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Characters of wreath products of compact groups with the infinite symmetric group and characters of their canonical subgroups

By

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Abstract

Characters of wreath products $G = \mathfrak{S}_\infty(T)$ of any compact groups T with the infinite symmetric group $\mathfrak{S}_\infty$ are studied. It is proved that the set $E(G)$ of all normalized characters is equal to the set $F(G)$ of all normalized *factorizable* continuous positive definite class functions. A general explicit formula of $f_A \in E(G)$ is given corresponding to a parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$. Similar results are obtained for certain canonical subgroups of G.

Introduction

Let $\mathfrak{S}_\infty(T) = \mathfrak{S}_\infty \ltimes D_\infty(T)$ be the wreath product of a compact group T with the infinite symmetric group $\mathfrak{S}_\infty$, where $D_\infty(T) = \prod'_{i\in\boldsymbol{N}} T_i$ is the restricted direct product of $T_i = T$. In this paper we give explicitly characters of all the factor representations of finite type of $\mathfrak{S}_\infty(T)$, and give a general character formula. Since a character determines a quasi-equivalence class of factor representations of finite type, we have thus classified all such quasi-equivalence classes. Let us explain in more detail.

1. For a Hausdorff topological group G, denote by $K(G)$ the set of all continuous positive definite class functions on G, and by $K_{\le 1}(G)$ and $K_1(G)$ the sets of $f \in K(G)$ satisfying respectively $f(e) \le 1$ and $f(e) = 1$ at the identity element $e \in G$. Let $E(G) = \mathrm{Extr}(K_1(G))$ be the set of extremal points of the convex set $K_1(G)$. Then a character of a factor representation of finite type of G is canonically in 1-1 correspondence with an $f \in E(G)$ (Theorem 1.1 quoted from [HH5]), and we call elements in $E(G)$ *characters* of G. This is our background.

2. Let N be a subgroup of G with the relative topology, and denote by $K_1(N,G)$ the set of functions in $K(N)$ invariant under G and put $E(N,G) = \mathrm{Extr}(K_1(N,G))$. Then the restriction of an $f \in E(G)$ is always in $E(N,G)$

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(Theorem 1.3(i)). A kind of converse assertion is also assured in a certain restricted case containing the case of $G = \mathfrak{S}_\infty(T)$ and its canonical subgroups N (Theorem 1.3(ii)). A proof of the former assertion in the general setting is given in Section 14 (Theorem 14.1), and another proof for the converse assertion in the case of $G = \mathfrak{S}_\infty(T)$ is given in Section 15 (Theorem 15.1). These results assure that $E(N)$ is obtained from $E(G)$ by restriction for $G = \mathfrak{S}_\infty(T)$ and its canonical subgroups N.

3. From now on, let $G = \mathfrak{S}_\infty(T)$. An element $g \in G$ is a pair (d, σ) of $d = (t_i)_{i \in \boldsymbol{N}} \in D_\infty(T)$ with $t_i \in T_i = T$ and $\sigma \in \mathfrak{S}_\infty$. Then we put $\mathrm{supp}(d) = \{i \in \boldsymbol{N}; t_i \neq e_T\}$ and $\mathrm{supp}(g) = \mathrm{supp}(d) \cup \mathrm{supp}(\sigma)$, where e_T denotes the identity element of T. An $f \in K(G)$ is called *factorizable* if $f(g_1 g_2) = f(g_1) f(g_2)$ for any $g_1, g_2 \in G$ with disjoint supports. Let $F(G)$ be the set of all factorizable $f \in K_1(G)$. Then we prove $E(G) \subset F(G)$ (Lemma 4.1) and $E(G) \supset F(G)$ (Lemma 4.4), and so $E(G) = F(G)$.

In the case of a finite group T, these inclusions were both proved by using the fact that the convex set $K_{\leq 1}(G)$ is compact in the weak topology $\sigma(L^\infty(G), L^1(G))$ (cf. [HH3]). But in the case of infinite compact group T, the proofs for Lemmas 4.1 and 4.4 are both different from those in [HH3].

4. To obtain all characters $f \in E(G)$, we proceed as follows. First, take a simple positive definite function F on G, and an increasing sequence $G_N \nearrow G$ of compact subgroups. Take a centralization F^{G_N} of F with respect to G_N as

$$F^{G_N}(g) = \int_{G_N} F(g' g {g'}^{-1}) \, d\mu_{G_N}(g') \qquad (g \in G),$$

where μ_{G_N} denotes the normalized Haar measure on G_N. Then consider the pointwise limit $f = \lim_{N \to \infty} F^{G_N}$. If it exists as a continuous function, it is positive definite and invariant under G, and so $f \in K_1(G)$.

As starting point of such process, we take a diagonal matrix element F of elementary induced representation $\rho = \mathrm{Ind}_H^G \pi$ of G of a unitary representation π of a subgroup H, or a similar one. Well choosing $\{G_N\}_{N \geq 1}$ and (π, H), we can actually get a big family of normalized factorizable positive definite class functions f_A depending on a parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \,;\, \mu \Big)$, which is determined from asymptotic data of $\{G_N\}_{N \geq 1}$ and data of (π, H). Let $E'(G)$ be the set of all such f_A. Then clearly $E'(G) \subset F(G)$. This process is carried out in Sections 9–12, and is similar to that in the case of a finite group T in [HH3].

5. In Section 13, we study how a factorizable $f \in F(G)$ can depend on a set of parameters. Taking a partial 'Fourier transform' of f on $G = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with respect to $D_n(T)$, we get a series of positive definite class functions $\mathcal{F}_{\zeta,\varepsilon,n}(f)$ on $\mathfrak{S}_n, n \geq 2$. Then, appealing to Korollar 1 to Satz 2 in [Tho2], we can prove that $F(G) \subset E'(G)$ and so $F(G) = E'(G)$ (Theorem 13.1).

Thus we obtain $E'(G) = F(G) = E(G)$ as sets, and also get the general explicit character formula valid for any characters of G. Furthermore, by this explicit parametrization of characters of G, we see that the set of characters $E(G)$ is compact in the topology τ_{cu} of compact uniform convergence (Theorem

13.2).

6. In Sections 14–16, we also study the cases of canonical subgroups of $G = \mathfrak{S}_\infty(T)$ such as $G' = \mathfrak{A}_\infty(T)$ and, in case T is abelian, such as $G^S = \mathfrak{S}^S_\infty(T)$ and $G'^S = \mathfrak{A}^S_\infty(T)$ for a closed subgroup S of T, which are defined in Section 2. To obtain all the characters of these subgroups from the result for G, we prepare a general theorem for reduction of characters to normal subgroups.

Let G be a topological group and N its normal subgroup. Denote by $K_1(N,G)$ the set of continuous positive definite functions on N normalized as $f(e) = 1$ which are G-invariant, and by $E(N,G) := \mathrm{Extr}(K_1(N,G))$ the set of extremal points of the convex set $K_1(N,G)$.

Theorem 14.1. *Let G be a Hausdorff topological group and N its normal subgroup with the relative topology.*

(i) *For an $F \in K_1(G)$, let $f = F|_N$ be its restriction on N. It belongs to $K_1(N,G)$, and if $f = a_1 f_1 + a_2 f_2$ with $a_i > 0, f_i \in K_1(N,G)$, then there exist extensions $F_i \in K_1(G)$ of f_i for $i = 1, 2$, such that $F = a_1 F_1 + a_2 F_2$.*

(ii) *For any $F \in E(G)$, its restriction $f = F|_N$ belongs to $E(N,G)$.*

For $G = \mathfrak{S}_\infty(T)$, let N be one of the above canonical subgroups of G. Then $K_1(N,G) = K_1(N)$ and so $E(N,G) = E(N)$. Hence Theorem 14.1 asserts that the restriction $E(G) \ni F \mapsto f = F|_N$ maps $E(G)$ into $E(N)$.

7. The present paper is organized as follows. After several preparations in Sections 2–3 for $G = \mathfrak{S}_\infty(T)$ and its canonical subgroups, the explicit formula for the character f_A of G is given in Theorem 5.1. When the compact group T is abelian, the formula takes a little simpler form as is given in Theorem 6.1. The character formula for canonical subgroups G^S of G is given in Theorem 7.1.

