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Real K-homology of complex projective spaces

By

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Introduction

The real K-homology theory is one of a few examples of generalized homology theories which take values in the category of comodules over the associated Hopf algebroid, which are not complex oriented in the sense of Adams [1], namely the real K-cohomology of the infinite dimensional complex projective space does not have a structure of formal group law induced by the group structure $m : \boldsymbol{C}P^\infty \times \boldsymbol{C}P^\infty \to \boldsymbol{C}P^\infty$. However, the real K-homology of the infinite dimensional complex projective space has the Pontrjagin ring structure which is regarded as a virtual dual of non-existent structure of formal group law ([4]). From this point of view, the ring structure of the real K-homology of the infinite dimensional complex projective space might be of some interest. The aim of this paper is to determine the module structure of the real K-homology of complex projective spaces over the coefficient ring KO_* and to describe the ring structure of the real K-homology of the infinite complex projective space.

In the first section, we prepare some necessary results in the following sections. Next, we determine the "conjugation map" on $K_*(\boldsymbol{C}P^l)$ induced by the map $BU(n) \to BU(n)$ which classifies the complex conjugate of the canonical bundle. We make some analysis on the conjugation map in section three and define certain elements of $K_*(\boldsymbol{C}P^\infty)$ which generates the image of the complexification map $KO_*(\boldsymbol{C}P^\infty) \to K_*(\boldsymbol{C}P^\infty)$. In section four, we determine the KO_*-module structure of $K_*(\boldsymbol{C}P^l)$ by using the Atiyah-Hirzebruch spectral sequence. It turns out that the complexification map $\boldsymbol{c} : \widetilde{KO}_*(\boldsymbol{C}P^l) \to \widetilde{K}_*(\boldsymbol{C}P^l)$ is injective if l is even or ∞. By virtue of this fact, we can describe the ring structure of $KO_*(\boldsymbol{C}P^\infty)$ by examining the image of $\boldsymbol{c}$ in the last section.

1. Preliminaries

We first recall the Bott periodicity

$$
\begin{aligned}
&O \simeq \Omega(\boldsymbol{Z} \times BO), \quad O/U \simeq \Omega O, \quad U/Sp \simeq \Omega(O/U), \quad \boldsymbol{Z} \times BSp \simeq \Omega(U/Sp)\\
&Sp \simeq \Omega(\boldsymbol{Z} \times BSp), \quad Sp/U \simeq \Omega Sp, \quad U/O \simeq \Omega(Sp/U), \quad \boldsymbol{Z} \times BO \simeq \Omega(U/O).
\end{aligned}
$$

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Thus the KO-spectrum $KO = (\varepsilon_n : SKO_n \to KO_{n+1})_{n\in \boldsymbol{Z}}$ is given as follows.

$$\begin{array}{llll} KO_{8n} = \boldsymbol{Z} \times BO, & KO_{8n+1} = U/O, & KO_{8n+2} = Sp/U, & KO_{8n+3} = Sp, \\ KO_{8n+4} = \boldsymbol{Z} \times BSp, & KO_{8n+5} = U/Sp, & KO_{8n+6} = O/U, & KO_{8n+7} = O. \end{array}$$

We also recall that $K^* = \boldsymbol{Z}[t, t^{-1}]$, $KO^* = \boldsymbol{Z}[\alpha, x, y, y^{-1}]/(2\alpha, \alpha^3, \alpha x, x^2 - 4y)$, where t, α, x and y are generators of $K^{-2} = \pi_2(K) \cong \boldsymbol{Z}$, $KO^{-1} = \pi_1(KO) \cong \boldsymbol{Z}/2\boldsymbol{Z}$, $KO^{-4} = \pi_4(KO) \cong \boldsymbol{Z}$, $KO^{-8} = \pi_8(KO) \cong \boldsymbol{Z}$. Note that t, α are the homotopy classes of the inclusion maps $S^2 = \boldsymbol{C}P^1 \to BU = K_0$, $S^1 = \boldsymbol{R}P^1 \to BO = KO_0$ to the bottom cells.

Let us denote by $h_2 : S^3 \to S^2$ the Hopf map, by $j : S^3 = Sp(1) \to Sp$, $i : S^2 = Sp(1)/U(1) \to Sp/U$ the inclusion maps of the bottom cells, and by $p : Sp \to Sp/U$ the quotient map. Then

$$\begin{array}{ccc} S^3 & \xrightarrow{h_2} & S^2 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ Sp & \xrightarrow{p} & Sp/U \end{array}$$

commutes.

Lemma 1.1. *The homotopy class of $ih_2 = pj$ generates $\pi_3(Sp/U) \cong \boldsymbol{Z}/2\boldsymbol{Z}$. Hence ih_2 represents $\alpha \in \pi_1(KO) \cong \pi_3(KO_2)$.*

Proof. By the commutativity of the above diagram, we have the following commutative diagram.

$$\begin{array}{ccc} \pi_3(S^3) & \underset{\cong}{\xrightarrow{h_{2*}}} & \pi_3(S^2) \\ \cong\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle i_*} \\ \pi_3(Sp) & \xrightarrow{p_*} & \pi_3(Sp/U) \end{array}$$

Since $p_* : \pi_3(Sp) \to \pi_3(Sp/U)$ is surjective, the assertion follows. □

Lemma 1.2. *Let $\eta_s : S^{2s-1} \to S^{2s-2} = \boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2}$ be the attaching map of the $2s$-cell of $\boldsymbol{C}P^s/\boldsymbol{C}P^{s-2}$ $(s \geq 2)$. Then, η_s is null homotopic if s is odd and it is homotopic to $S^{2s-4}h_2$ if s is even.*

Proof. Let g_j $(j = 2s-2, 2s)$ be the generators of $H^j(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-2}; \boldsymbol{F}_2)$. Since

$$Sq^2 g_{2s-2} = \begin{cases} g_{2s} & s \text{ is even} \\ 0 & s \text{ is odd} \end{cases},$$

the assertion follows. □

Lemma 1.3. *For integers n and m such that $n \geq 2$, the composition of the suspension $\sigma : \widetilde{KO}_{m-1}(S^n) \to \widetilde{KO}_m(S^{n+1})$ and $(S^{n-2}h_2)_* : \widetilde{KO}_m(S^{n+1}) \to \widetilde{KO}_m(S^n)$ coincides with the multiplication map by α.*

Proof. Let $f : S^{m+q-1} \to KO_q \wedge S^n$ be a map which represents an element ξ of $\widetilde{KO}_{m-1}(S^n)$. Then

$$S^{m+q} \xrightarrow{Sf} S^1 \wedge KO_q \wedge S^n \xrightarrow{T\wedge 1_{S^n}} KO_q \wedge S^1 \wedge S^n$$
$$= KO_q \wedge S^{n+1} \xrightarrow{1_{KO_q}\wedge S^{n-2}h_2} KO_q \wedge S^n$$

represents $(S^{n-2}h_2)_*\sigma(\xi)$. Hence the following composition also represents $(S^{n-2}h_2)_*\sigma(\xi)$.

$$S^{m+q+2} \xrightarrow{S^3 f} S^2 \wedge S^1 \wedge KO_q \wedge S^n \xrightarrow{1_{S^2}\wedge T\wedge 1_{S^n}}$$
$$S^2 KO_q \wedge S^1 \wedge S^n \xrightarrow{(\varepsilon_{q+1}S\varepsilon_q)\wedge S^{n-2}h_2} KO_{q+2} \wedge S^n$$

Choose $ih_2 : S^3 \to Sp/U$ as a representative of α. Let $(\mu_{p,q} : KO_p \wedge KO_q \to KO_{p+q})_{p,q\in \boldsymbol{Z}}$ be the product structure of KO and $(\iota_p : S^p \to KO_p)_{p\in\boldsymbol{Z}}$ the unit. Since $i : S^2 \to Sp/U = KO_2$ is identified with ι_2,

$$\begin{array}{ccc} S^2 \wedge KO_q & \xrightarrow{i\wedge 1_{KO_q}} & (Sp/U)\wedge KO_q \\ \Big\downarrow{\scriptstyle S\varepsilon_q} & & \Big\downarrow{\scriptstyle \mu_{2,q}} \\ SKO_{q+1} & \xrightarrow{\varepsilon_{q+1}} & KO_{q+2} \end{array}$$

is homotopy commutative. Hence $\alpha\xi$ is represented by the following composition.

$$S^{m+q+2} = S^3 \wedge S^{m+q-1} \xrightarrow{h_2\wedge f} S^2 \wedge KO_q \wedge S^n \xrightarrow{(\varepsilon_{q+1}S\varepsilon_q)\wedge 1_{S^n}} KO_{q+2} \wedge S^n$$

Since Sh_2 is order 2 in $\pi_4(S^3)$, $h_2 \wedge 1_{S^n} : S^{n+3} = S^3 \wedge S^n \to S^2 \wedge S^n = S^{n+2}$ is homotopic to $1_{S^n} \wedge h_2 : S^{n+3} = S^n \wedge S^3 \to S^n \wedge S^2 = S^{n+2}$ for $n \geq 0$. This implies that

$$S^3 \wedge KO_q \wedge S^n \xrightarrow{h_2\wedge 1_{KO_q}\wedge 1_{S^n}} S^2 \wedge KO_q \wedge S^n$$

is homotopic to

$$S^2 \wedge S^1 \wedge KO_q \wedge S^n \xrightarrow{1_{S^2}\wedge T\wedge 1_{S^n}} S^2 KO_q \wedge S^1 \wedge S^n \xrightarrow{1_{S^2 KO_q}\wedge S^{n-2}h_2} S^2 KO_q \wedge S^n,$$

which shows $(S^{n-2}h_2)_*\sigma(\xi) = \alpha\xi$.

□

Let us denote by $u_i \in \widetilde{KO}_i(S^i)$ $(i \geq 0)$ the canonical generators, that is, u_i's are given by $u_0 = 1$, $\sigma(u_i) = u_{i+1}$.

