphic to $(T(i)^\bullet)\langle -l\rangle[l]$ for some i and l. This completes the proof. $\square$

Our aim is to show that $\mathfrak{T}$ has enough projective objects. However, to do this it is more convenient to switch to a different language and to prove a more general result.

Let $B = \oplus_{i\in\mathbb{Z}} B_i$ be a basic positively graded $\Bbbk$-algebra such that $\dim_\Bbbk B_i < \infty$ for all $i \geq 0$. Denote by $\mathfrak{B}$ the category of linear complexes of projective B-modules, and by $\tilde{\mathfrak{B}}$ the category, whose objects are all sequences $\mathcal{P}^\bullet$ of projective B-modules, such that $\mathcal{P}^i \in \mathrm{add}(P\langle -i\rangle)$ for all $i \in \mathbb{Z}$, and whose morphisms are all morphisms of graded sequences (consisting of homogeneous maps of degree 0). The objects of $\tilde{\mathfrak{B}}$ will be called *linear sequences of projective modules.*

Denote by $\mu : B_1 \otimes_{B_0} B_1 \to B_2$ the multiplication map and by $\mu^* : B_2^* \to B_1^* \otimes_{B_0^*} B_1^*$ the dual map. Define the *quadratic dual* $B^!$ of B as the quotient of the free positively graded tensor algebra $B_0[B_1^*]$ modulo the homogeneous ideal generated by $\mu^*(B_2^*)$ (see [BGS, 2.8]).

A graded module, $M = \oplus_{i\in\mathbb{Z}} M_i$, over a graded algebra is called *locally finite* provided that $\dim M_i < \infty$ for all i. Note that a locally finite module does not need to be finitely generated. For a graded algebra, C, we denote by $C-$lfmod the category of all locally finite graded C-modules (with morphisms being homogeneous maps of degree 0).

The following statement was proved in [MS, Theorem 2.4]. For the sake of completeness we present a short version of the proof.

Theorem 8. *There is an equivalence of categories,* $\overline{F} : B_0[B_1^*]-\mathrm{lfmod} \to \tilde{\mathfrak{B}}$*, which induces an equivalence,* $F : B^!-\mathrm{lfmod} \to \mathfrak{B}$*.*

Proof. Let P denote the projective generator of B. We construct the functor $\overline{F}$ in the following way: Let $X = \oplus_{j\in\mathbb{Z}} X_j \in B_0[B_1^*]-$lfmod. We define $\overline{F}(X) = \mathcal{P}^\bullet$, where $\mathcal{P}^j = P\langle j\rangle \otimes_{B_0} X_j$, $j \in \mathbb{Z}$. To define the differential $d_j : \mathcal{P}^j \to \mathcal{P}^{j+1}$ we note that $P \cong {}_BB$ and use the following bijections:

(6.1)
$$\begin{aligned}
\{M \in B_0[B_1^*]-\mathrm{lfmod} : M|_{B_0} = X|_{B_0}\} &\cong \text{ (since } B_0[B_1^*] \text{ is free)}\\
\prod_{j\in\mathbb{Z}} \hom_{B_0-B_0}\left(B_1^*\langle j+1\rangle, \mathrm{Hom}_{\Bbbk}\left(X_j, X_{j+1}\right)\right) &\cong \text{ (by adjoint associativity)}\\
\prod_{j\in\mathbb{Z}} \hom_{B_0}\left(X_j, B_1\langle j+1\rangle \otimes_{B_0} X_{j+1},\right) &\cong \text{ (because of grading)}\\
\prod_{j\in\mathbb{Z}} \hom_{B_0}\left(X_j, B\langle j+1\rangle \otimes_{B_0} X_{j+1},\right) &\cong \text{ (by projectivity of } {}_BB)\\
\prod_{j\in\mathbb{Z}} \hom_{B}\left(B\langle j\rangle \otimes_{B_0} X_j, B\langle j+1\rangle \otimes_{B_0} X_{j+1}\right)&.
\end{aligned}$$

Thus, starting from the fixed X, the equalities of (6.1) produce for each $j \in Z$ a unique map from the space $\hom_B\left(B\langle j\rangle \otimes_{B_0} X_j, B\langle j+1\rangle \otimes_{B_0} X_{j+1}\right)$, which defines the differential in $\mathcal{P}^\bullet$.

Tensoring with the identity map on ${}_BB$ the correspondence $\overline{F}$, defined above on objects, extends to a functor from $B_0[B_1^*]-$lfmod to $\tilde{\mathfrak{B}}$. Since the space $\hom({}_BB, {}_BB) \cong B_0$ is a direct sum of several copies of $\Bbbk$, it follows by a direct calculation that $\overline{F}$ is full and faithful. It is also easy to derive from the construction that $\overline{F}$ is dense. Hence it is an equivalence of categories $B_0[B_1^*]-$lfmod and $\tilde{\mathfrak{B}}$.

Now the principal question is: when $\overline{F}(X)$ is a complex? Let

$$\begin{aligned}
d_j &: B\langle j\rangle \otimes_{B_0} X_j \to B\langle j+1\rangle \otimes_{B_0} X_{j+1},\\
d_{j-1} &: B\langle j-1\rangle \otimes_{B_0} X_{j-1} \to B\langle j\rangle \otimes_{B_0} X_j
\end{aligned}$$

be as constructed above. Let further

$$\begin{aligned}
\delta_j &: X_j \to B_1\langle j+1\rangle \otimes_{B_0} X_{j+1},\\
\delta_{j-1} &: X_{j-1} \to B_1\langle j\rangle \otimes_{B_0} X_j
\end{aligned}$$

be the corresponding maps, given by (6.1). Then $d_j d_{j-1} = 0$ if and only if

$$\left(\mu \otimes \mathrm{Id}_{X_{j+1}}\right) \circ (\mathrm{Id}_{B_1} \otimes \delta_j) \circ \delta_{j-1} = 0.$$

