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On splitting of certain Jacobian varieties

By

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Abstract

We give three examples of non-hyperelliptic curves of genus 4 whose Jacobian varieties are isomorphic to products of four elliptic curves. Two of the examples belong to one-parameter families of curves whose Jacobian varieties are isomorphic to products of two 2-dimensional complex tori. By constructing analogous families, we prove that for each $n > 1$, there is a one-parameter family of non-hyperelliptic curves of genus $2n$ whose Jacobian varieties are isomorphic to products of two n-dimensional tori.

1. Introduction

1.1. Introduction

The Jacobian variety of a closed Riemann surface, or a complete algebraic curve over $\mathbb{C}$ (in this paper, we call a closed Riemann surface simply a curve) is the moduli space of line bundles of degree 0 on the curve and it has a structure of a principally polarised Abelian variety (hereafter P.P.A.V.) The Jacobian variety is never isomorphic to a non-trivial product of P.P.A.V's of lower dimension as a P.P.A.V; however, it can be isomorphic to the product of complex tori disregarding the polarisation. Such a Jacobian variety is said to be splitting.

For curves of genus 2, Hayashida and Nishi [5] found many examples of splitting Jacobian varieties by using number theory. Since then, the case of genus 2 is well studied. For curves of genus 3, Klein's curve is known to have a splitting Jacobian variety (see [2]) and for curves of genus 4, Bring's curve is known to have a splitting Jacobian variety (see [7]). Ekedahl and Serre [3] gave examples of splitting Jacobian varieties of curves with various genera and Earle [1] gave one-parameter families of hyperelliptic curves with splitting Jacobian varieties of arbitrary even genus.

In this paper, we shall give certain new examples of splitting Jacobian varieties. In Sections 2, 3 and 4, examples of non-hyperelliptic curves of genus 4, of which Jacobian varieties are isomorphic to products of four elliptic curves, will be given. In Sections 2 and 4, we shall also give one-parameter families

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of curves, of which Jacobian varieties are isomorphic to products of two 2-dimensional complex tori and furthermore, in Section 2, we shall show that a similar family of curves exists for arbitrary even genus.

1.2. Automorphism and period matrix

Let C be a curve (a closed Riemann surface), and $g > 0$ be its genus, $\{\omega_1, \dots, \omega_g\}$ be a basis of holomorphic 1-forms on C, and $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ be a canonical basis of $H_1(C, \mathbb{Z})$. Throughout this paper, a topological 1-cycle and a class in $H_1(C, \mathbb{Z})$ determined by the cycle are not distinguished for the sake of simplicity.

The period matrix $\Pi(C)$ of the curve C is defined as follows:

$$\pi_j = \begin{pmatrix} \int_{\lambda_j} \omega_1 \\ \int_{\lambda_j} \omega_2 \\ \vdots \\ \int_{\lambda_j} \omega_g \end{pmatrix} \quad \pi_{g+j} = \begin{pmatrix} \int_{\mu_j} \omega_1 \\ \int_{\mu_j} \omega_2 \\ \vdots \\ \int_{\mu_j} \omega_g \end{pmatrix} \quad (j = 1 \dots g)$$

$$\Pi(C) = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_{2g} \end{pmatrix}.$$

The Jacobian variety $J(C)$ of the curve C is isomorphic to $\mathbb{C}^g/\Lambda(\Pi(C))$, where $\Lambda(\Pi(C))$ is the lattice in $\mathbb{C}^g$ generated by $2g$ row vectors of $\Pi(C)$.

Let $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$ be a $g \times 2g$ matrix. Assuming that M_2 is invertible, we have $M_2^{-1}M = \begin{pmatrix} M_2^{-1}M_1 & E \end{pmatrix}$, where E is the unit matrix. The complex tori $\mathbb{C}^g/\Lambda(M)$ and $\mathbb{C}^g/\Lambda(M_2^{-1}M)$ are isomorphic. The matrix $\begin{pmatrix} M_2^{-1}M_1 & E \end{pmatrix}$ is called the normalised form of the matrix M.

Since we use a canonical basis of $H_1(C, \mathbb{Z})$ to define the period matrix, the period matrix $\Pi(C) = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$ of the curve C can always be normalised. Let $\begin{pmatrix} Z & E \end{pmatrix}$ be the normalised form of $\Pi(C)$. It is known that Z is a symmetric matrix and its imaginary part $\mathbf{Im}(Z)$ is positive definite. A period matrix of this form is called a normalised period matrix.

Assume that C has an automorphism φ. It induces an automorphism $\hat{\varphi}$ of $H^1(C, \mathbb{Z})$ and $\hat{\varphi}$ maps a canonical basis to a canonical basis. A symplectic matrix expression $M_\varphi \in Sp(2g, \mathbb{Z})$ of this action is given by

$$\hat{\varphi}(\lambda_1, \lambda_2, \dots, \mu_{g-1}, \mu_g) = (\lambda_1, \lambda_2, \dots, \mu_{g-1}, \mu_g)M_\varphi.$$

If $\Pi'(C)$ is the period matrix of C with respect to the canonical basis $\{\hat{\varphi}(\lambda_1), \dots, \hat{\varphi}(\lambda_g), \hat{\varphi}(\mu_1), \dots, \hat{\varphi}(\mu_g)\}$, then $\Pi'(C) = \Pi(C)M_\varphi$. Let $\begin{pmatrix} Z & E \end{pmatrix}$ be the normalised form of $\Pi(C)$, and $\begin{pmatrix} Z' & E \end{pmatrix}$ be the normalised form of $\Pi'(C)$, then $Z = Z'$ and this gives a following relation:

$$Z = Z' = (\alpha Z + \beta)(\gamma Z + \delta)^{-1},$$

where

$$M_\varphi = {}^t\!\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Thus Z is a fixed point of the action of M_φ on $\mathfrak{S}_g$ given by

$$M_\varphi(T) = (\alpha T + \beta)(\gamma T + \delta)^{-1},$$

where $\mathfrak{S}_g$ is the Siegel upper half plane of degree g, the space of symmetric matrices of which imaginary parts are positive definite.

1.3. Case of genus 2

Consider the hyperelliptic curve C of genus 2 defined by

$$C : y^2 = (x^3 - a^3)(x^3 - a^{-3}).$$

The curve C admits the following three automorphisms:

$$\varphi_1 : \begin{cases} x \mapsto \omega x \\ y \to y \end{cases}$$

$$\varphi_2 : \begin{cases} x \mapsto 1/x \\ y \to y/x^2 \end{cases} \quad (\omega = e^{\frac{2\pi i}{3}}).$$

$$\iota : \begin{cases} x \to x \\ y \mapsto -y \end{cases}$$

Let us regard C as a two-sheeted covering over x-plane $\mathbb{P}^1$. Then we may choose a canonical base $\lambda_1, \lambda_2, \mu_1, \mu_2$ as in Fig. 1. Let $\hat{\varphi}_1$ be the map on

