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3-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ Part II, $G = E_7$, Case 5

By

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According to M. Hara [1], there are five cases of 3-graded decompositions $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ of simple Lie algebra $\mathfrak{g}$ of type E_7. In the preceding papers [2] and [3], we gave the group realization of Lie sualgebras $\mathfrak{g}_{ev} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \mathfrak{g}_0$ and $\mathfrak{g}_{ed} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_3$ of $\mathfrak{g}$ of Cases 1, 2, 3 and 4. In the present paper, we give the group realization of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ of Case 5. We rewrite the results of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ of the remainder Case 5.

Case 5	$\mathfrak{g}$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$	$\mathfrak{g}_0$ $\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$
	${\mathfrak{e}_7}^C$	$\mathfrak{sl}(8, C)$ $\mathfrak{sl}(3, C) \oplus \mathfrak{sl}(6, C)$	$C \oplus \mathfrak{sl}(3, C) \oplus \mathfrak{sl}(5, C)$ $30, 15, 5$
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(8, \boldsymbol{R})$ $\mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(6, \boldsymbol{R})$	$\boldsymbol{R} \oplus \mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(5, \boldsymbol{R})$ $30, 15, 5$

Our results of Case 5 are as follows:

Case 5	G	G_{ev} G_{ed}	G_0
	${E_7}^C$	$SL(8, C)/\boldsymbol{Z}_2$ $(SL(3, C) \times SL(6, C))/\boldsymbol{Z}_3$	$(C^* \times SL(3, C) \times SL(5, C))/\boldsymbol{Z}_{30}$
	$E_{7(7)}$	$SL(8, \boldsymbol{R})/\boldsymbol{Z}_2 \times 2$ $SL(3, \boldsymbol{R}) \times SL(6, \boldsymbol{R})$	$(\boldsymbol{R}^+ \times SL(3, \boldsymbol{R}) \times SL(5, \boldsymbol{R})) \times 2$

This paper is a continuation of [3], so the numbering of sections starts from 4.5.

Together with the preceding papers [2], [3] and the present paper, the group realization of Hara's table [1] of 3-graded decompositions of type E_7 have been

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completed. The group realizations in the cases of types G_2, F_4 and E_6 are already given in [6], so the group realizations of Hara's table [1] of exceptional Lie algebras are completed except type E_8.

4.5. Subgroups of type $\boldsymbol{A}_7{}^C, \boldsymbol{C} \oplus \boldsymbol{A}_2{}^C \oplus \boldsymbol{A}_4{}^C$ and $\boldsymbol{A}_2{}^C \oplus \boldsymbol{A}_5{}^C$ of $\boldsymbol{E}_7{}^C$

We use the same notations as that in [2], [3], [4], [5] and [6]. Here although the definitions of the C-linear transformations w_3, λ and ι of $\mathfrak{P}^C$ are already given in [2] and [3], we express just those definitions again. The C-linear transformations w_3, λ and ι of $\mathfrak{P}^C$ are defined by

$$\begin{aligned} w_3(X, Y, \xi, \eta) &= (w_3X, w_3Y, \xi, \eta), \\ \lambda(X, Y, \xi, \eta) &= (Y, -X, -\eta, \xi), \\ \iota(X, Y, \xi, \eta) &= (-iX, iY, -i\xi, i\eta), \ \ (X, Y, \xi, \eta) \in \mathfrak{P}^C, \end{aligned}$$

respectively, where w_3 of the right side is defined by $w_3 : \mathfrak{C}^C = \boldsymbol{C}^C \oplus (\boldsymbol{C}^C)^3 \to \mathfrak{C}^C = \boldsymbol{C}^C \oplus (\boldsymbol{C}^C)^3$, $w_3(a + \boldsymbol{m}) = a + w_1\boldsymbol{m}$, $w_1 = e^{2\pi e_1/3}$.

