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3-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ Part II, $G = E_7$, Cases 2, 3 and 4

By

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According to M. Hara [1], there are five cases of 3-graded decompositions $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ of simple Lie algebras $\mathfrak{g}$ of type E_7. In the preceding paper [2], we gave the group realization of Lie subalgebras $\mathfrak{g}_{ev} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \mathfrak{g}_0$ and $\mathfrak{g}_{ed} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_3$ of $\mathfrak{g}$ of Case 1. In the present paper, we give the group realization of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ of Cases 2, 3 and 4. We rewrite the results of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ of Cases 2, 3 and 4.

Case 2	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$\mathfrak{sl}(2, C) \oplus \mathfrak{so}(12, C)$ $C \oplus \mathfrak{sl}(7, C)$	$C \oplus C \oplus \mathfrak{sl}(6, C)$ $26, 16, 6$
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{so}(6, 6)$ $\boldsymbol{R} \oplus \mathfrak{sl}(7, \boldsymbol{R})$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{sl}(6, \boldsymbol{R})$ $26, 16, 6$

Case 3	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$C \oplus {\mathfrak{e}_6}^C$ $C \oplus \mathfrak{so}(12, C)$	$C \oplus C \oplus \mathfrak{so}(10, C)$ $17, 16, 10$
	$\mathfrak{e}_{7(7)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(6)}$ $\boldsymbol{R} \oplus \mathfrak{so}(6, 6)$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(5, 5)$ $17, 16, 10$
	$\mathfrak{e}_{7(-25)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(-26)}$ $\boldsymbol{R} \oplus \mathfrak{so}(2, 10)$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(1, 9)$ $17, 16, 10$

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Case 4	$\mathfrak{g}$	$\mathfrak{g}_{ev}$	$\mathfrak{g}_0$
		$\mathfrak{g}_{ed}$	$\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$C \oplus {\mathfrak{e}_6}^C$	$C \oplus C \oplus \mathfrak{so}(10, C)$
		$\mathfrak{sl}(2, C) \oplus C \oplus \mathfrak{so}(10, C)$	$26, 16, 1$
	$\mathfrak{e}_{7(7)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(6)}$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(5, 5)$
		$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R} \oplus \mathfrak{so}(5, 5)$	$26, 16, 1$
	$\mathfrak{e}_{7(-25)}$	$\boldsymbol{R} \oplus \mathfrak{e}_{6(-26)}$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(1, 9)$
		$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R} \oplus \mathfrak{so}(1, 9)$	$26, 16, 1$

Our results of Cases 2, 3 and 4 are as follows:

Case 2	G	G_{ev}	G_0
		G_{ed}	
	${E_7}^C$	$(SL(2, C) \times Spin(12, C))/\boldsymbol{Z}_2$	$(C^* \times C^* \times SL(6, C))/(\boldsymbol{Z}_6 \times \boldsymbol{Z}_6)$
		$(C^* \times SL(7, C))/\boldsymbol{Z}_7$	
	$E_{7(7)}$	$(SL(2, \boldsymbol{R}) \times spin(6, 6))/\boldsymbol{Z}_2 \times 2$	$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(6, \boldsymbol{R})) \times 2$
		$(\boldsymbol{R}^+ \times SL(7, \boldsymbol{R})) \times 2$	

Case 3	G	G_{ev}	G_0
		G_{ed}	
	${E_7}^C$	$(C^* \times {E_6}^C)/\boldsymbol{Z}_3$	$(C^* \times C^* \times Spin(10, C))/\boldsymbol{Z}_{12}$
		$(C^* \times Spin(12, C))/\boldsymbol{Z}_2$	
	$E_{7(7)}$	$(\boldsymbol{R}^+ \times E_{6(6)}) \times 2$	$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(5, 5)) \times 2$
		$(\boldsymbol{R}^+ \times spin(6, 6)) \times 2$	
	$E_{7(-25)}$	$(\boldsymbol{R}^+ \times E_{6(-26)}) \times 2$	$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(1, 9)) \times 2$
		$\boldsymbol{R}^+ \times spin(2, 10)$	

Case 4	G	G_{ev}	G_0
		G_{ed}	
	${E_7}^C$	$(C^* \times {E_6}^C)/\boldsymbol{Z}_3$	$(C^* \times C^* \times Spin(10, C))/\boldsymbol{Z}_{12}$
		$(SL(2, C) \times C^* \times Spin(10, C))/\boldsymbol{Z}_4$	
	$E_{7(7)}$	$(\boldsymbol{R}^+ \times E_{6(6)}) \times 2$	$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(5, 5)) \times 2$
		$(Sl(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times spin(5, 5)) \times 2$	
	$E_{7(-25)}$	$(\boldsymbol{R}^+ \times E_{6(-26)}) \times 2$	$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(1, 9)) \times 2$
		$(SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times Spin(1, 9)) \times 2$	

This paper is a continuation of [2], so the numbering of sections and theorems start from 4.2. We use the same notations as that in [2].

4. Group E_7

The connected universal linear Lie groups ${E_7}^C, E_{7(7)}$ and $E_{7(-25)}$ are given by

$$\begin{aligned}{E_7}^C &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\},\\ E_{7(7)} &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{P}') \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\},\\ E_{7(-25)} &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{P}) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}\end{aligned}$$

(although the definitions of ${E_7}^C$ and $E_{7(7)}$ are already given in [2]), where $\mathfrak{P} = \mathfrak{J} \oplus \mathfrak{J} \oplus \boldsymbol{R} \oplus \boldsymbol{R}$ ($\mathfrak{J}$ is the exceptional $\boldsymbol{R}$-Jordan algebra).

Here, we shall arrange mappings $\gamma, \gamma', \gamma_1, \sigma, \iota, \lambda, \kappa, \mu, \phi$ and φ used in this paper. By using the mapping $\varphi_2 : Sp(1, \boldsymbol{H}^C) \times Sp(1, \boldsymbol{H}^C) \to {G_2}^C$ defined by

$$\varphi_2(p, q)(a + be_4) = qa\overline{q} + (pb\overline{q})e_4, \quad a + be_4 \in \boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4 = \mathfrak{C}^C,$$

the C-linear transformations γ, γ' and γ_1 of $\mathfrak{C}^C$ are defined by

$$\gamma = \varphi_2(1, -1), \quad \gamma' = \varphi_2(e_1, e_1), \quad \gamma_1 = \varphi_2(e_2, e_2),$$

respectively. Then $\gamma, \gamma', \gamma_1 \in {G_2}^C \subset {E_7}^C$ and $\gamma^2 = {\gamma'}^2 = {\gamma_1}^2 = 1$. The C-linear transformation σ of $\mathfrak{J}^C$ is defined by

$$\sigma X = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

Then $\sigma \in {F_4}^C \subset {E_7}^C$ and $\sigma^2 = 1$. Next, the C-linear transformations ι and λ of $\mathfrak{P}^C$ are defined by

$$\begin{aligned}&\iota(X, Y, \xi, \eta) = (-iX, iY, -i\xi, i\eta),\\ &\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi), \qquad (X, Y, \xi, \eta) \in \mathfrak{P}^C,\end{aligned}$$

respectively. Then $\iota, \lambda \in {E_7}^C$ and $\iota^4 = \lambda^4 = 1$. Further, the C-linear mappings κ and μ of $\mathfrak{P}^C$ are defined by

$$\begin{aligned}\kappa(X, Y, \xi, \eta) &= \left(\begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\overline{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta\right),\\ \mu(X, Y, \xi, \eta) &= \left(\begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\overline{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\overline{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1\right),\end{aligned}$$

$(X, Y, \xi, \eta) \in \mathfrak{P}^C$, respectively. For $A \in SL(2, C)$, we define the C-linear transformation $\phi(A)$ of $\mathfrak{P}^C$ by

$$\phi(A)(X, Y, \xi, \eta) = (X', Y', \xi', \eta'),$$

$$\begin{pmatrix} \xi_1{}' \\ \eta' \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta_1{}' \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta_1 \end{pmatrix}, \quad \begin{pmatrix} \eta_2{}' \\ \xi_2{}' \end{pmatrix} = A \begin{pmatrix} \eta_2 \\ \xi_2 \end{pmatrix}, \quad \begin{pmatrix} \eta_3{}' \\ \xi_3{}' \end{pmatrix} = A \begin{pmatrix} \eta_3 \\ \xi_3 \end{pmatrix},$$

$$\begin{pmatrix} x_1{}' \\ y_1{}' \end{pmatrix} = ({}^tA^{-1}) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2{}' \\ y_2{}' \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} x_3{}' \\ y_3{}' \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}.$$

Then $\phi(A) \in {E_7}^C$. Finally we shall explain the mapping $\varphi : SU(8, \boldsymbol{C}^C) \to {E_7}^C$. Let $g : \mathfrak{J}^C \to \mathfrak{J}(4, \boldsymbol{H}^C)$ be the C-linear mapping defined by

$$g(M + \boldsymbol{a}) = \begin{pmatrix} \dfrac{1}{2}\operatorname{tr}(M) & i\boldsymbol{a} \\ i\boldsymbol{a}^* & M - \dfrac{1}{2}\operatorname{tr}(M)E \end{pmatrix}, \quad M + \boldsymbol{a} \in \mathfrak{J}(3, \boldsymbol{H}^C) \oplus (\boldsymbol{H}^C)^C = \mathfrak{J}^C.$$

By using the mapping g, we define the C-linear isomorphism $\chi : \mathfrak{P}^C \to \mathfrak{S}(8, \boldsymbol{C}^C) = \{S \in M(8, \boldsymbol{C}^C) \,|\, {}^tS = -S\}$ by

$$\chi(X, Y, \xi, \eta) = k_J\Big(gX - \frac{\xi}{2}E\Big) + e_1 k_J\Big(g(\gamma Y) - \frac{\eta}{2}E\Big),$$

where $k_J : \mathfrak{J}(4, \boldsymbol{H}^C) \to \mathfrak{S}(8, \boldsymbol{C}^C)$ is C-linear mapping defined by $k_J\Big(\big(a + be_2\big)\Big) = \Big(\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}\Big) J$, $a, b \in \boldsymbol{C}^C$, $J = \operatorname{diag}(J, J, J, J)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now, we define the mapping $\varphi : SU(8, \boldsymbol{C}^C) \to ({E_7}^C)^{\lambda\gamma}$ by

$$\varphi(A)P = \chi^{-1}(A(\chi P){}^tA), \quad P \in \mathfrak{P}^C,$$

then we have an isomorphism

$$SU(8, \boldsymbol{C}^C)/\boldsymbol{Z}_2 \cong ({E_7}^C)^{\lambda\gamma}, \quad \boldsymbol{Z}_2 = \{E, -E\}$$

(see [3, Theorem 4.5.3] for details).

4.2. Subgroups of type ${\boldsymbol{A}_1}^C \oplus {\boldsymbol{D}_6}^C, \boldsymbol{C} \oplus \boldsymbol{C} \oplus {\boldsymbol{A}_5}^C$ and $\boldsymbol{C} \oplus {\boldsymbol{A}_6}^C$ of ${\boldsymbol{E}_7}^C$

ι is conjugate to λ in ${E_7}^C$. Indeed, let $\delta_2 = \exp\Big(\varPhi\Big(0, -\dfrac{\pi i}{4}E, -\dfrac{\pi i}{4}E, 0\Big)\Big)$. Then $\delta_2 \in {E_7}^C$ and δ_2 satisfies

$${\delta_2}^{-1}\iota\delta_2 = \lambda.$$

Moreover, δ_2 satisfies $\delta_2\tau\lambda = \tau\lambda\delta_2$ and $\delta_2\gamma_1 = \gamma_1\delta_2$. Hence $\tau\lambda\iota\gamma_1$ is conjugate to $-\tau\gamma_1$ under δ_2. Indeed,

$${\delta_2}^{-1}(\tau\lambda\iota\gamma_1)\delta_2 = \tau\lambda{\delta_2}^{-1}\iota\delta_2\gamma_1 = \tau\lambda\lambda\gamma_1 = -\tau\gamma_1.$$

Furthermore, γ is conjugate to γ_1 in $E_7{}^C$. Indeed, let δ_1 be the C-linear transformation of $\mathfrak{C}^C$ satisfying

$$1 \to 1,\ e_1 \to e_4,\ e_2 \to e_2,\ e_3 \to e_6,\ e_4 \to e_1,\ e_5 \to -e_5,\ e_6 \to e_3,\ e_7 \to -e_7,$$

then $\delta_1 \in G_2{}^C \subset F_4{}^C \subset E_6{}^C \subset E_7{}^C, {\delta_1}^2 = 1$ and δ_1 satisfies

$$\delta_1\gamma\delta_1 = \gamma_1.$$

Hence we have

$$E_{7(7)} = (E_7{}^C)^{\tau\gamma} \cong (E_7{}^C)^{\tau\gamma_1} = (E_7{}^C)^{-\tau\gamma_1} \cong (E_7{}^C)^{\tau\lambda\iota\gamma_1}.$$

In the Lie algebra $\mathfrak{e}_7{}^C$, let

$$Z = i\Phi(G_{45} - G_{67}, -E, E, 0).$$

Theorem 4.2.1. *The 3-graded decomposition of* $\mathfrak{e}_{7(7)} = (\mathfrak{e}_7{}^C)^{\tau\lambda\iota\gamma_1}$ *(or* $\mathfrak{e}_7{}^C$*),*

$$\mathfrak{e}_{7(7)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\operatorname{ad} Z, Z = i\Phi(G_{45} - G_{67}, -E, E, 0)$*, is given by*

$$\mathfrak{g}_0 = \left\{\begin{array}{l} iG_{01},\ G_{02},\ iG_{03},\ iG_{12},\ G_{13},\ iG_{23}, \\ iG_{45}, G_{46} - G_{57},\ i(G_{47} + G_{56}),\ iG_{67}, \\ \tilde{A}_k(1), i\tilde{A}_k(e_1), \tilde{A}_k(e_2), i\tilde{A}_k(e_3),\ i(\check{E}_k - \hat{E}_k), \\ i(\check{F}_k(1) - \hat{F}_k(1)),\quad \check{F}_k(e_1) - \hat{F}_k(e_1), \\ i(\check{F}_k(e_2) - \hat{F}_k(e_2)), \check{F}_k(e_3) - \hat{F}_k(e_3),\ \ k = 1,2,3 \end{array}\right\} 37$$

$$\mathfrak{g}_{-1} = \left\{\begin{array}{l} G_{04} + iG_{05},\ G_{06} - iG_{07},\ iG_{14} - G_{15},\ iG_{16} + G_{17}, \\ G_{24} + iG_{25},\ G_{26} - iG_{27},\ iG_{34} - G_{35},\ iG_{36} + G_{37}, \\ \tilde{A}_k(e_4 + ie_5),\ \tilde{A}_k(e_6 - ie_7), \\ \check{F}_k(e_4 + ie_5) - \hat{F}_k(e_4 + ie_5), \check{F}_k(e_6 - ie_7) - \hat{F}_k(e_6 - ie_7), \\ 2i\tilde{F}_k(e_4 - ie_5) + \check{F}_k(e_4 - ie_5) + \hat{F}_k(e_4 - ie_5), \\ 2i\tilde{F}_k(e_6 + ie_7) + \check{F}_k(e_6 + ie_7) + \hat{F}_k(e_6 + ie_7),\ k = 1,2,3 \end{array}\right\} 26$$

$$\mathfrak{g}_{-2} = \left\{\begin{array}{l} (G_{46} + G_{57}) - i(G_{47} - G_{56}), \\ 2i\tilde{F}_k(1) + \check{F}_k(1) + \hat{F}_k(1),\quad 2\tilde{F}_k(e_1) + i\check{F}_k(e_1) + i\hat{F}_k(e_1), \\ 2i\tilde{F}_k(e_2) + \check{F}_k(e_2) + \hat{F}_k(e_2), 2\tilde{F}_k(e_3) + i\check{F}_k(e_3) + i\hat{F}_k(e_3), \\ 2iE_k \vee E_k + \check{E}_k + \hat{E}_k + i\mathbf{1},\ k = 1,2,3 \end{array}\right\} 16$$

$$\mathfrak{g}_{-3} = \left\{\begin{array}{l} 2i\tilde{F}_k(e_4 + ie_5) + \check{F}_k(e_4 + ie_5) + \hat{F}_k(e_4 + ie_5), \\ 2i\tilde{F}_k(e_6 - ie_7) + \check{F}_k(e_6 - ie_7) + \hat{F}_k(e_6 - ie_7),\ k = 1,2,3 \end{array}\right\} 6$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau,\quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau,\quad \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3})\tau.$$