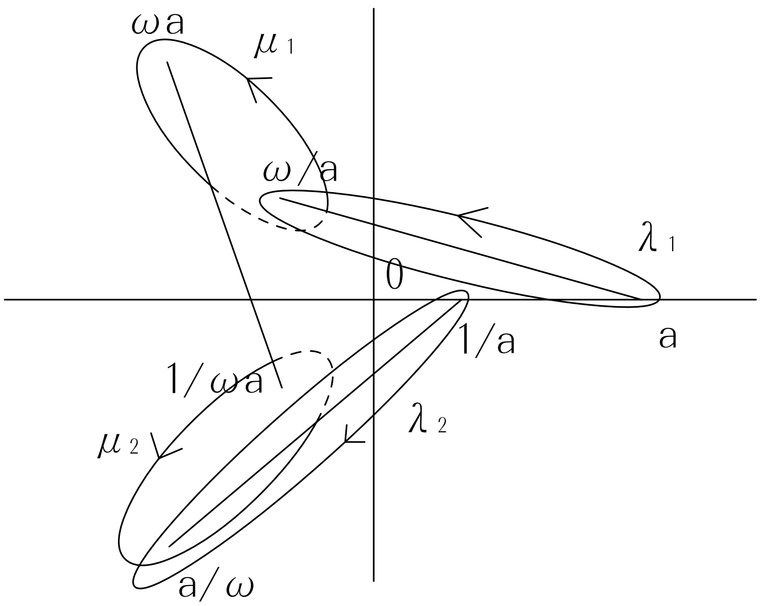

Figure 1.

$H^1(C, \mathbb{Z})$ induced by φ_1, then

$$\begin{aligned}\hat{\varphi}_1(\lambda_1) &= -\lambda_1 + \lambda_2 \\ \hat{\varphi}_1(\lambda_2) &= -\lambda_1 \\ \hat{\varphi}_1(\mu_1) &= \mu_2 \\ \hat{\varphi}_1(\mu_2) &= -\mu_1 - \mu_2.\end{aligned}$$

Thus the symplectic matrix corresponding to $\hat{\varphi}_1$ is given as follows:

$$\hat{\varphi}_1(\lambda_1, \lambda_2, \mu_1, \mu_2) = (\lambda_1, \lambda_2, \mu_1, \mu_2)\,{}^t\!\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

If $\Pi(C) = \begin{pmatrix} Z & E \end{pmatrix}$ is the normalised period matrix of C, then Z is a fixed point of the action of the above matrix;

$$Z = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} Z \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^{-1}.$$

Solving the above equation, we get

$$Z = \begin{pmatrix} 2z & z \\ z & 2z \end{pmatrix}.$$

Here z depends on the parameter a. Put

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

The matrix A is an element of $SL(4, \mathbb{Z})$. Multiplying the period matrix from right by A (this corresponds to the non-symplectic change of a homology basis), we have

$$\begin{pmatrix} Z & E \end{pmatrix} A = \begin{pmatrix} 3z & z & 1 & 1 \\ 3z & 2z & 1 & 2 \end{pmatrix}$$

and then normalising this, that is, multiplying this from left by the inverse matrix of the latter half of this matrix (this corresponds to the change of a basis of 1-forms), we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} Z & E \end{pmatrix} A = \begin{pmatrix} 3z & 0 & 1 & 0 \\ 0 & z & 0 & 1 \end{pmatrix}.$$

The lattice Λ_1 generated by the first and third rows of the above matrix and the lattice Λ_2 generated by the second and fourth rows are linearly independent in $\mathbb{C}^2$. This means

$$\begin{aligned} J(C) &\cong \mathbb{C}^2/\Lambda(\Pi(C)) \\ &\cong \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2 \end{aligned}$$

and we see the Jacobian variety of C splits into a product of two elliptic curves.

1.4. Complex multiplication

Let E be an elliptic curve. Assume that an n-dimension complex torus T is isogenous to E^n, n-th product of E. It is known that the following result holds (see [6]).

Theorem 1.1. *If E has a complex multiplication, then there exist elliptic curves $E_1 \dots E_n$ such that T is isomorphic to a product $E_1 \times E_2 \times \dots \times E_n$.*

An immediate consequence is the following criterion.

Corollary 1.1. *Let J be a Jacobian variety and $\Pi = \begin{pmatrix} Z & E \end{pmatrix}$ be its normalised period matrix. If every element of Z is contained in the same imaginary quadratic fields then J is isogenous to a product of elliptic curves.*

Proof. Assume that the elements of Z are contained in $\mathbb{Q}(\sqrt{-m})$, then there exists $n \in \mathbb{N}$ such that every element of nZ contains in $\mathbb{Z}(\sqrt{-m})$. Put $Z' = \operatorname{diag}(\sqrt{-m} \dots \sqrt{-m})$ and $\Pi' = \begin{pmatrix} Z' & E \end{pmatrix}$. Let Λ be the lattice generated by the row vectors of Π and Λ' be the lattice generated by the row vectors of Π', then the multiplying map $n : \mathbb{C}^d \to \mathbb{C}^d$ $(d = \dim J)$ induces a surjective map $\hat{n} : \mathbb{C}^d/\Lambda \to \mathbb{C}^d/\Lambda'$. This shows that $J = \mathbb{C}^d/\Lambda$ is isogenous to the product of elliptic curves. □

For example, consider the hyperelliptic curve

$$C' : y^2 = x^6 - 1.$$

We can calculate the period matrix $\Pi(C')$ of C' by the same method as the one in the previous section.

$$\Pi(C') = \begin{pmatrix} \frac{2}{\sqrt{3}}i & \frac{1}{\sqrt{3}}i & 1 & 0 \\ \frac{1}{\sqrt{3}}i & \frac{2}{\sqrt{3}}i & 0 & 1 \end{pmatrix}.$$

We have two proofs to show that the Jacobian variety $J(C')$ of C' splits. First, the period matrix has the same form $\begin{pmatrix} 2z & z & 1 & 0 \\ z & 2z & 0 & 1 \end{pmatrix}$ as in previous section and we know this type of a Jacobian variety splits. Second, every element of $\Pi(C')$ contains in the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$ and from the above corollary we can also conclude that $J(C')$ splits.

1.5. Canonical embedding of curve of genus 4

Every non-hyperelliptic curve of genus g can be canonically embedded in $\mathbb{P}^{g-1}$. If $g = 4$, we can embed a curve in $\mathbb{P}^3$ and the classical theory says that the curve is the intersection of a quadratic surface (or a quadric) S_1 of rank 3 or 4 and a cubic surface S_2. Conversely, a smooth intersection C of a quadratic surface and a cubic surface is a canonical curve and thus non-hyperelliptic. If a quadratic surface is nonsingular, then it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Hence, the curve C can be viewed as a curve in $\mathbb{P}^1 \times \mathbb{P}^1$.

If the curve admits an automorphism, it induces a linear transformation on the vector space of holomorphic 1-forms. Since the canonical embedding is an embedding with respect to these 1-forms, every automorphism of a canonically embedded curve is represented by a projective transformation on $\mathbb{P}^{g-1}$. If $g = 4$, an automorphism can be represented as an element of $GL(4, \mathbb{C})$.