In the Lie algebra $\mathfrak{e}_7{}^C$, let

$$Z = i\varPhi\Big(- (G_{45} + G_{67}), -\frac{1}{2}(3E_1 + E_2 + E_3), \frac{1}{2}(3E_1 + E_2 + E_3), 0\Big).$$

Theorem 4.5. *The 3-graded decomposition of* $\mathfrak{e}_{7(7)} = (\mathfrak{e}_7{}^C)^{\tau\lambda\iota\gamma_1}$ *(or* $\mathfrak{e}_7{}^C$*),*

$$\mathfrak{e}_{7(7)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $adZ, Z = i\varPhi\Big(- (G_{45} + G_{67}), -\frac{1}{2}(3E_1 + E_2 + E_3), \frac{1}{2}(3E_1 + E_2 + E_3), 0\Big)$*, is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01},\ G_{02}, iG_{03},\ iG_{12}, G_{13},\ iG_{23}, \\ iG_{45},\ G_{46} + G_{57},\ i(G_{47} - G_{56}),\ iG_{67}, \\ \tilde{A}_1(1),\ i\tilde{A}_1(e_1),\ \tilde{A}_1(e_2),\ i\tilde{A}_1(e_3), \\ i\check{E}_k - i\hat{E}_k,\ i\check{F}_1(1) - i\hat{F}_1(1),\ i\check{F}_1(e_k) - i\hat{F}_1(e_k),\ k = 1, 2, 3, \\ -2i\tilde{F}_1(e_4 - ie_5) + \check{F}_1(e_4 - ie_5) + \hat{F}_1(e_4 - ie_5), \\ -2i\tilde{F}_1(e_6 - ie_7) + \check{F}_1(e_6 - ie_7) + \hat{F}_1(e_6 - ie_7), \\ -2i\tilde{F}_1(e_4 + ie_5) - \check{F}_1(e_4 + ie_5) - \hat{F}_1(e_4 + ie_5), \\ -2i\tilde{F}_1(e_6 + ie_7) - \check{F}_1(e_6 + ie_7) - \hat{F}_1(e_6 + ie_7), \\ -2i\tilde{A}_2(1 - ie_1) + \check{F}_2(1 - ie_1) - \hat{F}_2(1 - ie_1), \\ -2i\tilde{A}_2(e_2 - ie_3) + \check{F}_2(e_2 - ie_3) - \hat{F}_2(e_2 - ie_3), \\ -2i\tilde{A}_2(1 + ie_1) - \check{F}_2(1 + ie_1) + \hat{F}_2(1 + ie_1), \\ -2i\tilde{A}_2(e_2 + ie_3) - \check{F}_2(e_2 + ie_3) + \hat{F}_2(e_2 + ie_3), \\ -2i\tilde{A}_3(1 + ie_1) + \check{F}_3(1 + ie_1) - \hat{F}_3(1 + ie_1), \\ -2i\tilde{A}_3(e_2 + ie_3) + \check{F}_3(e_2 + ie_3) - \hat{F}_3(e_2 + ie_3), \\ -2i\tilde{A}_3(1 - ie_1) - \check{F}_3(1 - ie_1) + \hat{F}_3(1 - ie_1), \\ -2i\tilde{A}_3(e_2 - ie_3) - \check{F}_3(e_2 - ie_3) + \hat{F}_3(e_2 - ie_3) \end{array} \right\} 33$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} G_{04} - iG_{05},\, G_{06} - iG_{07},\, iG_{14} + G_{15}, iG_{16} + G_{17}, \\ G_{24} - iG_{25},\, G_{26} - iG_{27},\, iG_{34} + G_{35}, iG_{36} + G_{37}, \\ \tilde{A}_1(e_4 - ie_5),\, \tilde{A}_1(e_6 - ie_7), \\ \dfrac{2}{3}i(-E_1 + 2E_2 - E_3)^{\sim} + \check{E}_2 + \hat{E}_2 + i\mathbf{1}, \\ \dfrac{2}{3}i(-E_1 - E_2 + 2E_3)^{\sim} + \check{E}_3 + \hat{E}_3 + i\mathbf{1}, \\ i\check{F}_1(e_4 + ie_5) - i\hat{F}_1(e_4 + ie_5),\, i\check{F}_1(e_6 + ie_7) - i\hat{F}_1(e_6 + ie_7), \\ 2i\tilde{F}_1(1) + \check{F}_1(1) + \hat{F}_1(1), \\ 2i\tilde{F}_1(e_k) + \check{F}_1(e_k) + \hat{F}_1(e_k),\ k = 1,2,3, \\ 2i\tilde{F}_2(1 + ie_1) + \check{F}_2(1 + ie_1) + \hat{F}_2(1 + ie_1), \\ 2i\tilde{F}_2(e_2 - ie_3) + \check{F}_2(e_2 - ie_3) + \hat{F}_2(e_2 - ie_3), \\ 2i\tilde{F}_3(1 - ie_1) + \check{F}_3(1 - ie_1) + \hat{F}_3(1 - ie_1), \\ 2i\tilde{F}_3(e_2 - ie_3) + \check{F}_3(e_2 - ie_3) + \hat{F}_3(e_2 - ie_3), \\ 2i\tilde{A}_2(e_k) - \check{F}_2(e_k) + \hat{F}_2(e_k),\, k = 4,5,6,7, \\ 2i\tilde{A}_3(e_k) - \check{F}_3(e_k) + \hat{F}_3(e_k),\, k = 4,5,6,7 \end{array} \right\} 30$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} (G_{46} - G_{57}) + i(G_{47} + G_{56}), \\ 2i\tilde{F}_1(e_4 - ie_5) + \check{F}_1(e_4 - ie_5) + \hat{F}_1(e_4 - ie_5), \\ 2i\tilde{F}_1(e_6 - ie_7) + \check{F}_1(e_6 - ie_7) + \hat{F}_1(e_6 - ie_7), \\ 2i\tilde{F}_2(e_k) + \check{F}_2(e_k) + \hat{F}_2(e_k),\, k = 4,5,6,7, \\ 2i\tilde{F}_3(e_k) + \check{F}_3(e_k) + \hat{F}_3(e_k),\, k = 4,5,6,7, \\ -2i\tilde{A}_2(1 - ie_1) + \check{F}_2(1 - ie_1) - \hat{F}_2(1 - ie_1), \\ -2i\tilde{A}_2(e_2 + ie_3) + \check{F}_2(e_2 + ie_3) - \hat{F}_2(e_2 + ie_3), \\ -2i\tilde{A}_3(1 + ie_1) - \check{F}_3(1 + ie_1) + \hat{F}_3(1 + ie_1), \\ -2i\tilde{A}_3(e_2 + ie_3) - \check{F}_3(e_2 + ie_3) + \hat{F}_3(e_2 + ie_3) \end{array} \right\} 15$$