The method of proofs of these theorems is explained in Section 8.

Sections 9–13 are principally devoted to prove Theorems 5.1 and 6.1.

Sections 14–16 are devoted to the cases of canonical subgroups of G.

8. All the characters of the infinite symmetric group $\mathfrak{S}_\infty$ itself have been given early in [Tho2], and this case is reexamined in [VeKe], [Oko], [KeOl], [Bia] etc. and recently in [Hir3]–[Hir4]. The case with T a finite abelian group, studied in [HH1], contains the cases of infinite Weyl groups $W_{\mathbf{B}_\infty}$ and $W_{\mathbf{D}_\infty}$, and the limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r,1,n)$ of complex reflexion groups. The case of $\mathfrak{S}_\infty(T)$ with T any finite group or the discrete case is worked out in [HH2]–[HH3].

Contents

1. Characters of factor representations of finite type

1.1. Characters and continuous positive definite class functions

We begin with a theorem in the general theory of representations of topological groups, which gives us an important background for our study.

Let G be a Hausdorff topological group, $K(G)$ the set of continuous positive definite class functions on G, and $K_1(G)$ the set of $f \in K(G)$ normalized as $f(e) = 1$ at the identity element $e \in G$, and $E(G) = \mathrm{Extr}\big(K_1(G)\big)$ the set of extremal points in the convex set $K_1(G)$.

On the other hand, let π be a continuous unitary representation (= UR) of G, and $\mathfrak{U} = \pi(G)''$ the von Neumann algebra generated by $\pi(G) = \{\pi(g); g \in G\}$. Then π is called *factorial* if $\mathfrak{U}$ is a factor. If the factor is of finite type, there exists a unique faithful finite normal trace t on the set $\mathfrak{U}^+$ of non-negative elements in $\mathfrak{U}$, normalized as $t(I) = 1$ at the identity operator I. The unique extention of t to a linear form on $\mathfrak{U}$ is denoted by ϕ, and the function

$$f(g) = \phi(\pi(g)) \qquad (g \in G) \tag{1.1}$$

is called a *character* of π. It naturally belongs to $K_1(G)$.

Theorem 1.1 ([HH5, Theorem 1.6.1]). *For a Hausdorff topological group G, let* $\mathrm{URff}(G)$ *be the set of all quasi-equivalence classes $[\pi]$ of continuous unitary representations π of G, factorial of finite type. Then the map $[\pi] \to f$ through* (1.1) *above gives a canonical bijective correspondence between* $\mathrm{URff}(G)$ *and $E(G)$.*

The inverse map is given by $E(G) \ni f \to [\pi_f] \in \mathrm{URff}(G)$, where π_f denotes the Gelfand-Raikov representation in [GeRa] *associated to f.*

In this connection, every element f in $E(G)$ is called a *character* of G of finite type.

Remark 1.1. In [Dix, 17.3], the above canonical bijection is asserted under the condition that G is locally compact and unimodular. This point is not mentioned in [Voic].

1.2. Topologies in the space of continuous positive definite class functions

Let $K_{\leq 1}(G) \supset K_1(G)$ be the set of $f \in K(G)$ such that $f(e) \leq 1$. Then the set of extremal points of $K_{\leq 1}(G)$ is the union of $E(G)$ and 0. In the case where G is locally compact and unimodular, it is known that the convex set $K_{\leq 1}(G)$ is compact in the weak topology $\sigma(L^\infty(G), L^1(G))$ (cf. [Dix, 17.3]). We tried to extend this result to the case where $G = \lim_{n\to\infty} G_n$ is the inductive limit of a countable inductive system $G_1 \to G_2 \to \cdots \to G_n \to \cdots$ of locally compact groups, where each homomorphism from G_n into G_{n+1} is assumed to be homeomorphic. In [TSH], this kind of inductive system is called a *countable LCG inductive system* and there were proved that G with the inductive limit topology τ_{ind} becomes a topological group and that G has sufficiently many continuous positive definite functions and accordingly sufficiently many URs.

For this kind of groups G in general, we have not yet succeeded to prove that $K_{\leq 1}(G)$ is compact with respect to a certain natural topology. For our group $G = \mathfrak{S}_\infty(T)$, if this is true, we can prove that an $f \in K_1(G)$ is *extremal* if and only if it is *factorizable*, or $E(G) = F(G)$, by applying the integral expression theorem of Choquet-Bishop-K. de Leeuw [BiLe] to the convex set $K_{\leq 1}(G)$ and the set $\mathrm{Extr}\big(K_{\leq 1}(G)\big) = E(G) \cup \{0\}$ of its extremal points.

In the case where T is finite, this is the case because the inductive limit topology τ_{ind} is discrete, and the classical result in [Dix, 17.3] can be applied. Thus we have succeeded to give all the characters in [HH2]–[HH3].

However, in the case where T is infinite, the inductive limit group $G = \mathfrak{S}_\infty(T)$ of $G_n = \mathfrak{S}_n(T)$ equipped with τ_{ind} is no more locally compact, and we do not know if $K_{\leq 1}(G)$ is compact. (For the compactness of $E(G)$ in $K_{\leq 1}(G)$, see Theorem 13.2.)

Let $\mathcal{P}(G)$ be the set of continuous positive definite functions on G, and $\mathcal{P}_1(G)$ be the subset of $\varphi \in \mathcal{P}(G)$ normalized as $\varphi(e) = 1$. Then $\mathcal{P}_1(G) \supset K_1(G)$, and we know that, for a locally compact group G, the weak topology $\sigma(L^\infty(G), L^1(G))$ on $\mathcal{P}_1(G)$ is equivalent to the topology τ_{cu} of uniform convergence on every compact subsets (cf. [Dix, 13.5.2]). We call τ_{cu} the *compact uniform topology* in short.

Theorem 1.2. *Let T be a compact group, and $G = \mathfrak{S}_\infty(T)$, which is considered as $\lim_{n\to\infty} G_n, G_n = \mathfrak{S}_n(T)$, and equipped with τ_{ind}. Then, on $\mathcal{P}_1(G)$ and $K_1(G)$, the compact uniform topology τ_{cu} is metrizable and complete.*

1.3. Restriction of characters to a normal subgroup

To study the characters of certain canonical normal subgroups of $\mathfrak{S}_\infty(T)$, we need in Section 14 the following result in a general theory. Let G be a topological group and N its normal subgroup with the relative topology. Denote by $K_1(N, G)$ the set of all $f \in K_1(N)$ which are G-invariant, that is, $f(g\xi g^{-1}) = f(\xi)$ $(\xi \in N, g \in G)$, and by $E(N, G)$ the set of extremal points $\mathrm{Extr}\big(K_1(N, G)\big)$.

Theorem 1.3. *Let G be a Hausdorff topological group and N its normal subgroup.*

(i) *For any $F \in E(G)$, its restriction $f = F|_N$ onto N belongs to $E(N, G)$.*

(ii) *Assume that G is a union of countable compact subsets $C_n \nearrow G$ and that the topology on G is the inductive limit of topologies τ^{C_n} on C_n. Assume further N is open in G. Then, any $f \in E(N, G)$ is the restriction of an $F \in E(G)$.*

The assertion (i) is proved in Theorems 14.1, and (ii) is proved in Theorem 15.1 in the special case of $G = \mathfrak{S}_\infty(T)$ with T a compact group. Here let us prove (ii) under the above general situation. To do so, we prepare some generality.