For the induced differential mapping $\varphi_* : \mathfrak{su}(8, \boldsymbol{C}^C) \to \mathfrak{e}_7{}^C$ of $\varphi\colon SU(8, \boldsymbol{C}^C)$

$\to {E_7}^C$, we have

$$
\begin{aligned}
\varphi_*(\mathrm{diag}(e_1,-e_1,0,0,0,0,0,0)) &= \Phi(-G_{45}+G_{67},0,0,0),\\
\varphi_*(\mathrm{diag}(0,e_1,-e_1,0,0,0,0,0)) &= \Phi\Big(-G_{67},\frac{1}{2}(E_2+E_3),-\frac{1}{2}(E_2+E_3),0\Big),\\
\varphi_*(\mathrm{diag}(0,0,e_1,-e_1,0,0,0,0)) &= \Phi(G_{45}+G_{67},0,0,0),\\
\varphi_*(\mathrm{diag}(0,0,0,e_1,-e_1,0,0,0)) &= \Phi\Big(\frac{1}{2}(G_{01}+G_{23}-G_{45}-G_{67}),\frac{1}{2}(E_1-E_2),\\
&\qquad -\frac{1}{2}(E_1-E_2),0\Big),\\
\varphi_*(\mathrm{diag}(0,0,0,0,e_1,-e_1,0,0)) &= \Phi(-G_{01}-G_{23},0,0,0),\\
\varphi_*(\mathrm{diag}(0,0,0,0,0,e_1,-e_1,0)) &= \Phi\Big(G_{23},\frac{1}{2}(E_2-E_3),-\frac{1}{2}(E_2-E_3),0\Big),\\
\varphi_*(\mathrm{diag}(0,0,0,0,0,0,e_1,-e_1)) &= \Phi(G_{01}-G_{23},0,0,0).
\end{aligned}
$$

From the facts above, we have also

$$
\begin{aligned}
\Phi(G_{01},0,0,0) &= \varphi_*(\mathrm{diag}(0,0,0,0,-e_1/2,e_1/2,e_1/2,-e_1/2)),\\
\Phi(G_{23},0,0,0) &= \varphi_*(\mathrm{diag}(0,0,0,0,-e_1/2,e_1/2,-e_1/2,e_1/2)),\\
\Phi(G_{45},0,0,0) &= \varphi_*(\mathrm{diag}(-e_1/2,e_1/2,e_1/2,-e_1/2,0,0,0,0)),\\
\Phi(G_{67},0,0,0) &= \varphi_*(\mathrm{diag}(e_1/2,-e_1/2,e_1/2,-e_1/2,0,0,0,0)),\\
\Phi(0,E_1,-E_1,0) &= \varphi_*(\mathrm{diag}(e_1/2,e_1/2,e_1/2,e_1/2,-e_1/2,-e_1/2,-e_1/2,-e_1/2)),\\
\Phi(0,E_2,-E_2,0) &= \varphi_*(\mathrm{diag}(e_1/2,e_1/2,-e_1/2,-e_1/2,e_1/2,e_1/2,-e_1/2,-e_1/2)),\\
\Phi(0,E_3,-E_3,0) &= \varphi_*(\mathrm{diag}(e_1/2,e_1/2,-e_1/2,-e_1/2,-e_1/2,-e_1/2,e_1/2,e_1/2)).
\end{aligned}
$$

Since $iZ = \Phi(-G_{45}+G_{67},E,-E,0) = \varphi_*(\mathrm{diag}(5e_1/2,e_1/2,-e_1/2,-e_1/2,$ $-e_1/2,\ -e_1/2,-e_1/2,\ -e_1/2))$, by using the mapping $\varphi : SU(8,\boldsymbol{C}^C) \to {E_7}^C$, we have

$$
\begin{aligned}
z_2 &= \exp\frac{2\pi i}{2}Z = \varphi(\mathrm{diag}(e_1,e_1,-e_1,-e_1,-e_1,-e_1,-e_1,-e_1)) = -\gamma,\\
z_4 &= \exp\frac{2\pi i}{4}Z = \varphi(\mathrm{diag}(-w_8,w_8,w_8^{-1},w_8^{-1},w_8^{-1},w_8^{-1},w_8^{-1},w_8^{-1})),\\
z_3 &= \exp\frac{2\pi i}{3}Z = \varphi(\mathrm{diag}(-w_1,-{w_1}^2,-w_1,-w_1,-w_1,-w_1,-w_1,-w_1))\\
&= \varphi(\mathrm{diag}(w_1,{w_1}^2,w_1,w_1,w_1,w_1,w_1,w_1)),
\end{aligned}
$$

where $w_8 = e^{2\pi e_1/8}, w_1 = e^{2\pi e_1/3}$.

$z_2 = -\gamma$ is conjugate to

$$
{z_2}' = \sigma
$$

in ${E_7}^C$. Indeed, let $\delta_3 = \varphi(B)$, where B is

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -e_1 & 0 & 0 & 0 & e_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -e_1 & 0 & 0 & 0 & e_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -e_1 & 0 & 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -e_1 & 0 & 0 & 0 & e_1 \end{pmatrix} \in SU(8, \boldsymbol{C}^C).$$

Then $\delta_3 \in {E_7}^C$ and δ_3 satisfies ${\delta_3}^{-1}(-\gamma_1)\delta_3 = \sigma$. Now, we consider the element $\delta_1\delta_3$, then we have

$$(\delta_1\delta_3)^{-1}(-\gamma)(\delta_1\delta_3) = \sigma.$$

z_3 is conjugate to

$${z_3}' = \varphi(\mathrm{diag}({w_1}^2, w_1, w_1, w_1, w_1, w_1, w_1, w_1))$$

under the action of $\varphi\left(\mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0, 0, 0, 0, 0, 0\right)\right) \in \varphi\Big(SU(8, \boldsymbol{C}^C)\Big) \subset {E_7}^C$.

Hereafter, we use ${z_2}'$ and ${z_3}'$ instead of z_2 and z_3, respcetively.

Since $({\mathfrak{e}_7}^C)_{ev} = ({\mathfrak{e}_7}^C)^{{z_2}'}, ({\mathfrak{e}_7}^C)_0 = ({\mathfrak{e}_7}^C)^{z_4}, ({\mathfrak{e}_7}^C)_{ed} = ({\mathfrak{e}_7}^C)^{{z_3}'}$, we shall determine the structures of groups

$$({E_7}^C)_{ev} = ({E_7}^C)^{{z_2}'}, \quad ({E_7}^C)_0 = ({E_7}^C)^{z_4}, \quad ({E_7}^C)_{ed} = ({E_7}^C)^{{z_3}'}.$$

Theorem 4.2.2. (1) $({E_7}^C)_{ev} \cong (SL(2, C) \times Spin(12, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(E, 1), (-E, -\sigma)\}$.

(2) $({E_7}^C)_0 \cong (C^* \times C^* \times SL(6, C))/(\boldsymbol{Z}_6 \times \boldsymbol{Z}_6)$, $\boldsymbol{Z}_6 \times \boldsymbol{Z}_6 = \{({\omega_6}^k, {\omega_6}^l, {\omega_6}^k {\omega_6}^l E) \,|\, k, l = 0, 1, \ldots, 5\}, \omega_6 = e^{2\pi i/6}$.

(3) $({E_7}^C)_{ed} \cong (C^* \times SL(7, C))/\boldsymbol{Z}_7$, $\boldsymbol{Z}_7 = \{({\omega_7}^k, {\omega_7}^k E) \,|\, k = 0, 1, \ldots, 6\}$, $\omega_7 = e^{2\pi i/7}$.

Proof. (1) Let $Spin(12, C) = \{\alpha \in {E_7}^C \,|\, \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu\} = ({E_7}^C)^{\kappa,\mu}$. We define a mapping $\psi : SL(2, C) \times Spin(12, C) \to ({E_7}^C)^\sigma$ by

$$\psi(A, \beta) = \phi(A)\beta.$$

Then ψ is well-defined and is a surjective homomorphism. Ker $\psi = \{(E, 1), (-E, -\sigma)\} = \boldsymbol{Z}_2$. Hence we have $({E_7}^C)_{ev} = ({E_7}^C)^\sigma \cong (SL(2, C) \times Spin(12, C))/\boldsymbol{Z}_2$ (see [3, Thorem 4.6.13] for details).

(2) We define a mapping $\varphi : S(U(1, \boldsymbol{C}^C) \times U(1, \boldsymbol{C}^C) \times U(6, \boldsymbol{C}^C)) \to ({E_7}^C)^{z_4}$ by

$$\varphi(b_1, b_2, B)P = \chi^{-1}((b_1, b_2, B)(\chi P)^t(b_1, b_2, B)), \quad P \in \mathfrak{P}^C,$$

as the restriction mapping of $\varphi : SU(8, \boldsymbol{C}^C) \to {E_7}^C$. Then φ is well-defined and is a homomorphism. Ker $\varphi = \{(1, 1, E), (-1, -1, -E)\} = \boldsymbol{Z}_2$. Since $({E_7}^C)^{z_4}$ is connected and $\dim_C(({\mathfrak{e}_7}^C)_0) = 37$ (Theorem 4.2.1) $= (1 + 1 + 36) - 1 = \dim_C(\mathfrak{s}(\mathfrak{u}(1, \boldsymbol{C}^C) \oplus \mathfrak{u}(1, \boldsymbol{C}^C) \oplus \mathfrak{u}(6, \boldsymbol{C}^C)))$, φ is onto. Thus we have

$$\begin{aligned}({E_7}^C)_0 &\cong S(U(1, \boldsymbol{C}^C) \times U(1, \boldsymbol{C}^C) \times U(6, \boldsymbol{C}^C))/\boldsymbol{Z}_2 \\ &\cong S(C^* \times C^* \times GL(6, C))/\boldsymbol{Z}_2.\end{aligned}$$

Since the mapping $h : C^* \times C^* \times SL(6, C) \to S(C^* \times C^* \times GL(6, C))$,

$$h(d_1, d_2, D) = ({d_1}^6, {d_2}^6, (d_1 d_2)^{-1} D)$$

induces an isomorphism $S(C^* \times C^* \times GL(6, C)) \cong (C^* \times C^* \times SL(6, C))/(\boldsymbol{Z}_6 \times \boldsymbol{Z}_6)$, $\boldsymbol{Z}_6 \times \boldsymbol{Z}_6 = \{({\omega_6}^k, {\omega_6}^l, {\omega_6}^k {\omega_6}^l E) \mid k, l = 0, 1, \ldots, 5\}$. Thus we have $({E_7}^C)_0 = ({E_7}^C)^{z_4} \cong (C^* \times C^* \times SL(6, C))/(\boldsymbol{Z}_6 \times \boldsymbol{Z}_6)$.

(3) We define a mapping $\varphi : S(U(1, \boldsymbol{C}^C) \times U(7, \boldsymbol{C}^C)) \to ({E_7}^C)^{{z_3}'}$ by

$$\varphi(b, B)P = \chi^{-1}((b, B)(\chi P)^t(b, B)), \quad P \in \mathfrak{P}^C,$$

as the restriction mapping of $\varphi : SU(8, \boldsymbol{C}^C) \to {E_7}^C$. Then φ is well-defined and is a homomorphism. Ker $\varphi = \{(1, E), (-1, -E)\} \cong \boldsymbol{Z}_2$. Since $({E_7}^C)^{z_3}$ is connected and $\dim_C(({\mathfrak{e}_7}^C)_{ed}) = 37 + 6 \times 2$ (Theorem 4.2.1) $= 49 = (1 + 49) - 1 = \dim_C(\mathfrak{s}(\mathfrak{u}(1, \boldsymbol{C}^C) \oplus \mathfrak{u}(7, \boldsymbol{C}^C)))$, φ is onto. Therefore we have

$$\begin{aligned}({E_7}^C)_{ed} &\cong S(U(1, \boldsymbol{C}^C) \times U(7, \boldsymbol{C}^C))/\boldsymbol{Z}_2 \\ &\cong S(C^* \times GL(7, C))/\boldsymbol{Z}_2.\end{aligned}$$

Since the mapping $h : C^* \times SL(7, C) \to S(C^* \times GL(7, C))$,

$$h(d, D) = (d^7, d^{-1} D)$$

induces an isomorphism $S(C^* \times GL(7, C)) \cong (C^* \times SL(7, C))/\boldsymbol{Z}_7$, $\boldsymbol{Z}_7 = \{({\omega_7}^k, {\omega_7}^k E) \mid k = 0, 1, \ldots, 6\}$. Thus we have $({E_7}^C)_{ev} = ({E_7}^C)^{{z_3}'} \cong (C^* \times SL(7, C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_7)$ $(\boldsymbol{Z}_2 = \{(1, E), (-1, E)\}) \cong (C^*/\boldsymbol{Z}_2 \times SL(7, C))/\boldsymbol{Z}_7$ $(\boldsymbol{Z}_2 = \{1, -1\}) \cong (C^* \times SL(7, C))/\boldsymbol{Z}_7$. □

4.2.1. Subgroups of type $\boldsymbol{A}_{1(1)} \oplus \boldsymbol{D}_{6(6)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{5(5)}$ and $\boldsymbol{R} \oplus \boldsymbol{A}_{6(6)}$ of $\boldsymbol{E}_{7(7)}$

Since $(\mathfrak{e}_{7(7)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1} = ({\mathfrak{e}_7}^C)^{\sigma} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1}, (\mathfrak{e}_{7(7)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1} = ({\mathfrak{e}_7}^C)^{z_4} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1}, (\mathfrak{e}_{7(7)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1} = ({\mathfrak{e}_7}^C)^{{z_3}'} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1}$, we shall determine the structures of groups

$$(E_{7(7)})_{ev} = ({E_7}^C)_{ev} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1} = ({E_7}^C)^{\sigma} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1},$$
$$(E_{7(7)})_0 = ({E_7}^C)_0 \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1} = ({E_7}^C)^{z_4} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1},$$
$$(E_{7(7)})_{ed} = ({E_7}^C)_{ed} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1} = ({E_7}^C)^{{z_3}'} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1}.$$

To define the element $\rho \in {E_7}^C$, we use the mapping $\phi_6 : Sp(1, \boldsymbol{H}^C) \times SU^*(6, \boldsymbol{C}^C) \to {E_6}^C$ by

$$\phi_6(p, A)(M + \boldsymbol{n}) = (hA)M(hA)^* + p\boldsymbol{n}(hA)^{-1},$$
$$M + \boldsymbol{n} \in \mathfrak{J}(3, \boldsymbol{H}^C) \oplus (\boldsymbol{H}^C)^3 = \mathfrak{J}^C,$$

where $k : M(3, \boldsymbol{H}^C) \to \{P \in M(6, \boldsymbol{C}^C) \,|\, JP = \overline{P}J\} \Big(J = \mathrm{diag}(J, J, J), J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Big)$ is defined by $k\Big(a + be_2\Big) = \left(\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}\right)$, $a, b \in \boldsymbol{C}^C$ and $h = k^{-1}$. Furthermore, by using the mapping $f : SL(6, C) \to SU^*(6, \boldsymbol{C}^C), f(A) = \varepsilon A - \overline{\varepsilon} JAJ, \varepsilon = (1 + ie_1)/2$, we can define the mapping $\varphi_6 : Sp(1, \boldsymbol{H}^C) \times SL(6, C) \to {E_6}^C$ by $\varphi_6 = \phi_6 f$.