$$\mathfrak{g}_{-3} = \left\{ \begin{array}{l} \dfrac{2}{3}i(2E_1 - E_2 - E_3)^{\sim} + \check{E}_1 + \hat{E}_1 + i\mathbf{1}, \\ 2i\tilde{F}_2(1 - ie_1) + \check{F}_2(1 - ie_1) + \hat{F}_2(1 - ie_1), \\ 2i\tilde{F}_2(e_2 + ie_3) + \check{F}_2(e_2 + ie_3) + \hat{F}_2(e_2 + ie_3), \\ 2i\tilde{F}_3(1 + ie_1) - \check{F}_3(1 + ie_1) - \hat{F}_3(1 + ie_1), \\ 2i\tilde{F}_3(e_2 + ie_3) + \check{F}_3(e_2 + ie_3) + \hat{F}_3(e_2 + ie_3) \end{array} \right\} 5$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau, \quad \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3})\tau.$$

By using the differential mapping $\varphi_* : \mathfrak{su}(8, \boldsymbol{C}^C) \to {\mathfrak{e}_7}^C$ of the mapping $\varphi : SU(8, \boldsymbol{C}^C) \to {E_7}^C$, we have

$$\begin{aligned} iZ &= \varPhi((G_{45} + G_{67}), \frac{1}{2}(3E_1 + E_2 + E_3), -\frac{1}{2}(3E_1 + E_2 + E_3), 0) \\ &= \varphi_*(\mathrm{diag}(5e_1/4, 5e_1/4, 5e_1/4, -3e_1/4, -3e_1/4, -3e_1/4, -3e_1/4, -3e_1/4)). \end{aligned}$$