2. Curve of genus 4, case 1

2.1. Special case

Let C_1 be a curve in $\mathbb{P}^3$ defined by

$$C_1 : \begin{cases} X_0X_1 + X_2X_3 = 0 \\ (X_0^3 - X_3^3) + (X_2^3 - X_1^3) = 0. \end{cases}$$

The curve C_1 is a smooth intersection of the quadratic surface and the cubic surface. Hence, it is a non-hyperelliptic curve of genus 4. The curve C_1 admits the following four automorphisms: (They are written in terms of linear transformations of four variables X_0, X_1, X_2, X_3)

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\omega = e^{-2\pi i/3}).$$
$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad P_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The order of a group generated by the above automorphisms is 72 and according to [8], this is the maximal possible automorphism group of the curve of genus 4. Thus this gives the automorphism group of C_1 ($\mathbf{Aut}(C_1)$ is isomorphic to $G(9 \times 8)$ in [8]).

The surface $S : X_0X_1 + X_2X_3 = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the map

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & S \\ [z_0 : z_1] \times [w_0 : w_1] & \mapsto & [z_1w_0 : z_0w_1 : -z_1w_1 : z_0w_0]. \end{array}$$

Through this map, C_1 is isomorphic to the curve defined by an equation

$$z^3 = \frac{1 + w^3}{1 - w^3}$$

in $\mathbb{P}^1 \times \mathbb{P}^1$, where $z = z_1/z_0, w = w_1/w_0$.

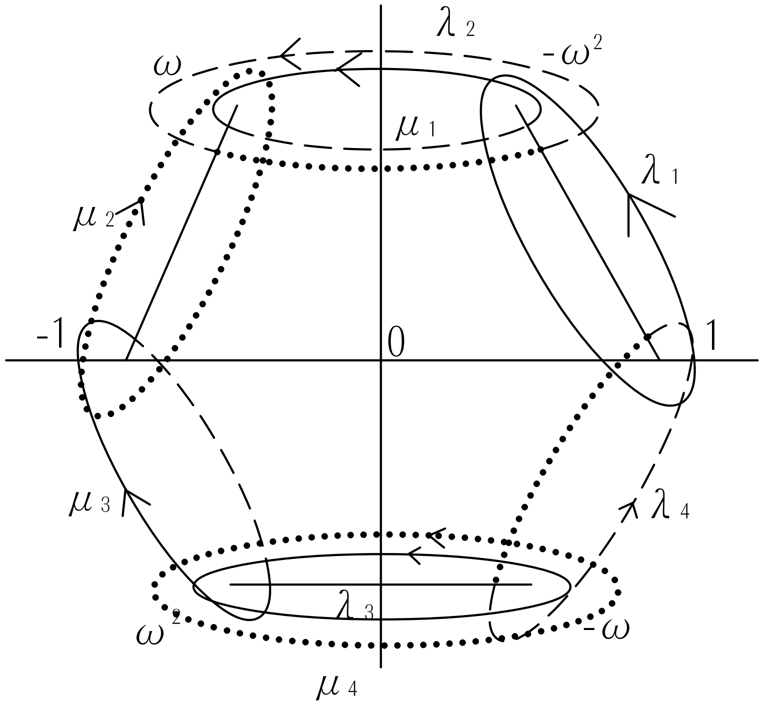

Figure 2.

The automorphisms P_1, P_2, P_3, P_4 act on (z, w) as

$$P'_1 : \begin{cases} z \mapsto \omega z \\ w \mapsto w \end{cases} \quad P'_2 : \begin{cases} z \mapsto z \\ w \mapsto \omega w \end{cases}$$

$$P'_3 : \begin{cases} z \mapsto \dfrac{1}{z} \\ w \mapsto -w \end{cases} \quad P'_4 : \begin{cases} z \mapsto -w \\ w \mapsto -z. \end{cases}$$

Let us consider a configuration of Fig. 2. Here we regard C_1 as a three-sheeted covering over w-plane $\mathbb{P}^1$. Each style (normal, dotted or broken) of curved lines lie on a different sheet of the covering and the automorphism P_1 (which corresponds to the change of sheets) maps "normal lines" to "dotted lines", "dotted lines" to "broken lines" and "broken lines" to "normal lines". For example, $P_1(\lambda_3) = \mu_4$.

The lines λ_j and μ_j $(j = 1, 2, 3, 4)$ in Fig. 2 give a canonical basis of $H_1(C, \mathbb{Z})$. Let $M_{P_k}(k = 1, 2, 3, 4)$ be the symplectic matrices corresponding to automorphisms P_k with respect to this basis. Then, we have

$$M_{P_1} = {}^t\begin{pmatrix} O & \begin{matrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{matrix} & -E \end{pmatrix},$$

$$M_{P_2} = {}^t\begin{pmatrix} \begin{matrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} & O \\ O & \begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{matrix} \end{pmatrix},$$

$$M_{P_3} = {}^t\begin{pmatrix} \begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \\ O & \begin{matrix} 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix} \end{pmatrix},$$

$$M_{P_4} = {}^t\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The fixed point matrices of the actions of M_{P_1}, M_{P_2} and M_{P_4} in the Siegel upper half plane can be written as

$$Z = \begin{pmatrix} -2a' & b'-1 & a' & b' \\ b'-1 & -2a' & 1-2b' & a' \\ a' & 1-2b' & -2a' & b'-1 \\ b' & a' & b'-1 & -2a' \end{pmatrix},$$

where a' and b' are indeterminants. Since Z is also fixed by M_{P_3}, this gives the relation $a'^2 = b'^2 - b'$.

If we choose another canonical basis

$$\begin{aligned} \bar{\lambda_1} &= \lambda_1 - \mu_4, & \bar{\mu_1} &= \mu_1, \\ \bar{\lambda_2} &= \lambda_2 + \mu_3, & \bar{\mu_2} &= \mu_2, \\ \bar{\lambda_3} &= \lambda_3 + \mu_2, & \bar{\mu_3} &= \mu_3, \\ \bar{\lambda_4} &= \lambda_4 - \mu_1, & \bar{\mu_4} &= \mu_4, \end{aligned}$$

and rewrite M_{P_k} with respect to the new basis, then the fixed point matrices of the actions of these rewritten symplectic matrices can be written in a form

$$Z' = \begin{pmatrix} -2a & b & a & b \\ b & -2a & -2b & a \\ a & -2b & -2a & b \\ b & a & b & -2a \end{pmatrix}, \quad a^2 = b^2 + b,$$

with $a = a', b = b' - 1$. Thus we get the one-parameter family of matrices fixed by the matrices $M_{P_1}, M_{P_2}, M_{P_3}$ and M_{P_4} and the period matrix $\Pi(C_1)$ can be written as

$$\Pi(C_1) = \begin{pmatrix} Z' & E \end{pmatrix}.$$

Choosing a matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in SL(8, \mathbb{Z})$$

and multiplying the period matrix from right by A, we get

$$\Pi(C_1)A = \begin{pmatrix} -3a & -3b & a & b & -1 & 0 & 1 & 0 \\ 3b & 3a & -2b & -2a & 0 & 1 & 0 & -2 \\ 3a & 3b & -2a & -2b & 1 & 0 & -2 & 0 \\ 0 & 0 & b & a & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By normalising this matrix, we get

$$\begin{pmatrix} \begin{matrix} 3a & 3b \\ 3b & 3a \end{matrix} & O & \begin{matrix} 1 & \\ & 1 \end{matrix} & \\ O & \begin{matrix} a & b \\ b & a \end{matrix} & & \begin{matrix} 1 & \\ & 1 \end{matrix} \end{pmatrix}.$$

