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Holomorphic $\mathbb{C}$-fibrations and pseudoconvexity of general order

By

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Abstract

We consider a domain D in $\mathbb{C}^n$ such that there is a Stein manifold E which is a $\mathbb{C}$-fibration over D. Simple examples show that D does not need to be Stein. However it cannot be arbitrarily and, in fact, we prove that D is pseudoconvex of order $n-2$.

1. Introduction

Let X be a complex space with countable topology. By a (*holomorphic*) $\mathbb{C}$-*fibration* over X we mean a locally analytically trivial fibre bundle $E \to X$ with fiber $\mathbb{C}$ (see [At]). Such examples appear naturally in the following way (see [Gr]): Let $\xi \in H^1(X, \mathcal{O}_X)$ be a non-vanishing cohomology class, if it exists. With respect to a trivializing Stein open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, ξ is reprezented by a 1-cocycle $\{\xi_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O})$. On the disjoint union of $U_i \times \mathbb{C}$ we introduce the equivalence relation by identifying the points (x,t) and $(x, t + \xi_{ij}(x))$ for $x \in U_i \cap U_j$, $t \in \mathbb{C}$, and $i, j \in I$. The quotient space thus obtained $E(\xi)$ is a complex space which is a $\mathbb{C}$-fibration over X. Since ξ is non-trivial, there is no holomorphic section of $E(\xi)$. Now, it may happen that, although X is not Stein, $E(\xi)$ is Stein; *e.g.* $X = \mathbb{C}^2 \setminus \{0\}$ and ξ given on $\mathbb{C}^* \times \mathbb{C}^*$ by $(z_1, z_2) \mapsto 1/z_1 z_2$ (see Section 3 for details).

Motivated by this Professor Tetsuo Ueda asked one of the authors (at the Hayama Conference in Complex Analysis, Japan, in the spring of 1995) to study convexity properties of open subsets of $\mathbb{C}^n$ on which there are Stein holomorphic $\mathbb{C}$-fibrations. This article stemmed from answering his question, *viz.* Corollary 1.

For a complex space Y we define property $\mathcal{H}_q$ as follows: Every holomorphic map from a q-th Hartogs figure to Y is extended holomorphically to its envelope. (See Section 2 for details.)

Theorem 1.1. *Let $E \to X$ be a holomorphic $\mathbb{C}$-fibration over a complex space X. If E satisfies $\mathcal{H}_q$ with $q \geq 2$, then so does X.*

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Corollary 1.1. *Let D be an open subset of $\mathbb{C}^n$. If there exists a Stein $\mathbb{C}$-fibration $E \to D$, then D is pseudoconvex of order $n-2$.*

In fact a more general result is true namely:

Theorem 1.2. *Let D be an open subset of a Stein manifold M of pure dimension n. If there exists a $\mathbb{C}$-fibration $E \to D$ such that E is weakly q-complete with $q \geq 2$, then D is q-complete with corners.*

We note that weak q-completeness here is defined with respect to continuous q-plurisubharmonic functions. It is important to notice that this holds true if E is q-complete with corners.

2. Preliminaries

To set the stage, for $r > 0$ and $z_0 \in \mathbb{C}$, put $\Delta(z_0; r) = \{z \in \mathbb{C}\,;\, |z - z_0| < r\}$, $\Delta(r) = \Delta(r; 0)$ and $\Delta = \Delta(1)$. Let q be a positive integer. Let k be a positive integer. Set $n = q + k$. Let r_1, r_2, s_1, s_2 be positive real numbers such that $r_2 < r_1$ and $s_2 < s_1$.

Define the open subset Ω_q in $\mathbb{C}^n = \mathbb{C}^q \times \mathbb{C}^k$ by setting:

$$(*) \qquad \Omega_q := \Big((\Delta^q(r_1) \setminus \overline{\Delta}^q(r_2)) \times \Delta^k(s_1)\Big) \cup \Big(\Delta^q(r_1) \times \Delta^k(s_2)\Big).$$

Its envelope of holomorphy $\widehat{\Omega}_q$ is $\Delta^q(r_1) \times \Delta^k(s_2)$. The set Ω_q is called a "q-th Hartogs figure" and $(\widehat{\Omega}_q, \Omega_q)$ is referred to as a "Hartogs pair of order q". If one wants to stress the ambient space as is the case in the subsequent Definition 2.2, one writes $\Omega_{q,n}$ instead of Ω_q; similarly for $\widehat{\Omega}_{q,n}$.