The last equality, in turn, is equivalent to the fact that the global composition of morphisms in the following diagram is zero:

$$B_2^* \xrightarrow{\mu^*} B_1^* \otimes B_1^* \xrightarrow{b} \mathrm{Hom}_{\Bbbk}(X_j, X_{j+1}) \otimes \mathrm{Hom}_{\Bbbk}(X_{j-1}, X_j) \xrightarrow{c} \mathrm{Hom}_{\Bbbk}(X_{j-1}, X_{j+1}),$$

where the map b is given by two different applications of (6.1) and c denotes the usual composition. Hence $\overline{F}(X)$ is a complex if and only if $\mathrm{Im}(\mu^*)X = 0$ or, equivalently, $X \in B^! -\mathrm{lfmod}$. $\square$

It is clear that the equivalence, constructed in the proof of Theorem 8, sends the auto-equivalence $\langle 1 \rangle$ on $B^! -\mathrm{lfmod}$ (resp. on $B_0[B_1^*] -\mathrm{lfmod}$) to the auto-equivalence $\langle -1 \rangle [1]$ on $\mathfrak{B}$ (resp. on $\tilde{\mathfrak{B}}$).

Now we are back to the original setup of this section.

Corollary 6. *Let $R = R(A) = \oplus_{i \geq 0} R_i$. Then the category $\mathfrak{T}$ is equivalent to* $R^! -\mathrm{lfmod}$.

Proof. Apply first the graded Ringel duality and then Theorem 8. $\square$

Corollary 7. *Assume that $R = R(A)$ is Koszul. Set $\Lambda = (E(R(A)))^{opp}$. Then*

(1) $\mathfrak{T}$ *is equivalent to the category* $\Lambda -\mathrm{lfmod}$.

(2) *The category $\mathfrak{T}^b$ is equivalent to* $\Lambda -\mathrm{gmod}$.

Proof. If an algebra $C = \oplus_{i \geq 0} C_i$ is Koszul then, by [BGS, Section 2.9], we have $(E(C))^{opp} \cong C^!$. Now everything follows from Corollary 6. $\square$

Corollary 7 motivates the further study of the categories $\mathfrak{T}$ and $\mathfrak{T}^b$. We start with a description of the first extension spaces between the simple objects in $\mathfrak{T}$. Surprisingly enough, this result can be obtained without any additional assumptions.

Lemma 6. *Let $i, j \in \{1, \dots, n\}$ and $l \in \mathbb{Z}$. Then $\mathrm{ext}^1_{\mathfrak{T}}(T(i)^\bullet, T(j)^\bullet \langle -l \rangle [l]) \neq 0$ implies $l = -1$. Moreover,*

$$\mathrm{ext}^1_{\mathfrak{T}}(T(i)^\bullet, T(j)^\bullet \langle 1 \rangle [-1]) \cong \hom_A(T(i), T(j)\langle 1 \rangle).$$

Proof. A direct calculation, using the definition of the first extension via short exact sequences and the abelian structure on $\mathfrak{T}$. $\square$

Recall from [CPS, DR1] that an associative algebra is quasi-hereditary if and only if its module category is a highest weight category. Our goal is to establish some conditions under which $\mathfrak{T}^b$ becomes a highest weight category. To prove that a category is a highest weight category one has to determine the (co)standard objects.

Proposition 5.

(1) *Assume that* $\Delta(i)$ *admits an LT-coresolution,* $\Delta(i) \hookrightarrow \mathcal{T}(\Delta(i))^\bullet$, *for all* i. *Then* $\mathrm{ext}^1_{\mathfrak{T}}\left(\mathcal{T}(\Delta(i))^\bullet, T(j)^\bullet\langle -l\rangle[l]\right) = 0$ *for all* $l \in \mathbb{Z}$ *and* $j \leq i$.

(2) *Assume that* $\nabla(i)$ *admits an LT-resolution,* $\mathcal{T}(\nabla(i))^\bullet \twoheadrightarrow \nabla(i)$, *for all* i. *Then we have* $\mathrm{ext}^1_{\mathfrak{T}}\left(T(j)^\bullet\langle -l\rangle[l], \mathcal{T}(\nabla(i))^\bullet\right) = 0$ *for all* $l \in \mathbb{Z}$ *and* $j \leq i$.

Proof. By duality, it is certainly enough to prove only the first statement. Using the induction with respect to the quasi-hereditary structure it is even enough to show that $\mathcal{T}(\Delta(n))^\bullet$ is projective in $\mathfrak{T}$. By Lemma 6 we can also assume that $l < 0$. Let

$$0 \to T(j)^\bullet\langle -l\rangle[l] \to \mathcal{X}^\bullet \to \mathcal{T}(\Delta(n))^\bullet \to 0 \tag{6.2}$$

be a short exact sequence in $\mathfrak{T}$. Let further $d^\bullet$ denote the differential in $\mathcal{T}(\Delta(n))^\bullet$. Consider the short exact sequence

$$0 \to \ker(d^{-l}) \to \mathcal{T}(\Delta(n))^{-l} \to \ker(d^{-l+1}) \to 0. \tag{6.3}$$

Since $\mathcal{T}(\Delta(n))^\bullet$ is a tilting coresolution of a standard module, it follows that all modules in (6.3) have standard filtration. Hence, applying $\mathrm{Hom}_A({}_-, T(j))$ to (6.3), and using the fact that $T(j)$ has a costandard filtration, we obtain the surjection

$$\mathrm{Hom}_A\left(\mathcal{T}(\Delta(n))^{-l}, T(j)\right) \twoheadrightarrow \mathrm{Hom}_A\left(\ker(d^{-l}), T(j)\right),$$

which induces the graded surjection

$$\hom_A\left(\mathcal{T}(\Delta(n))^{-l}, T(j)\langle -l\rangle\right) \twoheadrightarrow \hom_A\left(\ker(d^{-l}), T(j)\langle -l\rangle\right).$$

The last surjection allows one to perform a base change in $\mathcal{X}^\bullet$, which splits the sequence (6.2). This proves the statement. □