We define $\rho \in {E_7}^C$ by

$$\rho = \varphi_6(1, \mathrm{diag}(1, -1, 1, -1, 1, 1)).$$

Theorem 4.2.1.1. (1) $(E_{7(7)})_{ev} \cong (SL(2, \boldsymbol{R}) \times spin(6, 6))/\boldsymbol{Z}_2 \times \{1, \rho\}$, $\boldsymbol{Z}_2 = \{(E, 1), (-E, -\sigma)\}$.

(2) $(E_{7(7)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(6, \boldsymbol{R})) \times \{1, \gamma'\}$.

(3) $(E_{7(7)})_{ed} \cong (\boldsymbol{R}^+ \times SL(7, \boldsymbol{R})) \times \{1, \gamma'\}$.

Proof. (1) Since δ_2 satisfies ${\delta_2}^{-1}\sigma\delta_2 = \sigma$ and ${\delta_2}^{-1}(\tau\lambda\iota\gamma_1)\delta_2 = -\tau\gamma_1$, we have

$$(E_{7(7)})_{ev} = ({E_7}^C)^{\sigma} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1} \cong ({E_7}^C)^{\sigma} \cap ({E_7}^C)^{\tau\gamma_1}.$$

So we shall determine the structure of the group $(E_{7(7)})_{ev} \cong ({E_7}^C)^{\sigma} \cap ({E_7}^C)^{\tau\gamma_1}$. Now, for $\alpha \in (E_{7(7)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^{\sigma}$, there exist $A \in SL(2, C)$ and $\beta \in Spin(10, C)$ such that $\alpha = \psi(A, \beta) = \phi(A)\beta$ (Theorem 4.2.2.(1)). From $\tau\gamma_1\alpha\gamma_1\tau = \alpha$, that is, $\tau\gamma_1\phi(A)\beta\gamma_1\tau = \phi(A)\beta$, we have $\phi(\tau A)\tau\gamma_1\beta\gamma_1\tau = \phi(A)\beta$. Hence

$$\begin{cases} \phi(\tau A) = \phi(A) \\ \tau\gamma_1\beta\gamma_1\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \phi(\tau A) = -\phi(A) \\ \tau\gamma_1\beta\gamma_1\tau = -\sigma\beta. \end{cases}$$

In the former case, from $\tau A = A$, we have $A \in SL(2, \boldsymbol{R})$. We shall determine the structure of the group $\{\beta \in Spin(12, C) \,|\, \tau\gamma_1\beta\gamma_1\tau = \beta\} = Spin(12, C)^{\tau\gamma_1} = ((E_7{}^C)^{\kappa,\mu})^{\tau\gamma_1}$. The group $((E_7{}^C)^{\kappa,\mu})^{\tau\gamma_1}$ acts on the $\boldsymbol{R}$-vector space

$$\begin{aligned} V^{6,6} &= (\mathfrak{P}^C)_{\kappa,\tau\gamma_1} = \{P \in \mathfrak{P}^C \,|\, \kappa P = P, \tau\gamma_1 P = P\} \\ &= \left\{ P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \;\middle|\; \begin{array}{l} \xi_2, \xi_3, \eta_1, \eta \in \boldsymbol{R}, \\ x_1 \in (\mathfrak{C}^C)_{\tau\gamma_1} = \mathfrak{C}' \end{array} \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}\{\mu P, P\} = \eta_1\eta - \xi_2\xi_3 + x_1\overline{x}_1.$$

Since the group $Spin(12, C)^{\tau\gamma_1}$ is connected, we can define the mapping $\pi : Spin(12, C)^{\tau\gamma_1} \to O(V^{6,6})^0 = O(6,6)^0$ (which is the connected component subgroup of $O(6,6)$) by $\pi(\alpha) = \alpha | V^{6,6}$. Ker $\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim(\mathfrak{spin}(12, C)^{\tau\gamma_1}) = \dim((\mathfrak{e}_{7(7)})_{ev}) - \dim(\mathfrak{sl}(2, \boldsymbol{R})) = (37 + 16 \times 2) - 3$ (Theorem 4.2.1) $= 66 = \dim(\mathfrak{so}(6,6))$, π is onto. Hence we have $Spin(12, C)^{\tau\gamma_1}/\boldsymbol{Z}_2 = O(6,6)^0$. Therefore $Spin(12, C)^{\tau\gamma_1}$ is $spin(6,6)$ as a covering group of $O(6,6)^0$. Hence the group of the former case is $(SL(2, \boldsymbol{R}) \times spin(6,6))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(E, 1), (-E, -\sigma)\}$. In the latter case, $A = iI$ $(I = \mathrm{diag}(1, -1)), \beta = \phi(-iI)\rho$ satisfy the given condition and $\psi(iI, \phi(-iI)\rho) = \rho$. Thus we have $(E_{7(7)})_{ed} \cong (SL(2, \boldsymbol{R}) \times spin(6,6))/\boldsymbol{Z}_2 \times \{1, \rho\}$.

(2) For $\alpha \in (E_{7(7)})_0 \subset (E_7{}^C)_0$, there exists $(b_1, b_2, B) \in S(U(1, \boldsymbol{C}^C) \times U(1, \boldsymbol{C}^C) \times U(6, \boldsymbol{C}^C))$ such that $\alpha = \varphi(b_1, b_2, B)$ (Theorem 4.2.2.(2)). Since $\varphi : SU(8, \boldsymbol{C}^C) \to E_7{}^C$ satisfies

$$\tau\varphi(A)\tau = \varphi(I_2(\tau A)I_2), \qquad \gamma_1\varphi(A)\gamma_1 = \varphi(JAJ),$$
$$\lambda\varphi(A)\lambda^{-1} = \varphi(I_2 A I_2), \qquad \iota\varphi(A)\iota^{-1} = \varphi(J\overline{A}J),$$

$(I_2 = \mathrm{diag}(-1, -1, 1, \ldots, 1) \in SU(8, \boldsymbol{C}^C))$, we have

$$\tau\lambda\iota\gamma_1\varphi(A)\gamma_1\iota^{-1}\lambda^{-1}\tau = \varphi(\tau\overline{A}), \quad A \in SU(8, \boldsymbol{C}^C).$$

From $\tau\lambda\iota\gamma_1\alpha\gamma_1\iota^{-1}\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\iota\gamma_1\varphi(b_1, b_2, B)\gamma_1\iota^{-1}\lambda^{-1}\tau = \varphi(b_1, b_2, B)$, we have $\varphi(\tau\overline{b}_1, \tau\overline{b}_2, \tau\overline{B}) = \varphi(b_1, b_2, B)$. Hence

$$\begin{cases} \tau\overline{b}_1 = b_1 \\ \tau\overline{b}_2 = b_2 \\ \tau\overline{B} = B \end{cases} \quad \text{or} \quad \begin{cases} \tau\overline{b}_1 = -b_1 \\ \tau\overline{b}_2 = -b_2 \\ \tau\overline{B} = -B. \end{cases}$$

In the former case, $b_1, b_2 \in U(1, \boldsymbol{C}')$ and $B \in U(6, \boldsymbol{C}')$. Hence the group of the first case is

$$\begin{aligned} &S(U(1, \boldsymbol{C}') \times U(1, \boldsymbol{C}') \times U(6, \boldsymbol{C}'))/\boldsymbol{Z}_2, \quad \boldsymbol{Z}_2 = \{(1, 1, E), (-1, -1, -E)\} \\ &\quad \cong S(\boldsymbol{R}^* \times \boldsymbol{R}^* \times GL(6, \boldsymbol{R}))/\boldsymbol{Z}_2. \end{aligned}$$

As a similar way to Theorem 4.2.2.(2), $S(\boldsymbol{R}^* \times \boldsymbol{R}^* \times GL(6,\boldsymbol{R})) \cong (\boldsymbol{R}^* \times \boldsymbol{R}^* \times SL(6,\boldsymbol{R}))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$, $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2 = \{(1,1,E),(-1,1,E),(1,-1,E),(-1,-1,E)\}$. Hence the group of the first case is $(\boldsymbol{R}^* \times \boldsymbol{R}^* \times SL(6,\boldsymbol{R}))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2) \cong \boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(6,\boldsymbol{R})$. In the latter case, $(e_1, -e_1, e_1 I)$ $(I = \mathrm{diag}(1,-1,1,-1,1,-1))$ satisfies the given condition and $\varphi(e_1,-e_1,e_1 I) = \gamma'$. Thus we have $(E_{7(7)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(6,\boldsymbol{R})) \times \{1,\gamma'\}$.

(3) For $\alpha \in (E_{7(7)})_{ed} \subset ({E_7}^C)_{ed}$, there exists $(b,B) \in S(U(1,\boldsymbol{C}^C) \times U(7,\boldsymbol{C}^C))$ such that $\alpha = \varphi(b,B)$ (Theorem 4.2.2.(3)). From $\tau\lambda\iota\gamma_1\alpha\gamma_1\iota^{-1}\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\iota\gamma_1\varphi(b,B)\gamma_1\iota^{-1}\lambda^{-1}\tau = \varphi(b,B)$, we have $\varphi(\tau\overline{b},\tau\overline{B}) = \varphi(b,B)$. Hence

$$\begin{cases} \tau\overline{b} = b \\ \tau\overline{B} = B \end{cases} \quad \text{or} \quad \begin{cases} \tau\overline{b} = -b \\ \tau\overline{B} = -B. \end{cases}$$

In the former case, $b \in U(1,\boldsymbol{C}')$ and $B \in U(6,\boldsymbol{C}')$. Hence the group of the first case is

$$\begin{aligned} &S(U(1,\boldsymbol{C}') \times U(7,\boldsymbol{C}'))\boldsymbol{Z}_2, \quad \boldsymbol{Z}_2 = \{(1,E),(-1,-E)\} \\ &\quad \cong S(\boldsymbol{R}^* \times GL(7,\boldsymbol{R}))/\boldsymbol{Z}_2. \end{aligned}$$

As a similar way to Theorem 4.2.2.(3), $S(\boldsymbol{R}^* \times GL(7,\boldsymbol{R})) \cong (\boldsymbol{R}^* \times SL(7,\boldsymbol{R}))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1,E),(-1,E)\}$. Hence the group of the first case is $(\boldsymbol{R}^* \times SL(7,\boldsymbol{R}))/\boldsymbol{Z}_2 \cong \boldsymbol{R}^+ \times SL(7,\boldsymbol{R})$. In the latter case, $(e_1, e_1 I')$ $(I' = (-1,I))$ satisfies the given condition and $\varphi(e_1,e_1 I') = \gamma'$. Thus we have $(E_{7(7)})_{ev} \cong (\boldsymbol{R}^+ \times SL(7,\boldsymbol{R})) \times \{1,\gamma'\}$. □

4.3. Subgroups of type $\boldsymbol{C} \oplus {\boldsymbol{E}_6}^C, \boldsymbol{C} \oplus \boldsymbol{C} \oplus {\boldsymbol{D}_5}^C$ and $\boldsymbol{C} \oplus {\boldsymbol{D}_6}^C$ of ${\boldsymbol{E}_7}^C$

We add the mappings $\phi_1(\theta)$ and $\phi_2(\nu)$ used in the following sections. For $\theta, \nu \in C^*$, the C-linear transformation $\phi_1(\theta)$ of $\mathfrak{P}^C$ and the C-linear transformation $\phi_2(\nu)$ of $\mathfrak{J}^C$ are defined by

$$\phi_1(\theta)(X,Y,\xi,\eta) = (\theta^{-1}X, \theta Y, \theta^3\xi, \theta^{-3}\eta), \quad (X,Y,\xi,\eta) \in \mathfrak{P}^C,$$

$$\phi_2(\nu)X = \begin{pmatrix} \nu^4\xi_1 & \nu x_3 & \nu\overline{x}_2 \\ \nu\overline{x}_3 & \nu^{-2}\xi_2 & \nu^{-2}x_1 \\ \nu x_2 & \nu^{-2}\overline{x}_1 & \nu^{-2}\xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C,$$

respectively. Then $\phi_1(\theta) \in {E_7}^C$ and $\phi_2(\nu) \in {E_6}^C \subset {E_7}^C$.

In the Lie algebra ${\mathfrak{e}_7}^C$, let

$$Z = \varPhi\Big(4(E_1 \vee E_1), 0, 0, -\frac{5}{2}\Big).$$

Theorem 4.3.1. *The 3-graded decomposition of* $\mathfrak{e}_{7(7)} = ({\mathfrak{e}_7}^C)^{\tau\gamma}$ *(or* ${\mathfrak{e}_7}^C$*),*

$$\mathfrak{e}_{7(7)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\operatorname{ad} Z, Z = \Phi\Big(4(E_1 \vee E_1), 0, 0, -\dfrac{5}{2}\Big)$*, is given by*

$$\mathfrak{g}_0 = \left\{\begin{array}{l} iG_{kl},\ 0 \le k < 4 \le l \le 7,\ G_{kl},\ otherwise, \\ \tilde{A}_1(e_k),\ \tilde{F}_1(e_k),\ 0 \le k \le 3,\ \ i\tilde{A}_1(e_k),\ i\tilde{F}_1(e_k),\ 4 \le k \le 7, \\ (E_2 - E_3)^{\sim},\ E_1 \vee E_1,\ \mathbf{1} \end{array}\right\} \ 47$$

$$\mathfrak{g}_{-1} = \Big\{\check{F}_2(e_k),\ \check{F}_3(e_k),\ 0 \le k \le 3, i\check{F}_2(e_k),\ i\check{F}_3(e_k),\ 4 \le k \le 7,\ \hat{E}_1\Big\} \ 17$$

$$\mathfrak{g}_{-2} = \left\{\begin{array}{l} \tilde{A}_2(e_k) + \tilde{F}_2(e_k), \quad \tilde{A}_3(e_k) - \tilde{F}_3(e_k), \quad 0 \le k \le 3, \\ i\tilde{A}_2(e_k) + i\tilde{F}_2(e_k),\ i\tilde{A}_3(e_k) - i\tilde{F}_3(e_k),\ 4 \le k \le 7 \end{array}\right\} \ 16$$

$$\mathfrak{g}_{-3} = \{\check{F}_1(e_k),\ 0 \le k \le 3,\ i\check{F}_1(e_k),\ 4 \le k \le 3,\ \check{E}_2, \check{E}_3\} \ 10$$

$$\mathfrak{g}_1 = \lambda(\mathfrak{g}_{-1})\lambda^{-1}, \quad \mathfrak{g}_2 = \lambda(\mathfrak{g}_{-2})\lambda^{-1}, \quad \mathfrak{g}_3 = \lambda(\mathfrak{g}_{-3})\lambda^{-1}.$$

Since $\Phi(4(E_1 \vee E_1), 0, 0, 2) = -2\kappa$, for $t \in \boldsymbol{R}$ we have

$$\exp\big(\Phi(4it(E_1 \vee E_1), 0, 0, 2it)\big)(X, Y, \xi, \eta)$$
$$= \left(\begin{pmatrix} e^{2it}\xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & e^{-2it}\xi_2 & e^{-2it}x_1 \\ x_2 & e^{-2it}\overline{x}_1 & e^{-2it}\xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2it}\eta_1 & y_3 & \overline{y}_2 \\ \overline{y}_3 & e^{2it}\eta_2 & e^{2it}y_1 \\ y_2 & e^{2it}\overline{y}_1 & e^{2it}\eta_3 \end{pmatrix}, e^{2it}\xi, e^{-2it}\eta\right).$$

