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On the bounded condition of an o-minimal structure

By

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Abstract

We will show that the theory of ordered divisible vector spaces over an ordered field satisfies the bounded condition treated in [5].

1. Introduction

Grothendieck rings for some first-order structures have been calculated by many authors ([1], [2], [4], [5], [6], [7]). In particular, in [5], the bounded condition of o-minimal expansion of ordered abelian groups was introduced as a condition to decide the Grothendieck ring of the category is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. The same result is proved in [7] independently.

Let $\mathcal{G} = (G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group. We say a definable set $M \subseteq G^n$ is *bounded* if $M \subseteq [-b, b]^n$ for some positive $b \in G$, where $[-b, b]^n = \{(x_1, \ldots, x_n) \in G^n | -b \leq x_i \leq b\}$.

Definition 1.1 (Bounded Condition). Let $\mathcal{G} = (G, <, +, 0, \ldots)$ be an o-minimal expansion of an ordered abelian group. We say that $\mathcal{G}$ satisfies the *bounded condition* if the following property holds:

For all bounded definable sets $M \subseteq G^m$ and definable sets $N \subseteq G^n$, if M is definable isomorphic to N, then N is bounded.

We now give an example of o-minimal expansion of an ordered abelian group where satisfies the bounded condition.

Example 1.2 (Ordered Divisible Abelian Group). Let $\mathcal{L}_{og} = \{<, +, -, 0\}$, where $<$ is a binary relation symbol, $+, -$ are binary function symbols, and 0 is a constant symbol. The theory of ordered divisible abelian groups in the language $\mathcal{L}_{og}$ is given by the following sentences. This theory is often denoted by ODAG.

1. The axioms for ordered abelian groups.
2. For each $n \geq 1$, the axiom $\forall y \exists x \ (y = \underbrace{x + \cdots + x}_{n-\text{times}})$.

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It is known every model of $\mathcal{G} \models$ ODAG satisfies the bounded condition. Therefore, it has the bounded Euler characteristic χ_b(see [5, Definition 24]) on $\mathrm{Def}(\mathcal{G}, \mathcal{L}_{\mathrm{og}})$ because it has well geometric properties for cells. In this sense the bounded condition is also a necessary condition for good geometric properties.

We proved that the theory of ordered divisible abelian groups satisfied the bounded condition in [5]. In the present paper, we will show that the theory of ordered divisible vector spaces over an ordered field satisfies the bounded condition.

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2. Preliminaries of model theory

First, let us introduce several kinds of the main definitions and results which form the basic of model theory. This chapter is based on D. Marker [8].

Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-theory* T is a set of $\mathcal{L}$-sentences. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is a *model* of T, which is denoted by $\mathcal{M} \models T$, if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$.

Definition 2.1. A *universal sentence* is one of the form $\forall \overline{v} \phi(\overline{v})$, where ϕ is a quantifier-free formula. We denote by $T_\forall$ the set all of universal sentences which are logical consequences of T, namely,

$$T_\forall := \{\phi \mid \phi \text{ is a universal } \mathcal{L}\text{-sentence and } T \models \phi\}.$$

First, let us consider the following lemma concerning $T_\forall$.

Lemma 2.2. *Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Then $\mathcal{A} \models T_\forall$ if and only if there exists $\mathcal{M} \models T$ with $\mathcal{A} \subseteq \mathcal{M}$.*

Proof. First assume that there exists $\mathcal{M} \models T$ with $\mathcal{A} \subseteq \mathcal{M}$. By definition of $T_\forall$, for all $\forall v_1 \dots \forall v_n \phi(v_1, \dots, v_n) \in T_\forall$, $T \models \forall v_1 \dots \forall v_n \phi(v_1, \dots, v_n)$. Because $\mathcal{M} \models \forall v_1 \dots \forall v_n \phi(v_1, \dots, v_n)$, for all $a_1, \dots, a_n \in A$, $\mathcal{M} \models \phi(a_1, \dots, a_n)$. Since ϕ is a quantifier-free formula, $\mathcal{A} \models \phi(a_1, \dots, a_n)$.

Conversely, suppose $\mathcal{A}$ is a model of $T_\forall$. Let us begin with the following claim.