[ADL, Theorem 1] states that a standard Koszul quasi-hereditary algebra is Koszul (which means that if standard modules admit linear projective resolutions and costandard modules admit linear injective resolutions, then simple modules admit both linear projective and linear injective resolutions). An analogue of this statement in our case is the following:

Theorem 9. *Let* A *be an SCT algebra and* R *be the Ringel dual of* A. *Then*

(1) $R^!$ *is quasi-hereditary with respect to the usual order on* $\{1, 2, \dots, n\}$, *or, equivalently,* $\mathfrak{T}^b \simeq R^!\text{-gmod}$ *is a highest weight category;*

(2) $\mathcal{T}(\Delta(i))^\bullet$, $i = 1, \dots, n$, *are standard objects in* $\mathfrak{T}^b$;

(3) $\mathcal{T}(\nabla(i))^\bullet$, $i = 1, \dots, n$, *are costandard objects in* $\mathfrak{T}^b$;

Assume further that the algebra $R^!$ *is SCT. Then*

(4) *simple* A*-modules are* T*-Koszul, in particular, for every* $i = 1, \dots, n$ *there exists a linear complex,* $\mathcal{T}(L(i))^\bullet$, *of tilting modules, which is isomorphic to* $L(i)$ *in* $D^b(A\text{-gmod})$;

(5) $\mathcal{T}(L(i))^\bullet$, $i = 1, \dots, n$, *are tilting objects with respect to the quasi-hereditary structure on* $\mathfrak{T}^b$.

Proof. The algebra R is quasi-hereditary with respect to the opposite order on $\{1, 2, \dots, n\}$. It is standard Koszul, in particular, Koszul by [ADL, Theorem 1]. Hence its Koszul dual, which is isomorphic to $(R^!)^{opp}$ by [BGS, 2.10], is quasi-hereditary with respect to the usual order on $\{1, 2, \dots, n\}$ by [ADL, Theorem 2]. This certainly means that $R^!$ is quasi-hereditary with respect to the usual order on $\{1, 2, \dots, n\}$. From Corollary 7 we also obtain $R^!-\mathrm{gmod} \simeq \mathfrak{T}^b$. This proves the first statement.

That the objects $\mathcal{T}(\Delta(i))^\bullet$, $i = 1, \dots, n$, are standard and the objects $\mathcal{T}(\nabla(i))^\bullet$, $i = 1, \dots, n$, are costandard follows from Proposition 5 and [DR2, Theorem 1]. This proves (2) and (3).

Now we can assume that $R^!$ is an SCT-algebra. In particular, it is quasi-hereditary, and hence the category $\mathfrak{T}^b$ must contain tilting objects with respect to the corresponding highest weight structure. By [Ri, Proof of Lemma 3], the tilting objects in $\mathfrak{T}^b$ can be constructed via a sequence of universal extensions, which starts with some standard object and proceeds by extending other (shifted) standard objects by objects, already constructed on previous steps. The assumption that $R^!$ is SCT means that new standard objects should be shifted by $\langle -l \rangle [l]$ with $l > 0$. From the second statement of our theorem, which we have already proved above, it follows that the standard objects in $\mathfrak{T}^b$ are exhausted by $\mathcal{T}(\Delta(i))^\bullet$, $i = 1, \dots, n$, and their shifts. The homology of $\mathcal{T}(\Delta(i))^\bullet$ is concentrated in position 0 and in non-negative degrees. It follows that the homology of the tilting object in $\mathfrak{T}^b$, which we obtain, using this construction, will be concentrated in non-positive positions and in non-negative degrees.

On the other hand, a dual construction, that is the one, which uses costandard objects, implies that the homology of the same tilting object in $\mathfrak{T}^b$ will be concentrated in non-negative positions and in non-positive degrees. This means that the homology of an indecomposable tilting object in $\mathfrak{T}^b$ is concentrated in position 0 and in degree 0 and hence is a simple A-module. This proves the two last statements of our theorem and completes the proof. $\square$

In the next section we will show that all the above conditions are satisfied for the associative algebras, associated with the blocks of the BGG category $\mathcal{O}$.

We remark that, under conditions of Theorem 9, in the category $\mathfrak{T}$ the standard and costandard A-modules remain standard and costandard objects respectively via their tilting (co)resolutions. Tilting A-modules become simple objects, and simple A-modules become tilting objects via $\mathcal{T}(L(i))^\bullet$.

An SCT algebra A for which $R(A)^!$ is SCT will be called *balanced*. The results of this section allow us to formulae a new type of duality for balanced algebras (in fact, this just means that we can perform in one step the following path $A \rightsquigarrow R \rightsquigarrow R^! \rightsquigarrow R(R^!)$, which consists of already known dualities for quasi-hereditary algebras).

Corollary 8. *Let A be balanced and $\mathcal{T}(L(i))^\bullet$, $i = 1, \dots, n$, be a complete list of indecomposable tilting objects in $\mathfrak{T}^b$, constructed in Theorem* 9(5).

Then $\langle -1\rangle[1]$ *induces a (canonical)* $\mathbb{Z}$*-action on the algebra*

$$\overline{C}(A) = \mathrm{End}_A\left(\oplus_{l\in\mathbb{Z}} \oplus_{i=1}^{n} \mathcal{T}(L(i))^{\bullet}\langle -l\rangle[l]\right),$$

which makes $\overline{C}(A)$ *into the covering of the quotient algebra* $C(A)$*. The algebra* $C(A)$ *is balanced and* $C(C(A)) \cong A$*.*

Proof. From Theorem 9 it follows that $C(A) \cong (R(R^!))^{opp}$. Further, from Lemma 6 and the assumption that $R(A)^!$ is SCT it follows that the grading on both $R^!$ and $C(A)$, induced from $\mathfrak{T}$, is positive. In particular, Theorem 7 and [ADL, Theorem 2] now imply that $C(A)$ is balanced. Since both Ringel and Koszul dualities are involutive, we also have $A \cong (R(R(C(A))^!))^{opp}$. $\square$

Corollary 9. *Let* A *be balanced. Then* A *is standard Koszul and* $C(A) \cong (A^!)^{opp} \cong E(A)$*.*

Proof. A is standard Koszul by Theorem 7, in particular, it is Koszul by [ADL, Theorem 1]. Further, since no homotopy is possible in $\mathfrak{T}$, it follows that

$$\mathrm{ext}_A^l\left(L(i), L(j)\langle -l\rangle\right) \cong \mathrm{Hom}_{\mathfrak{T}}\left(\mathcal{T}(L(i))^{\bullet}, \mathcal{T}(L(j))^{\bullet}\langle -l\rangle[l]\right).$$

The last equality is obviously compatible with the $\mathbb{Z}$-actions and the compositions on both sides, which implies that the Koszul dual $(A^!)^{opp}$ of A is isomorphic to $C(A)$. $\square$

And now we can formulate, probably, the most surprising result of this section.