Hence

$$\begin{aligned} z_2 &= \exp\frac{2\pi i}{2}Z = \varphi(\mathrm{diag}({w_8}^5, {w_8}^5, {w_8}^5, {w_8}^5, {w_8}^5, {w_8}^5, {w_8}^5, {w_8}^5)) \\ &= -\lambda\gamma,\ w_8 = e^{\pi e_1/4}, \end{aligned}$$

$$z_4 = \exp\frac{2\pi i}{4}Z = \varphi(\mathrm{diag}({w_{16}}^5, {w_{16}}^5, {w_{16}}^5, {w_{16}}^{-3}, {w_{16}}^{-3}, {w_{16}}^{-3}, {w_{16}}^{-3}, {w_{16}}^{-3})),\ w_{16} = e^{\pi e_1/8},$$

$$z_3 = \exp\frac{2\pi i}{3}Z = \varphi(\mathrm{diag}(e^{5\pi e_1/6}, e^{5\pi e_1/6}, e^{5\pi e_1/6}, -e_1, -e_1, -e_1, -e_1, -e_1)).$$

z_3 is conjugate to $-w_3$ in ${E_7}^C$:

$$z_3 \sim -w_3.$$

Indeed, $Z = i\varPhi(-(G_{45}+G_{67}), -\frac{1}{2}(3E_1+E_2+E_3), \frac{1}{2}(3E_1+E_2+E_3), 0)$ is conjugate to $Z' = i\varPhi(-(G_{01}+G_{23}), -\frac{1}{2}(3E_1+E_2+E_3), \frac{1}{2}(3E_1+E_2+E_3), 0)$ under the action $\delta = \exp\Big(\frac{\pi}{2}\varPhi((G_{04}+G_{15}+G_{26}+G_{37}), 0, 0, 0)\Big) \in {E_7}^C$, that is,

$$Z' = \delta^{-1}Z\delta.$$

Since we have $iZ' = \varPhi(G_{01}+G_{23}, \frac{1}{2}(3E_1+E_2+E_3), -\frac{1}{2}(3E_1+E_2+E_3), 0) = \varphi_*(\mathrm{diag}(5e_1/4, 5e_1/4, e_1/4, e_1/4, -7e_1/4, e_1/4, -3e_1/4, -3e_1/4)$, so

$$\begin{aligned}{z_3}' &= \exp\frac{2\pi}{3}iZ' \\ &= \varphi(\mathrm{diag}(-e_1{w_1}^2, -e_1{w_1}^2, -e_1w_1, -e_1w_1, -e_1{w_1}^2, -e_1w_1, -e_1, -e_1)) \\ &= \varphi((\mathrm{diag}({w_1}^2, {w_1}^2, w_1, w_1, {w_1}^2, w_1, 1, 1))\varphi((\mathrm{diag}(-e_1, -e_1, -e_1, -e_1, \\ &\quad -e_1, -e_1, -e_1, -e_1)) \\ &= -\varphi(\mathrm{diag}({w_1}^2, {w_1}^2, w_1, w_1, {w_1}^2, w_1, 1, 1)).\end{aligned}$$

On the other hand, we know that w_3 is given by

$$w_3 = \exp\Big(\frac{2\pi i}{3}\varPhi((2G_{23}-G_{45}-G_{67}), 0, 0, 0)\Big)$$

(cf. [2]). Since $\varPhi((2G_{23}-G_{45}-G_{67}), 0, 0, 0) = \varphi_*(\mathrm{diag}(0, 0, -e_1, e_1, -e_1, e_1, -e_1, e_1))$, we see that

$$w_3 = \varphi(\mathrm{diag}(1, 1, {w_1}^2, w_1, {w_1}^2, w_1, {w_1}^2, w_1)).$$

Let $\delta' = \varphi(P)$, where

$$P = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in SU(8, \boldsymbol{C}^C),$$

then we have ${\delta'}^{-1}(-{z_3}')\delta' = w_3$. Thus we have $z_3 = \delta\delta'(-w_3){\delta'}^{-1}\delta^{-1}$, that is, $z_3 \sim -w_3$.