Claim. *Let $\mathcal{L}_A$ be a language $\mathcal{L} \cup \{a\}_{a \in A}$ where each a is a new constant symbol. Let $\mathrm{Diag}(\mathcal{A})$ be the set of $\phi(a_1, \dots, a_n)$, where $(a_1, \dots, a_n) \in A^n$ and ϕ is either an atomic $\mathcal{L}$-formula or negation of an atomic $\mathcal{L}$-formula with $\mathcal{A} \models \phi(a_1, \dots, a_n)$. Then $T \cup \mathrm{Diag}(\mathcal{A})$ is satisfiable as an $\mathcal{L}_A$-theory.*

Suppose the contrary. Then, by the compactness theorem, there is a finite subset $\Delta = (\psi_1, \dots, \psi_n) \subseteq \mathrm{Diag}(\mathcal{A})$ such that $T \cup \Delta$ is not satisfiable. Let $\overline{c} = (c_1, \dots, c_m)$ be the new constant symbols from A used in $\psi_1, \dots, \psi_n$ and say $\psi_i = \phi_i(\overline{c})$, where ϕ_i is a quantifier-free $\mathcal{L}$-formula. Because T is an $\mathcal{L}$-theory, the constants in $\overline{c}$ do not occur in T. Hence if $T \cup \{\exists\overline{v} \bigwedge \phi_i(\overline{v})\}$ is satisfiable, then by interpreting $\overline{c}$ as witnesses to the existential formula, $T \cup \Delta$ would be satisfiable. Thus $T \models \forall\overline{v} \bigvee \neg\phi_i(\overline{v})$. This formula is universal. Thus, $\forall\overline{v} \bigvee \neg\phi_i(\overline{v}) \in T_\forall$, which is contradict to $\mathcal{A} \models T_\forall$.

By above Claim, there is an $\mathcal{M} \models T \cup \mathrm{Diag}(\mathcal{A})$ as an $\mathcal{L}_A$-structure. It is clear that $\mathcal{M}$ is a model of T as an $\mathcal{L}$-structure. Let $j : A \to M$ by $j(a) = a^{\mathcal{M}}$. Then j is an $\mathcal{L}$-embedding. Thus $\mathcal{A} \subseteq \mathcal{M}$. □

We say that a theory T has *algebraically prime models* if for any $\mathcal{A} \models T_\forall$ there exist $\mathcal{M} \models T$ and an embedding $i : \mathcal{A} \to \mathcal{M}$ with the following universal property; for all $\mathcal{N} \models T$ and embeddings $j : \mathcal{A} \to \mathcal{N}$, there is $h : \mathcal{M} \to \mathcal{N}$ with $j = h \circ i$.

If $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$, we say that $\mathcal{M}$ is *simply closed* in $\mathcal{N}$, which is denoted by

$$\mathcal{M} \prec_s \mathcal{N},$$

if for any quantifier-free formula $\phi(\overline{v}, w)$ and any $\overline{a} \in M$, $\mathcal{N} \models \exists w\phi(\overline{a}, w)$ implies $\mathcal{M} \models \exists w\phi(\overline{a}, w)$.

Proposition 2.3. *Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. We assume the following; for all quantifier-free formulas $\phi(\overline{v}, w)$, if $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{M}, \mathcal{A} \subseteq \mathcal{N}$, $\overline{a} \in A$, and there is $b \in M$ such that $\mathcal{M} \models \phi(\overline{a}, b)$, then there is $c \in N$ such that $\mathcal{N} \models \phi(\overline{a}, c)$. Then T has quantifier elimination.*

Proof. See [8, Chapter 3, Corollary 3.1.6]. □

The following transformation of the above proposition is suitable for our purpose.

Corollary 2.4. *Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory such that*
1. *T has algebraically prime models and that,*
2. *$\mathcal{M} \prec_s \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ are models of T.*

Then T has quantifier elimination.