Especially, we have

$$\exp\big(\Phi(4\pi i(E_1 \vee E_1), 0, 0, 2\pi i)\big) = 1, \quad \exp\big(\Phi\big(2\pi i(E_1 \vee E_1), 0, 0, \pi i)\big) = -\sigma,$$
$$\exp\left(\Phi\Big(\frac{8\pi i}{3}(E_1 \vee E_1), 0, 0, \frac{4\pi i}{3}\Big)\right) = \kappa_3,$$

where κ_3 is the C-linear transformation of $\mathfrak{P}^C$ defined by

$$\kappa_3(X, Y, \xi, \eta) = \left(\begin{pmatrix} \omega^2\xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \omega\xi_2 & \omega x_1 \\ x_2 & \omega\overline{x}_1 & \omega\xi_3 \end{pmatrix}, \begin{pmatrix} \omega\eta_1 & y_3 & \overline{y}_2 \\ \overline{y}_3 & \omega^2\eta_2 & \omega^2 y_1 \\ y_2 & \omega^2\overline{y}_1 & \omega^2\eta_3 \end{pmatrix}, \omega^2\xi, \omega\eta\right),$$

where $\omega = e^{2\pi i/3}$. This κ_3 is nothing but $\phi\left(\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}\right)$ using $\phi : SL(2, C) \to {E_7}^C$. For $\Phi(0, 0, 0, it)$, we have

$$\exp(\Phi(0, 0, 0, it))(X, Y, \xi, \eta) = (e^{-it/3}X, e^{it/3}Y, e^{it}\xi, e^{-it}\eta).$$

Hence $\exp(\Phi(0, 0, 0, it)) = \phi_1(e^{it/3})$. Since $iZ = \Phi\Big(4i(E_1 \vee E_1), 0, 0, 2i\Big) + \Phi\Big(0, 0, 0, -\dfrac{9}{2}i\Big)$, furthermore $\Phi\Big(4i(E_1 \vee E_1), 0, 0, 2i\Big)$ and $\Phi\Big(0, 0, 0, -\dfrac{9}{2}i\Big)$ commute, we have

$$
\begin{aligned}
z_2 &= \exp\frac{2\pi i}{2}Z = \exp\big(\varPhi\big(4\pi i(E_1\vee E_1),0,0,2\pi i)\big)\exp\Big(\varPhi\Big(0,0,0,-\frac{9}{2}\pi i\Big)\Big)\\
&= \iota,\\
z_4 &= \exp\frac{2\pi i}{4}Z = \exp\big(\varPhi\big(2\pi i(E_1\vee E_1),0,0,\pi i)\big)\exp\Big(\varPhi\Big(0,0,0,-\frac{9}{4}\pi i\Big)\Big)\\
&= -\sigma\iota_8, \quad \iota_8 = \phi_1(e^{-3\pi i/4}),\\
z_3 &= \exp\frac{2\pi i}{3}Z = \exp\Big(\varPhi\Big(\frac{8\pi i}{3}(E_1\vee E_1),0,0,\frac{4\pi i}{3}\Big)\Big)\exp\varPhi\Big(0,0,0,-3\pi i\Big)\\
&= -\kappa_3.
\end{aligned}
$$

Since $(\mathfrak{e}_7{}^C)_{ev} = (\mathfrak{e}_7{}^C)^{z_2} = (\mathfrak{e}_7{}^C)^{\iota}, (\mathfrak{e}_7{}^C)_0 = (\mathfrak{e}_7{}^C)^{z_4} = (\mathfrak{e}_7{}^C)^{\sigma\iota_8}, (\mathfrak{e}_7{}^C)_{ed} = (\mathfrak{e}_7{}^C)^{z_3} = (\mathfrak{e}_7{}^C)^{\kappa_3}$, we shall determine the structures of groups

$$
\begin{gathered}
(E_7{}^C)_{ev} = (E_7{}^C)^{z_2} = (E_7{}^C)^{\iota}, \quad (E_7{}^C)_0 = (E_7{}^C)^{z_4} = (E_7{}^C)^{\sigma\iota_8},\\
(E_7{}^C)_{ed} = (E_7{}^C)^{z_3} = (E_7{}^C)^{\kappa_3}.
\end{gathered}
$$

Theorem 4.3.2. (1) $(E_7{}^C)_{ev} \cong (C^*\times E_6{}^C)/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(1,1),(\omega,\omega 1),(\omega^2,\omega^2 1)\}$, $\omega = e^{2\pi i/3}$.

(2) $(E_7{}^C)_0 \cong (C^*\times C^*\times Spin(10,C))/\boldsymbol{Z}_{12}$, $\boldsymbol{Z}_{12} = \{(\omega_{12}{}^{-4k},\omega_{12}{}^k,\phi_1(\omega_{12}{}^{4k})\phi_2(\omega_{12}{}^{-k}))\,|\,k=0,1,\ldots,11\}$, $\omega_{12} = e^{2\pi i/12}$.

(3) $(E_7{}^C)_{ed} \cong (C^*\times Spin(12,C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1,1),(-1,-\sigma)\}$.

Proof. (1) We define a mapping $\varphi_3 : C^*\times E_6{}^C \to (E_7{}^C)^{\iota}$ by

$$\varphi_3(\theta,\beta) = \phi_1(\theta)\beta.$$

Then φ_3 is well-defined and is a homomorphism. Ker $\varphi_3 = \{(1,1),(\omega,\phi_1(\omega^2)),(\omega^2,\phi_1(\omega))\} = \boldsymbol{Z}_3$. ($\phi_1(\omega^2)$ and $\phi_1(\omega)$ are nothing but the central elements $\omega 1$ and $\omega^2 1$ of $E_6{}^C$, respectively. So we may write Ker $\varphi_3 = \{(1,1),(\omega,\omega 1),(\omega^2,\omega^2 1)\}$). Since $(E_7{}^C)^{\iota}$ is connected and $\dim_C((\mathfrak{e}_7{}^C)_{ev}) = 47+16\times 2$ (Theorem 4.3.1) $= 79 = 1+78 = \dim_C(C\oplus\mathfrak{e}_6{}^C)$, φ_3 is onto. Thus we have $(E_7{}^C)_{ev} = (E_7{}^C)^{\iota} \cong (C^*\times E_6{}^C)/\boldsymbol{Z}_3$ (cf. [3, Theorem 4.4.4]).

(2) Let $Spin(10,C) = (E_6{}^C)_{E_1} = (E_7{}^C)_{(E_1,0,1,0),(-E_1,0,1,0)}$. We define a mapping $\varphi_4 : C^*\times C^*\times Spin(10,C) \to (E_7{}^C)^{\sigma\iota_8}$ by

$$\varphi_4(\theta,\nu,\beta) = \phi_1(\theta)\phi_2(\nu)\beta.$$

Then φ_4 is well-defined, that is, $\varphi_4(\theta,\nu,\beta)$ commutes with $\sigma\iota_8$. Furthermore, since $\phi_1(\theta)$, $\phi_2(\nu)$ and β commute with each other, φ_4 is a homomorphism. The kernel of φ_4 is

$$\text{Ker }\varphi_4 = \{(\omega_{12}{}^{-4k},\omega_{12}{}^k,\phi_1(\omega_{12}{}^{4k})\phi_2(\omega_{12}{}^{-k}))\,|\,k=0,1,\ldots,11\} = \boldsymbol{Z}_{12}.$$

Indeed, let $(\theta,\nu,\beta)\in$ Ker φ_4. Then $\varphi_4(\theta,\nu,\beta)P = P$ for any $P\in\mathfrak{P}^C$. Especially, for $P = (E_1,0,1,0)\in\mathfrak{P}^C$, we have$(\theta^{-1}\nu^4 E_1,0,\theta^3,0) = (E_1,0,1,0)$.

Hence $\theta^{-1}\nu^4 = 1, \theta^3 = 1$, that is, $\nu^4 = \theta, \theta^3 = 1$, so we have $\nu^{12} = 1$. Thus $\mathrm{Ker}\,\varphi_4 = \boldsymbol{Z}_{12}$ is obtained. Since $({E_7}^C)^{\sigma\iota_8}$ is connected and $\dim_C(({\mathfrak{e}_7}^C)_0) = 47$ (Theorem 4.3.1) $= 1 + 1 + 45 = \dim_C(C \oplus C \oplus \mathfrak{spin}(10, C))$, φ_4 is onto. Thus we have $({E_7}^C)_0 = ({E_7}^C)^{\sigma\iota_8} \cong (C^* \times C^* \times Spin(10, C))/\boldsymbol{Z}_{12}$.

(3) Let C^* be the subgroup $\left\{ a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in C^* \right\}$ of $SL(2, C)$, and we define a mapping $\psi : C^* \times Spin(12, C) \to ({E_7}^C)^{\kappa_3}$ by

$$\psi(a, \beta) = \phi(a)\beta$$

as the restriction mapping of $\psi : SL(2, C) \times Spin(12, C) \to ({E_7}^C)^{\sigma}$ defined in Theorem 4.2.2.(1). Then ψ is well-defined and a is homomorphism. $\mathrm{Ker}\,\psi = \{(1, 1), (-1, -\sigma)\} = \boldsymbol{Z}_2$. Since $({E_7}^C)^{\kappa_3}$ is connected and $\dim_C(({\mathfrak{e}_7}^C)_{ed}) = 47 + 10 \times 2$ (Theorem 4.3.1) $= 67 = 1 + 66 = \dim_C(C \oplus \mathfrak{spin}(12, C))$, ψ is onto. Thus we have $({E_7}^C)_{ed} = ({E_7}^C)^{\kappa_3} \cong (C^* \times Spin(12, C))/\boldsymbol{Z}_2$. (cf. [5, Theorem 4.22.(2)]). □

4.3.1. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{E}_{6(6)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{5(5)}$ and $\boldsymbol{R} \oplus \boldsymbol{D}_{6(6)}$ of $\boldsymbol{E}_{7(7)}$

We use the same noatation as that in **4.3**. Since $(\mathfrak{e}_{7(7)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma} = ({\mathfrak{e}_7}^C)^{\iota} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma}$, $(\mathfrak{e}_{7(7)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^{\tau\gamma} = ({\mathfrak{e}_7}^C)^{\sigma\iota_8} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma}$, $(\mathfrak{e}_{7(7)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma} = ({\mathfrak{e}_7}^C)^{\kappa_3} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma}$, we shall determine the structures of groups

$$\begin{aligned} (E_{7(7)})_{ev} &= ({E_7}^C)_{ev} \cap ({E_7}^C)^{\tau\gamma} = ({E_7}^C)^{\iota} \cap ({E_7}^C)^{\tau\gamma}, \\ (E_{7(7)})_0 &= ({E_7}^C)_0 \cap ({E_7}^C)^{\tau\gamma} = ({E_7}^C)^{\sigma\iota_8} \cap ({E_7}^C)^{\tau\gamma}, \\ (E_{7(7)})_{ed} &= ({E_7}^C)_{ed} \cap ({E_7}^C)^{\tau\gamma} = ({E_7}^C)^{\kappa_3} \cap ({E_7}^C)^{\tau\gamma}. \end{aligned}$$

Theorem 4.3.1.1. (1) $(E_{7(7)})_{ev} \cong (\boldsymbol{R}^+ \times E_{6(6)}) \times \{1, -1\}$.
(2) $(E_{7(7)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(5, 5)) \times \{1, -1\}$.
(3) $(E_{7(7)})_{ed} \cong (\boldsymbol{R}^+ \times spin(6, 6)) \times \{1, \rho\}$.

Proof. (1) For $\alpha \in (E_{7(7)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^{\iota}$, there exist $\theta \in C^*$ and $\beta \in {E_6}^C$ such that $\alpha = \varphi_3(\theta, \beta) = \phi_1(\theta)\beta$ (Theorem 4.3.2.(1)). From $\tau\gamma\alpha\gamma\tau = \alpha$, that is, $\tau\gamma\phi_1(\theta)\beta\gamma\tau = \phi_1(\theta)\beta$, we have $\phi_1(\tau\theta)\tau\gamma\beta\gamma\tau = \phi_1(\theta)\beta$. Hence

$$\left\{ \begin{array}{l} \phi_1(\tau\theta) = \phi_1(\theta) \\ \tau\gamma\beta\gamma\tau = \beta, \end{array} \right. \quad \left\{ \begin{array}{l} \phi_1(\tau\theta) = \phi_1(\omega)\phi_1(\theta) \\ \tau\gamma\beta\gamma\tau = \phi_1(\omega^2)\beta \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \phi_1(\tau\theta) = \phi_1(\omega^2)\phi_1(\theta) \\ \tau\gamma\beta\gamma\tau = \phi(\omega)\beta. \end{array} \right.$$

In the first case, $\tau\theta = \theta$, that is, $\theta \in \boldsymbol{R}^*$ and $\beta \in ({E_6}^C)^{\tau\gamma_1} = E_{6(6)}$. Hence the group of the first case is $\boldsymbol{R}^* \times E_{6(6)}$. The second and the third cases are impossible, because there exists no $\theta \in C^*$ satisfying $\theta = \omega^k\theta$ $(k = 1, 2)$.

Thus we have $(E_{7(7)})_{ev} \cong \boldsymbol{R}^* \times E_{6(6)} = (\boldsymbol{R}^+ \times E_{6(6)}) \times \{1, -1\}$ (note that $\varphi_3(-1, 1) = -1$).

(2) For $\alpha \in (E_{7(7)})_0 \subset ({E_7}^C)_0 = ({E_7}^C)^{\sigma\iota_8}$, there exist $\theta, \nu \in C^*$ and $\beta \in Spin(10, C)$ such that $\alpha = \varphi_4(\theta, \nu, \beta) = \phi_1(\theta)\phi_2(\nu)\beta$ (Theorem 4.3.2.(2)). From $\tau\gamma\alpha\gamma\tau = \alpha$, that is, $\tau\gamma\phi_1(\theta)\phi_2(\nu)\beta\gamma\tau = \phi_1(\theta)\phi_2(\nu)\beta$, we have $\phi_1(\tau\theta)$ $\phi_2(\tau\nu)\tau\gamma\beta\gamma\tau = \phi_1(\theta)\phi_2(\nu)\beta$. Hence

$$\begin{cases} \phi_1(\tau\theta) = \phi_1(\theta) \\ \phi_2(\tau\nu) = \phi_2(\nu) \\ \tau\gamma\beta\gamma\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \phi_1(\tau\theta) = \phi_1(\omega^{-4k})\phi_1(\theta) \\ \phi_2(\tau\nu) = \phi_2(\omega^k)\phi_2(\nu) \\ \tau\gamma\beta\gamma\tau = \phi_1(\omega^{4k})\phi_2(\omega^{-k})\beta, \quad k = 1, \ldots, 11. \end{cases}$$

In the former case, from $\tau\theta = \theta, \tau\nu = \nu$, we have $\theta, \nu \in \boldsymbol{R}^*$. We shall determine the structure of the group $\{\beta \in Spin(10, C) \,|\, \tau\gamma\beta\gamma\tau = \beta\} = Spin(10, C)^{\tau\gamma} = (({E_6}^C)_{E_1})^{\tau\gamma}$. The group $(({E_6}^C)_{E_1})^{\tau\gamma}$ acts on the $\boldsymbol{R}$-vector space

$$\begin{aligned} V^{5,5} &= \{X \in \mathfrak{J}^C \,\big|\, 4E_1 \times (E_1 \times X) = X, \tau\gamma X = X\} \\ &= \left\{ X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_2, \xi_3 \in \boldsymbol{R}, x_1 \in (\mathfrak{C})_{\tau\gamma} = \mathfrak{C}' \right\} \end{aligned}$$

with the norm

$$(E_1, X, X) = x_1\overline{x}_1 - \xi_2\xi_3.$$

Since the group $Spin(10, C)^{\tau\gamma}$ is connected, we can define a homomorphism $\pi : Spin(10, C)^{\tau\gamma} \to O(V^{5,5})^0 = O(5, 5)^0$ (which is the connected component subgroup of $O(5, 5)$) by $\pi(\alpha) = \alpha | V^{5,5}$. Ker $\pi = \{1, \sigma\}$. Since $\dim(((({\mathfrak{e}_6}^C)_{E_1})^{\tau\gamma})$ $= \dim((\mathfrak{e}_{7(7)})_0) - \dim \boldsymbol{R} - \dim \boldsymbol{R} = 47 - 1 - 1$ (Theorem 4.3.1) $= 45 = \dim(\mathfrak{o}(5, 5))$, π is onto. Hence we have $Spin(10, C)^{\tau\gamma}/\boldsymbol{Z}_2 \cong O(5, 5)^0$. Therefore $Spin(10, C)^{\tau\gamma}$ is $spin(5, 5)$ as a double covering group of $O(5, 5)^0$. Hence the group of the former case is $(\boldsymbol{R}^* \times \boldsymbol{R}^* \times spin(5, 5))/\boldsymbol{Z}_2$ ($\boldsymbol{Z}_2 = \{(1, 1, 1), (1, -1, \sigma)\}$) $\cong \boldsymbol{R}^* \times \boldsymbol{R}^+ \times spin(5, 5)$. The other cases are impossible, because there exists no $\theta \in C^*$ satisfying $\tau\theta = \omega^{-4k}\theta$ $(k = 1, \ldots, 11)$. Thus we have $(E_{7(7)})_0 \cong \boldsymbol{R}^* \times \boldsymbol{R}^+ \times spin(5, 5) = (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(5, 5)) \times \{1, -1\}$ (note that $\varphi_4(-1, 1, 1) = -1$).