For $s \geq 2$, consider the cofiber sequence $\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2} \xrightarrow{\iota} \boldsymbol{C}P^s/\boldsymbol{C}P^{s-2} \xrightarrow{\kappa} \boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}$. We have the long exact sequences associated with this cofiber sequence.

$$\cdots \to \widetilde{KO}_{n+1}(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}) \xrightarrow{\partial} \widetilde{KO}_n(\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2}) \xrightarrow{\iota_*} \widetilde{KO}_n(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-2})$$
$$\xrightarrow{\kappa_*} \widetilde{KO}_n(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}) \to \cdots$$

Lemma 1.4. *The connecting homomorphism*

$$\partial : \widetilde{KO}_{n+1}(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}) \to \widetilde{KO}_n(\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2})$$

is given by

$$\partial(u_{2s}) = \begin{cases} \alpha u_{2s-2} & s \text{ is even} \\ 0 & s \text{ is odd} \end{cases}.$$

Proof. Since the composition

$$\widetilde{KO}_n(S^{2s-1}) \xrightarrow{\sigma} \widetilde{KO}_{n+1}(S^{2s}) = \widetilde{KO}_{n+1}(\boldsymbol{C}P^s/\boldsymbol{C}P^{s-1}) \xrightarrow{\partial} \widetilde{KO}_n(\boldsymbol{C}P^{s-1}/\boldsymbol{C}P^{s-2})$$

coincides with the map induced by the attaching map η_s, the first formula follows from Lemma 1.2 and 1.3. □

The following result is known.

Proposition 1.1. *The complexification map* $\boldsymbol{c} : KO^*(X) \to K^*(X)$, *the realization map* $\boldsymbol{r} : K^*(X) \to KO^*(X)$ *and the conjugation map* $\Psi^{-1} : K^*(X) \to K^*(X)$ *are natural transformation of cohomology theories having the following properties.*

1) $\boldsymbol{c}$ *is a homomorphism of graded rings which maps* $\alpha \in KO^{-1}$ *to* 0, $x \in KO^{-4}$ *to* $2t^2$ *and* $y \in KO^{-8}$ *to* t^4.

2) $\boldsymbol{r}$ *is a homomorphism of graded abelian groups which maps* $t^{4i} \in K^{-8i}$ *to* $2y^i$, $t^{4i+1} \in K^{-8i-2}$ *to* $\alpha^2 y^i$ *and* $t^{4i+2} \in K^{-8i-4}$ *to* xy^i *for* $i \in \boldsymbol{Z}$.

3) Ψ^{-1} *is a ring homomorphism which maps* $t \in K^{-1}$ *to* $-t$.

4) $\boldsymbol{rc} = 2id_{KO^*(X)}$, $\boldsymbol{cr} = id_{K^*(X)} + \Psi^{-1}$ *and* $\Psi^{-1}\Psi^{-1} = id_{K^*(X)}$ *hold.*

We denote by $B : \widetilde{K}_n(X) \to \widetilde{K}_{n+2}(X)$ the Bott periodicity map $B(a) = ta$ and by $\boldsymbol{\alpha} : \widetilde{KO}_n(X) \to \widetilde{KO}_{n-1}(X)$, the multiplication map by $\alpha \in KO_1$. A fiber sequence $U/O \to BO \to BU$ gives a cofiber sequence $\Sigma KO \to KO \xrightarrow{\boldsymbol{c}} K$ of spectra. The following result is known.

Proposition 1.2. *There is a long exact sequence*

$$\cdots \to \widetilde{K}_{n+1}(X) \xrightarrow{\boldsymbol{r}B^{-1}} \widetilde{KO}_{n-1}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}_n(X) \xrightarrow{\boldsymbol{c}} \widetilde{K}_n(X) \xrightarrow{\boldsymbol{r}B^{-1}} \widetilde{KO}_{n-2}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}_{n-1}(X) \to \cdots.$$

Corollary 1.1. *Let* X *be a space such that* $K_1(X) = \{0\}$ ($X = \boldsymbol{C}P^l$ *for example*). *There is an exact sequence*

$$0 \to \widetilde{KO}_{2n-1}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}_{2n}(X) \xrightarrow{\boldsymbol{c}} \widetilde{K}_{2n}(X) \xrightarrow{\boldsymbol{r}B^{-1}} \widetilde{KO}_{2n-2}(X) \xrightarrow{\boldsymbol{\alpha}} \widetilde{KO}_{2n-1}(X) \to 0.$$

2. Conjugation in $K_*(\boldsymbol{C}P^l)$

Lemma 2.1. *Let E be a ring spectrum and $\psi : E \to E$ be a map of ring spectra. For a space X, consider the Kronecker pairing $\langle\ ,\ \rangle : E^*(X) \otimes_{E_*} E_*(X) \to E_*$. Then, $\psi(\langle \xi, a\rangle) = \langle \psi(\xi), \psi(a)\rangle$ for $\xi \in E^*(X)$ and $a \in E_*(X)$.*

Proof. Let $g : X \to E_n$ be the map which represents $\xi \in E^*(X)$ and $f : S^{k+m} \to E_k \wedge X$ the map which represents $a \in E_m(X)$. Then, $\psi_n g : X \to E_n$ and $(\psi_k \wedge 1_X)f : f : S^{k+m} \to E_k \wedge X$ represent $\psi(\xi)$, $\psi(a)$, respectively. The assertion follows from the homotopy commutativity of the following diagram. Here $\mu_{k,n} : E_k \wedge E_n \to E_{k+n}$ denote the ring structure of E.

$$\begin{array}{ccccccc}
S^{k+m} & \xrightarrow{f} & E_k \wedge X & \xrightarrow{1_{E_k}\wedge g} & E_k \wedge E_n & \xrightarrow{\mu_{k,n}} & E_{k+n} \\
 & & \downarrow{\scriptstyle \psi_k\wedge 1_X} & & \downarrow{\scriptstyle \psi_k\wedge\psi_n} & & \downarrow{\scriptstyle \psi_{k+n}} \\
 & & E_k \wedge X & \xrightarrow{1_{E_k}\wedge \psi_n g} & E_k \wedge E_n & \xrightarrow{\mu_{k,n}} & E_{k+n}
\end{array}$$

□

Let us denote by η_l the canonical complex line bundle over $\boldsymbol{C}P^l$. Put $\mu_l = \eta_l - 1 \in \widetilde{K}^0(\boldsymbol{C}P^l)$. Then, $K^*(\boldsymbol{C}P^l) = K^*[\mu_l]/(\mu_l^{l+1})$ and $\Psi^{-1}(\mu_l) = (1+\mu_l)^{-1} - 1$. We denote by $\beta_i \in H_{2i}(\boldsymbol{C}P^l; \boldsymbol{Z})$ the dual of $u^i \in H^{2i}(\boldsymbol{C}P^l; \boldsymbol{Z})$. Then β_i generates $H_{2i}(\boldsymbol{C}P^l; \boldsymbol{Z})$ which is isomorphic to $\boldsymbol{Z}$. The Atiyah-Hirzebruch spectral sequence $E^2_{p,q}(K; \boldsymbol{C}P^l) = H_p(\boldsymbol{C}P^l; K_q) \Rightarrow K_{p+q}(\boldsymbol{C}P^l)$ collapses and $K_*(\boldsymbol{C}P^l)$ is a free K_*-module generated by $\beta_0, \beta_1, \ldots, \beta_l$, where $\beta_i \in K_{2i}(\boldsymbol{C}P^l)$ is the dual of $t^{-i}\mu_l^i \in K^{2i}(\boldsymbol{C}P^l)$ ([1]). In order to calculate $\Psi^{-1}(\beta_j)$, we use the following fact.

Lemma 2.2. *For a positive integer i, the following equality holds in $\boldsymbol{Z}[[z]]$.*

$$(1 - (1+z)^{-1})^i = \sum_{k=i}^{\infty} (-1)^{k-i} \binom{k-1}{i-1} z^k$$

Proof. By the Taylor expansion of $(1+z)^{-i}$, we have

$$(1-(1+z)^{-1})^i = z^i(1+z)^{-i} = z^i \sum_{s=0}^{\infty} (-1)^s \binom{i+s-1}{i-1} z^s = \sum_{k=i}^{\infty} (-1)^{k-i} \binom{k-1}{i-1} z^k.$$

□

Proposition 2.1. *$\Psi^{-1} : K_*(\boldsymbol{C}P^l) \to K_*(\boldsymbol{C}P^l)$ is given as follows.*

$$\Psi^{-1}(\beta_0) = \beta_0, \qquad \Psi^{-1}(\beta_j) = \sum_{k=1}^{j} \binom{j-1}{k-1} t^{j-k} \beta_k \quad \text{if } j \geq 1$$

Proof. We put $\Psi^{-1}(\beta_j) = \sum_{k=0}^{j} c_{kj}\beta_k$ ($c_{kj} \in K_{2j-2k}$). By Lemma 2.1, we have

$$\left\langle \Psi^{-1}\left(t^{-i}\mu_l^i\right), \Psi^{-1}(\beta_j)\right\rangle = \Psi^{-1}\left(\left\langle t^{-i}\mu_l^i, \beta_j\right\rangle\right) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \cdots (1).$$

For $i = 0$, since $\langle 1, \Psi^{-1}(\beta_j)\rangle = \sum_{k=0}^{j} c_{kj}\langle 1, \beta_k\rangle = c_{0j}$, we have $c_{00} = 1$ and $c_{0j} = 0$ if $j > 0$. Suppose that $i > 0$. By Lemma 2.2,

$$\begin{aligned} \Psi^{-1}(t^{-i}\mu_l^i) &= (-t)^{-i}((1+\mu_l)^{-1} - 1)^i \\ &= t^{-i}(1 - (1+\mu_l)^{-1})^i \\ &= t^{-i}\sum_{s=i}^{\infty}(-1)^{s-i}\binom{s-1}{i-1}\mu_l^s. \end{aligned}$$