Proof. Suppose that $\mathcal{M}, \mathcal{N}$ are models of T, $\mathcal{H}$ is a common substructure of $\mathcal{M}$ and $\mathcal{N}$, $\overline{h} \in H, m \in M$, and $\mathcal{M} \models \phi(\overline{h}, m)$ where ϕ is a quantifier-free $\mathcal{L}$-formula. Since T has algebraically prime models and $\mathcal{H} \subseteq \mathcal{M}$, by using Lemma 2.2, we get $\mathcal{U} \models T$ and an embedding $i : \mathcal{H} \to \mathcal{U}$ such that for all $\mathcal{G} \models T$ and embeddings $j : \mathcal{H} \to \mathcal{G}$, there is $h : \mathcal{U} \to \mathcal{G}$ with $j = h \circ i$. Considering inclusion map $\mathcal{H} \hookrightarrow \mathcal{M}$, we can embed $\mathcal{U}$ into $\mathcal{M}$. Since ϕ is a quantifier-free, by second property of T, we get $\mathcal{U} \models \exists w\phi(\overline{h}, w)$. Considering inclusion map $\mathcal{H} \hookrightarrow \mathcal{N}$, we can embed $\mathcal{U}$ into $\mathcal{N}$ and then $\mathcal{N} \models \exists w\phi(\overline{h}, w)$. Thus there exists $n \in N$ such that $\mathcal{N} \models \phi(\overline{h}, n)$. Therefore, by Proposition 2.3, T has quantifier elimination. □

Next we define the theory of ordered divisible vector spaces over an ordered field $(F, >_F)$.

Definition 2.5. Let us fix $\mathcal{L}_F := \mathcal{L}_{og} \cup \{\lambda \mid \lambda \in F\}$, where λ is a unary function symbol for each $\lambda \in F$ to be interpreted as multiplication by the scalar. The theory T_{Flin} of ordered divisible vector spaces in the language $\mathcal{L}_F$ is defined by the following sentences:

1. The axioms for ordered divisible abelian groups.
2. The axioms for vector spaces over F.
3. For each $\lambda >_F 0$, $\forall x(x > 0 \rightarrow \lambda x > 0)$.

The next lemma show that T_{Flin} has algebraically prime models.

Lemma 2.6. *Let V be an ordered vector space over F. Then there is an ordered divisible vector space W over F, which is called the* ordered divisible hull *of V over F, and an embedding $\varphi : V \rightarrow W$ such that if $\chi : V \rightarrow W'$ is an embedding of V into an ordered divisible vector space W' over F, then there is a unique homomorphism $\psi : W \rightarrow W'$ with $\chi = \psi \circ \varphi$.*

Proof. We set $X = \{(g, n) \mid g \in V, n \in \mathbb{N}, n > 0\}$. We define an equivalence relation $\sim$ on X by $(g, n) \sim (h, m)$ if and only if $mg = nh$. We define W to be the quotient $X/\sim$. For (g, n), let $[(g, n)]$ or g/n denote the equivalence class of (g, n).

We can define $+$ and scalar $\lambda\cdot$ for each $\lambda \in F$ on W by

$$[(g, n)] + [(h, m)] = [(mg + nh, mn)],$$
$$\lambda \cdot [(g, n)] = [(\lambda g, n)].$$

It is easy to see that W is a vector space over F.

Suppose that $g/m \in W$ and $n > 0$, then

$$n[(g, mn)] = [(ng, mn)] = [(g, m)],$$

which show that W is divisible.

We can define an order $<$ on W by

$$g/n < h/m \Longleftrightarrow mg < nh.$$

It is clear that $(W, \mathcal{L}_F)$ is an ordered divisible vector space over F.

We can embed V into W by the map $\varphi(g) = g/1$. Suppose that W' is an ordered divisible vector space over F and $\chi : V \rightarrow W'$ is an embedding. Let $\psi : W \rightarrow W'$ by $\psi(g/n) := \chi(g)/n$. This ψ is a well-defined embedding and $\chi = \psi \circ \varphi$. □

Lemma 2.7. *Suppose that $\mathcal{V}$, $\mathcal{W}$ are models of T_{Flin} with $\mathcal{V} \subseteq \mathcal{W}$. Then $\mathcal{V} \prec_s \mathcal{G}$.*

Proof. Suppose that $\phi(v, \overline{w})$ is a quantifier-free formula, $\overline{a} \in V$, and that for some $b \in W, \mathcal{W} \models \phi(b, \overline{a})$. Because $\phi(v, \overline{w})$ is quantifier-free formula, there are atomic or negated atomic formulas $\theta_{i,j}(v, \overline{w})$ such that

$$\phi(v, \overline{w}) \leftrightarrow \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \theta_{i,j}(v, \overline{w}).$$

Because $\mathcal{W} \models \phi(b, \overline{a})$, $\mathcal{W} \models \bigwedge_{j=1}^{m} \theta_{i,j}(b, \overline{a})$ for some i. Thus without loss of generality, we may assume that $\phi(v, \overline{w})$ is a conjunction of atomic and negated atomic formulas.