For a topological groups G, let $\mathfrak{F}(G)$ be the space of functions ψ on G such that $\psi(g) = 0$ except finite number of $g \in G$, with the convolution $\psi_1 * \psi_2(g) := \sum_{h \in G} \psi_1(gh^{-1})\psi_2(h)$ and the conjugation $\psi^*(g) := \overline{\psi(g^{-1})}$. Put $f(\psi) = \sum_{g \in G} f(g)\psi(g)$ for $f \in K(G)$. For two elements $f_1, f_2 \in K(G)$, we introduce a partial order $f_1 \geq f_2$ by $f_1(\psi^* * \psi) \geq f_2(\psi^* * \psi)$ $(\psi \in \mathfrak{F}(G))$, and we say that f_2 is *majorized* by f_1 if $\lambda f_1 \geq f_2$ for some $\lambda > 0$. We denote this by $f_1 \succeq f_2$. Put

$$
(1.2) \qquad \begin{aligned} K(G; f) &:= \{f' \in K_{\leq 1}(G);\ f' \leq f\}, \\ K(G; \preceq f) &:= \{f' \in K_{\leq 1}(G);\ f' \preceq f\}. \end{aligned}
$$

Then, for an $f \in K_1(G)$, they are convex subsets of $K_1(G)$, and

$$
(1.3) \qquad K(G; \preceq f) = \bigcup\nolimits_{\lambda \geq 1} K(G; \lambda f).
$$

Lemma 1.4. *For an element $f \in K_1(G)$, take an $f' \leq f, \neq 0$. Then f' is extremal in $K_{\leq 1}(G)$ or $f' \in E(G)$ if and only if f' is extremal in $K(G; \lambda f)$ for every $\lambda \geq 1$ or for $\lambda = 1, 2, 3, \ldots$, that is, $f' \in \bigcap_{\lambda \in \boldsymbol{N}} \mathrm{Extr}\big(K(G; \lambda f)\big)$.*

Proof. Let $f'' \in K_1(G)$ be $f'' \leq f$. If f'' is not extremal, there exist $f_i \in K_{\leq 1}(G)$ and $\lambda_i > 0$ such that $f'' = \lambda_1 f_1 + \lambda_2 f_2$, $\lambda_1 + \lambda_2 = 1$. Then, $f_i \leq \lambda_i^{-1} f$ and $f_i \in K(G; \lambda f)$ with $\lambda = \max(\lambda_1^{-1}, \lambda_2^{-1})$, and so f' is not extremal in $K(G; \lambda f)$. Conversely if f' is not extremal in some $K(G; \lambda f)$, then it is so in $K_1(G)$. $\square$

Lemma 1.5. *For an $f \in K(G)$, functions $f' \in K(G; f)$ are uniformly equicontinous.*

Proof. For any $g, h \in G$, put $\psi = \delta_g - \delta_h$ with the delta-function δ_g supported by a point g, then

$$
f(\psi^* * \psi) = 2f(e) - f(gh^{-1}) - f(hg^{-1}) = 2\big(f(e) - \Re(f(gh^{-1}))\big),
$$

and so

$$
|f'(g) - f'(h)|^2 \leq 2f'(e)\big(f'(e) - \Re(f'(gh^{-1}))\big) \leq 2f(e)\big(f(e) - \Re(f(gh^{-1}))\big).
$$

$\square$

Lemma 1.6. *Assume that G is a union of countable compact subsets $C_n \nearrow G$ and that the topology on G is the inductive limit of topologies τ^{C_n} on C_n. Then, for an $f \in K_1(G)$, the convex subset $K(G;f)$ of the space $C(G)$ of continuous functions on G is compact in the topology of uniform convergence on every C_n.*

Proof. Put $\mathcal{C} = \{C_n; n \geq 1\}$ and denote by $\tau_{\mathcal{C}}$ the topology in $C(G)$ of uniform convergence on every $C_n \in \mathcal{C}$. Then $\tau_{\mathcal{C}}$ is metrizable and complete. The subset $K(G;f)$ is $\tau_{\mathcal{C}}$-closed and so complete. It is equicontinous by Lemma 1.5.

Take a sequence $(F_k)_{k\geq 1}$ of $K(G;f)$. Since F_k's are equicontinuous, on every compact set C_n, we can choose successively a subsequence converging uniformly on C_n for $n \geq 1$. Since $G = \bigcup_{n\geq 1} C_n$, taking the diagonal subsequence, we have a subsequence converging uniformly on every C_n. Then the limit function F_∞ is continuous on C_n, and by assumption, so is on G. Hence $F_\infty \in K(G;f)$, and so $K(G;f)$ is compact. □

Proof of Theorem 1.3 (ii). Take an arbitrary $f \in E(N,G)$. Extend it onto G by putting zero outside N and denote it by $\widetilde{f}$. Then, since N is open, $\widetilde{f}$ is continuous and $\widetilde{f} \in K_1(G)$.

Take a convex set $K(G;\lambda\widetilde{f}) = \{f' \in K_1(G); f' \leq \lambda\widetilde{f}\}$ for $\lambda = m \in \boldsymbol{N}$. Then $f'' \in K_1(G)$ is extremal or $f'' \in E(G)$ if and only if f'' is extremal in every $K(G;m\widetilde{f}), m = 1,2,\ldots$. Since $K(G;m\widetilde{f})$ are increasing as $m \to \infty$, their subsets of non-extremal points are increasing. Put $E'_m = \mathrm{Extr}\big(K(G;m\widetilde{f})\big)\setminus\{0\}$, then $\bigcap_{m\in\boldsymbol{N}} E'_m \subset E(G)$.

Now we can apply Choquet-Bishop-K. de Leeuw theorem (Theorem 5.6 in [BiLe]) of integral expression for a compact convex set as $\widetilde{f} = \int_{E'_m} F\, d\mu_m(F)$, where μ_m is a measure on E'_m. The integral converges in the topology $\tau_{\mathcal{C}}$, and so, by restricting on each $C_n \cap N$, we get on the whole N an expression of f as $f = \int_{E'_m} \big(F|_N\big)\, d\mu_m(F)$ with $F|_N \in K_1(N,G)$ for μ_m-almost all $F \in E'_m$. Since f is extremal in $K_1(N,G)$, we have $F|_N = f$ for μ_m-almost all $F \in E'_m$.

This means that f is obtained by restricting an $F \in \bigcap_{m\in\boldsymbol{N}} E'_m \subset E(G)$. □

Remark 1.2. The situation in (ii) is realized for $G = \mathfrak{S}_\infty(T)$ with a compact abelian group T and $G_n = \mathfrak{S}_n(T)$ as C_n, and N one of $G' := \mathfrak{A}_\infty(T)$, $G^S := \mathfrak{S}^S_\infty(T)$ and $G'^S := G' \cap G^S$ with an open subgroup S of T (see Section 2). In this case $E(N,G) = E(N)$.

2. Wreath products of compact groups with the infinite symmetric group

For a set I, we denote by $\mathfrak{S}_I$ the group of all finite permutations on I. A permutation σ on I is called *finite* if its support $\mathrm{supp}(\sigma) := \{\, i \in I \;;\; \sigma(i) \neq i \,\}$ is finite. We call the *infinite symmetric group* the permutation group $\mathfrak{S}_{\boldsymbol{N}}$ on the set of natural numbers $\boldsymbol{N}$. The index $\boldsymbol{N}$ is frequently replaced by ∞. The symmetric group $\mathfrak{S}_n$ is naturally imbedded in $\mathfrak{S}_\infty$ as the permutation group of the set $I_n := \{1, 2, \ldots, n\} \subset \boldsymbol{N}$.

Let T be a compact group. We consider a wreath product group $\mathfrak{S}_I(T)$ of T with a permutation group $\mathfrak{S}_I$ as follows:

$$\mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I\,, \quad D_I(T) = {\prod_{i\in I}}' T_i\,, \quad T_i = T\ (i \in I), \tag{2.1}$$

where the symbol $\prod'$ means the restricted direct product, and $\sigma \in \mathfrak{S}_I$ acts on $D_I(T)$ as

$$D_I(T) \ni d = (t_i)_{i\in I} \stackrel{\sigma}{\longmapsto} \sigma(d) = (t'_i)_{i\in I} \in D_I(T), \quad t'_i = t_{\sigma^{-1}(i)}\ (i \in I). \tag{2.2}$$

Identifying groups $D_I(T)$ and $\mathfrak{S}_I$ with their images in the semidirect product $\mathfrak{S}_I(T)$, we have $\sigma\, d\, \sigma^{-1} = \sigma(d)$. The groups $D_{I_n}(T)$ and $\mathfrak{S}_{I_n}(T)$ are denoted by $D_n(T)$ and $\mathfrak{S}_n(T)$ respectively, then $G := \mathfrak{S}_\infty(T)$ is an inductive limit of $G_n := \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$. Since T is compact, G_n is also compact, and the inductive system $(G_n)_{n\geq 1}$ is an example of countable LCG inductive systems in [TSH]. We introduce in G its inductive limit topology τ_{ind}. Then G with τ_{ind} becomes a topological groups (cf. **2.7** in [TSH]). By definition, a subset $B \subset G$ is τ_{ind}-open if and only if $B \cap G_n$ is open in G_n for any $n \geq 1$. A general theory of unitary representations of the inductive limit group G of a countable LCG inductive system is carried out in [TSH, §5] using continuous positive definite functions on the group.