Corollary 10. *Let* A *be balanced. Then the algebras* $R(A)$*,* $E(A)$ *and* $E(R(A))$ *and* $R(E(A))$ *are also balanced, moreover*

$$E(R(A)) \cong R(E(A))$$

as quasi-hereditary algebras. In other words, both the Ringel and Koszul dualities preserve the class of balanced algebras and commute on this class.

Proof. Follows from Theorem 9, Corollary 8 and Corollary 9. $\square$

The results, presented in this section motivate the following natural question: *is any SCT algebra balanced?*

7. The graded Ringel dual for the category $\mathcal{O}$

In this section we prove that the conditions of Theorem 6 are satisfied for the associative algebra, associated with a block of the BGG category $\mathcal{O}$. To do this we will use the graded approach to the category $\mathcal{O}$, worked out in [St1]. So, in this section we assume that A is the basic associative algebra of an indecomposable integral (not necessarily regular) block of the BGG category $\mathcal{O}$, [BGG]. The (not necessarily bijective) indexing set for simple modules will be the Weyl group W with the usual Bruhat order (such that the identity element is the maximal one and corresponds to the projective Verma=standard module). This algebra is Koszul by [BGS, So1], and thus we can fix on A the Koszul grading, which leads us to the situation, described in Section 5. Recall that a module, M, is called *rigid* provided that its socle and radical filtrations coincide, see for example [Ir2]. Our main result in this section is the following:

Theorem 10. $\mathrm{End}_A(T)$ *is positively graded, moreover, it is generated in degrees* 0 *and* 1. *Furthermore,* ∇ *admits an LT-resolution.*

Proof. From [FKM, Section 7] it follows that $T \cong \mathrm{Tr}_{P(w_0)}(P)$ and thus, by Lemma 4, there is a graded submodule, T' of P, which is isomorphic to T after forgetting the grading. Moreover, again by [FKM, Section 7], the restriction from P to T' induces an isomorphism of $\mathrm{End}_A(P)$ and $R = \mathrm{End}_A(T)$. So, to prove that $\mathrm{End}_A(T)$ is positively graded it is enough to show that $T' \cong T\langle -l\rangle$ for some l. Actually, we will show that this l equals the Loewy length of $\Delta(e)$.

Let θ_s denote the graded translation functor through the s-wall, see [St1, 3.2]. Let w_0 denote the longest element in the Weyl group. The socle of any Verma module in the category $\mathcal{O}$ is the simple Verma module $\Delta(w_0)$, see [Di, Chapter 7]. This gives, for some $l \in \mathbb{Z}$, a graded inclusion, $T(w_0)\langle -l\rangle \cong \Delta(w_0)\langle -l\rangle \hookrightarrow \Delta(e)$. Moreover, since Verma modules in $\mathcal{O}$ are rigid by [Ir2], and since their graded filtration in the Loewy one by [BGS, Proposition 2.4.1], it follows that this l equals the Loewy length of $\Delta(e)$. Now we would like to prove by induction that $T(w_0w)\langle -l\rangle \hookrightarrow P(w)$ for any $w \in W$. Assume that this is proved for some w and let s be a simple reflection such that $l(ws) > l(w)$. Translating through the s-wall we obtain $\theta_s T(w_0w)\langle -l\rangle \hookrightarrow \theta_s P(w)$. Further, the module $P(ws)$ is a direct summand of $\theta_s P(w)$ (after forgetting the grading). However, from [St1, Theorem 3.6] it follows that the inclusion $P(ws) \hookrightarrow \theta_s P(w)$ is homogeneous and has degree 0. The same argument implies that the inclusion $T(w_0ws) \hookrightarrow \theta_s T(w_0w)$ is homogeneous and has degree 0. This gives us the desired inclusion $T(w_0ws)\langle -l\rangle \hookrightarrow P(ws)$ of degree 0 and completes the induction. Adding everything up we obtain a graded inclusion of degree 0 from $T\langle -l\rangle$ to P.

Recall once more that the restriction from P to T induces an isomorphism of $\mathrm{End}_A(P)$ and $R = \mathrm{End}_A(T)$. Since $\mathrm{End}_A(P) = A$ is positively graded and is generated in degrees 0 and 1, we obtain that $\mathrm{End}_A(T)$ is positively graded and is generated in degrees 0 and 1 as well.

It is now left to prove the existence of an LT-resolution for ∇. Consider the minimal tilting resolution of ∇. In Section 5 we have defined the grading on T such that the canonical projection $T \to \nabla$ is a homogeneous map of degree 0. The kernel of this projection is thus graded and has a graded ∇-filtration. Proceeding by induction we obtain that the minimal tilting resolution of ∇ is graded. Let $R = R(A)$. Using the functor $F = \mathrm{Hom}_A(T, {}_-)$ we transfer this graded tilting resolution to a graded projective resolution of the direct sum $\Delta^{(R)}$ of standard R-modules. By [So2] we have $A \cong R$, moreover, we have just proved that the grading on R, which is induced from $A-$gmod, is the Koszul one. By [BGS, 3.11], the standard A-modules are Koszul, implying that the l-th term of the projective resolution of $\Delta^{(R)}$ is generated in degree l. Applying F^{-1} we thus obtain an LT-resolution of ∇. This completes the proof. $\square$

Catharina Stroppel gave an alternative argument for Theorem 10 (see Appendix), which uses graded twisting functors. The advantage of her approach is that it can be generalized also to the parabolic analogue of the category $\mathcal{O}$ defined in [RC].