We use $-w_3$ instead of z_3.

Since $({\mathfrak{e}_7}^C)_{ev} = ({\mathfrak{e}_7}^C)^{z_2} = ({\mathfrak{e}_7}^C)^{-\lambda\gamma} = ({\mathfrak{e}_7}^C)^{\lambda\gamma}, ({\mathfrak{e}_7}^C)_0 = ({\mathfrak{e}_7}^C)^{z_4}, ({\mathfrak{e}_7}^C)_{ed} = ({\mathfrak{e}_7}^C)^{-w_3} = ({\mathfrak{e}_7}^C)^{w_3}$, we shall determine the structure of groups

$$({E_7}^C)_{ev} = ({E_7}^C)^{z_2} = ({E_7}^C)^{-\lambda\gamma} = ({E_7}^C)^{\lambda\gamma}, \quad ({E_7}^C)_0 = ({E_7}^C)^{z_4},$$
$$({E_7}^C)_{ed} = ({E_7}^C)^{-w_3} = ({E_7}^C)^{w_3}.$$

Theorem 4.5.1. (1) $({E_7}^C)_{ev} \cong SL(8,C)/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{E, -E\}$.

(2) $({E_7}^C)_0 \cong (C^* \times SL(3,C) \times SL(5,C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_{15})$, $\boldsymbol{Z}_2 = \{(1,E,E), (-1, E,E)\}$, $\boldsymbol{Z}_{15} = \{({\omega_{15}}^k, {\omega_{15}}^{-5k}E,\ {\omega_{15}}^{3k}E) \,|\, k = 0,1,\ldots,14\}$, $\omega_{15} = e^{2\pi i/15}$.

(3) $({E_7}^C)_{ed} \cong (SL(3,C) \times SL(6,C))/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(E,E), (\omega E, \omega E), (\omega^2 E, \omega^2 E)\}$, $\omega = e^{2\pi i/3}$.

Proof. (1) The fact that $({E_7}^C)^{\lambda\gamma} \cong SL(8,C)/\boldsymbol{Z}_2$ is already used (cf. [3]). That is, the mapping $\varphi : SU(8, \boldsymbol{C}^C) \to ({E_7}^C)^{\lambda\gamma}$,

$$\varphi(A)P = \chi^{-1}(A(\chi P)\, {}^tA), \quad P \in \mathfrak{P}^C$$

induces the isomorphism $({E_7}^C)_{ev} = ({E_7}^C)^{\lambda\gamma} \cong SU(8, \boldsymbol{C}^C)/\boldsymbol{Z}_2 \cong SL(8,C)/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{E, -E\}$ (the last isomorphism is given by $f : SL(8,C) \to SU(8, \boldsymbol{C}^C)$, $f(A) = \iota A + \bar{\iota}\, {}^tA^{-1}, \iota = (1 + ie_1)/2$).

(2) We define a mapping $\varphi : S(U(3, \boldsymbol{C}^C) \times U(5, \boldsymbol{C}^C)) \to ({E_7}^C)^{z_4}$ by

$$\varphi(B_1, B_2)P = \chi^{-1}((B_1, B_2)(\chi P)^t(B_1, B_2)), \quad P \in \mathfrak{P}^C,$$

as the restriction mapping $\varphi : SU(8, \boldsymbol{C}^C) \to {E_7}^C$, where (B_1, B_2) means $\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \in SU(8, \boldsymbol{C}^C)$. Then φ is well-defined and is a homomorphism. $\mathrm{Ker}\,\varphi = \{(E,E), (-E,-E)\} = \boldsymbol{Z}_2$. Since $({E_7}^C)^{z_4}$ is connected and $\dim(({\mathfrak{e}_7}^C)_0) = 33$ (Theorem 4.5) $= 9 + 25 - 1 = \dim(\mathfrak{s}(\mathfrak{u}(3, \boldsymbol{C}^C) \oplus \mathfrak{u}(5, \boldsymbol{C}^C)))$, φ is onto. Thus we have

$$\begin{aligned}({E_7}^C)^{z_4} &\cong S(U(3, \boldsymbol{C}^C) \times U(5, \boldsymbol{C}^C))/\boldsymbol{Z}_2 \\ &\cong S(GL(3,C) \times GL(5,C))/\boldsymbol{Z}_2.\end{aligned}$$