The matrix shows that the Jacobian variety $J(C_1)$ of C_1 is isomorphic to the product of two 2-dimensional complex tori T_1 and T_2, where

$$T_1 = \mathbb{C}^2 / \left(\text{the lattice generated by } N_1 = \begin{pmatrix} 3a & 3b & 1 & 0 \\ 3b & 3a & 0 & 1 \end{pmatrix} \right)$$
$$T_2 = \mathbb{C}^2 / \left(\text{the lattice generated by } N_2 = \begin{pmatrix} a & b & 1 & 0 \\ b & a & 0 & 1 \end{pmatrix} \right).$$

Furthermore, T_1 and T_2 also split. To show this, take

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SL(4, \mathbb{Z})$$

and multiply N_1 by A_1 and N_2 by A_2 from right and then normalise the resulting matrices. Then we get

$$\begin{pmatrix} 3a+3b & 0 & 1 & 0 \\ 0 & \frac{a+b}{3b} & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{b}{a} & 0 & 1 & 0 \\ 0 & a+b & 0 & 1 \end{pmatrix}.$$

Here we use the equality $a^2 = b^2 + b$. Four values $3a + 3b, a + b/3b, a + b, b/a$ appearing in above matrices are related by

$$a + b = \frac{-\left(\frac{b}{a}\right)}{\left(\frac{b}{a}\right) - 1}$$
$$\frac{a+b}{3b} = \frac{\left(\frac{b}{a}\right) + 1}{3\left(\frac{b}{a}\right)}.$$

These relations show four elliptic curves with period matrices $\begin{pmatrix} 3a+3b & 1 \end{pmatrix}$, $\begin{pmatrix} a+b/3b & 1 \end{pmatrix}$, $\begin{pmatrix} a+b & 1 \end{pmatrix}$, $\begin{pmatrix} b/a & 1 \end{pmatrix}$ are isogenous.

Summarising these results, we obtain

Theorem 2.1. *The Jacobian variety $J(C_1)$ of the curve C_1 is isomorphic to the product of four elliptic curves, and they are isogenous to one another.*

However, we cannot say which values a and b corresponds to the Jacobian variety $J(C_1)$ by this calculation. This is because the dimension of the moduli space of 4-dimensional P.P.A.V's. is larger than the dimension of the moduli space of curves of genus 4.

2.2. One-parameter family case

Let $\{C_1(t)\}$ be a one-parameter family of curves of genus 4 in $\mathbb{P}^3$ defined by

$$C_1(t) : \begin{cases} X_0X_1 + X_2X_3 = 0 \\ (X_0^3 - X_3^3) + (tX_2^3 - X_1^3) = 0 \ (t \neq -1) \end{cases}$$

In $\mathbb{P}^1 \times \mathbb{P}^1$, the family can be defined by

$$z^3 = \frac{1+w^3}{1-tw^3}.$$

Note that the curve C_1 in the previous subsection is $C_1(1)$.

Every member of this family admits automorphisms P_1, P_2, P_4 defined in the previous subsection. The same argument as in the previous subsection shows that the period matrix of $C_1(t)$ takes the form

$$\begin{pmatrix} -2a & b & a & b & 1 & & & \\ b & -2a & -2b & a & & 1 & & \\ a & -2b & -2a & b & & & 1 & \\ b & a & b & -2a & & & & 1 \end{pmatrix}$$

and the Jacobian variety is isomorphic to the product of two 2-dimensional complex tori.

2.3. Higher genera case

We extend the result to the one for higher genera. Let $\{C_1^m(t)\}$ be a one-parameter family of curves of genus $(2m-2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by an equation

$$z^3 = \frac{1+w^m}{1-tw^m} \ (t \neq -1, m > 1).$$

Every member of this family admits automorphisms

$$P_{m,1} : \begin{cases} z \mapsto \omega z \\ w \mapsto w \end{cases}$$

$$P_{m,2} : \begin{cases} z \mapsto z \\ w \mapsto \zeta_m w, \end{cases}$$

where $\omega = e^{2\pi i/3}, \zeta_m = e^{2\pi i/m}$.

Proposition 2.1. *Each member of $\{C_1^m(t)\}$ is non-hyperelliptic for $m > 4$.*

Proof. If $C_1^m(t)$ is hyperelliptic, then $C_1^m(t)$ can be realised as a two-sheeted covering over $\mathbb{P}^1$ with $(4m-2)$ ramification points and every automorphism of $C_1^m(t)$ induces the automorphism of $\mathbb{P}^1$.

If 3 does not divide the number m then $(P_{m,1}P_{m,2})$ generates a cyclic subgroup of $\mathbf{Aut}(C_1^m(t))$ of order $3m$. Let x be local coordinates of $\mathbb{P}^1$ and t be a map induced by $(P_{m,1}P_{m,2})$ on $\mathbb{P}^1$. Every automorphism on $\mathbb{P}^1$ is a projective linear transformation and the automorphism t can be written as $t : x \mapsto \frac{ax+b}{cx+d}$. Since $t^{3m} = 1$, the matrix $M_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be diagonalised. Thus by choosing a suitable local coordinates x', t can be written as $t : x' \mapsto \zeta_{3m}^k x'$ for some

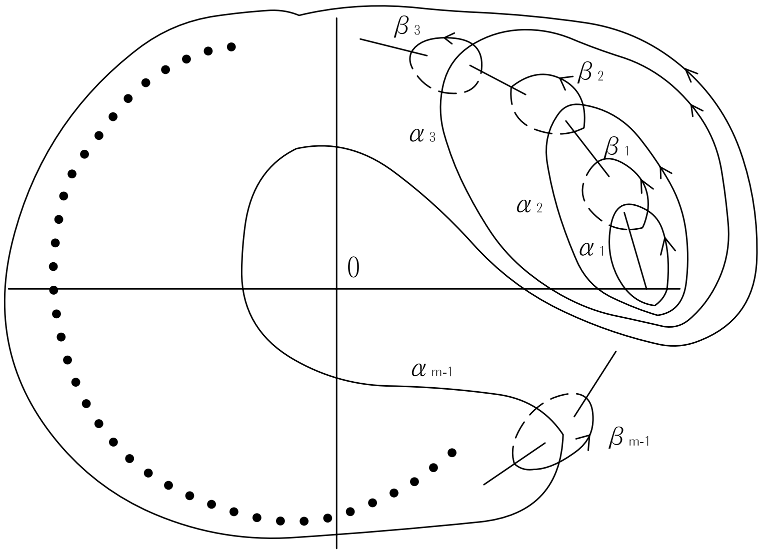

Figure 3.

k. Since the fixed points of t are 0 and ∞, the number of fixed points of $(P_{m,1}P_{m,2})^m$ is at most 4. However, $(P_{m,1}P_{m,2})^m = P_{m,1}$ or $P_{m,1}^2$ fixes $2m$ points. This is a contradiction.