The arguments, used in the proof of Theorem 10 also imply the following technical result:

Corollary 11.

(1) *The Loewy length* $\mathrm{l.l.}(P(w))$ *of* $P(w)$ *equals* $2\mathrm{l.l.}(\Delta(e)) - \mathrm{l.l.}(\Delta(w))$. *In particular, for the regular block of* $\mathcal{O}$ *we have* $\mathrm{l.l.}(P(w)) = l(w_0) + l(w) + 1$.

(2) *The Loewy length* $\mathrm{l.l.}(T(w))$ *of* $T(w)$ *equals* $2\mathrm{l.l.}(\Delta(w)) - 1$. *In particular, for the regular block of* $\mathcal{O}$ *we have* $\mathrm{l.l.}(T(w)) = 2(l(w_0) - l(w)) + 1$.

Proof. We start with the second statement. Recall that $\Delta(w) \hookrightarrow T(w)$, $T(w) \twoheadrightarrow \nabla(w)$, $[T(w) : L(w)] = 1$ and $L(w)$ is the simple top of $\Delta(w)$ and the simple socle of $\nabla(w)$. It follows that $\mathrm{l.l.}(T(w)) \geq \mathrm{l.l.}(\Delta(w)) + \mathrm{l.l.}(\nabla(w)) - 1 = 2\mathrm{l.l.}(\Delta(w)) - 1$ since $\mathcal{O}$ has a simple preserving duality. However, the graded filtration of the tilting module we have just constructed certainly has semi-simple subquotients (since A_0 is positively graded). All $\Delta(w')$ occurring in it have Loewy length less than or equal to that of $\Delta(w)$ and start in negative degrees since $\mathrm{End}_A(T)$ is positively graded. This implies that $\mathrm{l.l.}(T(w)) \leq 2\mathrm{l.l.}(\Delta(w)) - 1$ and completes the proof of the first part.

Since $P(w)$ has simple top, its graded filtration is the radical one by [BGS, Proposition 2.4.1]. However, from the proof of Theorem 10 and from the second part of this corollary, which we have just proved, it follows that the length of the graded filtration of $P(w)$ is exactly $2\mathrm{l.l.}(\Delta(e)) - \mathrm{l.l.}(\Delta(w))$.

The computations for the regular block follow from the results of [Ir1] and the proof is complete. $\square$

Corollary 12. *Let* $w \in W$. *Then the following conditions for* $T(w)$ *are equivalent*:

(a) $T(w)$ *is rigid.*

(b) $\mathrm{End}_A(T(w))$ *is commutative.*

(c) $T(w)$ *has simple top* (*or, equivalently, simple socle*).

(d) *The center of the universal enveloping algebra surjects onto* $\mathrm{End}_A(T(w))$.

(e) $T(w) \hookrightarrow P(w_0)$.

(f) $P(w_0) \twoheadrightarrow T(w)$.

(g) $[T(w) : \Delta(w')] \leq 1$ *for all* $w' \in W$.

(h) $[T(w) : \nabla(w')] \leq 1$ *for all* $w' \in W$.

(i) $[T(w) : \Delta(w_0)] = 1$.

(j) $[T(w) : \nabla(w_0)] = 1$.

We remark that, though $\Delta(w_0) \cong \nabla(w_0)$ is a simple module, the numbers $[T(w) : \Delta(w_0)]$ and $[T(w) : \nabla(w_0)]$ are not the composition multiplicities, but the multiplicities in the standard and the costandard filtrations of $T(w)$ respectively.

Proof. By [FKM, Section 7], $T(w) \hookrightarrow P(w_0 w)$ and the restriction induces an isomorphism for the endomorphism rings. Hence the equivalence of (b), (c), and (d) follows from the self-duality of $T(w)$ and [St2, Theorem 7.1]. That (c) implies (a) follows from [BGS, Proposition 2.4.1]. From the proof of Theorem 10 and [St1, Theorem 3.6] it follows that the highest and the lowest graded components of $T(w)$ are one-dimensional. Hence if $T(w)$ does not have simple top, its graded filtration, which is a Loewy one, does not coincide with the radical filtration and thus $T(w)$ is not rigid. This means that (a) implies (c). Since $L(w_0)$ is the socle of any Verma module, it follows that (f) is equivalent to (c). And, using the self-duality of both $T(w)$ and $P(w_0)$ we have that (f) is equivalent to (e).

The equivalence of (g) and (h) and the equivalence of (i) and (j) follows using the simple preserving duality on $\mathcal{O}$. Since $[P(w_0) : \Delta(w')] = 1$ for all w', we get that (f) implies (g). Let $T(w)$ be such that (g) is satisfied. Then, in particular, $[T(w) : \Delta(w_0)] \leq 1$. Since $L(w_0)$ is a simple socle of any Verma module, the self-duality of $T(w)$ implies $[T(w) : \Delta(w_0)] = 1$, which, in turn, implies that $T(w)$ has simple top, giving (c). Moreover, the same arguments shows that (i) implies (c). That (g) implies (i) is obvious, and the proof is complete. □

We remark that (in the case when the equivalent conditions of Corollary 12 are satisfied) the surjection of the center of the universal enveloping algebra onto $\mathrm{End}_A(T(w))$ is graded with respect to the grading on the center, considered in [So1].