The mapping $h : C^* \times SL(3,C) \times SL(5,C) \to S(GL(3,C) \times GL(5,C))$,

$$h(z, A, B) = \begin{pmatrix} z^5 A & 0 \\ 0 & z^{-3}B \end{pmatrix}$$

induces an isomorphism $(C^* \times SL(3,C) \times SL(5,C))/\boldsymbol{Z}_{15} \cong S(GL(3,C) \times GL(5,C))$, $\boldsymbol{Z}_{15} = \{({\omega_{15}}^k, {\omega_{15}}^{-5k}E, {\omega_{15}}^{3k}E) \,|\, k = 0,1,\ldots,14\}$. Thus we have $({E_7}^C)_0 = ({E_7}^C)^{z_4} \cong (C^* \times SL(3,C) \times SL(5,C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_{15})$.

(3) $({E_7}^C)^{-w_3} = ({E_7}^C)^{w_3} \cong (SL(3,C) \times SL(6,C))/\boldsymbol{Z}_3$ is shown in [2]. However, for later use, we review the outline of the proof. Let $SU(6, \boldsymbol{C}^C) = ({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3}$. By using the mapping $\varphi_{3,l} : SU(3, \boldsymbol{C}^C) \to {E_7}^C$, we define a mapping $\varphi_{w_3} : SU(3, \boldsymbol{C}^C) \times SU(6, \boldsymbol{C}^C) \to ({E_7}^C)^{w_3}$ by

$$\varphi_{w_3}(A, \beta) = \varphi_{3,l}(A)\beta.$$

Then φ_{w_3} induces the isomorphism $({E_7}^C)^{w_3} \cong (SU(3, \boldsymbol{C}^C) \times SU(6, \boldsymbol{C}^C))/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(E,E), (w_1E, w_1E), ({w_1}^2E, {w_1}^2E)\}\,(w_1 = e^{2\pi e_1/3}) \cong (SL(3,C) \times SL(6, C))/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(E,E), (\omega E, \omega E), (\omega^2 E, \omega^2 E)\}$ (cf. [2, Theorem 4.1.3]). □

4.5.1. Subgroups of type $\boldsymbol{A}_{7(7)}, \boldsymbol{R} \oplus \boldsymbol{A}_{2(2)} \oplus \boldsymbol{A}_{4(4)}$ and $\boldsymbol{A}_{2(2)} \oplus \boldsymbol{A}_{5(5)}$ of $\boldsymbol{E}_{7(7)}$

Since $(\mathfrak{e}_{7(7)})_{ev} = ({\mathfrak{e}_7}^C)_{ev} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1} = ({\mathfrak{e}_7}^C)^{\lambda\gamma} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1}$, $(\mathfrak{e}_{7(7)})_0 = ({\mathfrak{e}_7}^C)_0 \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1} = ({\mathfrak{e}_7}^C)^{z_4} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1}$, $(\mathfrak{e}_{7(7)})_{ed} = ({\mathfrak{e}_7}^C)_{ed} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1} = ({\mathfrak{e}_7}^C)^{w_3} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1}$, we shall determine the structure of groups

$$(E_{7(7)})_{ev} = ({E_7}^C)_{ev} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1} = ({E_7}^C)^{\lambda\gamma} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1},$$
$$(E_{7(7)})_0 = ({E_7}^C)_0 \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1} = ({E_7}^C)^{z_4} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1},$$
$$(E_{7(7)})_{ed} = ({E_7}^C)_{ed} \cap ({E_7}^C)^{\tau\lambda\iota\gamma_1} = ({E_7}^C)^{w_3} \cap ({\mathfrak{e}_7}^C)^{\tau\lambda\iota\gamma_1}.$$

Theorem 4.5.1.1. (1) $(E_{7(7)})_{ev} \cong SL(8, \boldsymbol{R})/\boldsymbol{Z}_2 \times \{1, \gamma_2\}, \boldsymbol{Z}_2 = \{E, -E\}$.
(2) $(E_{7(7)})_0 \cong (\boldsymbol{R}^+ \times SL(3, \boldsymbol{R}) \times SL(5, \boldsymbol{R})) \times \{1, \gamma_2\}$.
(3) $(E_{7(7)})_{ed} \cong SL(3, \boldsymbol{R}) \times SL(6, \boldsymbol{R})$.