If 3 divides the number m then $\mathbf{Aut}(C_1^m(t))$ has a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. But $\mathbf{Aut}(\mathbb{P}^1)$ never has a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. This is a contradiction. $\square$

Let us consider a configuration of Fig. 3. The meaning of normal, dotted and broken lines are the same as in Fig. 2

Define

$$\begin{array}{ll} \lambda_j = \alpha_j, & \mu_j = \beta_j, \\ \lambda_{(m-1)+j} = P_{m,1}(\alpha_j), & \mu_{(m-1)+j} = (P_{m,1})^2(\beta_j) \end{array} \quad (j = 1, \dots, m-1).$$

Then $\lambda_1, \dots, \lambda_{2m-2}, \mu_1, \dots, \mu_{2m-2}$ form a canonical basis. The symplectic matrices corresponding to the automorphisms with respect to this basis are given by

$$M_{P_{m,1}} = {}^t\begin{pmatrix} O & E_{m-1} & & \\ -E_{m-1} & -E_{m-1} & & O \\ & & -E_{m-1} & E_{m-1} \\ & O & -E_{m-1} & O \end{pmatrix},$$

$$M_{P_{m,2}} = {}^t\begin{pmatrix} Q_1 & O & & \\ O & Q_1 & & O \\ & & Q_2 & O \\ & O & O & Q_2 \end{pmatrix}$$

where

$$Q_1 = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ -1 & 0 & 0 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix}, \; Q_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ -1 & -1 & -1 & \dots & -1 \end{pmatrix}$$

$$E_{m-1} = (\text{unit matrix of degree } m-1).$$

Let $Z = (z_{j,k})$ be a fixed point matrix of the actions of automorphisms $M_{P_{m,1}}$ and $M_{P_{m,2}}$. The matrix Z has the following properties:

$$z_{j,k} = z_{k,j}, \; z_{j,k} = z_{(2m-2)+1-k,(2m-2)+1-j}.$$

Choose a matrix

$$S = \begin{pmatrix} E_{m-1} & E_{m-1} & & \\ Q_3 + E_{m-1} & Q_3 & \multicolumn{2}{c}{O} \\ & & E_{m-1} - Q_3 & Q_3 \\ \multicolumn{2}{c}{O} & -E_{m-1} & E_{m-1} \end{pmatrix},$$

where

$$Q_3 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}.$$

Multiplying $\Pi = \begin{pmatrix} Z & E_{2m-2} \end{pmatrix}$ from left by S and then normalising it, we get the matrix of the form

$$\begin{pmatrix} Z_1 & O & E_{m-1} & O \\ O & Z_2 & O & E_{m-1} \end{pmatrix}.$$

Hence the Jacobian variety $J(C_1^m(t))$ splits.

Theorem 2.2. *The Jacobian variety of $C_1^m(t)$ splits into a product of two $(m-1)$-dimensional complex tori.*

3. Curve of genus 4, case 2

3.1. Special case

Let C_2 be a curve of genus 4 in $\mathbb{P}^3$ defined by

$$C_2 : \begin{cases} X_0^2 + X_1^2 + X_2^2 = 0 \\ X_0 X_1 X_2 - X_3^3 = 0. \end{cases}$$

The curve C_2 admits the following three automorphisms: (They are written in terms of linear transformations of four variables X_0, X_1, X_2, X_3)

$$P_1 = \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$P_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\omega = e^{-2\pi i/3}).$$

The automorphism group $\mathbf{Aut}(C_2)$ is generated by the above three automorphisms and its order is 72 ($\mathbf{Aut}(C_2)$ is isomorphic to $G(8 \times 9)$ in [8] and this is the maximal possible automorphism group).

Put $z = (X_1 - iX_2)/X_3$. The mapping $z : C_2 \to \mathbb{P}^1$ is three-to-one and it ramifies at the points $z = 0, \infty, 1, -1, i, -i$. If we put $w = X_3/X_0$, we can embed C_2 into $\mathbb{P}^1 \times \mathbb{P}^1$ by (z, w). The image of this map is a curve defined by

$$w^3 = \frac{z^4 - 1}{4z^2} i.$$

This is a singular curve.

The automorphisms P_1, P_2 and P_3 act on (z, w) as follows:

$$P_1 : \begin{cases} z \mapsto z \\ w \mapsto \omega w \end{cases}$$
$$P_2 : \begin{cases} z \mapsto \dfrac{z-1}{z+1} \\ w \mapsto \dfrac{2zw}{z^2-1} \end{cases}$$
$$P_3 : \begin{cases} z \mapsto \dfrac{z-1}{z+1} i \\ w \mapsto \dfrac{2zw}{z^2-1} \end{cases}$$

Let us consider a configuration of Fig. 4. In Fig. 4, we regard C_2 as a three-sheeted covering over z-plane $\mathbb{P}^1$. Cycles α_j, $\beta_j (j = 1, 2, 3)$ in Fig. 4 are taken so as to pass through the same point $(z, w) = (1/2, \sqrt[3]{15/16 i})$.

Define

$$\begin{aligned} \lambda_1 &= \alpha_1 + (P_1)^2(\beta_2), & \mu_1 &= \alpha_2, \\ \lambda_2 &= \alpha_1 + P_1(\alpha_2) + (P_1)^2(\beta_2), & \mu_2 &= \beta_1, \\ \lambda_3 &= (P_1)^2(\beta_2), & \mu_3 &= P_1(\alpha_3), \\ \lambda_4 &= \alpha_3 + (P_1)^2(\beta_2), & \mu_4 &= \beta_3. \end{aligned}$$