In the standard terminology one considers $r_1 = s_1 = 1$, $r_2 = 1 - \varepsilon$ and $s_2 = \varepsilon$ for $\varepsilon > 0$ small enough.

A simple but important observation is that Ω_q is contractible. A homotopy between the identity map of Ω_q and the constant self map of Ω_q, $\Omega_q \ni z \mapsto 0$, is exhibited by

$$F : \Omega_q \times [0, 2] \to \Omega_q$$

defined by the following formula:

$$F(z_1, z_2; t) = \begin{cases} (z_1, (1-t)z_2), & \text{if} \quad 0 \leq t \leq 1; \\ ((2-t)z_1, 0), & \text{if} \quad 1 \leq t \leq 2. \end{cases}$$

Consequently Ω_q is simply connected and all cohomology groups $H^i(\Omega_q, \mathbb{Z})$, $i = 1, 2, \ldots$, vanish.

Definition 2.1. A complex space X has property $\mathcal{H}_q$ if any holomorphic map from a q-th Hartogs figure into X extends holomorphically to its envelope.

For instance, the standard Hartogs extension theorem says that a Stein space has property $\mathcal{H}_q$ for any q. However, the converse statement fails as shown by complex tori; see Proposition 2.1 from below. (Note also that the complex projective line $\mathbb{P}^1$ does not enjoy the property $\mathcal{H}_1$. For instance, the holomorphic map $f : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ defined by

$$(z_1, z_2) \mapsto \frac{z_1 z_2}{z_1^2 - z_2^3}$$

does not extend, not even as a continuous function.)

A map $\pi : \widetilde{X} \to X$ between complex spaces is said to be a holomorphic covering map if π is holomorphic and a topological covering map. Note that if $\pi : \widetilde{X} \to X$ is a holomorphic covering map, and f is a continuous map from a complex manifold M into $\widetilde{X}$, then f is holomorphic if and only if $\pi \circ f$ is holomorphic.

Proposition 2.1. *Let $\pi : \widetilde{X} \to X$ be a holomorphic covering map. Then X has the property $\mathcal{H}_q$ if and only if $\widetilde{X}$ has the property $\mathcal{H}_q$.*

Proof. Suppose $\widetilde{X}$ has property $\mathcal{H}_q$ and let $f : \Omega \to X$ be a holomorphic map, where $(\widehat{\Omega}, \Omega)$ is a q-Hartogs pair. Since Ω is simply connected, there is a lifting $g : \Omega \to \widetilde{X}$ of f. Let $\widehat{g} : \widehat{\Omega} \to \widetilde{X}$ be the holomorphic extension map of g. Thus $\pi \circ \widehat{g}$ is the desired holomorphic extension of f. The converse follows by a similar argument. $\square$

Definition 2.2. An open subset D of a complex manifold M of pure dimension n is said to be "pseudoconvex of order $n - q$" if, for every injective holomorphic map $F : \widehat{\Omega}_{q,n} \to M$ the condition $F(\Omega_{q,n}) \subset D$ implies that $F(\widehat{\Omega}_{q,n}) \subset D$.

Remark 1. The ordinary pseudoconvexity in $\mathbb{C}^n$ is recovered as "pseudoconvex of order $n - 1$".

Remark 2. It is easily seen that if D is an open subset of a connected complex manifold M with $n := \dim(M)$, then D is pseudoconvex of order $n - q$ provided that D has property $\mathcal{H}_q$.

Granting [Fr] we have:

Proposition 2.2. *$H^i(\Omega_q, \mathcal{O}) = 0$ for any positive integer $i \neq q$.*

Corollary 2.1. *Let q be an integer ≥ 2. Let $r \in (r_2, r_1)$ and $s \in (s_2, s_1)$. Set $U = \Delta^q(r) \times \Delta^k(s)$ and $W = \Omega_q \cup U$. Then one has:*
1. $H^1(W, \mathcal{O}) = 0$.
2. $H^2(W, \mathbb{Z}) = 0$.