Lemma 2.1. (i) *In the topology τ_{ind} on $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$, the subgroup $D_\infty(T)$ is open. Denote by τ^D_{ind} the inductive limit topology on $D_\infty(T)$ of the topologies on $D_n(T)$, then τ_{ind} on $\mathfrak{S}_\infty(T)$ is the product of τ^D_{ind} and the discrete topology $\tau^{\mathfrak{S}}_{disc}$ on $\mathfrak{S}_\infty$.*

(ii) *A function on $D_\infty(T)$ or on $\mathfrak{S}_\infty(T)$ is continuous with respect to its inductive limit topology if and only if its restriction on each subgroup $D_n(T)$ is continuous.*

Put $\Pi_I(T) = \prod_{i\in I} T_i$ be the direct product of $T_i = T$ over $i \in I$, and let τ_{prod} denote the product topology on $\Pi_I(T)$.

When T is a non-trivial finite group, the topology τ_{prod} on $\Pi_{\boldsymbol{N}}(T)$ is not discrete but totally disconnected, whereas the topology τ^D_{ind} on $D_\infty(T)$ is discrete. Thus τ_{ind} in $G = \mathfrak{S}_\infty(T)$ is discrete, and this case is worked out in [HH2]–[HH3].

When T is infinite, τ_{ind} is not discrete, and a subset $\{(d, \mathbf{1})\,;\, d \in D_\infty(T)\}$ $\cong D_\infty(T)$ is an open neighbourhood of the identity element e of G, where $\mathbf{1} \in \mathfrak{S}_\infty$ denotes the trivial permutation on $\boldsymbol{N}$.

Lemma 2.2. *Suppose T be compact and non-trivial.*

(i) *The subgroup $D_\infty(T)$ of $\Pi_{\boldsymbol{N}}(T)$ is not τ_{prod}-closed and so not compact.*

(ii) *The relative topology τ^D_{prod} on $D_\infty(T)$ induced from $\Pi_{\boldsymbol{N}}(T)$ is not locally compact and strongly weaker than τ^D_{ind}, and τ^D_{ind} is locally compact if and only if T is finite.*

A natural subgroup of $G = \mathfrak{S}_\infty(T)$ is given as a wreath product of T with the alternating group $\mathfrak{A}_\infty$ as $G' := \mathfrak{A}_\infty(T) = D_\infty(T) \rtimes \mathfrak{A}_\infty$. Furthermore, in the case where T is abelian, we put

$$P_I(d) = \prod_{i\in I} t_i \quad \text{for} \quad d = (t_i)_{i\in I} \in D_I(T), \tag{2.3}$$

and, for a subgroup S of T, we define a subgroup of $\mathfrak{S}_I(T)$ as

$$\mathfrak{S}_I^S(T) = D_I^S(T) \rtimes \mathfrak{S}_I \quad \text{with} \quad D_I^S(T) := \{\, d = (t_i)_{i\in I} \ ; \ P_I(d) \in S \,\}. \tag{2.4}$$

The subgroup $G^S := \mathfrak{S}_\infty^S(T)$ is normal in G. If S is open in T, G^S is open and $[G : G^S] = [T : S] < \infty$. A closed subgroup S is open in T if and only if the index $[T : S]$ is finite.

This kind of groups $\mathfrak{S}_\infty(T)$ and $\mathfrak{S}_\infty^S(T)$ with T abelian contain, as their special cases, the infinite Weyl groups of classical types, $W_{\mathbf{A}_\infty} = \mathfrak{S}_\infty$, $W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ and $W_{\mathbf{D}_\infty} = \mathfrak{S}_\infty^{\{e_T\}}(\boldsymbol{Z}_2)$, and moreover the inductive limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r,1,n)$ of complex reflexion groups $G(r,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_r)$ (cf. [ArKo], [Kaw], [Sho]).

In general, by Theorem 1.1, for a topological group G, the set $E(G)$ of all extremal elements of $K_1(G)$ is equal to the set of all characters of factor represetations of G of finite type, type I_n $(n < \infty)$ or II_1. When G is discrete, $K_1(G)$ itself is compact in the weak topology and $E(G)$ is closed in it. This is the case of $G = \mathfrak{S}_\infty(T)$ with T finite.

When T is compact and infinite, $G = \mathfrak{S}_\infty(T)$ is no longer locally compact, but is a limit of a countable LCG inductive system $G_n = \mathfrak{S}_n(T)$. The purpose of the present paper is to give explicitly all the characters of finite type, or all elements of $E(G)$ for $G = \mathfrak{S}_\infty(T)$ with T compact.

The case with a finite group T has been treated in [Hir3]–[Hir4] and [HH1]–[HH3]. Many of discussions in our previous papers in the discrete case can be transferred to the present case of a general compact group T. So we can treat both cases in a unified manner. For example, take a function F on T and consider $\frac{1}{|T|}\sum_{t\in T} F(t)$ or $\int_T F(t)\, d\nu_T(t)$, depending on whether T is finite or infinite, where $|T|$ denotes the number of elements in T, and ν_T denotes the nomalized Haar measure on the compact group T. In both cases, we can use the latter integration notation.

3. Structure of wreath product groups $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$

Fix a compact group T, and take the wreath product group $\mathfrak{S}_\infty(T)$ of T with the symmetric group $\mathfrak{S}_\infty$ as $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$, $D_\infty(T) := \prod'_{i\in\boldsymbol{N}} T_i$, $T_i = T$ $(i \in \boldsymbol{N})$. We identify frequently d and σ with their images in $\mathfrak{S}_\infty(T)$ respectively, then $\sigma d \sigma^{-1} = \sigma(d)$ and $(d,\sigma)(d',\sigma') = \big(d\cdot\sigma(d'), \sigma\sigma'\big)$ $(d, d' \in D_\infty(T), \sigma, \sigma' \in \mathfrak{S}_\infty)$.

Notation. For $d = (t_i)_{i\in I} \in D_I(T)$, $I \subset \boldsymbol{N}$, put $\mathrm{supp}_I(d) := \{\, i \in I \ ; \ t_i \neq e_T \,\}$ and we omit the suffix I if $I = \boldsymbol{N}$ or I is specified from the context.

3.1. Standard decomposition of elements and conjugacy classes

An element $g = (d, \sigma) \in G = \mathfrak{S}_\infty(T)$ is called *basic* in the following two cases:

CASE 1: σ is cyclic and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$;
CASE 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i\in \boldsymbol{N}}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{N}$.

The element $(d, \mathbf{1})$ in Case 2 is denoted by ξ_q, and put $\mathrm{supp}(\xi_q) := \mathrm{supp}(d) = \{\, q \,\}$.

For a cyclic permutation $\sigma = (i_1 \;\; i_2 \;\; \cdots \;\; i_\ell)$ of ℓ integers, we define its *length* as $\ell(\sigma) = \ell$, and for the identity permutation $\mathbf{1}$, put $\ell(\mathbf{1}) = 1$ for convenience. In this connection, ξ_q is also denoted by $(t_q, (q))$ with a trivial cyclic permutation (q) of length 1. In Cases 1 and 2, put $\ell(g) = \ell(\sigma)$ for $g = (d, \sigma)$, and $\ell(\xi_q) = 1$. It is very helpful for us to illustrate these basic elements by permutation matrices with entries from T or more correctly from the group algebra of T. For $g = (d, \sigma)$ with $d = (t_1, t_2, \ldots, t_\ell)$, $\sigma = (1 \;\; 2 \;\; 3 \;\; \cdots \;\; \ell)$, and $\xi_q = (t_q, (q))$, their expressions in matrix form are respectively

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & t_1 \\ t_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & t_3 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & t_{\ell-1} & 0 & 0 \\ 0 & \cdots & 0 & 0 & t_\ell & 0 \end{pmatrix}, \qquad \begin{array}{cc} \begin{pmatrix} e_T & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e_T & 0 & 0 & \cdots \\ 0 & \cdots & 0 & t_q & 0 & \cdots \\ 0 & \cdots & 0 & 0 & e_T & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} & \begin{array}{c} \\ \\ \\ q\text{-th.} \\ \\ \\ \end{array} \end{array}$$

An arbitrary element $g = (d, \sigma) \in G$, is expressed as a product of basic elements as

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m \tag{3.1}$$

with $g_j = (d_j, \sigma_j)$ in Case 1, in such a way that the supports of these components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$ $(1 \leq j \leq m)$, are mutually disjoint. This expression of g is unique up to the orders of ξ_{q_k}'s and g_j's , and is called a *standard decomposition* of g. For $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives a cycle decomposition of σ.