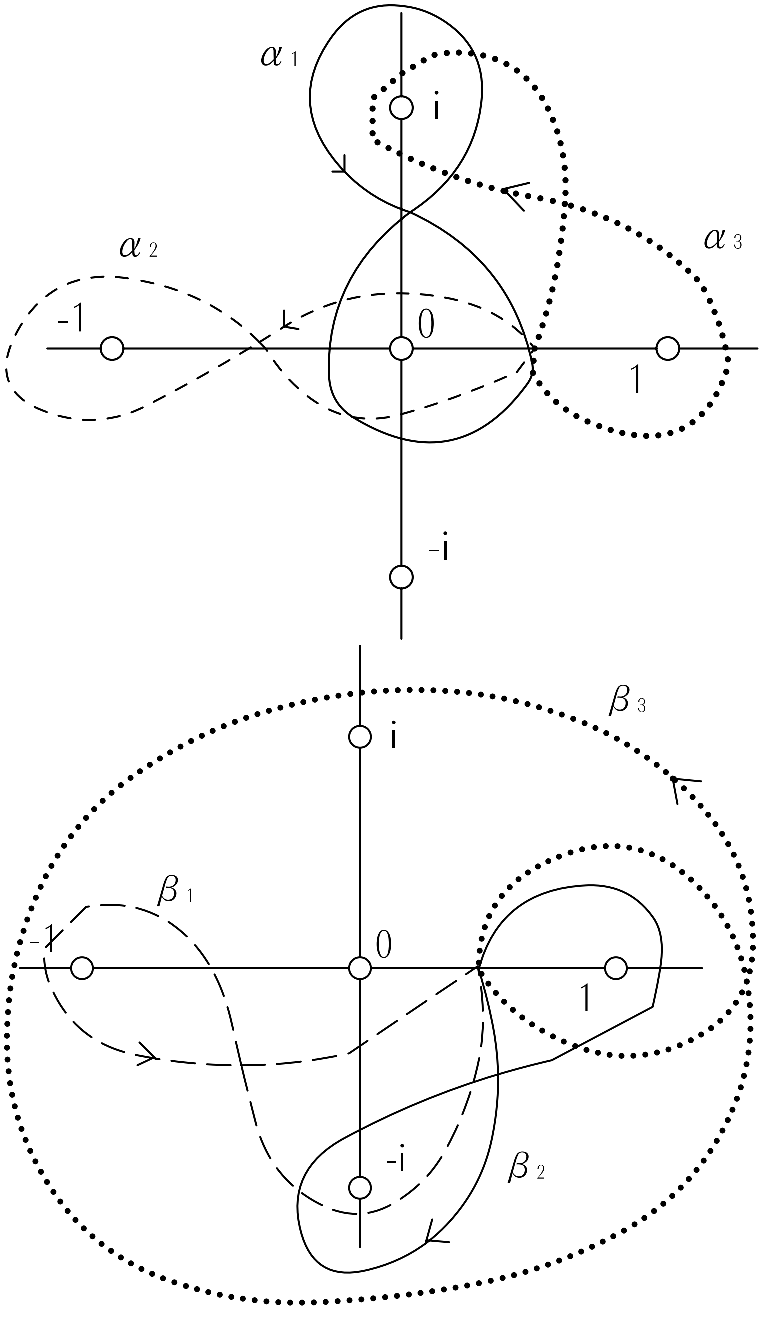

Figure 4.

Then λ_j and μ_j $(j = 1, 2, 3, 4)$ form a canonical basis. With respect to this basis, the symplectic matrix corresponding to P_3 has the form

$$M_{P_3} = {}^t\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To simplify calculations we change the basis as follows:

$$\begin{aligned} \bar{\lambda_1} &= \lambda_3, & \bar{\mu_1} &= \mu_1 + \mu_3 + \mu_4, \\ \bar{\lambda_2} &= -\lambda_3 + \lambda_4, & \bar{\mu_2} &= \mu_4, \\ \bar{\lambda_3} &= -\mu_1, & \bar{\mu_3} &= \lambda_1 - \lambda_3, \\ \bar{\lambda_4} &= \lambda_2, & \bar{\mu_4} &= \mu_2. \end{aligned}$$

Then the symplectic matrix corresponding to P_3 with respect to this new basis is given by

$$M'_{P_3} = {}^t\begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

The matrices corresponding to the other two automorphisms are given by

$$M'_{P_1} = {}^t\begin{pmatrix} -1 & -1 & 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \end{pmatrix},$$

$$M'_{P_2} = {}^t\begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Let us determine the period matrix of the curve C_2. Calculating the fixed point matrices under the actions of the matrices $M'_{P_1}, M'_{P_2}, M'_{P_3}$ and M'_{P_4}, we get four symmetric matrices:

$$Z_1(\zeta) = \begin{pmatrix} \zeta & -1 & \zeta & -\zeta-1 \\ -1 & -\zeta & 0 & 1 \\ \zeta & 0 & 0 & -\zeta-1 \\ -\zeta-1 & 1 & -\zeta-1 & 1 \end{pmatrix},$$

$$Z_2(\zeta) = \begin{pmatrix} \zeta & \frac{\zeta}{3\zeta+1} & \frac{\zeta^2}{3\zeta+1} & \frac{1}{3\zeta+1} \\ \frac{\zeta}{3\zeta+1} & \frac{38}{49}+\frac{36}{49}\zeta & \frac{15}{49}-\frac{9}{49}\zeta & \frac{49\zeta+196}{147\zeta+392} \\ \frac{\zeta^2}{3\zeta+1} & \frac{15}{49}-\frac{9}{49}\zeta & \frac{294\zeta+147}{147\zeta+392} & \frac{-49\zeta-147}{147\zeta+392} \\ \frac{1}{3\zeta+1} & \frac{49\zeta+196}{147\zeta+392} & \frac{-49\zeta-147}{147\zeta+392} & \frac{343\zeta+196}{147\zeta+392} \end{pmatrix},$$

where $\zeta = e^{-2\pi i/3}$ or $e^{-4\pi i/3}$.

By changing the canonical basis by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \in Sp(8,\mathbb{Z}),$$

the matrix $Z_1(\zeta)$ is changed into

$$Z'_1(\zeta) = \begin{pmatrix} -\zeta & 0 & 0 & 0 \\ 0 & \zeta & \zeta & -\zeta \\ 0 & \zeta & 0 & -\zeta \\ 0 & -\zeta & -\zeta & 0 \end{pmatrix}.$$

Thus we see that the principally polarised abelian variety $\mathbb{C}^4/\Lambda(\big(Z'_1(\zeta)\ E\big))$ is isomorphic to a product of an elliptic curve (as a 1-dimensional P.P.A.V.) and a 3-dimensional P.P.A.V. Therefore $\big(Z_1(\zeta) \quad E\big)$ cannot be a period matrix of the Jacobian variety and we conclude that the period matrix of C_2 has a form $\begin{pmatrix} Z_2(\zeta) & E \end{pmatrix}$. On the other hand, $\mathbf{Im}(Z_2(\zeta))$ is positive definite if and only if $\zeta = e^{-2\pi i/3}$; hence the period matrix is $\begin{pmatrix} Z_2(e^{-2\pi i/3}) & E \end{pmatrix}$.

Since every element of the period matrix is contained in the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$, by Corollary1.1, we obtain the following theorem.

Theorem 3.1. *The Jacobian variety $J(C_2)$ of C_2 is isomorphic to the product of four elliptic curves.*

3.2. One-parameter family case

Let $\{C_2(t)\}$ be a one-parameter family of curves defined by

$$C_2(t) : \begin{cases} X_0^2 + X_1^2 + X_2^2 - tX_3^2 = 0 \\ X_0X_1X_2 - X_3^3 = 0 \end{cases} \quad (t^3 \neq -27),$$

in $\mathbb{P}^3$.

The curve C_2 in the previous subsection is $C_2(0)$. Each member of this family admits automorphisms P_2 and P_3 defined in the previous subsection.

Put $S_{X_0,X_1} = (P_2)^2$. In terms of linear transformations of four variables X_0, X_1, X_2, X_3, the automorphism S_{X_0,X_1} can be written as

$$S_{X_0,X_1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The automorphism S_{X_0,X_1} has two fixed points $[1 : i : 0 : 0], [1 : -i : 0 : 0]$. Thus by the Hurwitz formula, the genus of the quotient curve $C_2'(t) = C_2(t)/\langle S_{X_0,X_1}\rangle$ is 2. The inhomogeneous equation of this curve is given by

$$C_2'(t) : y^2 = x^6 - \frac{t^2}{4}x^4 + \frac{t}{2}x^2 - \frac{1}{4}.$$

Indeed, define the map $\pi_{X_0,X_1} : C_2(t) \to C_2'(t)$ by

$$\begin{cases} x = \dfrac{X_0X_1}{X_3^2} \\ y = \dfrac{X_0^2X_1^2(X_0^2 - X_1^2)}{X_3^6}; \end{cases}$$

then this is a two-to-one map and each fibre consists of an orbit of S_{X_0,X_1}.