Proof. To show 1) we use Proposition 2.2 and the vanishing of cohomology for U with coefficients in $\mathcal{O}$. To proceed, from the Mayer–Vietoris sequence we retain the exact portion

$$H^0(\Omega_q, \mathcal{O}) \oplus H^0(U, \mathcal{O}) \to H^0(\Omega_q \cap U, \mathcal{O}) \to H^1(W, \mathcal{O}) \to 0.$$

But $H^0(U, \mathcal{O}) \to H^0(\Omega_q \cap U, \mathcal{O})$ is surjective, whence the desired result. The assertion in 2. is done similarly and therefore is omitted. $\square$

To conclude this section we recall three convexity notions. Let X be a complex space. First we have "q-convexity" in the sense of Andreotti and Grauert [AG]. (The normalization is such that "1-convex function" $\equiv$ "C^2-smooth, strictly plurisubharmonic function".)

A much weaker notion than this is q-convexity with corners: a continuous function $\varphi : X \to \mathbb{R}$ is q-convex with corners if, locally on open sets, φ equals maximum of finitely many q-convex function; see [DF1], [DF2] and [Pe].

A very flexible convexity property (close to q-convexity with corners but still weaker) is q-subpluriharmonicity which we now recall. (This is due to several authors, notably [Fu1] and [Fu2]; see also [HM] and [Sl].)

Let $\varphi : X \to [-\infty, \infty)$ be an upper-semicontinuous function. We say that φ is

- *subpluriharmonic* if, for any compact set $K \subset X$ and any pluriharmonic function h defined near K (*i.e.* h is locally the imaginary part of a holomorphic function), the inequality $\varphi \leq h$ on ∂K implies $\varphi \leq h$ on K.

- *q-plurisubharmonic* if, for any holomorphic map $F : \Delta^q \to X$, the function $\varphi \circ F : \Delta^q \to [-\infty, \infty)$ is subpluriharmonic (possibly $\equiv -\infty$).

A complex space X is said to be *weakly q-complete* (resp., *q-complete with corners*) if there exists a continuous exhaustion function $\varphi : X \to \mathbb{R}$ which is q-subpluriharmonic (resp., q-convex with corners).

Granting [Fu1], [Ma] and [Pe] we get:

Theorem 2.1. *For an open subset D of $\mathbb{C}^n$ the following statements are equivalent*:

1. *D has property $\mathcal{H}_q$*;
2. *D is pseudoconvex of order $n - q$*;
3. *$-\log \delta$ is q-plurisubharmonic*;
4. *D is q-complete with corners.*

Notice that the equivalence between assertions 1), 2) and 4) still hold when D is an open subset of a Stein manifold of pure dimension n as well as for $\mathbb{C}^n$.

3. Proofs of Theorems 1.1 and 1.2

First we prepare a few general facts on $\mathbb{C}$-fibrations over a complex space X. It is important to notice that $\mathrm{Aut}(\mathbb{C}) = \{f_{a,b}\,;\, a \in \mathbb{C}^*, b \in \mathbb{C}\}$, where $f_{a,b}(t) = at + b, t \in \mathbb{C}$. Therefore, if $E \to X$ is a holomorphic $\mathbb{C}$-fibration, then, with respect to a trivializing Stein open covering $\mathcal{U} = \{U_i\}_{i \in I}$, E is given by a 1-cocycle $\{a_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ and holomorphic functions $b_{ij} \in \mathcal{O}(U_{ij})$ such that, for all indices i, j, k for which the intersection $U_i \cap U_j \cap U_k$ is non empty, one has:

$$(\sharp) \qquad b_{ij} + a_{ij} b_{jk} = b_{ik}.$$

Reciprocally, given $\{a_{ij}\}$ and $\{b_{ij}\}$ as above, we get a $\mathbb{C}$-fibration over X taking the quotient of the disjoint union $U_i \times \mathbb{C}$ through the equivalence relation given by identifying the points (x, t) and $(x, b_{ij}(x) + ta_{ij}(x))$ for $x \in U_i \cap U_j$ and $t \in \mathbb{C}$.

Remark 3. It is easily seen that if $E \to X$ is a $\mathbb{C}$-fibration over a complex space X, then E is trivial if and only if E has a global holomorphic section.