(3) γ_1 and γ are conjugate under $\delta_1 \in {G_2}^C \subset {F_4}^C \subset {E_6}^C \subset {E_7}^C$: ${\delta_1}^{-1}\gamma_1\delta_1 = \gamma$ and δ_1 satisfies $\delta_1\kappa_3 = \kappa_3\delta_1, \delta_1\tau = \tau\delta_1$. Hence we have $({E_7}^C)^{\kappa_3} \cap ({E_7}^C)^{\tau\gamma} \cong ({E_7}^C)^{\kappa_3} \cap ({E_7}^C)^{\tau\gamma_1}$, so we shall determine the structure of the group $(E_{7(7)})_{ev} = ({E_7}^C)^{\kappa_3} \cap ({E_7}^C)^{\tau\gamma_1}$. Now, for $\alpha \in (E_{7(7)})_{ed} \subset ({E_7}^C)_{ed} = ({E_7}^C)^{\kappa_3}$, there exist $a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in C^* \subset SL(2, C)$ and $\beta \in Spin(12, C)$ such that $\alpha = \psi(a, \beta) = \phi(a)\beta$ (Theorem 4.3.2.(3)). From $\tau\gamma_1\alpha\gamma_1\tau = \alpha$, that

is, $\tau\gamma_1\phi(a)\beta\gamma_1\tau = \phi(a)\beta$, we have $\phi(\tau a)\tau\gamma_1\beta\gamma_1\tau = \phi(a)\beta$. Hence

$$\begin{cases} \phi(\tau a) = \phi(a) \\ \tau\gamma_1\beta\gamma_1\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \phi(\tau a) = -\phi(a) \\ \tau\gamma_1\beta\gamma_1\tau = -\sigma\beta. \end{cases}$$

In the former case, $\tau a = a$, that is, $a \in \boldsymbol{R}^*$ and the group $Spin(12, C)^{\tau\gamma_1}$ is $spin(6,6)$ (Theorem 4.3.2.(1)). Hence the group of the former case is $(\boldsymbol{R}^* \times spin(6,6))/\boldsymbol{Z}_2$ ($\boldsymbol{Z}_2 = \{(1,1), (-1,-\sigma)\}$) $\cong \boldsymbol{R}^+ \times spin(6,6)$. In the latter case, $a = iI$ and $\beta = \phi(-iI)\rho$ satisfy the given condition and $\psi(iI, \phi(-iI)\rho) = \rho$. Thus we have $(E_{7(7)})_{ed} \cong (\boldsymbol{R}^+ \times spin(6,6)) \times \{1, \rho\}$. □

4.3.2. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{E}_{6(-26)}$, $\boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{5(-45)}$ and $\boldsymbol{R} \oplus \boldsymbol{D}_{5(-26)}$ of $\boldsymbol{E}_{7(-25)}$

Theorem 4.3.2.1. *The 3-graded decomposition of* $\mathfrak{e}_{7(-25)} = ({\mathfrak{e}_7}^C)^\tau$,

$$\mathfrak{e}_{7(-25)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\operatorname{ad} Z, Z = \varPhi\Big(4(E_1 \vee E_1), 0, 0, -\dfrac{5}{2}\Big)$, *is given by*

$$\begin{aligned}
\mathfrak{g}_0 &= \left\{ \begin{array}{l} G_{kl},\ 0 \le k < l \le 7,\ G_{kl},\ \tilde{A}_1(e_k),\ \tilde{F}_1(e_k);\ 0 \le k \le 7, \\ (E_2 - E_3)^\sim,\ E_1 \vee E_1,\ \mathbf{1} \end{array} \right\} \ 47 \\
\mathfrak{g}_{-1} &= \Big\{ \check{F}_2(e_k),\ \check{F}_3(e_k),\ 0 \le k \le 7,\ \hat{E}_1 \Big\} \ 17 \\
\mathfrak{g}_{-2} &= \{ \tilde{A}_2(e_k) + \tilde{F}_2(e_k),\ \tilde{A}_3(e_k) - \tilde{F}_3(e_k),\ 0 \le k \le 7 \} \ 16 \\
\mathfrak{g}_{-3} &= \{ \check{F}_1(e_k),\ 0 \le k \le 7,\ \check{E}_2,\ \check{E}_3 \} \ 10 \\
\mathfrak{g}_1 &= \lambda(\mathfrak{g}_{-1})\lambda^{-1}, \quad \mathfrak{g}_2 = \lambda(\mathfrak{g}_{-2})\lambda^{-1}, \quad \mathfrak{g}_3 = \lambda(\mathfrak{g}_{-3})\lambda^{-1}.
\end{aligned}$$

We use the same notation as that in **4.3**. Since $(\mathfrak{e}_{7(-25)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^\tau = ({\mathfrak{e}_7}^C)^\iota \cap ({\mathfrak{e}_7}^C)^\tau, (\mathfrak{e}_{7(-25)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^\tau = ({\mathfrak{e}_7}^C)^{\sigma\iota_8} \cap ({\mathfrak{e}_7}^C)^\tau$, $(\mathfrak{e}_{7(-25)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^\tau = ({\mathfrak{e}_7}^C)^{\kappa_3} \cap ({\mathfrak{e}_7}^C)^\tau$, we shall determine the structures of groups

$$\begin{aligned}
(E_{7(-25)})_{ev} &= ({E_7}^C)_{ev} \cap ({E_7}^C)^\tau = ({E_7}^C)^\iota \cap ({E_7}^C)^\tau, \\
(E_{7(-25)})_0 &= ({E_7}^C)_0 \cap ({E_7}^C)^\tau = ({E_7}^C)^{\sigma\iota_8} \cap ({E_7}^C)^\tau, \\
(E_{7(-25)})_{ed} &= ({E_7}^C)_{ed} \cap ({E_7}^C)^\tau = ({E_7}^C)^{\kappa_3} \cap ({E_7}^C)^\tau.
\end{aligned}$$

Theorem 4.3.2.2. (1) $(E_{7(-25)})_{ev} \cong (\boldsymbol{R}^+ \times E_{6(-26)}) \times \{1, -1\}$.
(2) $(E_{7(-25)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(1,9)) \times \{1, -1\}$.
(3) $(E_{7(-25)})_{ed} \cong \boldsymbol{R}^+ \times spin(2,10)$.

Proof. (1) For $\alpha \in (E_{7(-25)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^\iota$, there exist $\theta \in C^*$ and $\beta \in {E_6}^C$ such that $\alpha = \varphi_3(\theta, \beta) = \phi_1(\theta)\beta$ (Theorem 4.3.2.(1)). From

$\tau\alpha\tau = \alpha$, that is, $\tau\phi_1(\theta)\beta\tau = \phi_1(\theta)\beta$, we have $\phi_1(\tau\theta)\tau\beta\tau = \phi_1(\theta)\beta$. Hence

$$\begin{cases} \phi_1(\tau\theta) = \phi_1(\theta) \\ \tau\beta\tau = \beta, \end{cases} \quad \begin{cases} \phi_1(\tau\theta) = \phi_1(\omega)\phi_1(\theta) \\ \tau\beta\tau = \phi_1(\omega^2)\beta \end{cases} \quad \text{or} \quad \begin{cases} \phi_1(\tau\theta) = \phi_1(\omega^2)\phi_1(\theta) \\ \tau\beta\tau = \phi_1(\omega)\beta. \end{cases}$$

In the first case, $\tau\theta = \theta$, that is, $\theta \in \boldsymbol{R}^*$ and $\beta \in ({E_6}^C)^\tau = E_{6(-26)}$. Therefore the group of the first case is $\boldsymbol{R}^* \times E_{6(-26)}$. The second and the third cases are impossible, because there exists no $\theta \in C$ satisfying $\tau\theta = \omega^k$ $(k = 1, 2)$. Thus we have $(E_{7(-25)})_{ev} \cong \boldsymbol{R}^* \times E_{6(-26)} = (\boldsymbol{R}^+ \times E_{6(-26)}) \times \{1, -1\}$.

(2) For $\alpha \in (E_{7(-25)})_0 \subset ({E_7}^C)_0 = ({E_7}^C)^{\sigma\iota_8}$, there exist $\theta, \nu \in C^*$ and $\beta \in Spin(10, C)$ such that $\alpha = \varphi_4(\theta, \nu, \beta) = \phi_1(\theta)\phi_2(\nu)\beta$ (Theorem 4.3.2.(2)). From $\tau\alpha\tau = \alpha$, that is, $\tau\phi_1(\theta)\phi_2(\nu)\beta\tau = \phi_1(\theta)\phi_2(\nu)\beta$, we have $\phi_1(\tau\theta)\phi_2(\tau\nu)$ $\tau\beta\tau = \phi_1(\theta)\phi_2(\nu)\beta$. Hence

$$\begin{cases} \phi_1(\tau\theta) = \phi_1(\theta) \\ \phi_2(\tau\nu) = \phi_2(\nu) \\ \tau\beta\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \phi_1(\tau\theta) = \phi_1(\omega^{-4k})\phi_1(\theta) \\ \phi_2(\tau\nu) = \phi_2(\omega^k)\phi_2(\nu) \\ \tau\beta\tau = \phi_1(\omega^{4k})\phi_2(\omega^{-k})\beta, \qquad k = 1, \cdots, 11. \end{cases}$$

In the former case, we have $\tau\theta = \theta, \tau\nu = \nu$, that is, $\theta, \nu \in \boldsymbol{R}^*$. We shall determine the structure of the group $\{\beta \in Spin(10, C) \,|\, \tau\beta\tau = \beta\} = Spin(10, C)^\tau = (({E_6}^C)_{E_1})^\tau$. The group $(({E_6}^C)_{E_1})^\tau$ acts on

$$\begin{aligned} V^{1,9} &= \{X \in \mathfrak{J}^C \,\big|\, 4E_1 \times (E_1 \times X) = X, \tau X = X\} \\ &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, \xi_2, \xi_3 \in \boldsymbol{R}, x_1 \in \mathfrak{C} \right\} \end{aligned}$$

with the norm

$$(E_1, X, X) = x_1\overline{x}_1 - \xi_2\xi_3.$$

Since the group $Spin(10, C)^\tau$ is connected, we can define a homomorphism $\pi : Spin(10, C)^\tau \to SO(V^{1,9}) = SO(1, 9)$ by $\pi(\alpha) = \alpha|V^{1,9}$. $\mathrm{Ker}\,\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim(({\mathfrak{e}_6}^C)_{E_1})^\tau) = \dim((\mathfrak{e}_{7(-25)})_0) - \dim\boldsymbol{R} - \dim\boldsymbol{R} = 47 - 1 - 1$ (Theorem 4.3.1) $= 45 = \dim(\mathfrak{o}(1, 9))$, π is onto. Hence $Spin(10, C)^\tau/\boldsymbol{Z}_2 \cong SO(1, 9)$, so $Spin(10, C)^\tau$ is $Spin(1, 9)$ as a double covering group of $SO(1, 9)$. Therefore the group of the former case is $(\boldsymbol{R}^* \times \boldsymbol{R}^* \times Spin(1, 9))/\boldsymbol{Z}_2$ $(\boldsymbol{Z}_2 = \{(1, 1, 1), (1, -1, \sigma)\}) \cong \boldsymbol{R}^* \times \boldsymbol{R}^+ \times Spin(1, 9)$. The other cases are impossible, because there exists no $\theta \in C$ satisfying $\tau\theta = \omega^{-4k}\theta$ $(k = 1, \ldots, 11)$. Thus we have $(E_{7(-25)})_0 \cong \boldsymbol{R}^* \times \boldsymbol{R}^+ \times Spin(1, 9) = (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(1, 9)) \times \{1, -1\}$.

(3) For $\alpha \in (E_{7(-25)})_{ed} \subset ({E_7}^C)_{ed} = ({E_7}^C)^{\kappa_3}$, there exist $a \in C^*$ and $\beta \in Spin(12, C)$ such that $\alpha = \psi(a, \beta) = \phi(a)\beta$ (Theorem 4.3.2.(3)). From $\tau\alpha\tau = \alpha$, that is, $\tau\phi(a)\beta\tau = \alpha$, we have $\phi(\tau a)\tau\beta\tau = \phi(a)\beta$. Hence

$$\begin{cases} \phi(\tau\theta) = \phi(\theta) \\ \tau\beta\tau = \beta \end{cases} \quad \text{or} \quad \begin{cases} \phi(\tau\theta) = -\phi(\theta) \\ \tau\beta\tau = -\sigma\beta \end{cases}$$

In the former case, we have $\tau\theta = \theta$, hence $\theta \in \boldsymbol{R}^*$. We shall determine the structure of the group $\{\beta \in Spin(12, C) \,|\, \tau\beta\tau = \beta\} = Spin(12, C)^\tau = (({E_7}^C)^{\kappa,\mu})^\tau$. The group $(({E_7}^C)^{\kappa,\mu})^\tau$ acts on the $\boldsymbol{R}$-vector space

$$\begin{aligned} V^{2,10} &= (\mathfrak{P}^C)_{\kappa,\tau} = \{P \in \mathfrak{P}^C \,|\, \kappa P = P, \tau P = P\} \\ &= \left\{ P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \middle| \begin{array}{l} \xi_2, \xi_3, \eta_1, \eta \in \boldsymbol{R}, \\ x_1 \in \mathfrak{C} \end{array} \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}\{\mu P, P\} = \eta_1\eta - \xi_2\xi_3 + x_1\overline{x}_1.$$

Since the group $Spin(12, C)^\tau$ is connected, we can define a homomorphism $\pi : Spin(12, C)^\tau \to O(V^{2,10})^0 = O(2, 10)^0$ (which is the connected component subgroup of $O(2, 10)$) by $\pi(\alpha) = \alpha | V^{2,10}$. $\operatorname{Ker} \pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim(({\mathfrak{e}_7}^C)^{\kappa,\mu}) = \dim((\mathfrak{e}_{7(-25)})_{ed}) - \dim(\mathfrak{sl}(2, \boldsymbol{R})) = (47 + 10 \times 2) - 3$ (Theorem 4.3.1) $= 54 = \dim(\mathfrak{o}(2, 10))$, π is onto. Hence $Spin(12, C)^\tau / \boldsymbol{Z}_2 \cong O(2, 10)^0$, so $Spin(12, C)^\tau$ is $spin(2, 10)$ as a double covering group of $O(2, 10)^0$. Therefore the group of the former case is $(\boldsymbol{R}^* \times spin(2, 10))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1, 1), (-1, -\sigma)\}$. The mapping $h : \boldsymbol{R}^* \times spin(2, 10) \to \boldsymbol{R}^+ \times spin(2, 10)$,

$$h(\theta, \beta) = \begin{cases} (\theta, \beta) & \text{for } \theta > 0 \\ (-\theta, -\sigma\beta) & \text{for } \theta < 0 \end{cases}$$

induces an isomorphism $(\boldsymbol{R}^* \times spin(2, 10))/\boldsymbol{Z}_2 \cong \boldsymbol{R}^+ \times spin(2, 10)$. The latter case is impossible. Indeed, since $\beta \in Spin(12, C)^\tau$ acts on $V^{2,10}$, β induces a matrix $B \in M(12, C)$ such that $\tau B = -B, {}^tBI_2B = I_2$. Put $B = iB', B' \in M(12, \boldsymbol{R})$, then ${}^tB'I_2B' = -I_2$, which is false, because the signatures of both sides are different. Thus we have $(E_{7(-25)})_{ed} \cong \boldsymbol{R}^+ \times spin(2, 10)$. □