To write down conjugacy class of $g = (d, \sigma)$, there appear products of components t_i of $d = (t_i)$, where the orders of taking products are crucial when T is not abelian. We denote by $[t]$ the conjugacy class of $t \in T$, and by $T/\!\sim$ the set of all conjugacy classes of T, and $t \sim t'$ denotes that $t, t' \in T$ are mutually conjugate in T. For a basic component $g_j = (d_j, \sigma_j)$ of g, let $\sigma_j = (i_{j,1} \;\; i_{j,2} \;\; \ldots \;\; i_{j,\ell_j})$ and put $K_j := \mathrm{supp}(\sigma_j) = \{\, i_{j,1},\, i_{j,2},\, \ldots,\, i_{j,\ell_j} \,\}$ with $\ell_j = \ell(\sigma_j) \geq 2$. For $d_j = (t_i)_{i\in K_j}$, we put

$$P_{\sigma_j}(d_j) := \left[t'_{\ell_j} t'_{\ell_j-1} \cdots t'_2 t'_1\right] \in T/\!\sim \qquad \text{with} \quad t'_k = t_{i_{j,k}} \;\; (1 \leq k \leq \ell_j). \tag{3.2}$$

The conjugacy class $P_{\sigma_j}(d_j)$ is well-defined, because, for $t_1, t_2, \ldots, t_\ell \in T$, we have $t_1 t_2 \cdots t_\ell \sim t_k t_{k+1} \cdots t_\ell t_1 \cdots t_{k-1}$ for any k.

Lemma 3.1. (i) *Let $\sigma \in \mathfrak{S}_\infty$ be a cycle, and put $K = \mathrm{supp}(\sigma)$. Then, an element $g = (d, \sigma) \in \mathfrak{S}_K(T)(=: G_K$ (put)) is conjugate in it to $g' = (d', \sigma) \in G_K$ with $d' = (t'_i)_{i \in K}, t'_i = e_T\ (i \neq i_0), [t'_{i_0}] = P_\sigma(d)$ for some $i_0 \in K$.*

(ii) *Identify $\tau \in \mathfrak{S}_\infty$ with its image in $G = \mathfrak{S}_\infty(T)$. Then we have, for $g = (d, \sigma)$, $\tau g \tau^{-1} = (\tau(d), \tau\sigma\tau^{-1})$ $(=: (d', \sigma')$ (put)), and $P_{\sigma'}(d') = P_\sigma(d)$.*

Applying this lemma to each basic components $g_j = (d_j, \sigma_j)$ of $g \in G$ in (3.1), we get the following result.

Theorem 3.2. *Let T be a compact group. Take an element $g \in G = \mathfrak{S}_\infty(T)$ and let its standard decomposition into basic elements be*

$$g = \xi_{q_1}\xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m$$

in (3.1) , with $\xi_{q_k} = (t_{q_k}, (q_k))$, and $g_j = (d_j, \sigma_j)$, σ_j cyclic, $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)$. Then the conjugacy class of g is determined by the set

$$\text{(3.3)} \qquad \begin{cases} [t_{q_k}] \in T/\!\sim \quad (1 \le k \le r)\ ; \\ (P_{\sigma_j}(d_j), \ell(\sigma_j)) \in (T/\!\sim) \times \boldsymbol{N}_{\ge 2}\ \ (1 \le j \le m), \end{cases}$$

where $\boldsymbol{N}_{\ge 2} = \{i \in \boldsymbol{N}; i \ge 2\}$.

3.2. The case where T is abelian

In the case where T is abelian, the set $T/\!\sim$ of conjugacy classes is equal to T itself. Take $g \in G$, and take its standard decompositon in (3.1). For $g_j = (d_j, \sigma_j)$, put $g'_j := (d'_j, \sigma_j)$, where $d'_j = (t'_i)_{i \in \boldsymbol{N}}$ with $t'_{i_0} = P(d_j) = \prod_{i \in K_j} t_i$ for some $i_0 \in K_j := \mathrm{supp}(\sigma_j)$, and $t'_i = e_T$ elsewhere.

Lemma 3.3. *Let T be abelian.* (i) *For a $g = (d, \sigma) \in \mathfrak{S}_\infty(T)$, let its standard decomposition be $g = \xi_{q_1}\xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m$ in (3.1). Define g'_j $(1 \le j \le m)$ as above and put $g' = \xi_{q_1}\xi_{q_2} \cdots \xi_{q_r} g'_1 g'_2 \cdots g'_m$. Then, g and g' are mutually conjugate in $\mathfrak{S}_\infty(T)$.*

(ii) *A complete set of parameters of the conjugacy classes of non-trivial elements g is given by*

$$\text{(3.4)} \qquad \{t'_1, t'_2, \ldots, t'_r\} \quad \text{and} \quad \{(u_j, \ell_j)\ ;\ 1 \le j \le m\},$$

where $t'_k = t_{q_k} \in T^* := T \setminus \{e_T\}$, $u_j = P(d_j) \in T$, $\ell_j = \ell(\sigma_j) \ge 2$, and $r + m > 0$.

3.3. Finite-dimensional irreducible representations

A finite dimensional continuous irreducible unitary representation (= IUR) of $G = \mathfrak{S}_\infty(T)$ is given as follows.

Lemma 3.4. *A finite-dimensional IUR π of $\mathfrak{S}_\infty(T)$ is a one-dimensional character, and is given in the form $\pi = \pi_{\zeta,\varepsilon}$ with*

$$\pi_{\zeta,\varepsilon}(g) = \zeta(P(d))\,(\mathrm{sgn}_{\mathfrak{S}})^{\varepsilon}(\sigma) \quad \text{for } g = (d, \sigma) \in \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty,$$

where ζ is a one-dimensional character of T, $P(d)$ is a product of components t_i of $d = (t_i)$, and $\mathrm{sgn}_{\mathfrak{S}}(\sigma)$ denotes the usual sign of σ and $\varepsilon = 0, 1$.

Note that, since $\zeta(P(d)) = \prod_{i\in\boldsymbol{N}} \zeta(t_i)$, the order of taking product for $P(d)$ has no meaning even if T is not abelian.

Proof. Let π be a finite-dimensional IUR of G. Assume that $\dim\pi > 1$. According to $I_n = \{1, 2, \ldots, n\} \nearrow \boldsymbol{N}$, the subgroup $G_n := \mathfrak{S}_{I_n}(T) = \mathfrak{S}_n(T)$ goes up to G. Hence, for n sufficiently large, the restriction $\pi|_{G_n}$ is already irreducible. Take a subset $J \subset \boldsymbol{N}$ disjoint with I_n, then any $g' \in G_J := \mathfrak{S}_J(T)$ commutes with $g \in G_n$, whence $\pi(g')$ should be a scalar operator. On the other hand, if $|J| = n$, the group G_J is conjugate to G_n in G, and so $\pi(g)$ is a scalar operator for any $g \in G_n$. This is a contradiction.