The curve $C_2'(t)$ has natural maps $\sigma'(t)$ and $\sigma''(t)$ to two elliptic curves

$$E'(t) : q'^2 = p'^3 - \frac{t^2}{4}p'^2 + \frac{t}{2}p' - \frac{1}{4},$$
$$E''(t) : q''^2 = p''^4 - \frac{t^2}{4}p''^3 + \frac{t}{2}p''^2 - \frac{1}{4}p'',$$

defined by

$$\sigma'(t) : \begin{cases} p' = x^2 \\ q' = y \end{cases} \quad \sigma''(t) : \begin{cases} p'' = x^2 \\ q'' = xy. \end{cases}$$

Let ω' and ω'' be holomorphic 1-forms on the elliptic curves $E'(t)$ and $E''(t)$ and let ω'_{X_0,X_1} and ω''_{X_0,X_1} be holomorphic 1-forms on $C_2(t)$ defined by

$$\omega'_{X_0,X_1} = (\sigma'(t) \cdot \pi_{X_0,X_1})^*(\omega'),$$
$$\omega''_{X_0,X_1} = (\sigma''(t) \cdot \pi_{X_0,X_1})^*(\omega'').$$

If we use the map $\pi_{X_1,X_2} : C_2(t) \to C'_2(t)$ defined by

$$\begin{cases} x = \dfrac{X_1 X_2}{X_3^2} \\ y = \dfrac{X_1^2 X_2^2 (X_1^2 - X_2^2)}{X_3^6} \end{cases}$$

instead of π_{X_0,X_1}, we can define $\omega'_{X_1,X_2}, \omega''_{X_1,X_2}$ similarly and $\omega'_{X_2,X_0}, \omega''_{X_2,X_0}$ as well. Observing the zeros of the forms, we know that $\omega'_{X_0,X_1}, \omega'_{X_1,X_2}, \omega'_{X_2,X_0}$ are the same form up to constant multiplication and $\omega''_{X_0,X_1}, \omega''_{X_1,X_2}, \omega''_{X_2,X_0}, \omega'_{X_0,X_1}$ form a basis of holomorphic 1-forms on $C_2(t)$. Thus we obtain the following theorem.

Theorem 3.2. *The Jacobian variety $J(C_2(t))$ of the curve $C_2(t)$ is isogenous to the product of four elliptic curves.*

If $t = 0$, two curves

$$E'(0) : q'^2 = p'^3 - \frac{1}{4},$$
$$E''(0) : q''^2 = p''^4 - \frac{1}{4} p''$$

are isomorphic and $E'(0)$ has a complex multiplication; hence, from Theorem 1.1 we infer the result in the previous subsection again.

4. Curve of genus 4, case 3

4.1. One-parameter family case

Let $\{H(t)\}$ be a one-parameter family of hyperelliptic curves defined by an equation

$$H(t) : y^2 = (x^5 - t^5)(x^5 - t^{-5}) \quad (t \neq 0, 1, -1).$$

Each member of the family admits the following three automorphisms:

$$P'_1 : \begin{cases} x \mapsto \zeta_5 x \\ y \mapsto y \end{cases} \quad (\zeta_5 = e^{2\pi i/5})$$

$$P'_2 : \begin{cases} x \mapsto 1/x \\ y \mapsto y/x^5 \end{cases}$$

$$\iota : \begin{cases} x \mapsto x \\ y \mapsto -y. \end{cases}$$

Let $\tilde{P}_1', \tilde{P}_2'$ and $\tilde{\iota}$ be the linear transformations on the vector space of holomorphic 1-forms on the curve $H(t)$ induced by the automorphisms P_1', P_2' and ι respectively. If we choose

$$\left\{\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}, \frac{x^3dx}{y}\right\}$$

as a basis of holomorphic 1-forms, the matrix expressions of $\tilde{P}_1, \tilde{P}_2$ and $\tilde{\iota}$ are given by

$$\tilde{P}_1' : \begin{pmatrix} \zeta_5 & 0 & 0 & 0 \\ 0 & \zeta_5^2 & 0 & 0 \\ 0 & 0 & \zeta_5^3 & 0 \\ 0 & 0 & 0 & \zeta_5^4 \end{pmatrix}, \quad \tilde{P}_2' : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\tilde{\iota} : \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is known that for every t, the Jacobian variety $J(H(t))$ of the curve $H(t)$ splits into the product of two 2-dimensional complex tori (See [1]).

In this subsection, we consider "non-hyperelliptic variant" of $H(t)$, that is, a non-hyperelliptic curve of which period matrix is fixed by the same symplectic matrices $M_{P_1'}$ and $M_{P_2'}$ as to the period matrix of $H(t)$ except the symplectic matrix M_ι corresponding to hyperelliptic involution. Let $\{C_3(t)\}$ be a one-parameter family of curves defined by the homogeneous equations

$$C_3(t) : \begin{cases} X_0X_3 + X_1X_2 = 0 \\ (X_0^2X_2 + X_3^2X_1) - t(X_1^2X_0 + X_2^2X_3) = 0 \end{cases} \qquad (t \neq 0).$$

Each member of $C_3(t)$ admits the following two automorphisms:

$$P_1 : \begin{pmatrix} \zeta_5 & 0 & 0 & 0 \\ 0 & \zeta_5^2 & 0 & 0 \\ 0 & 0 & \zeta_5^3 & 0 \\ 0 & 0 & 0 & \zeta_5^4 \end{pmatrix}, \quad P_2 : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We shall show that every member of this family also has a splitting Jacobian variety.

Put $z = X_1^2X_2/X_0^3$. Then $z : C_3(t) \to \mathbb{P}^1$ is a three-to-one map and it ramifies at the points $z = 0, \infty, t, -1/t$. If we take $w = X_2/X_0$, we can embed $C_3(t)$ into $\mathbb{P}^1 \times \mathbb{P}^1$ by (z, w). The image of this map is the curve defined by

$$w^5 = \frac{tz^2 - z^3}{1 + tz}.$$

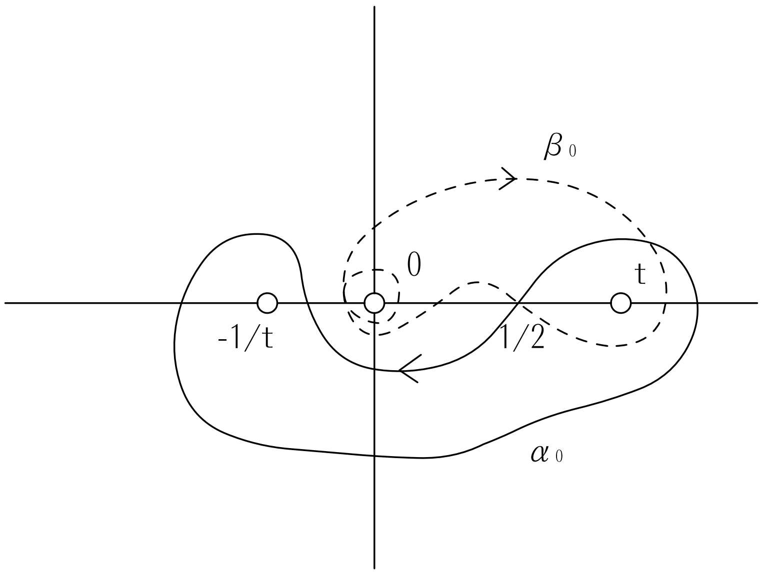

Figure 5.