Thus we have

$$\begin{aligned} \left\langle \Psi^{-1}(t^{-i}\mu_l^i), \Psi^{-1}(\beta_j)\right\rangle &= \left\langle t^{-i}\sum_{s=i}^{\infty}(-1)^{s-i}\binom{s-1}{i-1}\mu_l^s, \sum_{k=0}^{j} c_{kj}\beta_k \right\rangle \\ &= \sum_{s=i}^{\infty}\sum_{k=0}^{j}(-1)^{s-i}\binom{s-1}{i-1}t^{s-i}c_{kj}\left\langle t^{-s}\mu_l^s, \beta_k\right\rangle \\ &= \sum_{k=i}^{j}(-1)^{k-i}\binom{k-1}{i-1}t^{k-i}c_{kj} \end{aligned}$$

if $0 < i \leq j$. It follows from (1) that $c_{jj} = 1$ and, if $0 < i < j$,

$$\sum_{k=i}^{j-1}(-1)^{k-i}\binom{k-1}{i-1}t^{k-i}c_{kj} + (-1)^{j-i}\binom{j-1}{i-1}t^{j-i} = 0.$$

Therefore

$$c_{ij} = -\sum_{k=i+1}^{j-1}(-1)^{k-i}\binom{k-1}{i-1}t^{k-i}c_{kj} - (-1)^{j-i}\binom{j-1}{i-1}t^{j-i} \quad \cdots (2).$$

We show $c_{ij} = \binom{j-1}{i-1}t^{j-i}$ by the induction on $j - i$. It can be easily verified from (2) that $c_{j-1\,j} = \binom{j-1}{j-2}t$. Assume that $j - i = r$ and $c_{kj} = \binom{j-1}{k-1}t^{j-k}$ holds if $j - k < r$. Then,

$$
\begin{aligned}
c_{ij} &= -\sum_{k=i+1}^{j-1}(-1)^{k-i}\binom{k-1}{i-1}\binom{j-1}{k-1}t^{j-i} - (-1)^{j-i}\binom{j-1}{i-1}t^{j-i} \\
&= -\sum_{k=i+1}^{j}(-1)^{k-i}\binom{k-1}{i-1}\binom{j-1}{k-1}t^{j-i} \\
&= -\sum_{k=i+1}^{j}(-1)^{k-i}\binom{j-1}{i-1}\binom{j-i}{k-i}t^{j-i} \\
&= -\binom{j-1}{i-1}t^{j-i}\sum_{k=i+1}^{j}(-1)^{k-i}\binom{j-i}{k-i}
\end{aligned}
$$

Hence it remains to show that $\sum\limits_{k=i+1}^{j}(-1)^{k-i}\binom{j-i}{k-i} = -1$. But this follows from $\sum\limits_{p=0}^{j-i}(-1)^p\binom{j-i}{p} = ((-1)+1)^{j-i} = 0$. □

3. Eigen spaces of the conjugation map

We embed $K_*(\boldsymbol{C}P^l)$ into $K_*(\boldsymbol{C}P^\infty)$. Then $K_*(\boldsymbol{C}P^l)$ is a submodule of $K_*(\boldsymbol{C}P^\infty)$ spanned by $\beta_0, \beta_1, \dots, \beta_l$.

Recall the Pontrjagin ring structure on $K_*(\boldsymbol{C}P^\infty)$ ([1]). Let $m : \boldsymbol{C}P^\infty \times \boldsymbol{C}P^\infty \to \boldsymbol{C}P^\infty$ be the product map and $\eta \in K^0(\boldsymbol{C}P^\infty)$ the class of the canonical line bundle on $\boldsymbol{C}P^\infty$. Put $\mu = \eta - 1 \in K^*(\boldsymbol{C}P^\infty)$ and $\mu_1 = \mu \times 1, \mu_2 = 1 \times \mu \in K^*(\boldsymbol{C}P^\infty \times \boldsymbol{C}P^\infty)$. Since $m^* : K^*(\boldsymbol{C}P^\infty) \to K^*(\boldsymbol{C}P^\infty \times \boldsymbol{C}P^\infty)$ maps μ to $\mu_1 + \mu_2 + \mu_1\mu_2$, we have

$$
m^*\left((t^{-1}\mu)^k\right) = \sum_{i,j\geq 0,\; i+j\leq k} \frac{k!}{i!j!(k-i-j)!} t^{k-i-j}(t^{-1}\mu_1)^{k-j}(t^{-1}\mu_2)^{k-i}.
$$

Thus we have

$$
\left\langle m_*(\beta_i \otimes \beta_j), (t^{-1}\mu)^k \right\rangle = \frac{k!}{(k-i)!(k-j)!(i+j-k)!} t^{i+j-k}.
$$

Hence the Pontrjagin ring structure $m_* : K_*(\boldsymbol{C}P^\infty) \otimes K_*(\boldsymbol{C}P^\infty) \to K_*(\boldsymbol{C}P^\infty)$ is given by

$$
m_*(\beta_i \otimes \beta_j) = \sum_{i,j\leq k\leq i+j} \frac{k!}{(k-i)!(k-j)!(i+j-k)!} t^{i+j-k}\beta_k.
$$

For $x, y \in K_*(\boldsymbol{C}P^\infty)$, we denote $m_*(x \otimes y)$ by xy for short below.

Since $K_*(\boldsymbol{C}P^\infty)$ is torsion free, we can regard $K_*(\boldsymbol{C}P^\infty)$ as a subalgebra of $K_*(\boldsymbol{C}P^\infty) \otimes \boldsymbol{Q}$. Put $\tilde{\beta}_i = i!t^{-i}\beta_i$ and $z = \tilde{\beta}_1$. Then, $\beta_i\beta_1 = (i+1)\beta_{i+1} + it\beta_i$ implies a recursive formula $\tilde{\beta}_{i+1} = (z-i)\tilde{\beta}_i$. Hence we have

$$
\tilde{\beta}_i = z(z-1)\cdots(z-i+1).
$$

We set

$$\binom{z}{i} = \frac{1}{i!} z(z-1)\cdots(z-i+1).$$

The above argument shows $\beta_i = t^i \binom{z}{i}$ and $K_*(\boldsymbol{CP}^\infty) \subset K_* \otimes \boldsymbol{Q}[z]$. This implies the following.

Proposition 3.1. *$K_*(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q}$ is a polynomial algebra $K_* \otimes \boldsymbol{Q}[z]$ over $K_* \otimes \boldsymbol{Q} = \boldsymbol{Q}[t, t^{-1}]$ and $K_*(\boldsymbol{CP}^\infty)$ is the subalgebra of $K_* \otimes \boldsymbol{Q}[z]$ generated by $\binom{z}{i}$ for $i = 1, 2, 3, \ldots$.*

Remark 3.1. 1) $K_0(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q}$ is a polynomial algebra $\boldsymbol{Q}[z]$. $K_0(\boldsymbol{CP}^\infty)$ is the subalgebra of $\boldsymbol{Q}[z]$ generated by $\binom{z}{i}$ for $i = 1, 2, 3, \ldots$.

2) $\left\{ \binom{z}{i} \middle| \, i = 0, 1, 2, 3, \ldots \right\}$ is a basis of a free K_*-module $K_*(\boldsymbol{CP}^\infty)$ (resp. a free abelian group $K_0(\boldsymbol{CP}^\infty)$). Here we set $\binom{z}{0} = 1$.

Put $\bar{\Psi} = \Psi^{-1} \otimes id_{\boldsymbol{Q}} : K_*(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q} \to K_*(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q}$. Since $\bar{\Psi}(\beta_1) = \beta_1$ by Proposition 2.1, we have $\bar{\Psi}(z) = \bar{\Psi}(t^{-1}\beta_1) = -t^{-1}\bar{\Psi}(\beta_1) = -t^{-1}\beta_1 = -z$. Thus $\bar{\Psi}$ is a ring homomorphism given by $\bar{\Psi}(t) = -t$ and $\bar{\Psi}(z) = -z$.

It is clear that 1 and -1 are eigen values of $\bar{\Psi}$. Let us denote by W_* and Z_* the eigen spaces of $\bar{\Psi}$ corresponding to eigen values 1 and -1 respectively. We set $W_n = W_* \cap (K_n(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q})$ and $Z_n = Z_* \cap (K_n(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q})$. The following assertion is straightforward.

Proposition 3.2. *Basis of W_{4k}, W_{4k-2}, Z_{4k}, Z_{4k-2} are given by the following sets of monomials, respectively.*

$$\left\{ t^{2k} z^{2i} \middle| \, i = 0, 1, 2, \ldots \right\}, \quad \left\{ t^{2k-1} z^{2i-1} \middle| \, i = 1, 2, 3, \ldots \right\},$$
$$\left\{ t^{2k} z^{2i-1} \middle| \, i = 1, 2, 3, \ldots \right\}, \quad \left\{ t^{2k-1} z^{2i} \middle| \, i = 0, 1, 2, \ldots \right\}$$

We define $F_k(z) \in \widetilde{K}_0(\boldsymbol{CP}^\infty) \subset \boldsymbol{Q}[z]$ for $k = 1, 2, \ldots$ by

$$F_{2i-1}(z) = \sum_{j=i}^{2i-1} \binom{i-1}{j-i} \binom{z}{j}, \quad F_{2i}(z) = \sum_{j=i}^{2i} \left(\binom{i}{j-i} + \binom{i-1}{j-i-1} \right) \binom{z}{j}.$$

Proposition 3.3.

$$F_{2i-1}(z) = \binom{z+i-1}{2i-1} = \frac{1}{(2i-1)!} z(z^2-1^2)(z^2-2^2)\cdots(z^2-(i-1)^2)$$
$$F_{2i}(z) = \frac{z}{i}\binom{z+i-1}{2i-1} = \frac{2}{(2i)!} z^2(z^2-1^2)(z^2-2^2)\cdots(z^2-(i-1)^2)$$