Corollary 13. *Let $w \in W$, and s be a simple reflection. Then $\theta_s T(w) = T(w)\langle 1\rangle \oplus T(w)\langle -1\rangle$ if $l(ws) > l(w)$ and $\theta_s T(w) \in \mathrm{add}(T)$ (as a graded module) otherwise.*

Proof. In the case $l(ws) > l(w)$ the statement follows from [FKM, Section 7] and [St2, Section 8]. If $l(ws) < l(w)$ then Theorem 10 and [St2, Section 8] implies that $\theta_s T(w)$ has a graded Verma flag, and all Verma subquotients in this flag are of the form $\Delta(x)\langle k\rangle$, $k \geq 0$. The self-duality of $\theta_s T(w)$ now implies that $\theta_s T(w) \in \mathrm{add}(T)$. □

One more corollary of Theorem 10 is the following:

Proposition 6. *A is a balanced algebra, in particular, all standard, costandard, and simple A-modules are T-Koszul.*

Proof. That standard and costandard A-modules are T-Koszul follows from the fact that A is standard Koszul (see [ADL]) and Theorem 10. Hence A is SCT by Theorem 10 and Corollary 9. Further, the Koszul grading on $A-$mod induces on $R(A)^!-$mod the Koszul grading by [ADL, Theorem 3]. In particular, from Theorem 10 it follows that $R(A)^!$ is SCT, that is A is balanced. That simple A-modules are T-Koszul now follows from Theorem 9. □

With the same argument and using the result of Catharina Stroppel presented in the Appendix, one gets that the algebras of the blocks of the parabolic analogue of the category $\mathcal{O}$ in the sense of [RC] are also balanced.

We also remark that projective A-modules are not T-Koszul in general. For example, already for $\mathfrak{sl}_2$ we have $P(s_\alpha) \cong T(e)\langle -1\rangle$ and thus $P(s_\alpha)$ is not T-Koszul.

Corollary 14. *Let A be the associative algebra of the regular block of the category $\mathcal{O}$ endowed with Koszul grading. Then the category of linear bounded tilting complexes of A-modules is equivalent to $A-$*gmod.

Proof. Since A has a simple preserving duality, it is isomorphic to A^{opp}, moreover, A is Koszul self-dual by [So1] and Ringel self-dual by [So2]. Hence the necessary statement follows from Corollary 7. □

For singular blocks Corollary 7 and [BGS] imply that the category of linear bounded tilting complexes of A-modules is equivalent to the category of graded modules over the regular block of the parabolic category $\mathcal{O}$ with the same stabilizer (and vice versa).

8. Appendix (written by Catharina Stroppel)

In this appendix we reprove Theorem 10 in a way which implies the corresponding statement for the parabolic category $\mathcal{O}$. Our methods also provide an example for the theory developed in the paper in the context of properly stratified algebras. Since we do not use any new techniques, we refer mainly to the literature. We have to recall several constructions and definitions. We restrict ourselves to the case of the principal block to avoid even more notation.

For an algebra A we denote by mod$-A$ ($A-$mod$-A$ respectively) the category of finitely generated right A-modules (finitely generated A-bimodules). If A is graded, then we denote by gmod$-A$ and $A-$gmod$-A$ the corresponding categories of graded modules.

Let $\mathfrak{g}$ be a semisimple Lie algebra with fixed Borel and Cartan subalgebras $\mathfrak{b}$, $\mathfrak{h}$, Weyl group W with longest element w_0, and corresponding category $\mathcal{O}$. Let $\mathcal{O}_0$ be the principal block of $\mathcal{O}$ with the simple modules $L(x\cdot 0)$ of highest weight $x(\rho)-\rho$, where $x \in W$ and ρ denotes the half-sum of positive roots. Let $P(x\cdot 0)$ be the projective cover of $L(x\cdot 0)$.

Let $\mathcal{H}$ denote the category of Harish-Chandra bimodules with generalized trivial central character from both sides (as considered for example in [So3]). Let χ denote the trivial central character. For any $n \in \mathbb{Z}_{>0}$ we have the full subcategories $\mathcal{H}^n$ (and ${}^n\mathcal{H}$ respectively) of $\mathcal{H}$ given by objects X such that $X\mathrm{Ker}\chi^n = 0$ ($\mathrm{Ker}\chi^n X = 0$ respectively). There is an auto-equivalence η of $\mathcal{H}$, given by switching the left and right action of $U(\mathfrak{g})$ (see [Ja, 6.3]), and giving rise to equivalences $\mathcal{H}^n \cong {}^n\mathcal{H}$. For s a simple reflection we have translation functors through the s-wall: θ_s from the left hand side and θ_s^r from the right hand side (for a definition see [Ja, 6.33] or more explicitly [St2, 2.1]). In particular, $\eta\theta_s \cong \theta_s^r\eta$. Recall the equivalence ([BG, Theorem 5.9]) $\epsilon : \mathcal{H}^1 \cong \mathcal{O}_0$. We denote by $L_x = \epsilon^{-1}L(x \cdot 0)$ and consider it also as an object in $\mathcal{H}$. Note that $\eta L_x \cong L_{x^{-1}}$ (see [Ja, 6.34]). Let P_x^n and ${}^nP_x$ be the projective cover of L_x in $\mathcal{H}^n$ and ${}^n\mathcal{H}$ respectively. In particular, $\eta P_x^n \cong {}^nP_x$.

Recall the structural functor $\mathbb{V} : \mathcal{H} \to S(\mathfrak{h})\text{−mod−}S(\mathfrak{h})$ from [So3]. We equip the algebra $S = S(\mathfrak{h})$ with a $\mathbb{Z}$-grading such that $\mathfrak{h}$ is sitting in degree two. In [So3] it is proved that $\mathbb{V}P_x^n$ has a graded lift. By abuse of language, we denote the graded lift having $-l(x)$ as its lowest degree also by $\mathbb{V}P_x^n$. Let $A^n = \mathrm{End}_{S-\mathrm{gmod}-S}(\bigoplus_{x\in W}\mathbb{V}P_x^n)$. Then A^n is a graded algebra such that $\mathcal{H}^n \cong \text{mod−}A^n$. In particular, A^1 is the Koszul algebra corresponding to $\mathcal{O}_0$ ([BGS]). On the other hand we have ${}^nA = \mathrm{End}_{\mathcal{H}}(\bigoplus_{x\in W}{}^nP_x)$ and the corresponding equivalence ${}^n\mathcal{H} \cong \text{mod−}{}^nA$. Concerning the notation we will not distinguish between objects in $\mathcal{H}^n$ and mod−A^n or between objects in ${}^n\mathcal{H}$ and mod−${}^nA$. We fix a grading on ${}^nA$ such that η lifts to equivalences $\tilde{\eta} : \text{gmod−}A_n \cong \text{gmod−}A^n$ preserving the degrees in which a simple module is concentrated. More precisely, $\tilde{\eta}L(x) \cong L(x^{-1})$, where $L(x)$ denotes the graded lift of L_x, concentrated in degree zero, in the corresponding category.