Proposition 3.1. *Let X be a complex space such that $H^1(X, \mathcal{O}^*) = 0$. Then every $\mathbb{C}$-fibration over X is trivial.*

Proof. First we claim that $H^1(X, \mathcal{O}) = 0$. Indeed, granting Lojasiewicz's triangulability theorem [Lo], the topological space X has the homotopy type of a CW complex with countable many cells (X was supposed to be separable); hence $H^1(X, \mathbb{Z})$ is an abelian group with at most countable many generators. On the other hand $H^1(X, \mathcal{O})$ is a complex vector space and the field $\mathbb{C}$ is not countably generated as a $\mathbb{Z}$-module. The only possible dimension of $H^1(X, \mathcal{O})$ is then zero. Now, let $E \to X$ be a $\mathbb{C}$-fibration and consider a Stein open covering $\{U_i\}_i$ of X such that E is represented by $\{a_{ij}\} \in Z^1(\{U_i\}_i, \mathcal{O}^*)$ and $\{b_{ij}\} \in C^1(\{U_i\}, \mathcal{O})$ satisfying ($\sharp$). Since $H^1(X, \mathcal{O}^*) = 0$, there exists $\{\lambda_i\} \in C^0(\{U_i\}, \mathcal{O}^*)$ such that $a_{ij} = \lambda_i/\lambda_j$. Therefore $b_{ij}\lambda_i + b_{jk}\lambda_j = b_{ik}\lambda_i$; hence $\{b_{ij}\lambda_i\}_{ij}$ gives a 1-cocycle which is exact by the above claim. Thus there exists $\{\mu_i\} \in C^0(\{U_i\}_i, \mathcal{O})$ such that $b_{ij} = \lambda_i(\mu_i - \mu_j)$. It is then easy to see that the holomorphic map from $X \times \mathbb{C}$ into $(U_i \times \mathbb{C})/_\sim$ given by

$$(x, t) \mapsto (x, -\mu_i + t/\lambda_i(x))$$

defines the desired isomorphism. □

Now we are in a position to give the proof of Theorem 1.1.

Proof. Let $\pi : E \to X$ be a $\mathbb{C}$-fibration. Let $(\widehat{\Omega}, \Omega)$ be a Hartogs pair of order q with $q \geq 2$ and $f : \Omega \to X$ a holomorphic map. There is a canonical commutative diagram of holomorphic mappings

$$\begin{tikzcd} f^\star(E) \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ \Omega \arrow[r, "f"] & X. \end{tikzcd}$$

We want a holomorphic map $\widehat{f} : \widehat{\Omega} \to X$ which extends f. In order to do this, granting the above discussion and Corollary 2.1, the pull-back $f^\star(E)$ over Ω is trivial so that there is a holomorphic section $s : \Omega \to f^\star(E)$. This induces via the above commutative diagram a holomorphic map $g : \Omega \to E$ with $\pi \circ g = f$. On the other hand, as E has property $\mathcal{H}_q$, this g extends to a holomorphic map $\widehat{g} : \widehat{\Omega} \to E$. Therefore $\pi \circ \widehat{g}$ is the desired holomorphic extension of f from $\widehat{\Omega}$ into X. □

Remark 4. It is not difficult to see that, conversely, if X satisfies property $\mathcal{H}_q$, then so does E. Note that here we do not have the restriction $q \geq 2$.

Here we give the **proof of Theorem 1.2**

Proof. Let $(\widehat{\Omega}, \Omega)$ be a Hartogs pair of order q. Set $k = n - q$. There is no loss in generality to assume that

$$\Omega = \Big((\Delta^q \setminus \overline{\Delta}^q(1-\varepsilon)) \times \Delta^k \Big) \cup \Big(\Delta^q \times \Delta^k(\varepsilon) \Big),$$

so that $\widehat{\Omega} = \Delta^n$. Also we assume that $\Delta^n \subset M$ and $\Omega \subset D$. We want $\Delta^n \subset D$. This will follow from the following

Claim. Let $r \in (1-\varepsilon, 1)$. Then $\overline{\Delta}^q(r) \times \Delta^k \subset D$.