Proof. Put $\widetilde{F}_{2i-1}(z) = \binom{z+i-1}{2i-1}$, $\widetilde{F}_{2i}(z) = \frac{2}{i}\binom{z+i-1}{2i-1}$. It is easy to verify that $F_k(z) = \widetilde{F}_k(z)$ for $k = 1, 2, 3, 4$ and $\widetilde{F}_k(z+1) - 2\widetilde{F}_k(z) + \widetilde{F}_k(z-1) = \widetilde{F}_{k-1}(z)$

for $k \geq 2$. Since $\binom{z+1}{j} - 2\binom{z}{j} + \binom{z-1}{j} = \binom{z-1}{j-2}$, we have

$$
\begin{aligned}
F_{2i-1}(z+1) - 2F_{2i-1}(z) + F_{2i-1}(z-1) &= \sum_{j=i}^{2i-1} \binom{i-1}{j-i}\binom{z-1}{j-2} \\
&= \sum_{k=i-1}^{2i-2} \binom{i-1}{k-i+1}\binom{z-1}{k-1} \\
&= \sum_{k=i}^{2i-2} \binom{i-2}{k-i}\binom{z-1}{k-1} + \sum_{k=i-1}^{2i-3} \binom{i-2}{k-i+1}\binom{z-1}{k-1} \\
&= \sum_{j=i-1}^{2i-3} \binom{i-2}{j-i+1}\binom{z-1}{j} + \sum_{j=i-1}^{2i-3} \binom{i-2}{j-i+1}\binom{z-1}{j-1} \\
&= \sum_{j=i-1}^{2i-3} \binom{i-2}{j-i+1}\binom{z}{j} = F_{2i-3}(z)
\end{aligned}
$$

Assume $i \geq 3$ and $F_{2i-3}(z) = \widetilde{F}_{2i-3}(z)$. Then, $F_{2i-1}(z+1) - 2F_{2i-1}(z) + F_{2i-1}(z-1) = \widetilde{F}_{2i-1}(z+1) - 2\widetilde{F}_{2i-1}(z) + \widetilde{F}_{2i-1}(z-1)$.

Note that $F_{2i-1}(0) = F_{2i-1}(1) = F_{2i-1}(2) = 0$ if $i \geq 3$. Put $a_n = F_{2i-1}(n) - \widetilde{F}_{2i-1}(n)$ for $n = 0, 1, 2, \ldots$. Then $a_0 = a_1 = a_2 = 0$ and $a_{n+1} - a_n = a_n - a_{n-1}$ for $n = 1, 2, \ldots$. Hence $a_n - a_{n-1} = a_1 - a_0 = 0$ and $a_n = a_0 = 0$ for $n = 0, 1, 2, \ldots$. Therefore $F_{2i-1}(n) = \widetilde{F}_{2i-1}(n)$ for $n = 0, 1, 2, \ldots$. Since $F_{2i-1}(z)$ and $\widetilde{F}_{2i-1}(z)$ are polynomials of z, $F_{2i-1}(z) = \widetilde{F}_{2i-1}(z)$. Proof of $F_{2i}(z) = \widetilde{F}_{2i}(z)$ is similar. □

By the above result, we see that $F_{2i}(-z) = F_{2i}(z)$ and $F_{2i-1}(-z) = -F_{2i-1}(z)$. Hence $F_{2i}(z) \in W_* \cap \widetilde{K}_0(\boldsymbol{C}P^\infty)$ and $F_{2i-1}(z) \in Z_* \cap \widetilde{K}_0(\boldsymbol{C}P^\infty)$. More precisely, the following result holds.

Corollary 3.1. *Let $e = 0$ or 1. $\left\{ t^{2k-e}F_{2i-e}(z) \middle| \, i = 1, 2, \ldots \right\}$ is a basis of $W_* \cap \widetilde{K}_{4k-2e}(\boldsymbol{C}P^\infty)$ over $\boldsymbol{Z}$. Similarly, $\left\{ t^{2k-e}F_{2i+e-1}(z) \middle| \, i = 1, 2, \ldots \right\}$ is a basis of $Z_* \cap \widetilde{K}_{4k-2e}(\boldsymbol{C}P^\infty)$ over $\boldsymbol{Z}$.*

Proof. Note that $F_{2i}(z)$ (resp. $F_{2i-1}(z)$) is a polynomial of degree $2i$ (resp. $2i-1$) which is a linear combination of monomials $z^2, z^4, \ldots, z^{2i}$ (resp $z, z^3, \ldots, z^{2i-1}$). Hence z^{2i} (resp. z^{2i-1}) is a linear combination of

$$F_2(z), F_4(z), \ldots, F_{2i}(z) \text{ (resp. } \quad F_1(z), F_3(z), \ldots, F_{2i-1}(z)).$$

It follows that $\{F_{2i}(z) | \, i = 1, 2, 3, \ldots\}$ (resp. $\{F_{2i-1}(z) | \, i = 1, 2, 3, \ldots\}$) is a basis of $W_* \cap (\widetilde{K}_0(\boldsymbol{C}P^\infty) \otimes \boldsymbol{Q})$ (resp. $Z_* \cap (\widetilde{K}_0(\boldsymbol{C}P^\infty) \otimes \boldsymbol{Q})$) over $\boldsymbol{Q}$.

Suppose $\zeta \in W_* \cap \widetilde{K}_0(\boldsymbol{C}P^\infty)$. Then, $\zeta = \sum_{i=1}^{n} m_i F_{2i}(z)$ for some $m_i \in \boldsymbol{Q}$

$(i = 1, 2, \ldots, n)$. By the definition of $F_{2i}(z)$, we have

$$\zeta = \sum_{j=1}^{2n} \left(\sum_{\frac{j}{2} \le i \le j} \left(\binom{i}{j-i} + \binom{i-1}{j-i-1} \right) m_i \right) \binom{z}{j}.$$

Since $\left\{ \binom{z}{i} \middle| \, i \ge 1 \right\}$ is a basis of a free abelian group $\widetilde{K}_0(\boldsymbol{CP}^\infty)$,

$$m_j + \sum_{\frac{j}{2} \le i \le j-1} \left(\binom{i}{j-i} + \binom{i-1}{j-i-1} \right) m_i \in \boldsymbol{Z}.$$

Now, we can show that every m_j is an integer by induction on j. Hence $\{F_{2i}(z) |\, i = 1, 2, 3, \ldots\}$ generates $W_* \cap \widetilde{K}_0(\boldsymbol{CP}^\infty)$ over $\boldsymbol{Z}$.

Similarly, $\{F_{2i-1}(z) |\, i = 1, 2, 3, \ldots\}$ generates $Z_* \cap \widetilde{K}_0(\boldsymbol{CP}^\infty)$ over $\boldsymbol{Z}$.

Since the multiplication map $t\times : \widetilde{K}_n(\boldsymbol{CP}^\infty) \to \widetilde{K}_{n+2}(\boldsymbol{CP}^\infty)$ by t maps $W_{-e} \cap \widetilde{K}_n(\boldsymbol{CP}^\infty)$ isomorphically onto $W_{e-1} \cap \widetilde{K}_{n+2}(\boldsymbol{CP}^\infty)$, the assertion follows from the above result. □

Proposition 3.4. *In $\boldsymbol{Q}[z]$, the following formula holds.*

$$\binom{z}{j} = \sum_{1 \le i \le \frac{j+1}{2}} \frac{(-1)^{j-1}}{2} \left(\binom{j-i}{i-1} + \binom{j-i-1}{i-2} \right) F_{2i-1}(z) + \sum_{1 \le i \le \frac{j}{2}} \frac{(-1)^j}{2} \binom{j-i-1}{i-1} F_{2i}(z)$$

Here we set $\binom{j-2}{-1} = \begin{cases} 1 & j = 1 \\ 0 & j \ne 1 \end{cases}.$

Proof. Let q and m be positive integers. In $\boldsymbol{Q}[[z]]$, we have $(1-z)^{-q} = \sum\limits_{k \ge 0} (-1)^k \binom{-q}{k} z^k = \sum\limits_{k \ge 0} \binom{k+q-1}{q-1} z^k$. Since $(1-z)^{m-q} = (1-z)^{-q}(1-z)^m = \left(\sum\limits_{k \ge 0} \binom{k+q-1}{q-1} z^k \right) \left(\sum\limits_{r \ge 0} (-1)^r \binom{m}{m-r} z^r \right)$, the coefficient of z^p in $(1-z)^{m-q}$ is

$$\sum_{k \ge 0} (-1)^{p-k} \binom{q+k-1}{q-1} \binom{m}{k+m-p}.$$

On the other hand, the coefficient of z^p in $(1-z)^{m-q}$ is $\binom{q-m+p-1}{q-m-1}$ if $q > m$, $(-1)^p \binom{m-q}{p}$ if $q \le m$. Thus we have

$$\sum_{k \ge 0} (-1)^k \binom{k+q-1}{q-1} \binom{m}{k+m-p} = \begin{cases} (-1)^p \binom{q-m+p-1}{q-m-1} & q > m \\ \binom{m-q}{p} & q \le m \end{cases}.$$

Apply this for $(m, p, q) = (j-1, 2j-2i, i), (j-1, 2j-2i, i-1), (j, 2j-2i+1, i), (j, 2j-2i+1, i-1)$, $(j-1, 2j-2i+1, i), (j-1, 2j-2i+1, i-1), (j-$

$1, 2j-2i-1, i), (j, 2j-2i, i), (j-1, 2j-2i, i)$. We have the following formulas, where δ_{ij} denotes the Kronecker's delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

$$\sum_{2i-1, j \le k \le 2j-1} (-1)^{k-1} \binom{k-i}{i-1} \binom{j-1}{k-j} = \sum_{2i-1, j \le k \le 2j-1} (-1)^{k-1} \binom{k-i-1}{i-2} \binom{j-1}{k-j} = \delta_{ij},$$
$$\sum_{2i-1, j \le k \le 2j} (-1)^{k-1} \binom{k-i}{i-1} \binom{j}{k-j} = 0, \quad \sum_{2i-1, j \le k \le 2j} (-1)^{k-1} \binom{k-i-1}{i-2} \binom{j}{k-j} = \delta_{ij},$$
$$\sum_{2i-1, j \le k \le 2j} (-1)^{k-1} \binom{k-i}{i-1} \binom{j-1}{k-j-1} = -\delta_{ij},$$
$$\sum_{2i-1, j \le k \le 2j} (-1)^{k-1} \binom{k-i-1}{i-2} \binom{j-1}{k-j-1} = 0, \quad \sum_{2i, j \le k \le 2j-1} (-1)^{k} \binom{k-i-1}{i-1} \binom{j-1}{k-j} = 0,$$
$$\sum_{k \ge 2i} (-1)^{k} \binom{k-i-1}{i-1} \binom{j}{k-j} = \sum_{k \ge 2j} (-1)^{k} \binom{k-i-1}{i-1} \binom{j-1}{k-j-1} = \delta_{ij}.$$