If $\theta(v, w_1, \ldots, w_m)$ is an atomic formula, then for some elements $\lambda_1, \ldots, \lambda_l$, $\lambda \in F$,

$$\theta(v, w_1, \ldots, w_m) \leftrightarrow \sum_{k=1}^{m} \lambda_k w_k + \lambda v = 0$$

or

$$\theta(v, w_1, \ldots, w_m) \leftrightarrow \sum_{k=1}^{m} \lambda_k w_k + \lambda v > 0.$$

In particular, there is an element $g \in V$ such that $\theta(v, \overline{a})$ is either the form $\lambda v = g$ or $\lambda v > g$. Also note that every formula $\lambda v \neq g$ is equivalent to $\lambda v > g$ or $-\lambda v > -g$. Thus we may assume that

$$\phi(v, \overline{a}) \leftrightarrow \bigwedge_{i=1}^{s} (\lambda_i v = g_i) \wedge \bigwedge_{j=1}^{t} (\lambda_j v > h_j),$$

where $g_i, h_j \in V$ and $\lambda_i, \lambda_j \in F$.

Case 1. If $\lambda_i \neq 0$ for some $1 \leq i \leq s$.

Then

$$\phi(v, \overline{a}) \leftrightarrow \bigwedge_{i=1}^{s} (\lambda_i v = g_i) \wedge \bigwedge_{j=1}^{t} (\lambda_j v > h_j).$$

Because $\mathcal{W} \models \phi(b, \overline{a})$, we must have $b = \lambda^{-1} g_i \in V$.

Case 2. If $\lambda_i = 0$ for all $1 \leq i \leq s$.

Then we assume

$$\phi(v, \overline{a}) \leftrightarrow \bigwedge_{j=1}^{t} (\lambda_j v > h_j),$$

where $\lambda_j \neq 0$ $(j = 1, \ldots, t)$. Let $k_0 = \min\{\lambda^{-1} h_j \mid \lambda_j < 0\}$ and $k_1 = \max\{\lambda^{-1} h_j \mid \lambda_j > 0\}$. Then $c \in W$ satisfies $\phi(v, \overline{a})$ if and only if $k_1 < c < k_0$. Since $\mathcal{W} \models \phi(b, \overline{a})$, we must have $k_1 < k_0$, so that V is a dense linearly ordered set. Thus there exists $d \in V$ such that $k_1 < d < k_0$.

Consequently we have $\mathcal{V} \prec_s \mathcal{W}$. □

We are now ready to prove that T_{Flin} has quantifier elimination.

Theorem 2.8. *The theory T_{Flin} in the language $\mathcal{L}_F$ has quantifier elimination.*

Proof. Suppose that $\mathcal{A} \models (T_{Flin})_\forall$. By Lemma 2.2 there is an $\mathcal{M} \models T_{Flin}$ such that $\mathcal{A} \subseteq \mathcal{M}$. By Lemma 2.6, we can take the divisible hull $\mathcal{H}$ of $\mathcal{A}$. Hence T_{Flin} has algebraically prime models, so that T_{Flin} satisfies first property of Corollary 2.4.

By Lemma 2.7, T_{Flin} satisfies the second property of Corollary 2.4. Thus T_{Flin} has quantifier elimination. □

3. Preliminaries to o-minimal geometry

Let us recall the definition of o-minimal structure and two important results in the subject of o-minimality: the monotonicity theorem and the cell decomposition theorem.

Definition 3.1 (O-minimal Structure)**.** We say that a dense linearly ordered structure $(G, <, \ldots)$ without endpoints is an *o-minimal structure* if for any definable set $X \subseteq G$ there are finite many intervals $I_1, \ldots, I_m$ and a finite set X_0 such that

$$X = X_0 \cup I_1 \cup \cdots \cup I_m.$$

A theory T is said to be an *o-minimal theory* if every model of T is an o-minimal structure.

Proposition 3.2. *The theory T_{Flin} in the language $\mathcal{L}_F$ is an o-minimal theory.*

Proof. Let $\mathcal{V}$ be a model of T_{Flin}. We need to show every definable set

$$M = \{x \in V \mid \mathcal{V} \models \phi(x, a_1, \ldots, a_n)\}$$

is a finite union of points and intervals with endpoints in $V \cup \{\pm\infty\}$, where ϕ is a formula and $a_1, \ldots, a_n \in V$. By quantifier elimination,

$$M = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{i,j}$$

where $A_{i,j}$ is equal to either

$$\{x \in V \mid \lambda_{i,j} x = g_{i,j}\} \quad \text{or} \quad \{x \in V \mid \mu_{i,j} x > h_{i,j}\}$$

for some $g_{i,j}, h_{i,j} \in V$ and $\lambda_{i,j}, \mu_{i,j} \in F$. Solution sets of nontrivial equations yield finite sets and solution sets of the second form give rise to finite union of intervals. □

We work with a fixed but arbitrary o-minimal expansion of an ordered abelian group $(G, <, 0, +, -, \ldots)$ from here through the end of this section.