A one-dimensional IUR of $\mathfrak{S}_\infty$ equals to the trivial one $\boldsymbol{1}_{\mathfrak{S}_\infty}$ or the sign one $\mathrm{sgn}_{\mathfrak{S}}$, and its kernel contains $\mathfrak{A}_\infty$. Moreover, the subgroup $D^{\{e_T\}}(T) = \{d = (t_i)_{i\in\boldsymbol{N}} \in D(T); P(d) = e_T\}$ is contained in the commutator group $[G, G]$, because, for $d' = (t, e_T, e_T, \ldots) \in D(T)$ and a permutation $\sigma = (1\ \ 2)$, we have the commutator $d'\sigma\, {d'}^{-1}\sigma^{-1} = (t, t^{-1}, e_T, e_T, \ldots)$. Actually $[G, G] = D^{\{e_T\}}(T) \rtimes \mathfrak{A}_\infty$. Therefore π is essentially a character of $G/[G, G] \cong T \times \boldsymbol{Z}_2$. This proves our assertion. □

In the case where T is abelian and S an open subgroup of T, we have an open subgroup $G^S = \mathfrak{S}^S_\infty(T)$ of $G = \mathfrak{S}_\infty(T)$. We can prove $[G^S, G^S] = [G, G]$, and $G^S/[G^S, G^S] \cong S \times \boldsymbol{Z}_2$, and get similarly as Lemma 3.4 the following fact for G^S.

Lemma 3.5. *Assume that T is abelian and S an open subgroup of T. Then, a finite-dimensional IUR π of $G^S = \mathfrak{S}^S_\infty(T)$ is a one-dimensional character, and is given in the form*

$$\pi(g) = \zeta_S(P(d))\ (\mathrm{sgn}_{\mathfrak{S}})^\varepsilon\,(\sigma) \quad \textit{for}\ \ g = (d, \sigma) \in \mathfrak{S}^S_\infty(T) = D^S_\infty(T) \rtimes \mathfrak{S}_\infty,$$

where ζ_S is a one-dimensional character of S.

4. Factorizable positive definite class functions

4.1. Factorizability

Let T be a compact group, and f a continuous positive definite class function on $G = \mathfrak{S}_\infty(T)$ or $f \in K(G)$.

Definition 4.1. An $f \in K(G)$ is called *factorizable* if it has the following properties which are mutually equivelent:

(FTP) *For any $g = (d, \sigma) \in G$, let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m, \xi_q = \big(t_q, (q)\big)$, $g_j = (d_j, \sigma_j)$, be its standard decomposition. Then,*

$$f(g) = \prod_{1\le k\le r} f(\xi_{q_k}) \times \prod_{1\le j\le m} f(g_j). \tag{4.1}$$

(FTP′) *For any two elements g, g' with disjoint supports in $\boldsymbol{N}$,*

$$f(gg') = f(g)f(g'). \tag{4.2}$$

Let $F(G)$ be the set of all factorizable f in $K_1(G)$. In the case where T is finite, we prove $E(G) = F(G)$ in [HH3, Theorem 12], thanks to the fact that $K_{\le 1}(G)$ is weakly compact, by applying the integral expression theorem of Choquet-Bishop-K. de Leeuw ([Cho], [BiLe]) for a compact convex set and its subset of extremal points.

In the case where T is infinite, we do not know yet if $K_{\le 1}(G)$ is compact with respect to a certain natural topology, and so we cannot apply the same method as for the finite case. So we apply different methods using the detailed explicit structure of the group $G = \mathfrak{S}_\infty(T)$.

Lemma 4.1. *Let $G = \mathfrak{S}_\infty(T)$ with a compact group T. Then every character of G is factorizable, or $E(G) \subset F(G)$.*

Proof. Take an $f \in E(G)$. Suppose a $g \in G$ is decomposed as $g = g_1 g_2$ with disjoint supports $\mathrm{supp}(g_i) \subset \boldsymbol{N}$. To prove $f(g) = f(g_1)f(g_2)$, we proceed as follows. Put $K_1 = \mathrm{supp}(g_1), K_2 = \boldsymbol{N} \setminus K_1$, and $G_i = \mathfrak{S}_{K_i}(T)$, then $g_i \in G_i$ and G_1 is compact. Let us consider the restriction $f' = f|_{G'}$ on $G' := G_1 \times G_2$.

Let π be an IUR of G_1 identified with its equivalence class $[\pi] \in \widehat{G_1}$, and let $\widetilde{\chi_\pi}$ be its normalized character. Put

$$
\begin{aligned}
f_\pi(g_2) &= \int_{G_1} f'(g_1, g_2)\, \widetilde{\chi_\pi}({g_1}^{-1})\, d\mu_{G_1}(g_1) \\
&= \int_{G_1} f(g_1 g_2)\, \widetilde{\chi_\pi}({g_1}^{-1})\, d\mu_{G_1}(g_1),
\end{aligned}
\tag{4.3}
$$

where μ_{G_1} denotes the normalized Haar measure on G_1. Then $f_\pi \in K_{\le 1}(G_2)$ and so $f_\pi(e) \ge |f_\pi(g_2)|$ $(g_2 \in G_2)$, and the infinite series $\sum_{\pi \in \widehat{G_1}} f_\pi(e)$ is convergent because it is the value at $g_1 = e$ of the uniformly convergent Fourier expansion $\sum_{\pi \in \widehat{G_1}} f_\pi(e)\, \widetilde{\chi_\pi}(g_1)$ of $f'(g_1, e) = f(g_1)$ in $K_1(G_1)$ (cf. Lemma 13.2). Therefore the infinite series expansion

$$
f(g_1 g_2) = f'(g_1, g_2) = \sum_{\pi \in \widehat{G_1}} \widetilde{\chi_\pi}(g_1)\, f_\pi(g_2) \quad \big((g_1, g_2) \in G_1 \times G_2\big)
\tag{4.4}
$$

is uniformly convergent. In particular,

$$
f(g_2) = \sum_{\pi \in \widehat{G_1}} f_\pi(g_2) \qquad (g_2 \in G_2).
\tag{4.5}
$$

On the other hand, note that $G_2 \cong G = \mathfrak{S}_\infty(T)$ and that every conjugacy class C_2 of G_2 is an intersection with G_2 of a conjugacy class C of G: $C_2 = C \cap G_2$. Then we see that $f_2 := f|_{G_2}$ is in $E(G_2)$ since $f \in E(G)$. Therefore, from the sum expression (4.5), we see that each summand $f_\pi \in K_{\le 1}(G_2)$ is a constant multiple of f_2, whence $f_\pi = f_\pi(e)\, f_2$.

Thus we get from (4.4) the equality $f(g_1 g_2) = f(g_1) f(g_2)$. □

4.2. Parameters for a factorizable f and Property (FTP$''$)

Let us rewrite the conditions (FTP) and (FTP$'$) in another form. As is proved in Theorem 3.2, conjugacy classes of *basic elements* in G is given by the set Ω of the following objects ω :

$$\omega = ([t], \ell\,) \in (T/\!\sim) \times \boldsymbol{N}, \tag{4.6}$$

and the conjugacy class of $g \in G, \neq e$, with the above standard decomposition is determined by the collection

$$([t_{q_k}], \ell = 1) \quad (1 \le k \le r) \quad \text{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)\,) \quad (1 \le j \le m), \tag{4.7}$$

and the conjugacy class of $g = e$ by $\omega_0 = ([e_T], \ell = 1)$.

For $g \neq e$, denote by $n_\omega(g)$ the multiplicity of $\omega \in \Omega$ in this collection for g. We put $n_{\omega_0}(e) = 1$ and $n_{\omega_0}(g) = 0$ $(g \in G, \neq e)$ superfluously by definition.

Put $\boldsymbol{Z}_{\ge 0} := \{n \in \boldsymbol{Z}\,;\, n \ge 0\}$ and denote by $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$ the set of all $\boldsymbol{n} = (n_\omega)_{\omega\in\Omega}$, $n_\omega \in \boldsymbol{Z}_{\ge 0}$, with $n_\omega = 0$ for almost all ω, and $n_{\omega_0} = 1$ if $n_\omega = 0$ $(\forall \omega \neq \omega_0)$ and $n_{\omega_0} = 0$ otherwise. Then, $\boldsymbol{n}(g) := (n_\omega(g))_{\omega\in\Omega}$ is an element of $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$, and the correspondence

$$\Phi :\ [g] \ \mapsto\ \boldsymbol{n}(g) \in (\boldsymbol{Z}_{\ge 0})^{(\Omega)} \tag{4.8}$$

gives a bijective map from the set $G/\!\sim$ of all conjugacy classes of $g \in G$ onto $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$. We introduce in the latter the topology in $G/\!\sim$ through the map Φ.