The automorphisms P_1 and P_2 act on (z, w) as follows:

$$P_1 : \begin{cases} z \mapsto z \\ w \mapsto \zeta_5 w \end{cases}$$
$$P_2 : \begin{cases} z \mapsto -1/z \\ w \mapsto -1/w. \end{cases}$$

Let us consider a configuration in Fig. 5. Here we regard $C_3(t)$ as a three-sheeted covering over z-plane $\mathbb{P}^1$. Cycles α_0, β_0 in Fig. 5 are passing through the point $(z, w) = (1/2, \gamma)$, where γ is one of the numbers that satisfy the equation $\gamma^5 = (2t-1)/(4t+8)$. We denote $(P_1)^j(\alpha_0)$ by α_j and $(P_1)^j(\beta_0)$ by $\beta_j (j = 0, 1, 2, 3, 4)$.

Define

$$\begin{aligned} \lambda_1 &= \alpha_0, & \mu_1 &= \beta_0 + \alpha_1 + \alpha_2, \\ \lambda_2 &= \alpha_0 + \alpha_1, & \mu_2 &= \beta_1 + \alpha_2 + \alpha_3, \\ \lambda_3 &= \alpha_0 + \alpha_1 + \alpha_2, & \mu_3 &= \beta_2 + \alpha_3 + \alpha_4, \\ \lambda_4 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3, & \mu_4 &= \beta_3 + \alpha_4 + \alpha_0, \end{aligned}$$

then λ_j and μ_j $(j = 1, 2, 3, 4)$ form a canonical basis. With respect to this basis, the symplectic matrices corresponding to P_1, P_2 are given by

$$M_{P_1} = {}^t\begin{pmatrix} -1 & 1 & 0 & 0 & & & & \\ -1 & 0 & 1 & 0 & & O & & \\ -1 & 0 & 0 & 1 & & & & \\ -1 & 0 & 0 & 0 & & & & \\ & & & & 0 & 1 & 0 & 0 \\ & O & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & 1 \\ & & & & -1 & -1 & -1 & -1 \end{pmatrix}$$

$$M_{P_2} = {}^t\begin{pmatrix} -1 & 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & O & & \\ 0 & 1 & 0 & -1 & & & & \\ 0 & 1 & -1 & 0 & & & & \\ & & & & -1 & 0 & 0 & 0 \\ & O & & & 1 & 1 & 1 & 1 \\ & & & & 0 & 0 & 0 & -1 \\ & & & & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The fixed point matrices of the actions of M_{P_1} and M_{P_2} in the Siegel upper half plane are given by

$$Z = \begin{pmatrix} 2b & a & b & 2b-a \\ a & 2a & 2a-b & b \\ b & 2a-b & 2a & a \\ 2b-a & b & a & 2b \end{pmatrix},$$

where a and b are indeterminants. The period matrix $\Pi(C_3(t))$ of $C_3(t)$ can be written as $\begin{pmatrix} Z & E \end{pmatrix}$.

Choose a matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \in SL(8, \mathbb{Z})$$

and multiply $\Pi(C_3(t))$ by A from right. Then we get

$$\Pi(C_3(t))A = \begin{pmatrix} 2a-b & a & a+b & 4b-a & 1 & 0 & 1 & 0 \\ 4a-2b & 2a & 4a-b & a+b & 2 & 0 & 0 & 1 \\ 3a-b & 2a-b & 4a-b & a+b & 2 & -1 & 0 & 1 \\ a-b & b & a+b & 4b-a & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Normalise this matrix we get

$$\begin{pmatrix} 2a-b & a & & & 1 & & & \\ a-b & b & \multicolumn{2}{c}{O} & & 1 & & \\ & & 4a-b & a+b & & & 1 & \\ \multicolumn{2}{c}{O} & a+b & 4b-a & & & & 1 \end{pmatrix}.$$

Hence, the Jacobian variety $J(C_3(t))$ of $C_3(t)$ is isomorphic to a product of two 2-dimensional complex tori $T_1(t)$ and $T_2(t)$, where

$$T_1(t) = \mathbb{C}^2 / \left(\text{the lattice generated by } \begin{pmatrix} 2a-b & a & 1 & 0 \\ a-b & b & 0 & 1 \end{pmatrix} \right)$$

$$T_2(t) = \mathbb{C}^2 / \left(\text{the lattice generated by } \begin{pmatrix} 4a-b & a+b & 1 & 0 \\ a+b & 4b-a & 0 & 1 \end{pmatrix} \right).$$

Thus we obtain the following theorem.

Theorem 4.1. *The Jacobian variety of $C_3(t)$ splits into the product of two 2-dimensional complex tori.*

4.2. Bring's curve

The curve $C_3 = C_3(1)$ is called Bring's curve. The automorphism group of Bring's curve is isomorphic to S_5, the symmetric group of 5 letters. It is known that the Jacobian variety of the Bring's curve splits into a product of four mutually isogenous elliptic curves (see [4], [7]). We can show this fact by calculating the period matrix by using an additional automorphism.

The curve C_3 admits the automorphism

$$P_3 : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in addition to P_1 and P_2. The symplectic matrix corresponding to P_3 with respect to the canonical basis introduced in the previous subsection is

$$M_{P_3} = {}^t\begin{pmatrix} -1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \\ -1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

In the previous subsection, we calculate the fixed point matrix Z of actions of the matrices M_{P_1} and M_{P_2} and show that the Jacobian variety $J(C_3(t))$ splits into a product of two 2-dimensional complex tori $T_1(t)$ and $T_2(t)$.

For the curve C_3, the matrix Z is also fixed by M_{P_3}. This gives the new relation $2b - 3a = 1$. Thus we get the one-parameter family of matrices fixed by M_{P_1}, M_{P_2} and M_{P_3}.

Since $2b-3a = 1$, the Jacobian variety $J(C_3)$ of Bring's curve is isomorphic to a product of tori T_1' and T_2', where

$$T_1' = \mathbb{C}^2 \Big/ \left(\text{the lattice generated by } N_1' = \begin{pmatrix} \frac{1}{2}a - \frac{1}{2} & a & 1 & 0 \\ -\frac{1}{2}a - \frac{1}{2} & \frac{3}{2}a + \frac{1}{2} & 0 & 1 \end{pmatrix} \right)$$

$$T_2' = \mathbb{C}^2 \Big/ \left(\text{the lattice generated by } N_2' = \begin{pmatrix} \frac{5}{2}a - \frac{1}{2} & \frac{5}{2}a + \frac{1}{2} & 1 & 0 \\ \frac{5}{2}a + \frac{1}{2} & 5a + 2 & 0 & 1 \end{pmatrix} \right).$$

Choose

$$A_1 = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix} \in SL(4, \mathbb{Z})$$

and multiply N_1' by A_1 and N_2' by A_2 from right and then normalise the resulting matrices. Then we get

$$\begin{pmatrix} \tau & 0 & 1 & 0 \\ 0 & 5\tau & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5\tau & 0 & 1 & 0 \\ 0 & 5\tau & 0 & 1 \end{pmatrix},$$

where $\tau = \frac{1}{2}a + \frac{1}{2}$. Thus we obtain the following theorem.

Theorem 4.2. *The Jacobian variety $J(C_3)$ of Bring's curve splits into the product of four elliptic curves $E_\tau \times E_{5\tau} \times E_{5\tau} \times E_{5\tau}$.*

This theorem is a special case of Theorem 4.1 in [4].

By this calculation we cannot say which value a corresponds to the period matrix of C_3. In [7], the explicit period matrix is given by using Schottky's relation.

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