Theorem 3.3 (Monotonicity Theorem)**.** *Let $f : (a, b) \to G$ be a definable function on the interval (a, b). Then there are points $a_1 < \cdots < a_k$ in (a, b) such that on each subinterval (a_j, a_{j+1}), with $a_0 = a, a_{k+1} = b$, the function is either constant, or strictly monotone and continuous.*

Proof. See [3, Chapter 3, Theorem 1.2]. □

A *decomposition* of G^m is defined by induction on m as follows:
(I) A decomposition of $G^1 = G$ is a collection:

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\},$$

where $a_1 < \cdots < a_k$ are points in G.

(II) Suppose that a decomposition of G^m is already defined inductively, then a decomposition of G^{m+1} is a finite collection of pairwise disjoint cells $\{C_i\}$ such that $\bigcup C_i = G^{m+1}$ and the set of projections $\{\pi(C_i)\}$ is a decomposition of G^m, where $\pi : G^{m+1} \to G^m$ is the projection of first m-coordinates.

A decomposition $\mathcal{D}$ of G^m is called a *partition* of a set $M \subseteq G^m$ if each cell in $\mathcal{D}$ is either part of M or disjoint from M.

We are now ready to state the cell decomposition theorem.

Theorem 3.4 (Cell Decomposition Theorem). *Let $M_1, \dots, M_k \subseteq G^m$ be finitely many definable sets. Then there is a decomposition of G^m partitioning each of $M_1, \dots, M_k$.*

Proof. See [3, Chapter 3, Theorem 2.11]. □

For each definable set M in G^m, we put

$$C(M) := \{f : M \to G \mid f \text{ is definable and continuous}\},$$
$$C_\infty(M) := C(M) \cup \{\pm\infty\},$$

where we regard $\pm\infty$ as constant functions on G. For $f \in C(M)$, the graph of f is denoted by $\Gamma(f) \subseteq M \times G$.

Next we show the following useful properties of bounded definable sets.

Lemma 3.5. *Let $M \subseteq G^n$ be a bounded definable set with $\dim M = 1$. Then there exists a definable bijection $M \xrightarrow{\sim} D$ for some bounded definable set $D \subseteq G$.*

Proof. Since $\dim M = 1$, by Theorem 3.4 we have the following decomposition

$$M = C_1 \cup \cdots \cup C_l \cup C_{l+1} \cup \cdots \cup C_m, \quad C_i \cap C_j = \varnothing \ \ (i \neq j)$$

where $C_1, \dots, C_m$ are cells, $\dim C_1 = 1, \dots, \dim C_l = 1$ and $\dim C_{l+1} = 0, \dots, \dim C_m = 0$.

Claim. *Let $C \subseteq G^n$ be a cell such that $\dim C = 1$ and C is bounded. Then there exists the projection of n_ith-coordinate $p_{n_i} : G^n \to G$ for some $1 \leq n_i \leq n$ such that $p_{n_i}|_C : C \to p_{n_i}(C)$ is definably bijective. Here, note that $p_{n_i}(C)$ is a bounded interval.*

We prove this claim by induction on n. In the case where $n = 1$, since each C is equal to either an interval or a point, it is easy to see that the claim holds. Suppose that the claim is true for $n = k$, and we show that it holds for $n = k + 1$. Let $p_1 : G^{k+1} \to G$ be the projection to the first coordinate.

Case 1. $\dim p_1(C) = 0$.

Since $\dim p_1(C) = 0$, there are a point $a \in G$ and a cell $D \subseteq G^k$ such that $C = \{a\} \times D$. By inductive assumption, there is a projection $p_{n_i} : G^k \to G$ such that $p_{n_i}|D$ is bijective. Let τ be a projection such that $\tau : G^{k+1} \to G^k((x_1, \ldots, x_{k+1}) \mapsto (x_2, \ldots, x_{k+1}))$. Then $p_{n_i+1} = p_{n_i} \circ \tau$ and $p_{n_i+1}|C$ is a definably bijective function from C to $p_{n_i}(C)$.