Let us fix n. For s a simple reflection we denote by S^s the s-invariants in S. We define $\tilde{\theta}_s : \text{gmod−}A^n \to \text{gmod−}A^n$ as tensoring with the graded $A^n = \mathrm{End}_{S-\mathrm{gmod}-S}(\bigoplus_{x\in W}\mathbb{V}P_x^n)$ bimodule

$$\mathrm{Hom}_{S-\mathrm{gmod}-S}\left(\bigoplus_{x\in W}\mathbb{V}P_x^n, \bigoplus_{x\in W} S\otimes_{S^s}\mathbb{V}P_x^n\langle -1\rangle\right).$$

Because of [So3, Lemma 10], this is a graded lift (in the sense of [St1]) of the translation functor $\theta_s : \mathcal{H}^n \to \mathcal{H}^n$. As in [St1] we have the adjunction morphisms $\mathrm{ID}\langle 1\rangle \to \tilde{\theta}_s$ and $\tilde{\theta}_s \to \mathrm{ID}\langle -1\rangle$. Define $\tilde{\theta}_s^r = \eta\tilde{\theta}_s\eta : {}^nA\text{−gmod} \to {}^nA\text{−gmod}$. We have again the adjunction morphism $a_s^{(n)} : \mathrm{ID}\langle 1\rangle \to \tilde{\theta}_s^r$. Let $T_s^{(n)}$ denote the functor given by taking the cokernel of $a_s^{(n)}$. We fix a compatible system of surjections $P^n \twoheadrightarrow P^m$ for $n \geq m$. It gives rise to a system of graded projections $p_{n,m} : {}^nA \twoheadrightarrow {}^m A$ for $n \geq m$. Let ${}^\infty A = \varprojlim {}^nA$ and ${}^\infty T_s = \varprojlim T^{(n)} : \text{gmod−}{}^\infty A \to \text{gmod−}{}^\infty A$. Note that ${}^\infty T_s$ preserves the category gmod−A^1 (considered as a subcategory of ${}^\infty A$). In fact, it is a graded lift of Arkhipov's twisting functor (as considered in [AS], [KM]). Let $T_s :$ gmod−$A^1 \to$ gmod−A^1. For $x \in W$ with reduced expression $[x] = s_{i_1}s_{i_2}\cdots s_{i_r}$ set $T_{[x]} = T_{s_1}T_{s_2}\cdots T_{s_r}$. Set $A = A^1$.

Proposition 7. *Let x, $s \in W$ and let s be a simple reflection. Then the following holds*

(1) *The functor $T_{[x]}$ is* (*up to isomorphism*) *independent of the chosen reduced expression.*

(2) *Moreover, if $sx > x$ and $\Delta(x) \in$ gmod−A denotes the graded lift of the Verma module with simple head $L(x)$* (*concentrated in degree zero*), *then*

$T_s\Delta(x) \cong \Delta(sx)$ and $T_s\nabla(sx) \cong \nabla(x)$, where $\nabla(x)$ denotes the graded lift of the dual Verma module with socle $L(x)$ (concentrated in degree zero).

Proof. We consider now the adjunction morphism $b_s : \mathrm{ID} \to \tilde{\theta}_s^r$ between endofunctors on mod$-^n A$. Let $\tilde{T}_s$ denote the functor given by taking the cokernel of b_s, restricted to mod$-A$. Let $\tilde{T}_{[x]} = \tilde{T}_{s_1}\tilde{T}_{s_2}\cdots\tilde{T}_{s_r}$. Then $\tilde{T}_{[x]}$ does not depend on the chosen reduced expression ([Jo], [KM]). If we show that $\tilde{T}_{[x]}$ is indecomposable, then a graded lift is unique up to isomorphism and grading shift, and the statement follows say from the second part of the theorem. Set $G = \tilde{T}_x$. Let us prove the indecomposability: We claim that the canonical evaluation morphism $\mathrm{End}(G) \to \mathrm{End}_{\mathfrak{g}}(GP(w_0 \cdot 0))$, $\varphi \mapsto \varphi_{P(w_0\cdot 0)}$, is an isomorphism. Assume $\varphi_{P(w_0\cdot 0)} = 0$. Let P be a projective object in $\mathcal{O}_0$. Then there is a short exact sequence

$$P \to \oplus_I P(w_0 \cdot 0) \to Q \tag{8.1}$$

for some finite set I and some module Q having a Verma flag. (To see this consider the projective Verma module. It is the unique Verma submodule of $P(w_0 \cdot 0) = 0$, hence the desired sequence exists. The existence of the sequence for any projective object follows then using translation functors.) By [AS, Lemma 2.1], we get an exact sequence $GP \to \oplus_I GP(w_0 \cdot 0) \to GQ$. Hence $\varphi_{P(w_0\cdot 0)} = 0$ implies $\varphi_P = 0$ for any projective object P. Since G is right exact, it follows $\varphi = 0$. Let $g \in \mathrm{End}_{\mathfrak{g}}(GP(w_0 \cdot 0))$. Since $\mathrm{End}_{\mathfrak{g}}(GP(w_0 \cdot 0)) \cong \mathrm{End}_{\mathfrak{g}}(P(w_0 \cdot 0))$ ([AS, Proposition 5.3]), g defines an endomorphism of G when restricted to the additive category generated by $P(w_0 \cdot 0)$. Note that (by taking the injective hull of Q) the sequence (8.1) gives rise to an exact sequence