4.4. Subgroups of type $\boldsymbol{C} \oplus {\boldsymbol{E}_6}^C, \boldsymbol{C} \oplus \boldsymbol{C} \oplus {\boldsymbol{D}_5}^C$ and ${\boldsymbol{A}_1}^C \oplus \boldsymbol{C} \oplus {\boldsymbol{D}_6}^C$ of ${\boldsymbol{E}_7}^C$

In the Lie algebra ${\mathfrak{e}_7}^C$, let

$$Z = \varPhi\Big(-2iG_{01}, 0, 0, -\frac{3}{2}\Big).$$

Theorem 4.4.1. *The 3-graded decomposition of* $\mathfrak{e}_{7(7)} = ({\mathfrak{e}_7}^C)^{\tau\gamma_1}$ *(or* ${\mathfrak{e}_7}^C$*),*

$$\mathfrak{e}_{7(7)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\operatorname{ad} Z, Z = \varPhi\Big(-2iG_{01}, 0, 0, -\dfrac{3}{2}\Big)$*, is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01},\ iG_{23},\ G_{24},\ iG_{25},\ G_{26},\ iG_{27},\ iG_{34},\ G_{35},\ iG_{36},\ G_{37},\ iG_{45}, \\ G_{46},\ iG_{47},\ iG_{56},\ G_{57},\ iG_{67},\ (E_1 - E_2)^\sim,\ (E_2 - E_3)^\sim, \mathbf{1}, \\ \tilde{A}_1(e_2),\ i\tilde{A}_1(e_3),\ \tilde{A}_1(e_4),\ i\tilde{A}_1(e_5),\ \tilde{A}_1(e_6),\ i\tilde{A}_1(e_7), \\ \tilde{F}_1(e_2),\ i\tilde{F}_1(e_3),\ \tilde{F}_1(e_4),\ i\tilde{F}_1(e_5),\ \tilde{F}_1(e_6),\ i\tilde{F}_1(e_7), \\ \check{F}_2(1 - ie_1),\ \check{F}_2(e_2 - ie_3),\ \check{F}_2(e_4 - ie_5),\ \check{F}_2(e_6 - ie_7), \\ \check{F}_3(1 - ie_1),\ \check{F}_3(e_2 + ie_3),\ \check{F}_3(e_4 + ie_5),\ \check{F}_3(e_6 + ie_7), \\ \hat{F}_2(1 + ie_1),\ \hat{F}_2(e_2 + ie_3),\ \hat{F}_2(e_4 + ie_5),\ \hat{F}_2(e_6 + ie_7), \\ \hat{F}_3(1 + ie_1),\ \hat{F}_3(e_2 - ie_3),\ \hat{F}_3(e_4 - ie_5),\ \hat{F}_3(e_6 - ie_7) \end{array} \right\} 47$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} \tilde{A}_2(1 + ie_1),\ \tilde{A}_2(e_2 + ie_3),\ \tilde{A}_2(e_4 + ie_5),\ \tilde{A}_2(e_6 + ie_7), \\ \tilde{A}_3(1 + ie_1),\ \tilde{A}_3(e_2 - ie_3),\ \tilde{A}_3(e_4 - ie_5),\ \tilde{A}_3(e_6 - ie_7), \\ \tilde{F}_2(1 + ie_1),\ \tilde{F}_2(e_2 + ie_3),\ \tilde{F}_2(e_4 + ie_5),\ \tilde{F}_2(e_6 + ie_7), \\ \tilde{F}_3(1 + ie_1),\ \tilde{F}_3(e_2 - ie_3),\ \tilde{F}_3(e_4 - ie_5),\ \tilde{F}_3(e_6 - ie_7), \\ \check{F}_1(e_2),\ i\check{F}_1(e_3),\ \check{F}_1(e_4),\ i\check{F}_1(e_5),\ \check{F}_1(e_6),\ i\check{F}_1(e_7), \\ \hat{F}_1(1 + ie_1),\ \ \check{E}_k,\ k = 1, 2, 3 \end{array} \right\} 26$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} G_{02} - iG_{12},\ iG_{03} + G_{13},\ G_{04} - iG_{14},\ iG_{05} + G_{15}, \\ G_{06} - iG_{16},\ iG_{07} + G_{17},\ \tilde{A}_1(1 - ie_1),\ \tilde{F}_1(e_0 - ie_1), \\ \check{F}_2(1 + ie_1),\ \check{F}_2(e_2 + ie_3),\ \check{F}_2(e_4 + ie_5),\ \check{F}_2(e_6 + ie_7), \\ \check{F}_3(1 + ie_1),\ \check{F}_2(e_2 - ie_3),\ \check{F}_2(e_4 - ie_5),\ \check{F}_2(e_6 - ie_7) \end{array} \right\} 16$$

$$\mathfrak{g}_{-3} = \left\{ \hat{F}_1(1 - ie_1) \right\} 1$$

$$\mathfrak{g}_1 = \tau\lambda(\mathfrak{g}_{-1})\lambda^{-1}\tau, \quad \mathfrak{g}_2 = \tau\lambda(\mathfrak{g}_{-2})\lambda^{-1}\tau, \quad \mathfrak{g}_3 = \tau\lambda(\mathfrak{g}_{-3})\lambda^{-1}\tau.$$

For $a \in U(1, \boldsymbol{C}^C)$, we define the C-linear transformation $D(a)$ of $\mathfrak{J}^C$ by

$$D(a)X = \begin{pmatrix} \xi_1 & x_3 a & \overline{a x_2} \\ \overline{x_3 a} & \xi_2 & \overline{a} x_1 \overline{a} \\ a x_2 & a \overline{x}_1 a & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

Then $D(a) \in {F_4}^C \subset {E_6}^C \subset {E_7}^C$.

Since $iZ = \Phi\Big(2G_{01}, 0, 0, -\dfrac{3}{2}i\Big) = \Phi(2G_{01}, 0, 0, 0) + \Phi\Big(0, 0, 0, -\dfrac{3}{2}i\Big)$, furthermore $\Phi\Big(2G_{01}, 0, 0, 0\Big)$ and $\Phi\Big(0, 0, 0, -\dfrac{3}{2}i\Big)$ commute, we have

$$z_2 = \exp\frac{2\pi i}{2}Z = \exp(\Phi(2\pi G_{01}, 0, 0, 0))\exp\left(\Phi\Big(0, 0, 0, -\frac{3}{2}\pi i\Big)\right) = -\sigma\iota,$$

$$z_4 = \exp\frac{2\pi i}{4}Z = \exp(\Phi(\pi G_{01}, 0, 0, 0))\exp\left(\Phi\Big(0, 0, 0, -\frac{3}{4}\pi i\Big)\right) = \sigma_4\iota_4,$$

$$z_3 = \exp\frac{2\pi i}{3}Z = \exp\left(\Phi\Big(\frac{4\pi}{3}G_{01}, 0, 0, 0\Big)\right)\exp(\Phi(0, 0, 0, -\pi i)) = \sigma_3\iota_3,$$

where $\sigma_4 = D(e_1), \iota_4 = \phi_1(e^{-2\pi i/8}), \sigma_3 = D(e^{2\pi e_1/3}), \iota_3 = \phi_1(e^{-2\pi i/6})$.

$z_2 = -\sigma\iota$ is conjugate to

$$z_2{}' = \iota$$

in ${E_7}^C$. Indeed, let $\delta_4 = \phi(J)$, then we have

$$\delta_4{}^{-1}(-\sigma\iota)\delta_4 = \iota.$$

Next, we shall show that $z_4 = \sigma_4\iota_4$ is conjugate to

$$z_4{}' = -\sigma\iota_4{}^{-1}, \quad \iota_4{}^{-1} = \phi_1(e^{2\pi i/8})$$

in ${E_7}^C$. Indeed, δ_4 satisfies $\delta_4{}^{-1}\sigma_4\delta_4 = \sigma_4$ and $\delta_4{}^{-1}\iota_4\delta_4 = -\sqrt{\sigma}^{\,-1}\iota_4{}^{-1}$, where $\sqrt{\sigma} \in {E_6}^C$ is defined by

$$\sqrt{\sigma}X = \begin{pmatrix} \xi_1 & ix_3 & i\overline{x}_2 \\ i\overline{x}_3 & -\xi_2 & -x_1 \\ ix_2 & -\overline{x}_1 & -\xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

Hence we have

$$\delta_4{}^{-1}\sigma_4\iota_4\delta_4 = -\sqrt{\sigma}^{\,-1}\sigma_4\iota_4{}^{-1},$$

that is, $\sigma_4\iota_4$ is conjugate to $-\sqrt{\sigma}^{\,-1}\sigma_4\iota_4{}^{-1}$. Next, we shall show that $\sqrt{\sigma}\sigma_4$ is conjugate to σ in ${E_6}^C \subset {E_7}^C$. For this end, for the induced differential mapping $\varphi_{6*} : \mathfrak{sp}(1, \boldsymbol{H}^C) \times \mathfrak{sl}(6, C) \to {\mathfrak{e}_6}^C$ of φ_6, we have $G_{01} = \varphi_{6*}(0, \mathrm{diag}(0, 0, i/2, -i/2, -i/2, i/2))$ ([6]). Hence we have

$$\sqrt{\sigma}^{\,-1} = \varphi_6(1, \mathrm{diag}(-1, -1, i, i, i, i)), \quad \sigma_4 = \varphi_6(1, \mathrm{diag}(1, 1, i, -i, -i, i)).$$

So we have

$$\sqrt{\sigma}\sigma_4 = \varphi_6(1, \mathrm{diag}(-1, -1, -1, 1, 1, -1)),$$

which is conjugate to

$$\varphi_6(1, \mathrm{diag}(1, 1, -1, -1, -1, -1)) = \sigma.$$

Furthermore, this conjugation is given under $\varphi_6(1, SL(6, C)) \subset {E_6}^C \subset {E_7}^C$. Hence we see that $\sigma_4\iota_4$ is conjugate to $-\sigma\iota_4{}^{-1}$.

Finally, we shall show that $z_3 = \sigma_3\iota_3$ is conjugate to

$$z_3{}' = -\sigma_3$$

in ${E_7}^C$. Indeed, denote $\omega_6 = e^{2\pi i/6}$, then $\omega_1 = \omega_6{}^2$. First note that

$$\begin{aligned} \sigma_3\iota_3(X, Y, \xi, \eta) &= (\omega_6\sigma_3X, \omega_6{}^{-1}\sigma_3Y, -\xi, -\eta) = -(-\omega_6\sigma_3X, -\omega_6{}^{-1}\sigma_3Y, \xi, \eta) \\ &= -(\omega^2\sigma_3X, \omega\sigma_3Y, \xi, \eta) \\ &= -\omega^2 1(\sigma_3X, \sigma_3Y, \xi, \eta) \quad (\text{note that } \omega^2 1 \in {E_6}^C). \end{aligned}$$

Now, we use $G_{01} = \varphi_{6*}(0, \mathrm{diag}(0, 0, i/2, -i/2, -i/2, i/2))$ again, then we have

$$\begin{aligned} \sigma_3 &= \varphi_6(1, \mathrm{diag}(1, 1, e^{2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{2\pi i/3})) \\ &= \varphi_6(1, \mathrm{diag}(1, 1, \omega, \omega^2, \omega^2, \omega)). \end{aligned}$$

Since the central element $\omega^2 E$ of $SL(6,C)$ is transfered to the central element $\omega^2 1$ of ${E_6}^C$ by φ_6, we have

$$(\omega^2 1)\sigma_3 = \varphi_6(1, \mathrm{diag}(\omega^2, \omega^2, 1, \omega, \omega, 1)),$$

which is conjugate to

$$\varphi_6(1, \mathrm{diag}(1, 1, \omega, \omega^2, \omega^2, \omega)) = \sigma_3$$

under $\varphi_6(1, SL(6,C)) \subset {E_6}^C \subset {E_7}^C$. Hence we see that $\sigma_3 \iota_3$ is conjugate to $-\sigma_3$.

Hereafter, we use ${z_2}', {z_4}'$ and ${z_3}'$ instead of z_2, z_4 and z_3, respectively.

Since $({\mathfrak{e}_7}^C)_{ev} = ({\mathfrak{e}_7}^C)^{{z_2}'} = ({\mathfrak{e}_7}^C)^{\iota}, ({\mathfrak{e}_7}^C)_0 = ({\mathfrak{e}_7}^C)^{{z_4}'} = ({\mathfrak{e}_7}^C)^{\sigma {\iota_4}^{-1}}, ({\mathfrak{e}_7}^C)_{ed} = ({\mathfrak{e}_7}^C)^{{z_3}'} = ({\mathfrak{e}_7}^C)^{\sigma_3}$, we shall determine the structures of groups

$$({E_7}^C)_{ev} = ({E_7}^C)^{{z_2}'} = ({E_7}^C)^{\iota}, \quad ({E_7}^C)_0 = ({E_7}^C)^{{z_4}'} = ({E_7}^C)^{\sigma {\iota_4}^{-1}},$$
$$({E_7}^C)_{ed} = ({E_7}^C)^{{z_3}'} = ({E_7}^C)^{\sigma_3}.$$

Theorem 4.4.2. (1) $({E_7}^C)_{ev} \cong (C^* \times {E_6}^C)/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(1,1), (\omega, \omega 1), (\omega^2, \omega^2 1)\}$

(2) $({E_7}^C)_0 \cong (C^* \times C^* \times Spin(10,C))/\boldsymbol{Z}_{12}$, $\boldsymbol{Z}_{12} = \{({\omega_{12}}^{4k}, {\omega_{12}}^k, \phi_1({\omega_{12}}^{4k})\phi_2({\omega_{12}}^k)) \,|\, k = 0, 1, \ldots, 11\}$, $\omega_{12} = e^{2\pi i/12}$.

(3) $({E_7}^C)_{ed} \cong (SL(2,C) \times C^* \times Spin(10,C))/\boldsymbol{Z}_4, \boldsymbol{Z}_4 = \{(E,1,1), (E,-1,\sigma), (-E,-i,-D(e_1)), (-E,i,-\sigma D(e_1))\}$

Proof. (1) $({E_7}^C)_{ev} = ({E_7}^C)^{\iota} \cong (C^* \times {E_6}^C)/\boldsymbol{Z}_3$ is already shown in Theorem 4.3.2.(1).