Case 2. $\dim p_1(C) = 1$.

Let $\pi_q : G^{k+1} \to G^q(q = 1, \ldots, k+1)$ be the projection to the first q-coordinates. Since $p_1(C)$ is an interval, C is a $(1, 0, \ldots, 0)$-cell. Thus we have $\dim \pi_q(C) = 1$ for all $q = 1, \ldots, k+1$. Hence each cell $\pi_q(C)$ $(q = 2, \ldots, k+1)$ is the graph of a definable function $f_q \in C(\pi_{q-1}(C))$.

By using $f_2, \ldots, f_k$, we inductively define functions $g_2, \ldots, g_{k+1} : p_1(C) \to G$ as follows: $g_2(x) := f_2(x)$. If g_j is already given inductively, then we define g_{j+1} by $g_{j+1}(x) := f_{j+1}(x, g_2(x), \ldots, g_j(x))$ where $2 \leq j \leq k+1$ and $x \in p_1(C)$. Then for a definable function $g : p_1(C) \to G^k$ $(x \mapsto (g_2(x), \ldots, g_{k+1}(x)))$, $C = \Gamma(g)$. Thus we obtain a definable bijection $p_1|C : C \to p_1(C)$.

By Claim, each C_i $(i = 1, \ldots, l)$ is definably bijective to an interval of G and each C_i $(i = l+1, \ldots, m)$ is a point set. Thus we can define a definable bijection $M \to D$ for some bounded definable set $D \subseteq G$. □

Let σ be a permutation of $\{1, \ldots, m\}$ and A a subset of G^m. We set $x\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ for $x = (x_1, \ldots, x_m) \in G^m$ and $A\sigma = \{x\sigma \mid x \in A\}$.

Lemma 3.6. *Let $C \subseteq G^m$ be a non-bounded cell. Then there exists a non-bounded cell $C' \subseteq G^m$ such that the projection of first coordinate of C' is a non-bounded interval and C' is definably embedded into C.*

Proof. Since C is non-bounded, there exists the projection p_{n_i} of n_ith-coordinate such that $p_{n_i}(C)$ is a non-bounded interval. We denote the transposition $(1, n_i)$ by σ. Since symmetric group on $\{1, \ldots, m\}$ is generated by the transpositions $(i, i+1)$, there exist the transpositions $\tau_1, \ldots, \tau_n$ such that $\sigma = \tau_n \circ \cdots \circ \tau_1$. We give a proof only for the case $\sigma = \tau_2 \circ \tau_1$, but the generalization is straightforward. By using [3, Chapter 4, Proposition 2.13], there exist pairwise disjoint cells $C_1, \ldots, C_l$ such that

$$C = C_1 \cup \cdots \cup C_l \quad \text{and} \quad C_1\tau_1, \ldots, C_l\tau_1 \quad \text{are also cells.}$$

Since $p_{n_i}(C_{l_1})$ is a non-bounded interval for some $1 \leq l_1 \leq l$, $p_{\tau_1(n_i)}(C_{l_1}\tau_1)$ is a non-bounded interval. By using the proposition again for the non-bounded cell $C_{l_1}\tau_1$, we have pairwise disjoint cells $D_1, \ldots, D_{l'}$ such that

$$C_{l_1}\tau_1 = D_1 \cup \cdots \cup D_{l'} \quad \text{and} \quad D_1\tau_2, \ldots, D_{l'}\tau_2 \quad \text{are also cells.}$$

Since $p_{\tau_1(n_i)}(D_{l_2})$ is a non-bounded interval for some $1 \le l_2 \le l'$, we have a non-bounded interval $p_{\tau_2(\tau_1(n_i))}(D_{l_2}\tau_2) = p_1(D_{l_2}\tau_2)$ and a non-bounded cell $D_{l_2}\tau_2$. □

Corollary 3.7. *Let $M \subseteq G^m$ be a non-bounded definable set and $N \subseteq G^n$ a bounded definable set. If there exists a definable bijection $\theta : M \overset{\sim}{\to} N$, then $(0, +\infty)$ is definably bijective to D for some bounded definable set $D \subseteq G$.*

Proof. Let $\pi_q : G^n \to G^q$ be the projection to the first q-coordinates. By Theorem 3.4,

$$M = C_1 \cup \cdots \cup C_m, \;\; C_i \cap C_j = \varnothing \;\; (i \neq j),$$

where $C_1, \ldots, C_m$ are cells. Since M is a non-bounded definable set, we can choose a non-bounded cell C_i for some $1 \le i \le m$. Since C_i is non-bounded, there exists the projection of n_ith-coordinate $p_{n_i} : G^n \to G$ such that $p_{n_i}(C_i)$ is a non-bounded interval. By using Lemma 3.6, we assume that $\pi_1(C_i)$ is a non-bounded interval I.