Let A be a matrix whose $(i, 2j-1)$-entry is $\binom{j-1}{i-j}$, $(i, 2j)$-entry is $\binom{j}{i-j} + \binom{j-1}{i-j-1}$ and B a matrix whose $(2i-1, j)$-entry is $\frac{(-1)^{j-1}}{2} \left(\binom{j-i}{i-1} + \binom{j-i-1}{i-2} \right)$, $(2i, j)$-entry is $\frac{(-1)^{j}}{2} \binom{j-i-1}{i-1}$. Using the above equalities, it is straightforward to verify that BA is the unit matrix. □

Corollary 3.2. *The following equalities hold.*

$$\boldsymbol{cr}(t^{2k}\beta_{2j}) = \sum_{i=1}^{j} \binom{2j-i-1}{i-1} t^{2k+2j} F_{2i}(z)$$
$$\boldsymbol{cr}(t^{2k-1}\beta_{2j-1}) = -\sum_{i=1}^{j-1} \binom{2j-i-2}{i-1} t^{2k+2j-2} F_{2i}(z)$$
$$\boldsymbol{cr}(t^{2k-1}\beta_{2j}) = -\sum_{i=1}^{j} \left(\binom{2j-i}{i-1} + \binom{2j-i-1}{i-2} \right) t^{2k+2j-1} F_{2i-1}(z)$$
$$\boldsymbol{cr}(t^{2k}\beta_{2j-1}) = \sum_{i=1}^{j} \left(\binom{2j-i-1}{i-1} + \binom{2j-i-2}{i-2} \right) t^{2k+2j-1} F_{2i-1}(z)$$

Proof. By Proposition 3.4 and Corollary 3.1, we have

$$\Psi^{-1}\left(\binom{z}{j}\right) = -\sum_{1 \le i \le \frac{j+1}{2}} \frac{(-1)^{j-1}}{2} \left(\binom{j-i}{i-1} + \binom{j-i-1}{i-2} \right) F_{2i-1}(z)$$
$$+ \sum_{1 \le i \le \frac{j}{2}} \frac{(-1)^{j}}{2} \binom{j-i-1}{i-1} F_{2i}(z).$$

Hence

$$\binom{z}{j} + \Psi^{-1}\left(\binom{z}{j}\right) = \sum_{1\le i\le \frac{j}{2}} (-1)^j \binom{j-i-1}{i-1} F_{2i}(z)$$
$$\binom{z}{j} - \Psi^{-1}\left(\binom{z}{j}\right) = \sum_{1\le i\le \frac{j+1}{2}} (-1)^{j-1} \left(\binom{j-i}{i-1} + \binom{j-i-1}{i-2}\right) F_{2i-1}(z)$$

Substituting the above equalities into

$$\boldsymbol{cr}(t^n\beta_j) = \boldsymbol{cr}\left(t^{n+j}\binom{z}{j}\right) = t^{n+j}\left(\binom{z}{j} + (-1)^{n+j}\Psi^{-1}\left(\binom{z}{j}\right)\right),$$

the result follows. □

For each positive integer n, we inductively define sequences of integers $(a(n,0), a(n,1), \ldots, a(n,n-1))$, $(c(n,0), c(n,1), \ldots, c(n,n-1))$, $(d(n,0), d(n,1), \ldots, d(n,n-1))$ by $a(n,0) = 1$, $c(n,0) = n$, $d(n,0) = 1$ and

$$a(n,j) = -\sum_{i=0}^{j-1} \binom{n+j-2i-1}{n-j-1} a(n,i)$$
$$c(n,j) = -(n-j)\sum_{i=0}^{j-1}\left(\left(\binom{n-2i+j-1}{n-j-1} + \binom{n-2i+j-2}{n-j-2}\right)c(n,i) - \left(\binom{n-2i+j}{n-j-1} + \binom{n-2i+j-1}{n-j-2}\right)d(n,i)\right)$$
$$d(n,j) = -\sum_{i=0}^{j-1}\left(\left(\binom{n-2i+j-1}{n-j-1} + \binom{n-2i+j-2}{n-j-2}\right)c(n,i) - \left(\binom{n-2i+j}{n-j-1} + \binom{n-2i+j-1}{n-j-2}\right)d(n,i)\right).$$

The following result is a direct consequence of Corollary 3.2.

Proposition 3.5. *The following equalities hold.*

$$\boldsymbol{cr}\left(\sum_{j=0}^{n-1} a(n,j)t^{2j}\beta_{2n-2j}\right) = t^{2n}F_{2n}(z)$$
$$\boldsymbol{cr}\left(\sum_{j=0}^{n-1}\left(c(n,j)t^{2j}\beta_{2n-2j-1} + d(n,j)t^{2j-1}\beta_{2n-2j}\right)\right) = t^{2n-1}F_{2n-1}(z)$$

4. Real K-homology of complex projective spaces

Consider the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q}(KO;\boldsymbol{C}P^l) \cong H_p(\boldsymbol{C}P^l;KO_q) \Rightarrow KO_{p+q}(\boldsymbol{C}P^l).$$

E^2-term is a free KO_*-module generated by

$$\beta_0, \beta_1, \ldots, \beta_l \quad (\beta_j \in E^2_{2j,0}(KO; \boldsymbol{C}P^l)).$$

Lemma 4.1. *$d^2 : E^2_{p,q}(KO; \boldsymbol{C}P^l) \to E^2_{p-2,q+1}(KO; \boldsymbol{C}P^l)$ is given by*

$$d^2(\beta_j) = \begin{cases} \alpha\beta_{j-1} & j \text{ is positive and even} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We first note that the p-skeleton $(\boldsymbol{C}P^l)^p$ is $\boldsymbol{C}P^{[\frac{p}{2}]}$ if $p \leqq 2l$. Hence $E^1_{p,q}(KO, \boldsymbol{C}P^l) = 0$ if p is odd and $E^2_{p,q}(KO; \boldsymbol{C}P^l) = E^1_{p,q}(KO; \boldsymbol{C}P^l) = \widetilde{KO}_{p+q}(\boldsymbol{C}P^{\frac{p}{2}}/\boldsymbol{C}P^{\frac{p}{2}-1})$ if p is positive and even. If p is even, $d_2 : E^2_{p,q}(KO; \boldsymbol{C}P^l) \to E^2_{p-2,q+1}(KO; \boldsymbol{C}P^l)$ coincides with the connecting homomorphism

$$\partial : \widetilde{KO}_{p+q}(\boldsymbol{C}P^{\frac{p}{2}}/\boldsymbol{C}P^{\frac{p}{2}-1}) \to \widetilde{KO}_{p+q-1}(\boldsymbol{C}P^{\frac{p}{2}-1}/\boldsymbol{C}P^{\frac{p}{2}-2})$$

of the long exact sequence associated with the cofibration

$$\boldsymbol{C}P^{\frac{p}{2}-1}/\boldsymbol{C}P^{\frac{p}{2}-2} \to \boldsymbol{C}P^{\frac{p}{2}}/\boldsymbol{C}P^{\frac{p}{2}-2} \to \boldsymbol{C}P^{\frac{p}{2}}/\boldsymbol{C}P^{\frac{p}{2}-1}.$$

Then, the result follows from Lemma 1.4. □

By the above result, β_0, β_{2i-1} $(1 \le i \le \frac{l+1}{2})$, $2\beta_{2i}$, $\alpha^2\beta_{2i}$, $x\beta_{2i}$ $(1 \le i \le \frac{l}{2})$ are cycles of the E^2-term. We denote by $\beta_0 \in E^3_{0,0}(KO; \boldsymbol{C}P^l)$, $\beta_{2i-1,0} \in E^3_{4i-2,0}(KO; \boldsymbol{C}P^l)$, $\beta_{2i,0} \in E^3_{4i,0}(KO; \boldsymbol{C}P^l)$, $\beta_{2i-1,1} \in E^3_{4i,2}(KO; \boldsymbol{C}P^l)$, $\beta_{2i,1} \in E^3_{4i,4}(KO; \boldsymbol{C}P^l)$ the elements of the E^3-term corresponding to β_0, β_{2i-1}, $2\beta_{2i}$, $\alpha^2\beta_{2i}$, $x\beta_{2i}$, respectively. The next result follows from the definitions of these elements and Lemma 4.1.

Proposition 4.1. *The following relations hold for $1 \le i \le \frac{l}{2}$ in the E^3-term.*

$$\alpha\beta_{2i-1,0} = \alpha\beta_{2i,0} = \alpha\beta_{2i-1,1} = \alpha\beta_{2i,1} = 2\beta_{2i-1,1} = x\beta_{2i-1,1} = 0,$$
$$x\beta_{2i,0} = 2\beta_{2i,1}, \quad x\beta_{2i,1} = 2y\beta_{2i,0}$$

By Lemma 4.1, the kernel of d^2 is generated by β_0, β_{2i-1} $(1 \le i \le \frac{l+1}{2})$, $2\beta_{2i}$, $\alpha^2\beta_{2i}$, $x\beta_{2i}$ $(1 \le i \le \frac{l}{2})$ over KO_*. Moreover, image of d^2 is generated by $\alpha\beta_{2i-1}$ $(1 \le i \le \frac{l}{2})$ over KO_*. Thus we have the following.

Proposition 4.2. *E^3-term is generated by the following set of elements over KO_*.*

1) *If l is even,* $\left\{\beta_{2i-1,0}, \beta_{2i,0}, \beta_{2i-1,1}, \beta_{2i,1} \,\middle|\, 1 \le i \le \frac{l}{2}\right\} \cup \{\beta_0\}$.
2) *If l is odd,* $\left\{\beta_{2i-1,0}, \beta_{2i,0}, \beta_{2i-1,1}, \beta_{2i,1} \,\middle|\, 1 \le i \le \frac{l-1}{2}\right\} \cup \{\beta_0, \beta_{l,0}\}$.