Proof. (1) For $\alpha \in (E_{7(7)})_{ev} \subset ({E_7}^C)_{ev} = ({E_7}^C)^{\lambda\gamma}$, there exists $A \in SL(8, C)$ such that $\alpha = \varphi(A)$ (Theorem 4.5.1. (1)). Now, from $\tau\lambda\iota\gamma_1\alpha\gamma_1\iota^{-1}\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\iota\gamma_1\varphi(A)\gamma_1\iota^{-1}\lambda^{-1}\tau = \varphi(A)$, we have $\varphi(\tau A) = \varphi(A)$ (cf. [3, Theorem 4.2.1.1.(2)]). Hence

$$\tau A = A, \quad \text{or} \quad \tau A = -A.$$

In the former case, $A \in SL(8, \boldsymbol{R})$. Hence the group of the former case is $SL(8, \boldsymbol{R})/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{E, -E\}$. In the latter case, $A = iI, (I = \mathrm{diag}(-1, 1, -1, 1, -1, 1, -1, 1))$ satisfies the condition $\tau A = -A$ and $\varphi(iI) = \gamma_2$. Therefore $(E_{7(7)})_{ev} \cong SL(8, \boldsymbol{R})/\boldsymbol{Z}_2 \times \{1, \gamma_2\}$.

(2) For $\alpha \in (E_{7(7)})_0 \subset ({E_7}^C)_0$, there exists $(B_1, B_2) \in S(U(3, \boldsymbol{C}^C) \times U(5, \boldsymbol{C}^C))$ such that $\alpha = \varphi(B_1, B_2)$ (Theorem 4.5.1.(2)). From $\tau\lambda\iota\gamma_1\alpha\gamma_1\iota^{-1}\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\iota\gamma_1\varphi(B_1, B_2)\gamma_1\iota^{-1}\lambda^{-1}\tau = \varphi(B_1, B_2)$, we have $\varphi(\tau\overline{B}_1, \tau\overline{B}_2) = \varphi(B_1, B_2)$ (cf. [3, Theorem 4.2.1.1.(2)]). Hence

$$\begin{cases} \tau\overline{B}_1 = B_1 \\ \tau\overline{B}_2 = B_2, \end{cases} \quad \text{or} \quad \begin{cases} \tau\overline{B}_1 = -B_1 \\ \tau\overline{B}_2 = -B_2. \end{cases}$$

In the former case, $B_1 \in U(3, \boldsymbol{C}')$ and $B_2 \in U(5, \boldsymbol{C}')$. Hence the group of the former case is

$$S(U(3,\boldsymbol{C}') \times U(5,\boldsymbol{C}'))/\boldsymbol{Z}_2 \cong S(GL(3,\boldsymbol{R}) \times GL(5,\boldsymbol{R}))/\boldsymbol{Z}_2,$$
$$\boldsymbol{Z}_2 = \{(E,E),(-E,-E)\}.$$