If $\pi_2(C_i)$ is a (1,0)-cell $\Gamma(f)$ for some $f \in C(I)$, then we define a definable injection $\iota_2 : I \to \pi_2(C_i)$ by $\iota_2(x) := (x, f(x))$.

If $\pi_2(C_i)$ is a (1,1)-cell $\{(x, y) \in I \times G \mid g(x) < y < h(x)\}$ for some $g, h \in C_\infty(I)$, then we define a definable injection $\iota_2 : I \to \pi_2(C_i)$ by

$$\iota_2(x) := \begin{cases} (x, x) & \text{if } g = -\infty, h = +\infty, \\ (x, h(x) - a) & \text{if } g = -\infty, h \in C(I), \\ (x, g(x) + a) & \text{if } g \in C(I), h = +\infty, \\ (x, (g(x) + h(x))/2) & \text{if } g \in C(I), h \in C(I), \end{cases}$$

where a is a positive element of G. By continuing in this process, we have a sequence of definable injections

$$I \xrightarrow{\;\iota_2\;} \pi_2(C_i) \xrightarrow{\;\iota_3\;} \cdots \xrightarrow{\;\iota_{n-1}\;} \pi_{n-1}(C_i) \xrightarrow{\;\iota_n\;} C_i.$$

Let $\iota : I \to C_i$ be the composition of these definable injections. Since $\dim \theta(\iota(I)) = 1$ and $\theta(\iota(I)) \subseteq N$ is bounded, by Lemma 3.5, there is a bounded definable set $D \subseteq G$ such that D is definably bijective to $\theta(\iota(I))$. The interval $(0, +\infty)$ is definably embedded into I, we have a sequence of definable injections as follow:

$$(0, +\infty) \longrightarrow I \xrightarrow{\;\sim\;} \iota(I) \xrightarrow[\theta|_{\iota(I)}]{\;\sim\;} \theta(\iota(I)) \xrightarrow[\text{Lemma 3.5}]{\;\sim\;} D.$$

Hence we have a definable bijection between $(0, +\infty)$ and D. □

4. The bounded condition of the ordered divisible vector spaces over an ordered field F

In this section, we prove the main result of this paper.

Theorem 4.1. *Let $\mathcal{V}$ be a model of T_{Flin} in the language of $\mathcal{L}_F$. Then $\mathcal{V}$ satisfies the bounded condition.*

Proof. Suppose the contrariety. Then there are a non-bounded definable set $X \subseteq V^m$ and a bounded definable set $Y \subseteq V^n$ such that X is definably bijective to Y. By Corollary 3.7, there is a definable bijection $f : (0, +\infty) \to D$ where D is bounded definable set of V. By the monotonicity theorem 3.3, there are points $a_1 < \cdots < a_n$ in $(0, +\infty)$ such that on each subinterval (a_j, a_{j+1}) with $a_0 = 0, a_{n+1} = +\infty$, the function f is strictly monotone. Since T_{Flin} admits quantifier elimination, we may assume that $f(x) = \lambda x + c$ on $x \in (a_n, +\infty)$ for some $\lambda \in F(\lambda \neq 0)$ and $c \in V$.

Since D is bounded, there exist two points $d_1, d_2 \in V$ such that $d_2 < x < d_1$ for all $x \in D$.

If $\lambda > 0$, we can choose $x_0 \in (a_n, +\infty)$ such that $x_0 > (-c + d_1)/\lambda$. Then $f(x_0) = \lambda x_0 + c > d_1$. If $\lambda < 0$, we can choose $x_0 \in (a_n, +\infty)$ such that $x_0 > (-c + d_2)/\lambda$. Then $f(x_0) = \lambda x_0 + c < d_2$. They are contradicting to $f|_{(a_n, +\infty)} : (a_n, \infty) \to D$. □

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