Corollary 4.1. $E^3_{*,*}(KO; \boldsymbol{C}P^l) = E^\infty_{*,*}(KO; \boldsymbol{C}P^l)$

Proof. Since $E^3_{p,q}(KO; \boldsymbol{C}P^l) = \{0\}$ if $p+q$ is odd and $0 < p < 2l$, there is no possibility of non-trivial differentials. □

Corollary 4.2. *$\widetilde{KO}_n(\boldsymbol{CP}^l) = \{0\}$ if " l is even and n is odd." or " l is odd and $n \not\equiv 2l+1$ modulo 8."*

Applying Corollary 1.1 to the above result, we have the following.

Corollary 4.3. *$\boldsymbol{c} : \widetilde{KO}_{2n}(\boldsymbol{CP}^l) \to \widetilde{K}_{2n}(\boldsymbol{CP}^l)$ is injective if l is even or $n \not\equiv l+1$ modulo 4. In particular, $\widetilde{KO}_{2n}(\boldsymbol{CP}^l)$ is $\boldsymbol{Z}$-torsion free if l is even or $n \not\equiv l+1$ modulo 4.*

We define elements $\gamma_{i,s} \in KO_{2i+4s}(\boldsymbol{CP}^l)$ for $1 \le i \le 2\left[\frac{l}{2}\right]$, $s = 0, 1$ by

$$\gamma_{2n,s} = \boldsymbol{r}\left(\sum_{j=0}^{n-1} a(n,j)t^{2j+2s}\beta_{2n-2j}\right),$$

$$\gamma_{2n-1,s} = \boldsymbol{r}\left(\sum_{j=0}^{n-1} \left(c(n,j)t^{2j+2s}\beta_{2n-2j-1} + d(n,j)t^{2j+2s-1}\beta_{2n-2j}\right)\right).$$

If l is odd, we define an element $\gamma_{l,0} \in \widetilde{KO}_{2l}(\boldsymbol{CP}^l)$ as follows. Since $\boldsymbol{cr}(t^{l-1}F_l(z)) = 0$ by Corollary 3.1 and $\boldsymbol{c} : \widetilde{KO}_{2l-2}(\boldsymbol{CP}^l) \to \widetilde{K}_{2l-2}(\boldsymbol{CP}^l)$ is injective by Corollary 4.3, we have $\boldsymbol{r}(t^{l-1}F_l(z)) = 0$. It follows from Corollary 1.1 that $t^lF_l(z)$ is in the image of $\boldsymbol{c} : \widetilde{KO}_{2l}(\boldsymbol{CP}^l) \to \widetilde{K}_{2l}(\boldsymbol{CP}^l)$ which is injective by Corollary 4.3. Hence there exists an unique element $\gamma_{l,0} \in \widetilde{KO}_{2l}(\boldsymbol{CP}^l)$ that maps to $t^lF_l(z)$ by $\boldsymbol{c}$.

We put

$$\lambda_{2i-1} = t^{2i-1}F_{2i-1}(z) = \sum_{j=i}^{2i-1} \binom{i-1}{j-i} t^{2i-j-1}\beta_j \in \widetilde{K}_{4i-2}(\boldsymbol{CP}^l)$$

$$\lambda_{2i} = t^{2i}F_{2i}(z) = \sum_{j=i}^{2i} \left(\binom{i}{j-i} + \binom{i-1}{j-i-1}\right) t^{2i-j}\beta_j \in \widetilde{K}_{4i}(\boldsymbol{CP}^l).$$

Remark 4.1. 1) It follows from Proposition 3.3 that $\lambda_i \in K_*(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q} = \boldsymbol{Q}[t, t^{-1}, z]$ belongs to the subalgebra of $\boldsymbol{Q}[t, t^{-1}, z]$ generated by $\lambda_1 = tz$ and t^2.

2) By Proposition 3.4, we have the following equality in $K_*(\boldsymbol{CP}^l) \otimes \boldsymbol{Q}$.

$$\beta_j = \sum_{1 \le i \le \frac{j+1}{2}} \frac{(-1)^{j-1}}{2} \left(\binom{j-i}{i-1} + \binom{j-i-1}{i-2}\right) t^{j-2i+1}\lambda_{2i-1}$$
$$+ \sum_{1 \le i \le \frac{j}{2}} \frac{(-1)^j}{2} \binom{j-i-1}{i-1} t^{j-2i}\lambda_{2i}$$

(3.5) implies the following.

Lemma 4.2. *$\boldsymbol{c} : KO_*(\boldsymbol{CP}^l) \to K_*(\boldsymbol{CP}^l)$ maps $\gamma_{i,s}$ to $t^{2s}\lambda_i$.*

Lemma 4.3. *$2\beta_{2i} \in E^2_{4i,0}(KO;\boldsymbol{C}P^l)$, $\alpha^2\beta_{2i} \in E^2_{4i,2}(KO;\boldsymbol{C}P^l)$ and $x\beta_{2i} \in E^2_{4i,4}(KO;\boldsymbol{C}P^l)$ $(1 \le i \le \frac{l}{2})$ are permanent cycles corresponding to $\gamma_{2i,0}$, $\gamma_{2i-1,1}$ and $\gamma_{2i,1}$, respectively. Hence $\gamma_{2i,0} \in F_{4i,0} - F_{4i-1,1}$, $\gamma_{2i-1,1} \in F_{4i,2} - F_{4i-1,3}$ and $\gamma_{2i,1} \in F_{4i,4} - F_{4i-1,5}$.*

Proof. There is a map $\boldsymbol{r}^r : E^r_{p,q}(K;\boldsymbol{C}P^l) \to E^r_{p,q}(KO;\boldsymbol{C}P^l)$ of spectral sequences induced by $\boldsymbol{r} : K_*(\boldsymbol{C}P^l) \to KO_*(\boldsymbol{C}P^l)$. Since $\boldsymbol{r}^2(\beta_{2i}) = 2\beta_{2i}$, $\boldsymbol{r}^2(t\beta_{2i}) = \alpha^2\beta_{2i}$ and $\boldsymbol{r}^2(t^2\beta_{2i}) = x\beta_{2i}$ by 2) of (1.1), the result follows. $\square$

Lemma 4.4. *$\gamma_{2i-1,0}$ belongs to the image $F_{4i-2,0}$ of the map*

$$KO_{4i-2}(\boldsymbol{C}P^{2i-1}) \to KO_{4i-2}(\boldsymbol{C}P^l)$$

induced by the inclusion map for $1 \le i \le \frac{l+1}{2}$. On the other hand, $\gamma_{2i-1,0}$ does not belong to the image $F_{4i-3,1}$ of the map $KO_{4i-2}(\boldsymbol{C}P^{2i-2}) \to KO_{4i-2}(\boldsymbol{C}P^l)$.

Proof. Since $F_{2l,0} = KO_{2l}(\boldsymbol{C}P^l)$, it suffices to show $\gamma_{2i-1,0} \in F_{4i-2,0}$ for $1 \le i \le \frac{l}{2}$. Let us denote by $p : \boldsymbol{C}P^l \to \boldsymbol{C}P^l/\boldsymbol{C}P^{2i-1}$, $p' : \boldsymbol{C}P^{2i} \to \boldsymbol{C}P^{2i}/\boldsymbol{C}P^{2i-1} = S^{4i}$ the quotient maps and $\iota : \boldsymbol{C}P^{2i} \to \boldsymbol{C}P^l$, $\iota' : \boldsymbol{C}P^{2i}/\boldsymbol{C}P^{2i-1} \to \boldsymbol{C}P^l/\boldsymbol{C}P^{2i-1}$ the inclusion maps. We put

$$\tilde{\gamma}_{2i-1,0} = \sum_{j=0}^{n-1} \left(c(n,j)t^{2j}\beta_{2n-2j-1} + d(n,j)t^{2j-1}\beta_{2n-2j} \right).$$

$\tilde{\gamma}_{2i-1,0}$ is regarded as an element of $K_{4i-2}(\boldsymbol{C}P^{2i})$ and $\gamma_{2i-1,0} \in KO_{4i-2}(\boldsymbol{C}P^l)$ is the image of $\tilde{\gamma}_{2i-1,0}$ by the composition $K_{4i-2}(\boldsymbol{C}P^{2i}) \xrightarrow{\iota_*} K_{4i-2}(\boldsymbol{C}P^l) \xrightarrow{\boldsymbol{r}} KO_{4i-2}(\boldsymbol{C}P^l)$. Since $\beta_1, \beta_2, \ldots, \beta_{2i-1} \in K_{4i-2}(\boldsymbol{C}P^{2i})$ are in the image of $K_{4i-2}(\boldsymbol{C}P^{2i-1}) \to K_{4i-2}(\boldsymbol{C}P^{2i})$, $p'_* : K_{4i-2}(\boldsymbol{C}P^{2i}) \to K_{4i-2}(S^{4i})$ maps $\tilde{\gamma}_{2i-1,0}$ to $t^{-1}u_{4i}$. It follows from (1.1) that $\boldsymbol{r} : K_{4i-2}(S^{4i}) \to KO_{4i-2}(S^{4i})$ maps $t^{-1}u_{4i}$ to zero. By the commutativity of the following diagram, $p_*(\gamma_{2i-1,0}) = p_*(\boldsymbol{r}(\iota_*(\tilde{\gamma}_{2i-1,0}))) = \iota'_*(\boldsymbol{r}(p'_*(\tilde{\gamma}_{2i-1,0}))) = \iota'_*(\boldsymbol{r}(t^{-1}u_{4i})) = 0$.

$$\begin{array}{ccccc}
K_{4i-2}(S^{4i}) & \xleftarrow{p'_*} & K_{4i-2}(\boldsymbol{C}P^{2i}) & \xrightarrow{\iota_*} & K_{4i-2}(\boldsymbol{C}P^l) \\
\downarrow{\scriptstyle \boldsymbol{r}} & & \downarrow{\scriptstyle \boldsymbol{r}} & & \downarrow{\scriptstyle \boldsymbol{r}} \\
KO_{4i-2}(S^{4i}) & \xleftarrow{p'_*} & KO_{4i-2}(\boldsymbol{C}P^{2i}) & \xrightarrow{\iota_*} & KO_{4i-2}(\boldsymbol{C}P^l) \\
 & & \downarrow{\scriptstyle p'_*} & & \downarrow{\scriptstyle p_*} \\
 & & KO_{4i-2}(\boldsymbol{C}P^{2i}/\boldsymbol{C}P^{2i-1}) & \xrightarrow{\iota'_*} & KO_{4i-2}(\boldsymbol{C}P^l/\boldsymbol{C}P^{2i-1})
\end{array}$$

Hence $\gamma_{2i-1,0} \in \operatorname{Ker}\ p_* = \operatorname{Im}(KO_{4i-2}(\boldsymbol{C}P^{2i-1}) \to KO_{4i-2}(\boldsymbol{C}P^l))$.