The mapping $h : \boldsymbol{R}^* \times SL(3,\boldsymbol{R}) \times SL(5,\boldsymbol{R}) \to S(GL(3,\boldsymbol{R}) \times GL(5,\boldsymbol{R}))$,

$$h(z,A,B) = \begin{pmatrix} z^5 A & 0 \\ 0 & z^{-3}B \end{pmatrix}$$

induces an isomorphism $\boldsymbol{R}^* \times SL(3,\boldsymbol{R}) \times SL(5,\boldsymbol{R}) \cong S(GL(3,\boldsymbol{R}) \times GL(5,\boldsymbol{R}))$. Thus we have $({E_7}^C)_0 \cong (\boldsymbol{R}^* \times SL(3,\boldsymbol{R}) \times SL(5,\boldsymbol{R}))/\boldsymbol{Z}_2$ ($\boldsymbol{Z}_2 = \{1,E,E),(-1, E,E)\}$) $\cong \boldsymbol{R}^+ \times SL(3,\boldsymbol{R}) \times SL(5,\boldsymbol{R})$. In the latter case, iI satisfies the condition and $\varphi(iI) = \gamma_2$. Thus we have $(E_{7(7)})_0 \cong (\boldsymbol{R}^+ \times SL(3,\boldsymbol{R}) \times SL(5,\boldsymbol{R})) \times \{1,\gamma_2\}$

(3) For $\alpha \in (E_{7(7)})_{ed} \subset ({E_7}^C)_{ed} = ({E_7}^C)^{w_3}$, there exist $A \in SU(3,\boldsymbol{C}^C)$ and $\beta \in ({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3} \cong SU(6,\boldsymbol{C}^C)$ (cf. [2, Proposition 1.3.7]) such that $\alpha = \varphi_{w_3}(A,\beta) = \varphi_{3,l}(A)\beta$ (cf. [2, Theorem 4.1.3]). From $\tau\lambda\iota\gamma_1\alpha\gamma_1\iota^{-1}\lambda^{-1}\tau = \alpha$, that is, $\tau\lambda\iota\gamma_1\varphi_{3,l}(A)\beta\gamma_1\iota^{-1}\lambda^{-1}\tau = \varphi_{3,l}(A)\beta$. Hence

$$\text{(i)} \begin{cases} \tau\gamma_1 A = A, \\ \tau\lambda\iota\gamma_1\beta\gamma_1\iota^{-1}\lambda^{-1}\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau\gamma_1 A = w_1 A, \\ \tau\lambda\iota\gamma_1\beta\gamma_1\iota^{-1}\lambda^{-1}\tau = w_1\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \tau\gamma_1 A = {w_1}^2 A, \\ \tau\lambda\iota\gamma_1\beta\gamma_1\iota^{-1}\lambda^{-1}\tau = {w_1}^2\beta. \end{cases}$$

Case (i) From $\tau\gamma_1 A = A$, we have $A \in SU(3,\boldsymbol{C}')$. To determine the structure of the group $\{\beta \in SU(6,\boldsymbol{C}^C) \,|\, \tau\lambda\iota\gamma_1\beta\gamma_1\iota^{-1}\lambda^{-1}\tau = \beta\} = SU(6,\boldsymbol{C}^C)^{\tau\lambda\iota\gamma_1}$, we consider a correspondence

$$f : ({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1} \to ({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\lambda\iota\gamma_1}, \; f(\alpha) = {\delta_2}^{-1}\alpha\delta_2,$$

where $\delta_2 = \exp\varPhi\big(0, \dfrac{i\pi}{4}E, \dfrac{i\pi}{4}E, 0\big) \in E_7, {\delta_2}^{-1}\tau\gamma_1\delta_2 = -\tau\lambda\iota\gamma_1$. Then f gives an isomophism $({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\lambda\iota\gamma_1} \cong ({E_7}^C)^{w_3,\varepsilon_1,\varepsilon_2,\gamma_3,\tau\gamma_1} \cong SU(6,\boldsymbol{C}')$(cf. [2, Theorem 4.2 (3)]). Therefore the group of Case (i) is $SU(3,\boldsymbol{C}') \times SU(6,\boldsymbol{C}') \cong SL(3,\boldsymbol{R}) \times SL(6,\boldsymbol{R})$.

Case (ii) $\varphi(w_1E, w_1E) = 1$.

Case (iii) $\varphi({w_1}^2E, {w_1}^2E) = 1$.

Thus we have the rquired isomorphism $(E_{7(7)})_{ed} \cong SU(3,\boldsymbol{C}') \times SU(6,\boldsymbol{C}') \cong SL(3,\boldsymbol{R}) \times SL(6,\boldsymbol{R})$. □

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