In order to establish this, we consider the set S of those $s \in (0,1)$ such that $\overline{\Delta}^q(r) \times \overline{\Delta}^k(s) \subset D$. Now, to show this, by a standard argument we check that the set S is non-empty, open, and closed in $(0,1)$. Notice that the first two conditions are trivially fulfilled so one is left to check that S is closed.

Let $s \in (0,1)$ and $\{s_\nu\}$ a sequence of points in S increasing to s. Let $\xi \in \partial \Delta^k(s)$. We show that $\overline{\Delta}^q(r) \times \{\xi\}$ lies in D. For this, select a sequence of points $\xi_\nu \in \Delta^k(s_\nu)$ which converges to ξ. Set $K_\nu := \overline{\Delta}^q(r) \times \{\xi_\nu\}$ and T their union, which is a subset of W, where

$$W := \Omega \cup \Big(\Delta^q(r) \times \Delta^k(s) \Big).$$

Let φ be a continuous q-plurisubharmonic exhaustion function on E. From Corollary 2.1 it follows that E is trivial over W; hence there is a holomorphic section $\sigma : W \to E$ of the canonical projection $\pi : E \to D$. Notice that

$$bT := \cup_\nu \Big((\partial K_\nu) \times \{\xi_\nu\} \Big)$$

is a relatively compact set in W. Therefore, by the maximum principle for q-subpluriharmonic functions one deduces immediately that $c := \sup_{\sigma(T)} \varphi$ is finite. Thus T is contained in the compact subset $\pi(\{\varphi \leq c\})$ of D. This shows that S is closed concluding the claim, whence the theorem. $\square$

Example 3.1. Let $X = \mathbb{C}^2 \setminus \{0\}$ and ξ given with respect to the Stein covering $U_1 = \mathbb{C}^* \times \mathbb{C}$, $U_2 = \mathbb{C} \times \mathbb{C}^*$ by $\xi(z_1, z_2) = 1 + 1/z_1 z_2$. Clearly ξ defines a non zero cohomology class in $H^1(\mathbb{C}^2 \setminus \{0\}, \mathcal{O})$ and then, as alluded to in the introduction, it produces a $\mathbb{C}$-fibration $E \to \mathbb{C}^2 \setminus \{0\}$. We show that E is Stein.

In fact, E is obtained glueing $E_1 = U_1 \times \mathbb{C}$ and $E_2 = U_2 \times \mathbb{C}$ by identifying points (z, t) and $(z, t + 1/z_1 z_2)$ where $z \in U_1 \cap U_2$ and $t \in \mathbb{C}$. There is a canonical projection $\pi : E \to X$. Then we define two holomorphic mappings $f, g : E \to \mathbb{C}$ as follows: f is given by $(z, t) \mapsto t z_1$ on E_1 and $(z, t) \mapsto t z_1 + 1/z_2$ on E_2. Similarly, g is defined by $(z, t) \mapsto -z_2 t + 1/z_1$ on E_1 and $(z, t) \mapsto -t z_2$ on

E_2. Eventually, by a simple computation we check that the holomorphic map $(\pi, f, g) : E \to \mathbb{C}^4$ is a holomorphic embedding; its image is the hypersurface $\{z_1 w_2 - z_2 w_1 = 1\}$ in $\mathbb{C}^4$. In other words, this $E(\xi)$ is nothing else than the special linear group $\mathrm{Sl}(2, \mathbb{C})$. (This example is related to the construction of a compactification of $\mathbb{C}^3$; see [FuT] and [FN].)

Related to the above example we give:

Proposition 3.2. *Let A be a closed subset of an open set D in $\mathbb{C}^n$. If the Hausdorff $(2n-4)$-measure of A is zero, then there does not exist a Stein $\mathbb{C}$-fibration $E \to D \setminus A$.*

Proof. Assume in order to reach a contradiction that we have a Stein holomorphic $\mathbb{C}$-fibration $E \to D \setminus A$. By Corollary 1.1 it follows that D is pseudoconvex of order $n-2$. Since $h^{2n-4}(A) = 0$, it results from [Va] that A is analytic of pure dimension $n-2$ which contradicts the hypothesis. $\square$

Remark 5. This proposition improves Proposition 5 in [Abe] where the case when A is analytic of codimesion ≥ 3 is considered.

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