By (4.2), $\boldsymbol{c}(\gamma_{2i-1,0}) = \lambda_{2i-1} = \sum_{j=i}^{2i-1} \binom{i-1}{j-i} t^{2i-j-1}\beta_j \in F_{4i-2,0} - F_{4i-3,1}$. Therefore $\gamma_{2i-1,0} \notin F_{4i-3,1}$. $\square$

Lemma 4.5. *$\beta_{2i-1} \in E^2_{4i-2,0}(KO;\boldsymbol{CP}^l)$ is the permanent cycle corresponding to $\gamma_{2i-1,0}$.*

Proof. Since $E^2_{4i-2,0}(KO;\boldsymbol{CP}^l)$ is isomorphic to $\boldsymbol{Z}$ generated by β_{2i-1}, the above result implies that there exists $m \in \boldsymbol{Z}$ such that $m\beta_{2i-1}$ is the permanent cycle corresponding to $\gamma_{2i-1,0}$. Consider a map $\boldsymbol{c}^r : E^r_{p,q}(KO;\boldsymbol{CP}^l) \to E^r_{p,q}(K;\boldsymbol{CP}^l)$ of spectral sequences induced by $\boldsymbol{c} : KO_*(\boldsymbol{CP}^l) \to K_*(\boldsymbol{CP}^l)$. Since $\boldsymbol{c}^2(m\beta_{2i-1}) = m\beta_{2i-1}$ is the permanent cycle corresponding to $\boldsymbol{c}(\gamma_{2i-1,0})$ and $\boldsymbol{c}(\gamma_{2i-1,0}) = \lambda_{2i-1} \equiv \beta_{2i-1}$ modulo $F_{4i-3,1}$, $m\beta_{2i-1} \in E^2_{2i-1,0}(K;\boldsymbol{CP}^l)$ is the permanent cycle corresponding to $\beta_{2i-1} \in K_{4i-2}(\boldsymbol{CP}^l)$. Therefore we have $m = 1$. □

Let us denote by $\beta_0 \in KO_0(\boldsymbol{CP}^l)$ the unique element corresponding to $\beta_0 \in E^3_{0,0}(KO;\boldsymbol{CP}^l)$. Clearly, $\boldsymbol{c}(\beta_0) = \beta_0$.

Theorem 4.1. *$KO_*(\boldsymbol{CP}^l)$ is generated by the following set of elements over KO_*.*
1) *If l is even,* $\left\{\gamma_{2i-1,0}, \gamma_{2i,0}, \gamma_{2i-1,1}, \gamma_{2i,1} \,\middle|\, 1 \le i \le \frac{l}{2}\right\} \cup \{\beta_0\}$.
2) *If l is odd,* $\left\{\gamma_{2i-1,0}, \gamma_{2i,0}, \gamma_{2i-1,1}, \gamma_{2i,1} \,\middle|\, 1 \le i \le \frac{l-1}{2}\right\} \cup \{\beta_0, \gamma_{l,0}\}$.

By the definition of $\gamma_{i,s}$ and the above result, we have the following.

Corollary 4.4. *The image of $\boldsymbol{c} : \widetilde{KO}_{2n}(\boldsymbol{CP}^l) \to \widetilde{K}_{2n}(\boldsymbol{CP}^l)$ is spanned over $\boldsymbol{Z}$ by*
$\left\{t^{n-2i}\lambda_{2i} \,\middle|\, 1 \le i \le \frac{l}{2}\right\}$ *if n is even,* $\left\{t^{n-2i+1}\lambda_{2i-1} \,\middle|\, 1 \le i \le \frac{l+1}{2}\right\}$ *if n is odd.*

We also have the following result from (1.1), (4.1) and (4.2).

Corollary 4.5. *The image of $\boldsymbol{c} \otimes id_{\boldsymbol{Q}} : KO_*(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q} \to K_*(\boldsymbol{CP}^\infty) \otimes \boldsymbol{Q}$ is the subalgebra generated by tz and t^2.*

Theorem 4.2. *Relations $\alpha\gamma_{i,s} = 0$, $x\gamma_{i,s} = 2y^s\gamma_{i,1-s}$ hold for $1 \le i \le 2\left[\frac{l}{2}\right]$ and $s = 0, 1$ in $KO_*(\boldsymbol{CP}^l)$.*

Proof. First, assume that l is even. We have $\widetilde{KO}_n(\boldsymbol{CP}^l) = \{0\}$ for odd n. Hence $\alpha\gamma_{i,s} = 0$ for dimensional reason. By (4.2) and (1.1), $\boldsymbol{c}(x\gamma_{i,0}) = \boldsymbol{c}(2\gamma_{i,1}) = 2t^2\lambda_i$ and $\boldsymbol{c}(x\gamma_{i,1}) = \boldsymbol{c}(2y\gamma_{i,0}) = 2t^4\lambda_i$. Since $\boldsymbol{c} : \widetilde{K}_{2n}(\boldsymbol{CP}^l) \to \widetilde{KO}_{2n}(\boldsymbol{CP}^l)$ is injective by (4.3), we have $x\gamma_{i,0} = 2\gamma_{i,1}$ and $x\gamma_{i,1} = 2y\gamma_{i,0}$. If l is odd, since $\gamma_{i,s} \in KO_*(\boldsymbol{CP}^l)$ $(1 \le i \le 2\left[\frac{l}{2}\right],\ s = 0, 1)$ are the images of $\gamma_{i,s} \in KO_*(\boldsymbol{CP}^{l-1})$ by the map induced by the inclusion map, the same relations holds in $KO_*(\boldsymbol{CP}^l)$. □

Proposition 3.4 enable us to describe the realization map $\boldsymbol{r} : \widetilde{K}_*(\boldsymbol{CP}^l) \to \widetilde{KO}_*(\boldsymbol{CP}^l)$.

Proposition 4.3.

$$\begin{aligned}
\boldsymbol{r}(t^{4n-j}\beta_j) = & \sum_{1\le k\le \frac{j}{4}} (-1)^j \binom{j-2k-1}{2k-1} y^{n-k}\gamma_{4k,0} \\
& + \sum_{1\le k\le \frac{j+2}{4}} (-1)^j \binom{j-2k}{2k-2} y^{n-k}\gamma_{4k-2,1} \\
\boldsymbol{r}(t^{4n+2-j}\beta_j) = & \sum_{1\le k\le \frac{j}{4}} (-1)^j \binom{j-2k-1}{2k-1} y^{n-k}\gamma_{4k,1} \\
& + \sum_{1\le k\le \frac{j+2}{4}} (-1)^j \binom{j-2k}{2k-2} y^{n-k+1}\gamma_{4k-2,0} \\
\boldsymbol{r}(t^{4n+1-j}\beta_j) = & \sum_{1\le k\le \frac{j+1}{4}} (-1)^{j-1} \left(\binom{j-2k}{2k-1} + \binom{j-2k-1}{2k-2}\right) y^{n-k}\gamma_{4k-1,1} \\
& + \sum_{1\le k\le \frac{j+3}{4}} (-1)^{j-1} \left(\binom{j-2k+1}{2k-2} + \binom{j-2k}{2k-3}\right) y^{n-k+1}\gamma_{4k-3,0} \\
\boldsymbol{r}(t^{4n+3-j}\beta_j) = & \sum_{1\le k\le \frac{j+1}{4}} (-1)^{j-1} \left(\binom{j-2k}{2k-1} + \binom{j-2k-1}{2k-2}\right) y^{n-k+1}\gamma_{4k-1,0} \\
& + \sum_{1\le k\le \frac{j+3}{4}} (-1)^{j-1} \left(\binom{j-2k+1}{2k-2} + \binom{j-2k}{2k-3}\right) y^{n-k+1}\gamma_{4k-3,1}
\end{aligned}$$

Proof. Since

$$\boldsymbol{cr}(t^{m-j}\beta_j) = t^{m-j}\beta_j + \Psi^{-1}(t^{m-j}\beta_j) = t^m\binom{z}{j} + \bar{\Psi}\left(t^m\binom{z}{j}\right) = t^m\binom{z}{j} + (-t)^m\binom{-z}{j}$$

and

$$\begin{aligned}
\binom{z}{j} + \binom{-z}{j} &= \sum_{1\le i\le \frac{j}{2}} (-1)^j \binom{j-i-1}{i-1} F_{2i}(z) \\
\binom{z}{j} - \binom{-z}{j} &= \sum_{1\le i\le \frac{j+1}{2}} (-1)^{j-1} \left(\binom{j-i}{i-1} + \binom{j-i-1}{i-2}\right) F_{2i-1}(z)
\end{aligned}$$

by Proposition 3.4, the result follows from the definition of $\gamma_{i,s}$'s and the injectivity of $\boldsymbol{c}$. □

Remark 4.2. Let us denote by $\eta_L, \eta_R : KO_* \to KO_*KO$ be the left, right unit of Hopf algebroid (KO_*, KO_*KO). We denote by $\varphi : \widetilde{KO}_*(\boldsymbol{CP}^\infty) \to KO_*KO \otimes_{KO_*} \widetilde{KO}_*(\boldsymbol{CP}^\infty)$ the KO_*KO-comodule structure map and by $\psi : \widetilde{K}_*(\boldsymbol{CP}^\infty) \to K_*K \otimes_{K_*} \widetilde{K}_*(\boldsymbol{CP}^\infty)$ the K_*K-comodule structure map.