(2) Let $Spin(10,C) = ({E_6}^C)_{E_1} = ({E_7}^C)_{(E_1,0,1,0),(-E_1,0,1,0)}$. We define a mapping $\varphi_4 : C^* \times C^* \times Spin(10,C) \to ({E_7}^C)^{\sigma {\iota_4}^{-1}} = ({E_7}^C)_0$ by

$$\varphi_4(\theta, \nu, \beta) = \phi_1(\theta)\phi_2(\nu)\beta,$$

Although ${\iota_4}^{-1}$ is different from ι_8, by the same proof of Theorem 4.3.2.(2), we have $({E_7}^C)_0 \cong (C^* \times C^* \times Spin(10,C))/\boldsymbol{Z}_2$

(3) Let $Spin(10,C) = (({E_7}^C)^{\kappa,\mu})_{(F_1(1),0,0,0),(F_1(e_1),0,0,0)}$ (cf. [5, Proposition 4.7.(2)]). We define a mapping $\varphi_5 : SL(2,C) \times U(1, \boldsymbol{C}^C) \times Spin(10,C) \to ({E_7}^C)^{\sigma_3}$ by

$$\varphi_5(A, a, \beta) = \phi(A)D(a)\beta,$$

φ_5 is well-defined because $\sigma_3 = \varphi_5(E, w_1, 1), w_1 = e^{2\pi e_1/3}$. Since $D(a)$ commutes with $\phi(A)$ and β, φ_5 is a homomorphism. Ker $\varphi_5 = \{(E,1,1), (E,-1,$

$\sigma), (-E, e_1, -D(e_1)), (-E, -e_1, -\sigma D(e_1))\} = \boldsymbol{Z}_4$. Since $({E_7}^C)^{\sigma_3}$ is connected and $\dim_C(\mathfrak{sl}(2,C) \oplus \mathfrak{u}(1,\boldsymbol{C}^C) \oplus \mathfrak{spin}(10,C)) = 3 + 1 + 45 = 47 + 1 + 1 = \dim_C(({\mathfrak{e}_7}^C)_{ev})$ (Theorem 4.4.1), φ_5 is onto. Thus we have $({E_7}^C)_{ev} = ({E_7}^C)^{\sigma_3} \cong (SL(2,C) \times U(1,\boldsymbol{C}^C) \times Spin(10,C))/\boldsymbol{Z}_4 \cong (SL(2,C) \times C^* \times Spin(10,C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(E,1,1), (E,-1,\sigma), (-E,-i,-D(e_1)), (-E,i,-\sigma D(e_1))\}$ (note that by the isomorphism $f : U(1,\boldsymbol{C}^C) \to C^*, f(a) = (a+a^{-1})/2 + ((a-a^{-1})/2)ie_1$, e_1 is transformed to $-i$). □

4.4.1. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{E}_{6(6)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{5(5)}$ and $\boldsymbol{A}_1 \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{5(5)}$ of $\boldsymbol{E}_{7(7)}$

We use the same notation as that in **4.4**. Since $(\mathfrak{e}_{7(7)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1} = ({\mathfrak{e}_7}^C)^{\iota} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1}, (\mathfrak{e}_{7(7)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1} = ({\mathfrak{e}_7}^C)^{\sigma{\iota_4}^{-1}} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1}$, $(\mathfrak{e}_{7(7)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1} = ({\mathfrak{e}_7}^C)^{\sigma_3} \cap ({\mathfrak{e}_7}^C)^{\tau\gamma_1}$, we shall determine the structures of groups

$$\begin{aligned}(E_{7(7)})_{ev} &= ({E_7}^C)_{ev} \cap ({E_7}^C)^{\tau\gamma_1} = ({E_7}^C)^{\iota} \cap ({E_7}^C)^{\tau\gamma_1},\\ (E_{7(7)})_0 &= ({E_7}^C)_0 \cap ({E_7}^C)^{\tau\gamma_1} = ({E_7}^C)^{\sigma{\iota_4}^{-1}} \cap ({E_7}^C)^{\tau\gamma_1},\\ (E_{7(7)})_{ed} &= ({E_7}^C)_{ed} \cap ({E_7}^C)^{\tau\gamma_1} = ({E_7}^C)^{\sigma_3} \cap ({E_7}^C)^{\tau\gamma_1}.\end{aligned}$$

$\sigma' \in {F_4}^C \subset {E_6}^C \subset {E_7}^C$ is defined by

$$\sigma' X = \begin{pmatrix} \xi_1 & x_3 & -\overline{x}_2 \\ \overline{x}_3 & \xi_2 & -x_1 \\ -x_2 & -\overline{x}_1 & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

Theorem 4.4.1.1. (1) $(E_{7(7)})_{ev} \cong (\boldsymbol{R}^+ \times E_{6(6)}) \times \{1,-1\}$.
(2) $(E_{7(7)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(5,5)) \times \{1,-1\}$.
(3) $(E_{7(7)})_{ed} \cong (SL(2,\boldsymbol{R}) \times \boldsymbol{R}^+ \times spin(5,5)) \times \{1, \sigma', \rho, \sigma'\rho\}$.

Proof. (1) γ_1 and γ are conjugate under $\delta_1 \in {G_2}^C \subset {F_4}^C \subset {E_6}^C \subset {E_7}^C$: ${\delta_1}^{-1}\gamma_1\delta_1 = \gamma$ and δ_1 satisfies $\delta_1\iota = \iota\delta_1, \delta_1\tau = \tau\delta_1$. Hence we have $({E_7}^C)^{\iota} \cap ({E_7}^C)^{\tau\gamma_1} \cong ({E_7}^C)^{\iota} \cap ({E_7}^C)^{\tau\gamma}$, so we shall determine the structure of the group $(E_{7(7)})_{ev} = ({E_7}^C)^{\iota} \cap ({E_7}^C)^{\tau\gamma}$. Now, for $\alpha \in (E_{7(7)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^{\iota}$, there exist $\theta \in C^*$ and $\beta \in {E_6}^C$ such that $\alpha = \varphi_3(\theta,\beta) = \phi_1(\theta)\beta$ (Theorem 4.4.2.(1)). From $\tau\gamma\alpha\gamma\tau = \alpha$, that is, $\tau\gamma\phi_1(\theta)\beta\gamma\tau = \phi_1(\theta)\beta$, we have $\phi_1(\tau\theta)\tau\gamma\beta\gamma\tau = \phi_1(\theta)\beta$. Hence

$$\begin{cases} \phi_1(\tau\theta) = \phi_1(\theta) \\ \tau\gamma\beta\gamma\tau = \beta, \end{cases} \begin{cases} \phi_1(\tau\theta) = \phi_1(\omega)\phi_1(\theta) \\ \tau\gamma\beta\gamma\tau = \phi_1(\omega^2)\beta \end{cases} \text{ or } \begin{cases} \phi_1(\tau\theta) = \phi_1(\omega^2)\phi_1(\theta) \\ \tau\gamma\beta\gamma\tau = \phi_1(\omega)\beta. \end{cases}$$

In the first case $\tau\theta = \theta$, that is, $\theta \in \boldsymbol{R}^*$ and $\beta \in ({E_6}^C)^{\tau\gamma} = E_{6(6)}$. Hence the group of the first case is $\boldsymbol{R}^* \times E_{6(6)}$. The second and the third cases are impossible, because there exists no $\theta \in C^*$ satisfying $\tau\theta = \omega^k\theta$ $(k = 1, 2)$. Hence we have $(E_{7(7)})_{ev} \cong \boldsymbol{R}^* \times E_{6(6)} = (\boldsymbol{R}^+ \times E_{6(6)}) \times \{1, -1\}$.

(2) Although ${\iota_4}^{-1}$ is different from ι_8, by the same way as Theorem 4.3.1.1.(2), we have $(E_{7(7)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(5,5)) \times \{1, -1\}$.

(3) For $\alpha \in (E_{7(7)})_{ed} \subset ({E_7}^C)_{ed} = ({E_7}^C)^{\sigma_3}$, there exist $A \in SL(2, C), a \in U(1, \boldsymbol{C}^C)$ and $\beta \in Spin(10, C)$ such that $\alpha = \varphi_5(A, a, \beta) = \phi(A)D(a)\beta$ (Theorem 4.4.2.(3)). From $\tau\gamma_1\alpha\gamma_1\tau = \alpha$, that is, $\tau\gamma_1\phi(A)D(a)\beta\gamma_1\tau = \phi(A)D(a)\beta$, we have $\phi(\tau A)D(\tau\overline{a})\tau\gamma_1\beta\gamma_1\tau = \phi(A)D(a)\beta$. Hence

$$\text{(i)} \begin{cases} \phi(\tau A) = \phi(A) \\ D(\tau\overline{a}) = D(a) \\ \tau\gamma_1\beta\gamma_1\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \phi(\tau A) = \phi(A) \\ D(\tau\overline{a}) = D(-a) \\ \tau\gamma_1\beta\gamma_1\tau = \sigma\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \phi(\tau A) = \phi(-A) \\ D(\tau\overline{a}) = D(e_1 a) \\ \tau\gamma_1\beta\gamma_1\tau = -D(e_1)\beta \end{cases} \quad \text{or} \quad \text{(iv)} \begin{cases} \phi(\tau A) = \phi(-A) \\ D(\tau\overline{a}) = D(-e_1 a) \\ \tau\gamma_1\beta\gamma_1\tau = -\sigma D(e_1)\beta. \end{cases}$$

(i) From $\tau A = A$, $\tau\overline{a} = a$, we have $A \in SL(2, \boldsymbol{R}), a \in U(1, \boldsymbol{C}') \cong \boldsymbol{R}^*$, respectively. The group $\{\beta \in Spin(10, C) \,|\, \tau\gamma_1\beta\gamma_1\tau = \beta\} = Spin(10, C)^{\tau\gamma_1} = ((({E_7}^C)^{\kappa,\mu})_{(F_1(1),0,0,0),(F_1(e_1),0,0,0)})^{\tau\gamma_1}$ acts on the $\boldsymbol{R}$-vector space

$$\begin{aligned} V^{5,5} &= ((\mathfrak{P}^C)_{\kappa,\tau\gamma_1})_{(F_1(1),0,0,0),(F_1(e_1),0,0,0)} \\ &= \left\{ P \in \mathfrak{P}^C \,\middle|\, \begin{array}{l} \kappa P = P, \tau\gamma_1 P = P, \\ \{\mu(F_1(1),0,0,0), P\} = \{\mu(F_1(e_1),0,0,0), P\} = 0 \end{array} \right\} \\ &= \left\{ P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \,\middle|\, \begin{array}{l} \xi_2, \xi_3 \in C, \eta_1, \eta \in \boldsymbol{R}, \\ x_1 \in \mathfrak{C}', \\ (1, x_1) = (e_1, x_1) = 0 \end{array} \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}\{\mu P, P\} = \eta_1\eta - \xi_2\xi_3 + x_1\overline{x}_1.$$

Hence the group $Spin(10, C)^{\tau\gamma_1}$ is $spin(5,5)$ as in a similar way to (1). Therefore the group of (i) is $(SL(2, \boldsymbol{R}) \times \boldsymbol{R}^* \times spin(5,5))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(E, 1, 1), (E, -1, \sigma)\}$. The mapping $h : SL(2, \boldsymbol{R}) \times \boldsymbol{R}^* \times spin(5,5) \to SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times spin(5,5)$,

$$h(A, \theta, \beta) = \begin{cases} (A, \theta, \beta) & \text{for } \theta > 0 \\ (A, -\theta, \sigma\beta) & \text{for } \theta < 0 \end{cases}$$

induces an isomorphim $(SL(2, \boldsymbol{R}) \times \boldsymbol{R}^* \times spin(5,5))/\boldsymbol{Z}_2 \cong SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times spin(5,5)$.

(ii) $\varphi_4(E, e_1, \sigma' D(-e_1)) = \sigma'$.

(iii) $\varphi_4\Big(iI, \dfrac{1 - e_1}{\sqrt{2}}, \phi(-iI)D\Big(\dfrac{1 + e_1}{\sqrt{2}}\Big)\rho\Big) = \rho$.

(iv) $\varphi_4\Big(iI, \dfrac{1+e_1}{\sqrt{2}}, \phi(-iI)D\Big(\dfrac{1-e_1}{\sqrt{2}}\Big)\sigma'\rho\Big) = \sigma'\rho$.

Thus we have $(E_{7(7)})_{ed} \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times spin(5,5)) \times \{1, \sigma', \rho, \sigma'\rho\}$. □

4.4.2. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{E}_{6(-26)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{5(-27)}$ and $\boldsymbol{A}_1 \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{5(-27)}$ of $\boldsymbol{E}_{7(-25)}$

We define $\delta_5 \in {E_6}^C$ by $\delta_5 = \exp\big(\dfrac{\pi i}{2}\widetilde{F}_1(1)\big)$ and define a complex-conjugate linear transformation τ_1 of $\mathfrak{J}^C$ by

$$\tau_1 X = {\delta_5}^{-1}\tau\delta_5 X = \begin{pmatrix} \tau\xi_1 & -i\tau\overline{x}_2 & -i\tau x_3 \\ -i\tau x_2 & -\tau\xi_3 & -\tau\overline{x}_1 \\ -i\tau\overline{x}_3 & -\tau x_1 & -\tau\xi_2 \end{pmatrix}, \quad X \in \mathfrak{J}^C$$

([4, 3.4.4]). This τ_1 is naturally extented to the complex-conjugate linear transformation τ_1 of $\mathfrak{P}^C$ by

$$\tau_1(X, Y, \xi, \eta) = (\tau_1 X, \tau_1\sigma Y, \tau\xi, \tau\eta), \quad (X, Y, \xi, \eta) \in \mathfrak{P}^C.$$

In the Lie algebra ${\mathfrak{e}_7}^C$, we have

$$\tau_1 \varPhi(\phi, A, B, \nu)\tau_1 = \varPhi(\tau_1\phi\tau_1, \tau_1 A, \tau_1\sigma B, \tau\nu).$$

Since τ and τ_1 are related with $\tau_1 = {\delta_5}^{-1}\tau\delta_5$, we have

$$E_{6(-26)} = ({E_6}^C)^\tau \cong ({E_6}^C)^{\tau_1}, \quad E_{7(-25)} = ({E_7}^C)^\tau \cong ({E_7}^C)^{\tau_1}.$$

Lemma 4.4.2.1. *In the Lie algebra ${\mathfrak{e}_7}^C$, we have*

(1) $\tau_1 G_{0l}\tau_1 = -G_{0l}, \quad \tau_1 G_{kl}\tau_1 = G_{kl}$.

(2) $\begin{cases} \tau_1\widetilde{A}_1(a)\tau_1 = -\widetilde{A}_1(\tau\overline{a}), \tau_1\widetilde{A}_2(a)\tau_1 = -i\widetilde{F}_3(\tau\overline{a}), \tau_1\widetilde{A}_3(a)\tau_1 = \widetilde{F}_3(\tau\overline{a}), \\ \tau_1\widetilde{F}_1(a)\tau_1 = \widetilde{F}_1(\tau\overline{a}),\ \tau_1\widetilde{A}_2(a)\tau_1 = i\widetilde{A}_3(\tau\overline{a}),\ \tau_1\widetilde{A}_3(a)\tau_1 = -i\widetilde{A}_3(\tau\overline{a}). \end{cases}$

(3) $\begin{cases} \tau_1(\xi_1E_1 + \xi_2E_2 + \xi_3E_3)^\sim\tau_1 = ((\tau\xi_1)E_1 + (\tau\xi_2)E_2 + (\tau\xi)E_3)^\sim, \\ \xi_1 + \xi_2 + \xi_3 = 0. \end{cases}$

(4) $\begin{cases} \tau_1\check{E}_1\tau_1 = \check{E}_1, \quad \tau_1\check{E}_2\tau_1 = -\check{E}_3, \quad \tau_1\check{E}_3\tau_1 = -\check{E}_2, \\ \tau_1\hat{E}_1\tau_1 = \hat{E}_1, \quad \tau_1\hat{E}_2\tau_1 = -\hat{E}_3, \quad \tau_1\hat{E}_3\tau_1 = -\hat{E}_2. \end{cases}$

(5) $\begin{cases} \tau_1\check{F}_1(a)\tau_1 = -\check{F}_1(\tau\overline{a}), \tau_1\check{F}_2(a)\tau_1 = -i\check{F}_3(\tau\overline{a}), \tau_1\check{F}_3(a)\tau_1 = -i\check{F}_3(\tau\overline{a}), \\ \tau_1\hat{F}_1(a)\tau_1 = -\hat{F}_1(\tau\overline{a}),\ \tau_1\hat{F}_2(a)\tau_1 = i\hat{F}_3(\tau\overline{a}),\ \tau_1\hat{F}_3(a)\tau_1 = i\hat{F}_3(\tau\overline{a}). \end{cases}$

(6) $\tau_1\mathbf{1}\tau_1 = \mathbf{1}$.