For $\omega = ([t], \ell) \in \Omega$, put $\omega^{-1} := ([t^{-1}], \ell)$. Then, if ω is the conjugacy class of $\xi_q = (t_q, (q))$ or of $g_j = (d_j, \sigma_j)$, then ω^{-1} is that of ξ_q^{-1} or of g_j^{-1} respectively. Hence, $n_\omega(g^{-1}) = n_{\omega^{-1}}(g)$, and the transformation $[g] \mapsto [g^{-1}]$ in the set $G/\sim$ of conjugacy classes of elements in G induces an involutive transformation ι on $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$ given as

$$\begin{aligned}\iota\ :\ (\boldsymbol{Z}_{\ge 0})^{(\Omega)} \ni \boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \ \longmapsto& \\ \boldsymbol{n'} = (n'_\omega)_{\omega\in\Omega} \ \text{ with }\ & n'_\omega = n_{\omega^{-1}}\ (\omega \in \Omega).\end{aligned} \tag{4.9}$$

We put $\Omega_{re} := \{\,\omega \in \Omega\ ;\ \omega^{-1} = \omega\,\}$, $\Omega_c := \{\,\omega \in \Omega\,;\, \omega^{-1} \neq \omega\,\}$, then $\Omega = \Omega_{re} \sqcup \Omega_c$. Furthermore put $D_\omega := D = \{\,z \in \boldsymbol{C}; |z| \le 1\,\} \subset \boldsymbol{C}$ for $\omega \in \Omega_c$, and $I_\omega := [-1, 1] \subset \boldsymbol{R}$ for $\omega \in \Omega_{re}$, and

$$S := \prod_{\omega\in\Omega} D_\omega. \tag{4.10}$$

With the product topology τ_{prod}, S is compact, and on it we have two commuting involutions as

$$\begin{cases} \jmath(s) := (s'_\omega)_{\omega\in\Omega} \ \text{ with }\ s'_\omega := s_{\omega^{-1}}\ ; \\ \overline{s} := (\overline{s_\omega})_{\omega\in\Omega} \ \text{ (conjugate numbers)}, \end{cases} \tag{4.11}$$

for $s = (s_\omega)_{\omega\in\Omega}$. Then we put

$$S' := \{\, s \in S = \prod_{\omega\in\Omega} D_\omega\,;\, \jmath(s) = \overline{s}\}, \tag{4.12}$$

then for $s \in S'$, $s_{\omega^{-1}} = \overline{s_\omega}$ and so $s_\omega \in I_\omega$ for $\omega \in \Omega_{re}$.

For a continuous positive definite class function f on G, put

$$(4.13) \qquad s(f) = (s_\omega)_{\omega\in\Omega} \quad \text{with} \quad s_\omega = f(g_\omega),$$

where g_ω denotes a basic element in the class ω (put $g_{\omega_0} = e$ and $s_{\omega_0} = f(g_{\omega_0}) = f(e)$). Since $f \in K_{\leq 1}(G)$ has the symmetry

$$(\text{SYM1}) \qquad f(g^{-1}) \;=\; \overline{f(g)} \qquad (g \in G),$$

and since ω^{-1} is represented by g_ω^{-1}, there holds a symmetry condition for $s = s(f)$

$$(\text{SYM2}) \qquad \jmath(s) = \overline{s} \qquad (\text{or } s \in S'\,)$$

Define a positive definite class function $\overline{f}$ by $\overline{f}(g) = \overline{f(g)}$ $(g \in G)$, then $s(\overline{f}) = \overline{s(f)}$.

On the product space $S' \times (\boldsymbol{Z}_{\geq 0})^{(\Omega)}$, we define a function

$$(4.14) \qquad P(\boldsymbol{n}, s) \;=\; \prod_{\omega\in\Omega} {s_\omega}^{n_\omega} \qquad \text{with } {s_\omega}^0 = 1,$$

for $\boldsymbol{n} = (n_\omega)_{\omega\in\Omega}$, $s = (s_\omega)_{\omega\in\Omega}$. Then, $P(\iota(\boldsymbol{n}), s) = P(\boldsymbol{n}, \jmath(s)) = P(\boldsymbol{n}, \overline{s})$. Fixing an $s = (s_\omega) \in S'$, we get a function

$$\Psi_s(\boldsymbol{n}) := P(\boldsymbol{n}, s) \qquad \text{on} \quad (\boldsymbol{Z}_{\geq 0})^{(\Omega)} \cong G/\!\sim .$$

Similarly, fixing an $\boldsymbol{n}$, we get a function on S' by $P_{\boldsymbol{n}}(s) := P(\boldsymbol{n}, s)$ $(s \in S')$.

Converse to (4.13), for every $s \in S'$, we get a factorizable class function on G as

$$(4.15) \qquad f_s := \Psi_s \circ \Phi \quad \text{or} \quad f_s(g) \;=\; \Psi_s(\boldsymbol{n}(g)) = P(\boldsymbol{n}(g), s) = P_{\boldsymbol{n}(g)}(s),$$

where $\boldsymbol{n}(g) = \Phi([g])$ for $g \in G$. The function $P_{\boldsymbol{n}(g)}(s)$ satisfies a symmetry condition

$$(\text{SYM3}) \qquad P_{\iota(\boldsymbol{n})}(s) = \overline{P_{\boldsymbol{n}}(s)} \quad \text{for} \quad \boldsymbol{n} = \boldsymbol{n}(g) \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)}.$$

The condition (SYM3) is equivalent to (SYM1) for $f = f_s$. Thus the condition (FTP) above is rewritten as follows:

(FTP$''$) *There exists an* $s = (s_\omega)_{\omega\in\Omega}$ *in* S' *such that* $f = f_s$ *in* (4.15).

4.3. Factorizablity and Extremality

We prove the converse inclusion $F(G) \subset E(G)$ by Lemmas 4.3 and 4.4 below, and then they give together with Lemma 4.1 the following theorem.

Theorem 4.2. *Let* $G = \mathfrak{S}_\infty(T)$ *with a compact group* T. *An* $f \in K_1(G)$ *is extremal if and only if it is factorizable, that is,* $E(G) = F(G)$.

Let us prove $F(G) \subset E(G)$. Note that the subgroups $G_n = \mathfrak{S}_n(T)$ are compact and that any compact subset of G in the inductive limit topology τ_{ind} for $G = \lim_{n\to\infty} G_n$ is contained in some G_m. For an $f \in K_1(G)$, let

$$K(G; f) = \{f' \in K(G); f' \le f\}, \quad K(G; \preceq f) = \bigcup_{\lambda \ge 1} K(G; \lambda f)$$

be as in (1.2) in **1.3**. By Lemma 1.6, $K(G; f)$ is compact in $K_{\le 1}(G) \subset C(G)$ in the compact uniform topology τ_{cu}. Let $E'_m = \mathrm{Extr}\big(K(G; mf)\big) \setminus \{0\}$ be the set of non-zero extremal points of $K(G; mf)$ for $\lambda = m$. We know that f is extremal in $K_{\le 1}(G)$ if and only if f is so in $K(G; \preceq f)$, and by Lemma 1.4, that the intersection $\bigcap_{m\in\boldsymbol{N}} E'_m$ is equal to $E(G) \cap K(G; f)$.