Since $\eta_L(\alpha) = \eta_R(\alpha)$ ([3]) and $\widetilde{KO}_*(\boldsymbol{CP}^\infty)$ is α-torsion group, it follows from (4.3) that $KO_*KO \otimes_{KO_*} \widetilde{KO}_*(\boldsymbol{CP}^\infty)$ is $\boldsymbol{Z}$-torsion free. Hence the vertical maps of the following diagram is injective by (4.3) and the results on KO_*KO

([3]).

$$\begin{array}{ccc} \widetilde{KO}_*(\boldsymbol{C}P^\infty) & \xrightarrow{\varphi} & KO_*KO \otimes_{KO_*} \widetilde{KO}_*(\boldsymbol{C}P^\infty) \\ \downarrow \boldsymbol{c} & & \downarrow (\boldsymbol{c}\wedge\boldsymbol{c})\otimes\boldsymbol{c} \\ \widetilde{K}_*(\boldsymbol{C}P^\infty) & \xrightarrow{\psi} & K_*K \otimes_{K_*} \widetilde{K}_*(\boldsymbol{C}P^\infty) \end{array}$$

If we set $\psi(\beta_j) = \sum_{i=1}^{j} \alpha_{ij} \otimes \beta_i$ ($\alpha_{ij} \in K_{2j-2i}K$), then $\alpha_{jj} = 1$ and a relation $(j+1)\beta_{j+1} = \beta_1\beta_j - jt\beta_j$ implies a recursive formula on α_{ij}'s.

$$(j+1)\alpha_{ij+1} = (iv - ju)\alpha_{ij} + i\alpha_{i-1j} \quad \text{and} \quad \alpha_{ii} = 1 \quad \text{for} \quad j \geq i \geq 1$$

Here, we put $u = \eta_L(t)$ and $v = \eta_R(t)$ for the left, right unit $\eta_L, \eta_R : K_* \to K_*K$ of Hopf algebroid (K_*, K_*K). In particular, we have $(j+1)\alpha_{1j+1} = (v-ju)\alpha_{1j}$, hence $\alpha_{1j} = u^j v^{-1}\binom{u^{-1}v}{j}$ for $j \geq 1$. It seems to be difficult to give a good description of α_{ij} for $i \geq 2$. Using (4.2), 2) of (4.1) and above observation, it may be possible to determine $\psi(\gamma_{i,s})$ for small i.

5. Pontrjagin ring structure of $KO_*(\boldsymbol{C}P^\infty)$

The relation $\beta_i\beta_j = \sum_{i,j \leq k \leq i+j} \frac{k!}{(k-i)!(k-j)!(i+j-k)!} t^{i+j-k}\beta_k$ implies the following formula.

Proposition 5.1.

$$\binom{z}{i}\binom{z}{j} = \sum_{i,j \leq k \leq i+j} \frac{k!}{(k-i)!(k-j)!(i+j-k)!}\binom{z}{k}$$

We put

$$A_{i,j,k} = \sum_{p=i}^{2i-1}\sum_{q=j}^{2j-1}\sum_{p,q,2k \leq r \leq p+q} \frac{(-1)^r}{2}\binom{i-1}{p-i}\binom{j-1}{q-j}\binom{r}{p}\binom{p}{r-q}\binom{r-k-1}{k-1},$$

$$B_{i,j,k} = \sum_{p=i}^{2i-1}\sum_{q=j}^{2j}\sum_{p,q,2k \leq r \leq p+q} \frac{(-1)^{r-1}}{2}\binom{i-1}{p-i}\left(\binom{j}{q-j} + \binom{j-1}{q-j-1}\right) \times \binom{r}{p}\binom{p}{r-q}\left(\binom{r-k}{k-1} + \binom{r-k-1}{k-2}\right),$$

$$C_{i,j,k} = \sum_{p=i}^{2i}\sum_{q=j}^{2j}\sum_{p,q,2k \leq r \leq p+q} \frac{(-1)^r}{2}\left(\binom{i}{p-i} + \binom{i-1}{p-i-1}\right) \times \left(\binom{j}{q-j} + \binom{j-1}{q-j-1}\right)\binom{r}{p}\binom{p}{r-q}\binom{r-k-1}{k-1}.$$

Proposition 5.2. *The following relations hold in $K_*(\boldsymbol{C}P^\infty)$.*

$$F_{2i-1}(z)F_{2j-1}(z) = \sum_{k=1}^{i+j-1} A_{i,j,k}F_{2k}(z),$$
$$F_{2i-1}(z)F_{2j}(z) = \sum_{k=1}^{i+j} B_{i,j,k}F_{2k-1}(z),$$
$$F_{2i}(z)F_{2j}(z) = \sum_{k=1}^{i+j} C_{i,j,k}F_{2k}(z).$$

Proof. Since $F_{2i-1}(z)F_{2j-1}(z)$, $F_{2i}(z)F_{2j}(z)$ belong to $W_* \cap \widetilde{K}_0(\boldsymbol{C}P^\infty)$ and $F_{2i-1}(z)F_{2j}(z)$ belongs to $Z_* \cap \widetilde{K}_0(\boldsymbol{C}P^\infty)$, $F_{2i-1}(z)F_{2j-1}(z)$, $F_{2i}(z)F_{2j}(z)$ are linear combinations of $F_{2k}(z)$'s and $F_{2i-1}(z)F_{2j}(z)$ is a linear combination of $F_{2k-1}(z)$'s. The result follows from the definition of $F_i(z)$, (5.1) and Proposition 3.4. □

The above relations imply the next result which gives the product structure of $KO_*(\boldsymbol{C}P^\infty)$.

Theorem 5.1. *The following relations hold in $KO_*(\boldsymbol{C}P^\infty)$.*

$$\begin{aligned}
\gamma_{2i-1,0}\gamma_{2j-1,0} &= \sum_{0\le s\le \frac{i+j-2}{2}} A_{i,j,i+j-2s-1}y^s\gamma_{2i+2j-4s-2,0} \\
&\quad + \sum_{0\le s\le \frac{i+j-3}{2}} A_{i,j,i+j-2s-2}y^s\gamma_{2i+2j-4s-4,1}, \\
\gamma_{2i-1,0}\gamma_{2j-1,1} &= \sum_{0\le s\le \frac{i+j-2}{2}} A_{i,j,i+j-2s-1}y^s\gamma_{2i+2j-4s-2,1} \\
&\quad + \sum_{0\le s\le \frac{i+j-3}{2}} A_{i,j,i+j-2s-2}y^{s+1}\gamma_{2i+2j-4s-4,0}, \\
\gamma_{2i-1,1}\gamma_{2j-1,1} &= \sum_{0\le s\le \frac{i+j-2}{2}} A_{i,j,i+j-2s-1}y^{s+1}\gamma_{2i+2j-4s-2,0} \\
&\quad + \sum_{0\le s\le \frac{i+j-3}{2}} A_{i,j,i+j-2s-2}y^{s+1}\gamma_{2i+2j-4s-4,1}, \\
\gamma_{2i-1,0}\gamma_{2j,0} &= \sum_{0\le s\le \frac{i+j-1}{2}} B_{i,j,i+j-2s}y^s\gamma_{2i+2j-4s-1,0} \\
&\quad + \sum_{0\le s\le \frac{i+j-2}{2}} B_{i,j,i+j-2s-1}y^s\gamma_{2i+2j-4s-3,1}, \\
\gamma_{2i-1,1}\gamma_{2j,0} &= \sum_{0\le s\le \frac{i+j-1}{2}} B_{i,j,i+j-2s}y^s\gamma_{2i+2j-4s-1,1} \\
&\quad + \sum_{0\le s\le \frac{i+j-2}{2}} B_{i,j,i+j-2s-1}y^{s+1}\gamma_{2i+2j-4s-3,0},
\end{aligned}$$

$$\gamma_{2i-1,0}\gamma_{2j,1} = \sum_{0\le s\le \frac{i+j-1}{2}} B_{i,j,i+j-2s}y^s\gamma_{2i+2j-4s-1,1} + \sum_{0\le s\le \frac{i+j-2}{2}} B_{i,j,i+j-2s-1}y^{s+1}\gamma_{2i+2j-4s-3,0},$$

$$\gamma_{2i-1,1}\gamma_{2j,1} = \sum_{0\le s\le \frac{i+j-1}{2}} B_{i,j,i+j-2s}y^{s+1}\gamma_{2i+2j-4s-1,0} + \sum_{0\le s\le \frac{i+j-2}{2}} B_{i,j,i+j-2s-1}y^{s+1}\gamma_{2i+2j-4s-3,1},$$

$$\gamma_{2i,0}\gamma_{2j,0} = \sum_{0\le s\le \frac{i+j-1}{2}} C_{i,j,i+j-2s}y^s\gamma_{2i+2j-4s,0} + \sum_{0\le s\le \frac{i+j-2}{2}} C_{i,j,i+j-2s-2}y^s\gamma_{2i+2j-4s-2,1},$$

$$\gamma_{2i,0}\gamma_{2j,1} = \sum_{0\le s\le \frac{i+j-1}{2}} C_{i,j,i+j-2s}y^s\gamma_{2i+2j-4s,1} + \sum_{0\le s\le \frac{i+j-2}{2}} C_{i,j,i+j-2s-2}y^{s+1}\gamma_{2i+2j-4s-2,0},$$

$$\gamma_{2i,1}\gamma_{2j,1} = \sum_{0\le s\le \frac{i+j-1}{2}} C_{i,j,i+j-2s}y^{s+1}\gamma_{2i+2j-4s,0} + \sum_{0\le s\le \frac{i+j-2}{2}} C_{i,j,i+j-2s-2}y^{s+1}\gamma_{2i+2j-4s-2,1}$$

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