$$0 \to P \to \oplus_I P(w_0 \cdot 0) \to \oplus_{I'} P(w_0 \cdot 0)$$

for some finite sets I, I'. Using again [AS, Lemma 2.1] we get an exact sequence

$$0 \to GP \to \oplus_I GP(w_0 \cdot 0) \to \oplus_{I'} GP(w_0 \cdot 0).$$

Hence g defines an endomorphism g_P of P. Standard arguments show that this is independent of the chosen exact sequence. Since G is right exact, g extends uniquely to an endomorphism φ of G. By construction $\varphi_{P(w_0\cdot 0)} = g$. This proves the surjectivity. Since $\mathrm{End}_{\mathfrak{g}}(GP(w_0 \cdot 0)) \cong \mathrm{End}_{\mathfrak{g}}(P(w_0 \cdot 0))$ is a local ring, the functor G is indecomposable. This proves the first part of the proposition.

We have $\tilde{T}_s f(\Delta(x)) \cong f(\Delta(sx))$, where f denotes the grading forgetting functor. Hence, $T_s(\Delta(x)) \cong \Delta(sx)\langle k\rangle$ for some $k \in \mathbb{Z}$. On the other hand $\eta T_s \eta \Delta(x^{-1}) \cong \Delta(x^{-1}s)$ ([St1, Theorem 3.6]). Hence $k = 0$ and $T_s\Delta(x) \cong \Delta(sx)$. Forgetting the grading we have $\tilde{T}_s f(\nabla(sx)) \cong f(\nabla(x))$. On the other hand $\eta T_s \eta \nabla((sx)^{-1}) \cong \nabla(x^{-1})$ ([St1, Theorem 3.10]). The second part of the proposition follows. $\square$

Since $T_{[x]}$ does not depend on the chosen reduced expression, we denote it just T_x in the following. Let $P(x) \in \text{gmod}-A$ be the indecomposable projective module with simple head $L(x)$ concentrated in degree zero. Set $P = \bigoplus_{x \in W} P(x)$. Let $T(x)$ denote the graded lift of an indecomposable tilting module, characterized by the property that $\Delta(x)$ is a submodule and $\nabla(x)$ is a quotient. Let $T = \bigoplus_{x \in W} T(x)$.

Theorem 11. *Let $x \in W$. There is an isomorphism of graded algebras*

$$\mathrm{End}_A(P) \cong \mathrm{End}_A(T_x P).$$

For $x = w_0$ we get in particular

$$\mathrm{End}_A(P) \cong \mathrm{End}_A(T).$$

Proof. The first isomorphism follows directly from [AS, Lemma 2.1] and the definition of T_x. For the second we claim that $T_{w_0} P(y) \cong T(w_0 y)$. By Proposition 7 we have $T_{w_0} P(0) \cong \Delta(w_0)$. Hence, the statement is true for $y = e$. Using translation functors we directly get $T_{w_0} P(y) \cong T(w_0 y)\langle k \rangle$ for some $k \in \mathbb{Z}$. On the other hand $P(y)$ surjects onto $\Delta(y)$. Then $T_{w_0} P(y)$ surjects onto $T_{w_0} \Delta(y)$. The latter is isomorphic to $\nabla(w_0 y)$, since $\Delta(w_0) \cong \nabla(w_0)$. $\square$

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with corresponding parabolic subgroup $W_\mathfrak{p}$ of W. Let $\mathcal{O}_0^\mathfrak{p}$ be the full subcategory of $\mathcal{O}_0$ given by locally $\mathfrak{p}$-finite objects. If $P \in \mathcal{O}_0$ is a minimal projective generator, then its maximal quotient $P^\mathfrak{p}$ contained in $\mathcal{O}_0^\mathfrak{p}$ is a minimal projective generator of $\mathcal{O}_0^\mathfrak{p}$ and $\mathrm{End}_\mathfrak{g}(P^\mathfrak{p})$ inherits a grading from $A = \mathrm{End}_\mathfrak{g}(P)$. We will consider then the category $\text{gmod}-A^\mathfrak{p}$ as the full subcategory of $\text{gmod}-A$ given by all objects having only composition of the form $L(x)\langle k \rangle$, where $k \in \mathbb{Z}$ and $x \in W^\mathfrak{p}$, the set of shortest coset representative of $W_\mathfrak{p} \backslash W$. Let $\Delta^\mathfrak{p}(x) \in \text{gmod}-A^\mathfrak{p}$, $\nabla^\mathfrak{p}(x)$ be the standard graded lifts of the standard and costandard modules in $\mathcal{O}_0^\mathfrak{p}$ (which were denoted by $\Delta(x)$ and $\nabla(x)$ in Section 7). Let $T^\mathfrak{p}$ be the module T from Corollary 13 for the category $\text{gmod} - A^\mathfrak{p}$. Then Theorem 10 generalizes to the following.

Corollary 15. $\mathrm{End}_{A^\mathfrak{p}}(T^\mathfrak{p})$ *is positively graded, moreover, it is generated in degrees* 0 *and* 1*. Furthermore,* ∇ *admits an LT-resolution.*

Proof. Let $w = w_0^\mathfrak{p} \in W^\mathfrak{p}$ be the longest element. Then $\Delta^\mathfrak{p}(w)$ is a tilting module and canonically a quotient of $\Delta(w) \cong T_w \Delta(e) = T_w P(e)$. Using translation functors we get that $T^\mathfrak{p}$ is a quotient of $T_w P$. Hence, there is a surjection of graded algebras from $\mathrm{End}_A(T_w P) \cong \mathrm{End}_A(P)$ onto $\mathrm{End}_A(T^\mathfrak{p})$. Hence $\mathrm{End}(T_w P^\mathfrak{p}) \cong \mathrm{End}_{A^\mathfrak{p}}(T^\mathfrak{p})$ is positively graded and generated in degrees 0 and 1. The existence of the resolution follows using the same arguments as in the proof of Theorem 10. $\square$

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