Theorem 4.4.2.2. *The 3-graded decomposition of $\mathfrak{e}_{7(-25)} = ({\mathfrak{e}_7}^C)^{\tau_1}$,*

$$\mathfrak{e}_{7(-25)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}\, Z, Z = \varPhi\Big(-2iG_{01}, 0, 0, -\dfrac{3}{2}\Big)$, *is given by*

$$\mathfrak{g}_0 = \left\{\begin{array}{l} iG_{01},\ G_{kl},\ 2 \le k \le 7,\ i(E_1 - E_2)^{\sim},\ \ i(E_2 - E_3)^{\sim}, \mathbf{1}, \\ \tilde{A}_1(e_k),\ i\tilde{F}_1(e_k),\ 2 \le k \le 7, \\ \check{F}_2(1 - ie_1) - i\check{F}_3(1 - ie_1),\ \check{F}_2(e_k - ie_{k+1}) + i\check{F}_3(e_k + ie_{k+1}), \\ i\check{F}_2(1 - ie_1) - \check{F}_3(1 - ie_1),\ i\check{F}_2(e_k - ie_{k+1}) + \check{F}_3(e_k + ie_{k+1}), \\ \hat{F}_2(1 - ie_1) - i\hat{F}_3(1 - ie_1),\ \hat{F}_2(e_k - ie_{k+1}) + i\hat{F}_3(e_k + ie_{k+1}), \\ i\hat{F}_2(1 - ie_1) - \hat{F}_3(1 - ie_1),\ i\hat{F}_2(e_k - ie_{k+1}) + \hat{F}_3(e_k + ie_{k+1}), \\ \hfill k = 2, 4, 6 \end{array}\right\} 47$$

$$\mathfrak{g}_{-1} = \left\{\begin{array}{l} \check{F}_1(e_k),\ 2 \le k \le 7,\ i\check{F}_1(1 + ie_k), \check{E}_1,\ \check{E}_2 - \check{E}_3,\ i(\check{E}_2 + \check{E}_3), \\ \tilde{A}_2(1 + ie_1) - i\tilde{F}_3(1 + ie_1),\ \tilde{A}_2(e_k + ie_{k+1}) + i\tilde{F}_3(e_k - ie_{k+1}), \\ i\tilde{A}_2(1 + ie_1) - \tilde{F}_3(1 + ie_1),\ i\tilde{A}_2(e_k + ie_{k+1}) + \tilde{F}_3(e_k - ie_{k+1}), \\ \tilde{A}_3(1 + ie_1) + i\tilde{F}_2(1 + ie_1),\ \tilde{A}_3(e_k + ie_{k+1}) - i\tilde{F}_2(e_k - ie_{k+1}), \\ i\tilde{A}_3(1 + ie_1) + \tilde{F}_2(1 + ie_1),\ i\tilde{A}_3(e_k + ie_{k+1}) - \tilde{F}_2(e_k - ie_{k+1}), \\ \hfill k = 2, 4, 6 \end{array}\right\} 26$$

$$\mathfrak{g}_{-2} = \left\{\begin{array}{l} iG_{0k} + G_{1k},\ 2 \le k \le 7, i\tilde{A}_1(1 - ie_1),\ \tilde{F}_1(1 - ie_1), \\ \check{F}_2(1 + ie_1) - i\check{F}_3(1 + ie_1),\ \check{F}_2(e_k + ie_{k+1}) + i\check{F}_3(e_k - ie_{k+1}), \\ i\check{F}_2(1 + ie_1) - \check{F}_3(1 + ie_1),\ i\check{F}_2(e_k + ie_{k+1}) + \check{F}_3(e_k - ie_{k+1}), \\ \hfill k = 2, 4, 6 \end{array}\right\} 16$$

$$\mathfrak{g}_{-3} = \big\{\check{F}_1(e_0 - ie_1)\big\}\ 1$$

$$\mathfrak{g}_1 = \tau\lambda(\mathfrak{g}_{-1})\lambda^{-1}\tau, \quad \mathfrak{g}_2 = \tau\lambda(\mathfrak{g}_{-2})\lambda^{-1}\tau, \quad \mathfrak{g}_3 = \tau\lambda(\mathfrak{g}_{-3})\lambda^{-1}\tau.$$

We use the same notation as that in **4.4**. Since $(\mathfrak{e}_{7(-25)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^{\tau_1} = ({\mathfrak{e}_7}^C)^{\iota} \cap ({\mathfrak{e}_7}^C)^{\tau_1}, (\mathfrak{e}_{7(-25)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^{\tau_1} = ({\mathfrak{e}_7}^C)^{\sigma{\iota_4}^{-1}} \cap ({\mathfrak{e}_7}^C)^{\tau_1}$, $(\mathfrak{e}_{7(-25)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^{\tau_1} = ({\mathfrak{e}_7}^C)^{\sigma_3} \cap ({\mathfrak{e}_7}^C)^{\tau_1}$, we shall determine the structures of groups

$$(E_{7(-25)})_{ev} = ({E_7}^C)_{ev} \cap ({E_7}^C)^{\tau_1} = ({E_7}^C)^{\iota} \cap ({E_7}^C)^{\tau_1},$$
$$(E_{7(-25)})_0 = ({E_7}^C)_0 \cap ({E_7}^C)^{\tau_1} = ({E_7}^C)^{\sigma{\iota_4}^{-1}} \cap ({E_7}^C)^{\tau_1},$$
$$(E_{7(-25)})_{ed} = ({E_7}^C)_{ed} \cap ({E_7}^C)^{\tau_1} = ({E_7}^C)^{\sigma_3} \cap ({E_7}^C)^{\tau_1}.$$

Theorem 4.4.2.3. (1) $(E_{7(-25)})_{ev} \cong (\boldsymbol{R}^+ \times E_{6(-26)}) \times \{1, -1\}$.
(2) $(E_{7(-25)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(1, 9)) \times \{1, -1\}$.
(3) $(E_{7(-25)})_{ed} \cong (SL(2, \boldsymbol{R}) \times \boldsymbol{R}^+ \times Spin(1, 9)) \times \{1, \sigma'\}$.

Proof. (1) For $\alpha \in (E_{7(-25)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^{\iota}$, there exist $\theta \in C^*$ and $\beta \in {E_6}^C$ such that $\alpha = \varphi_3(\theta, \beta) = \phi_1(\theta)\beta$ (Theorem 4.4.2.(1)). The condition $\tau_1\alpha\tau_1 = \alpha$ is $\tau_1\phi_1(\theta)\beta\tau_1 = \phi_1(\theta)\beta$. $\phi_1(\theta)$ satisfies $\tau_1\phi_1(\theta)\tau_1 = \phi_1(\tau\theta)$, so we have $\phi_1(\tau\theta)\tau_1\beta\tau_1 = \phi_1(\theta)\beta$. Hence

$$\left\{\begin{array}{l} \phi_1(\tau\theta) = \phi_1(\theta) \\ \tau_1\beta\tau_1 = \beta, \end{array}\right. \quad \left\{\begin{array}{l} \phi_1(\tau\theta) = \phi_1(\omega)\phi_1(\theta) \\ \tau_1\beta\tau_1 = \phi_1(\omega^2)\beta \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} \phi_1(\tau\theta) = \phi_1(\omega^2)\phi_1(\theta) \\ \tau_1\beta\tau_1 = \phi_1(\omega)\beta. \end{array}\right.$$

In the first case, $\tau\theta = \theta$, that is, $\theta \in \boldsymbol{R}^*$ and $\beta \in ({E_6}^C)^{\tau_1} \cong ({E_6}^C)^{\tau} = E_{6(-26)}$. Hence the group of the first case is $\boldsymbol{R}^* \times E_{6(-26)}$. The second and the third cases are impossible, because there are no $\theta \in C^*$ satisfying $\tau\theta = \omega^k\theta$ $(k = 1, 2)$. Thus we have $(E_{7(-25)})_{ev} \cong \boldsymbol{R}^* \times E_{6(-26)} = (\boldsymbol{R}^+ \times E_{6(-26)}) \times \{1, -1\}$.

(2) Although the proof is similar to that of Theorem 4.3.2.2.(2), we will give the proof again. For $\alpha \in (E_{7(-25)})_0 \subset ({E_7}^C)_0 = ({E_7}^C)^{\sigma{\iota_4}^{-1}}$, there exist $\theta, \nu \in C^*$ and $\beta \in Spin(10, C)$ such that $\alpha = \varphi_4(\theta, \nu, \beta) = \phi_1(\theta)\phi_2(\nu)\beta$ (Theorem 4.3.2.(2)). The condition $\tau_1\alpha\tau_1 = \alpha$ is $\tau_1\phi_1(\theta)\phi_2(\nu)\beta\tau_1 = \phi_1(\theta)\phi_2(\nu)\beta$. $\phi_1(\theta), \phi_2(\nu)$ satisfy $\tau_1\phi_1(\theta)\tau_1 = \phi_1(\tau\theta), \tau_1\phi_2(\nu)\tau_1 = \phi_2(\tau\nu)$, so we have $\phi_1(\tau\theta)$ $\phi_2(\tau\nu)\tau_1\beta\tau_1 = \phi_1(\theta)\phi_2(\nu)\beta$. Hence

$$\left\{\begin{array}{l} \phi_1(\tau\theta) = \phi_1(\theta) \\ \phi_2(\tau\nu) = \phi_2(\nu) \\ \tau_1\beta\tau_1 = \beta \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} \phi_1(\tau\theta) = \phi_1(\omega^{-4k})\phi_1(\theta) \\ \phi_2(\tau\nu) = \phi_2(\omega^k)\phi_2(\nu) \\ \tau_1\beta\tau_1 = \phi_1(\omega^{4k})\phi_2(\omega^{-k})\beta, \quad k = 1, \cdots 11. \end{array}\right.$$

In the former case, from $\tau\theta = \theta, \tau\nu = \nu$, we have $\theta, \nu \in \boldsymbol{R}^*$. We shall determine the structure of the group $\{\beta \in Spin(10, C) \,|\, \tau_1\beta\tau_1 = \beta\} = Spin(10, C)^{\tau_1} = (({E_6}^C)_{E_1})^{\tau_1}$. The group $(({E_6}^C)_{E_1})^{\tau_1}$ acts on the $\boldsymbol{R}$-vector space

$$\begin{aligned} V^{1,9} &= \{X \in \mathfrak{J}^C \,\big|\, 4E_1 \times (E_1 \times X) = X, \tau_1 X = X\} \\ &= \left\{ X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & -\tau\xi_2 \end{pmatrix} \,\middle|\, \begin{array}{l} \xi_2 \in C, \\ x_1 = ix + y,\, x \in \boldsymbol{R},\, y \in \mathfrak{C}, \overline{y} = -y \end{array} \right\} \end{aligned}$$

with the norm

$$(E_1, X, X) = -\xi_2(\tau\xi_2) - x_1\overline{x}_1 = -\xi_2(\tau\xi_2) + x^2 - y\overline{y}.$$

Since the group $Spin(10, C)^{\tau_1}$ is connected, we can define a homomorphism $\pi : Spin(10, C)^{\tau_1} \to O(V^{1,9})^0 = O(1, 9)^0$ by $\pi(\alpha) = \alpha | V^{1,9}$. $\mathrm{Ker}\, \pi = \{1, \sigma\}$. Since $\dim(((\mathfrak{e}_6{}^C)_{E_1})^{\tau_1}) = \dim((\mathfrak{e}_{7(-25)})_0) - \dim \boldsymbol{R} - \dim \boldsymbol{R} = 47 - 1 - 1$ (Theorem 4.4.1) $= 45 = \dim(\mathfrak{o}(1, 9))$, π is onto. Hence we have $Spin(10, C)^{\tau_1}/\boldsymbol{Z}_2 \cong O(1, 9)^0$. Therefore $Spin(10, C)^{\tau_1}$ is $Spin(1, 9)$ as a double covering group of $O(1, 9)^0$. Hence the group of the former case is $(\boldsymbol{R}^* \times \boldsymbol{R}^* \times Spin(1, 9))/\boldsymbol{Z}_2$ ($\boldsymbol{Z}_2 = \{(1, 1, 1), (1, -1, \sigma)\}$) $\cong \boldsymbol{R}^* \times \boldsymbol{R}^+ \times Spin(1, 9)$. The other cases are impossible, because there exists no $\theta \in C^*$ satisfying $\tau\theta = \omega^{-4k}\theta$ $(k = 1, \ldots, 11)$. Thus we have $(E_{7(-25)})_0 \cong \boldsymbol{R}^* \times \boldsymbol{R}^+ \times Spin(1, 9) = (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(1, 9)) \times \{1, -1\}$.

(3) For $\alpha \in (E_{7(-25)})_{ed} \subset ({E_7}^C)_{ed} = ({E_7}^C)^{\sigma_3}$, there exist $A \in SL(2, C)$, $a \in U(1, \boldsymbol{C}^C)$ and $\beta \in Spin(10, C)$ such that $\alpha = \varphi_5(A, a, \beta) = \phi(A)D(a)\beta$ (Theorem 4.4.2.(3)). The condition $\tau_1\alpha\tau_1 = \alpha$ is $\tau_1\phi(A)D(a)\beta\tau_1 = \phi(A)D(a)\beta$. $D(a)$ satisfies $\tau_1 D(a)\tau_1 = D(\tau\overline{a})$, so we have $\phi(\tau A)D(\tau\overline{a})\tau_1\beta\tau_1 = \phi(A)D(a)\beta$.

Hence

$$\text{(i)}\begin{cases}\phi(\tau A)=\phi(A)\\ D(\tau\overline{a})=D(a)\\ \tau_1\beta\tau_1=\beta,\end{cases}\qquad \text{(ii)}\begin{cases}\phi(\tau A)=\phi(A)\\ D(\tau\overline{a})=D(-a)\\ \tau_1\beta\tau_1=\sigma\beta,\end{cases}$$

$$\text{(iii)}\begin{cases}\phi(\tau A)=\phi(-A)\\ D(\tau\overline{a})=D(e_1a)\\ \tau_1\beta\tau_1=-D(e_1)\beta\end{cases}\quad\text{or}\quad \text{(iv)}\begin{cases}\phi(\tau A)=\phi(-A)\\ D(\tau\overline{a})=D(-e_1a)\\ \tau_1\beta\tau_1=-\sigma D(e_1)\beta.\end{cases}$$

(i) From $\tau A=A$ and $\tau\overline{a}=a$, we have $A\in SL(2,\boldsymbol{R})$ and $a\in U(1,\boldsymbol{C}')\cong\boldsymbol{R}^*$, respectively. The group $\{\beta\in Spin(10,C)\,|\,\tau_1\beta\tau_1=\beta\}=Spin(10,C)^{\tau_1}=(((E_7{}^C)^{\kappa,\mu})_{(F_1(1),0,0,0),(F_1(e_1),0,0,0)})^{\tau_1}$ acts on the $\boldsymbol{R}$-vector space

$$\begin{aligned}V^{1,9}&=((\mathfrak{P}^C)_{\kappa,\tau_1})_{(F_1(1),0,0,0),(F_1(e_1),0,0,0)}\\
&=\left\{P\in\mathfrak{P}^C\,\middle|\,\begin{array}{l}\kappa P=P,\tau_1P=P,\\ \{\mu(F_1(1),0,0,0),P\}=\{\mu(F_1(e_1),0,0,0),P\}=0\end{array}\right\}\\
&=\left\{P=\left(\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\overline{x}_1&-\tau\xi_2\end{pmatrix},\begin{pmatrix}\eta_1&0&0\\0&0&0\\0&0&0\end{pmatrix},0,\eta\right)\,\middle|\,\begin{array}{l}\xi_2\in C,\eta_1,\eta\in\boldsymbol{R},\\ x_1\in\mathfrak{C},\\ (1,x_1)=(e_1,x_1)=0\end{array}\right\}\end{aligned}$$

with the norm

$$(P,P)_\mu=\frac{1}{2}\{\mu P,P\}=\eta_1\eta+\xi_2(\tau\xi_2)+x_1\overline{x}_1.$$

Hence the group $Spin(10,C)^{\tau_1}$ is $Spin(1,9)$ and the group of (i) is $SL(2,\boldsymbol{R})\times\boldsymbol{R}^+\times Spin(1,9)$ as in a similar way in Theorem 4.4.2.3.(2).

(ii) $\varphi(E,e_1,\sigma'D(-e_1))=\sigma'$.

(iii) and (iv) are impossible. Indeed, β satisfies $(\beta P,\beta P)_\mu=-(P,P)_\mu$, but this is false because the signatures of both sides are different.

Thus we have $(E_{7(-25)})_{ed}\cong(SL(2,\boldsymbol{R})\times\boldsymbol{R}^+\times Spin(1,9))\times\{1,\sigma'\}$.

□

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