Lemma 4.3. *Let $f' \neq 0$ be an extremal element in $K(G; mf)$ or $f' \in E'_m$. Then f' is factorizable. So $E'_m \subset F(G)$.*

Proof. We apply the proof of Lemma 4.1 to $f' \in E'_m$ instead of $f \in E(G)$ there. We keep to the notations there. For every $\pi \in \widehat{G_1}$, put

$$f'_\pi(g_2) = \int_{G_1} f'(g_1 g_2)\, \widetilde{\chi_\pi}({g_1}^{-1})\, d\mu_{G_1}(g_1), \tag{4.16}$$

Then $f'_\pi \in K_{\le 1}(G_2)$ and the infinite series expansion

$$f'(g_1 g_2) = \sum_{\pi\in\widehat{G_1}} \widetilde{\chi_\pi}(g_1)\, f'_\pi(g_2) \quad \big((g_1, g_2) \in G_1 \times G_2\big) \tag{4.17}$$

is uniformly convergent, and putting $f'_2 = f'|_{G_2}$, we have

$$f'_2(g_2) = \sum_{\pi\in\widehat{G_1}} f'_\pi(g_2) \quad (g_2 \in G_2), \qquad f'_2 \ge f'_\pi. \tag{4.18}$$

Note that $G_2 \cong G$ and that every conjugacy class C_2 of G_2 is an intersection with G_2 of a conjugacy class C of G: $C_2 = C \cap G_2$, and hence class functions on G correspond bijectively to those on G_2. Thus translating the situation from G to G_2, we see that $f'_2 \in E'_{m,2} = \mathrm{Extr}\big(K(G_2; mf_2)\big) \setminus \{0\}$, since $f' \in E'_m = \mathrm{Extr}\big(K(G; mf)\big) \setminus \{0\}$, where $f_2 = f|_{G_2}$. Therefore, from the sum expression (4.18), we see that each summand $f'_\pi \in K_{\le 1}(G_2)$ is a constant multiple of f'_2, whence $f'_\pi = f'_\pi(e)\, f'_2$. □

Lemma 4.4. *For $G = \mathfrak{S}_\infty(T)$, every continuous factorizable positive definite class function on G is a character, or $F(G) \subset E(G)$.*

Proof. Take an $f \in F(G)$. Note that the convex set $K(G; mf)$ is compact, then we can apply Choquet-Bishop-K. de Leeuw Theorem ([Cho], [BiLe]) to $f \in K(G; mf)$. Then f is expressed with a positive measure μ'_m on $E'_m = \mathrm{Extr}\big(K(G; mf)\big) \setminus \{0\}$

$$f = \int_{E'_m} f'\, d\mu'_m(f'),$$

where the integral converges in τ_{cu}. Since f is factorizable, there exists an $s^0 = (s^0_\omega)_{\omega\in\Omega} \in S'$ in (FTP$''$) such that $f = f_{s^0}$ or $s^0 = s(f)$, and since $E'_m \subset F(G)$ by Lemma 4.3, for any $f' \in E'_m$, there exists $s = (s_\omega)_{\omega\in\Omega} \in S'$ such that $f' = f_s$ or $s = s(f')$. (Later in Section 13, we see independently that the image $S'' \subset S'$ of $F(G)$ under the continuous map $f \to s = s(f)$ is compact. See, Theorem 13.2 or more originally (13.1)–(13.2) and Theorem 13.1.)

Note that, on the image $S''_m = \{s(f'); f' \in E'_m\}$ of E'_m in S', the topology induced from E'_m is stronger than or equal to $\tau_{prod}|_{S''_m}$, and that for any $g = \prod_{\omega\in\Omega} g_\omega^{\,n_\omega} \in G$, the map

$$E'_m \in f' = f_s \longmapsto f_s(g) = \prod_{\omega\in\Omega} s_\omega^{\,n_\omega} = P_{\boldsymbol{n}}(s)$$

is continuous, where $\boldsymbol{n} = \boldsymbol{n}(g) \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)}$. Thus we get

$$P_{\boldsymbol{n}}(s^0) = \int_{S''_m} P_{\boldsymbol{n}}(s)d\mu''_m(s), \tag{4.19}$$

with the measure μ''_m on S''_m transmitted from μ'_m on E'_m.

The set of continuous functions $\{P_{\boldsymbol{n}}; \boldsymbol{n} \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)}\}$ on $S' \supset S''_m$ separates points of S' and contains conjugate functions $\overline{P_{\boldsymbol{n}}} = P_{\iota(\boldsymbol{n})}$ as is seen from (SYM3). Therefore, by Stone-Weierstrass approximation theorem, we have the integral expression as in (4.19) for all continuous functions F on S' as $F(s^0) = \int_{S''_m} F(s)d\mu''_m(s)$. Hence the measure μ''_m should be supported by a single point set $\{s^0\}$. This means that $f = f_{s^0} \in E'_m$. So $f \in \bigcap_{m\in\boldsymbol{N}} E'_m \subset E(G)$. This is to be proved. $\square$

5. Characters of $\mathfrak{S}_\infty(T)$ with T any compact group

5.1. Character formula for factor representations of finite type of $\mathfrak{S}_\infty(T)$

Let $\widehat{T}$ be the dual of T consisting of all equivalence classes of continuous irreducible unitary representations (= IURs). We identify every equivalence class with one of its representative. Thus $\zeta \in \widehat{T}$ is an IUR and denote by χ_ζ its character: $\chi_\zeta(t) = \mathrm{tr}(\zeta(t))$ $(t \in T)$, then $\dim\zeta = \chi_\zeta(e_T)$. Put $G = \mathfrak{S}_\infty(T)$. For a $g \in G$, let its standard decomposition into basic components be

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m\,, \tag{5.1}$$

where the supports of components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) := \mathrm{supp}(\sigma_j)$ $(1 \leq j \leq m)$, are mutually disjoint. Furthermore, $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, with $\ell(\xi_{q_k}) = 1$ for $1 \leq k \leq r$, and σ_j is a cycle of length $\ell(\sigma_j) \geq 2$ and $\mathrm{supp}(d_j) \subset K_j = \mathrm{supp}(\sigma_j)$. For $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives a cycle decomposition of σ. For $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put $P_{\sigma_j}(d_j)$ as in (3.2).

For one-dimensional characters of $\mathfrak{S}_\infty$, we introduce simple notation as

$$\chi_\varepsilon(\sigma) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad (\sigma \in \mathfrak{S}_\infty\,;\ \varepsilon = 0,1). \tag{5.2}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T}, \varepsilon \in \{0,1\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}\,, \tag{5.3}$$

of decreasing sequences of non-negative real numbers

$$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}\,,\ \ \alpha_{\zeta,\varepsilon,1} \geq \alpha_{\zeta,\varepsilon,2} \geq \alpha_{\zeta,\varepsilon,3} \geq\ \cdots\ \geq 0\,;$$

and a set of non-negative $\mu_\zeta \geq 0$ $(\zeta \in \widehat{T})$, which altogether satisfy the condition

$$\sum_{\zeta\in\widehat{T}}\ \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| \ +\ \|\mu\| \ =\ 1\,, \tag{5.4}$$

$$\text{with}\quad \|\alpha_{\zeta,\varepsilon}\| = \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i}\,,\quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta\,.$$

Note that, under the condition (5.4), there exists a countable subset $\widehat{T}_0 \subset \widehat{T}$ such that $\alpha_{\zeta,\varepsilon} = \mathbf{0}$ and $\mu_\zeta = 0$ for $\zeta \notin \widehat{T}_0$.

Recall that $E(G)$ and $F(G)$ denote respectively the set of all characters and that of all factorisable elements in $K_1(G)$, and that $E(G) = F(G)$.

Theorem 5.1. *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact group T with $\mathfrak{S}_\infty$. For a parameter $A := \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$ in* (5.3)–(5.4)*, the following formula gives an element in $F(G) = E(G)$: for a $g \in G$, let* (5.1) *be its standard decomposition, then put*

$$f_A(g) = \prod_{1\leq k\leq r}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\ \sum_{i\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\right)\chi_\zeta(t_{q_k})\right\}$$
$$\times \prod_{1\leq j\leq m}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\ \sum_{i\in\boldsymbol{N}} \left(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\right)^{\ell(\sigma_j)} \chi_\varepsilon(\sigma_j)\right)\chi_\zeta\big(P_{\sigma_j}(d_j)\big)\right\}, \tag{5.5}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$.

Conversely any element in $F(G) = E(G)$ is given in the form of f_A.

Note 5.1. Let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}$ without the components $g_j = (d_j, \sigma_j)$ with $\ell(\sigma_j) \geq 2$, then the formula gives

$$f_A(g) = \prod_{1\leq k\leq r}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\ \sum_{i\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\right)\chi_\zeta(t_{q_k})\right\}.$$

If $t_{q_k} = e_T$, the corresponding term is

$$\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\right)\chi_\zeta(t_{q_k}) = \sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1,$$

by the equality condition (5.4), and the formula is valid even for $g = e$ with $f_A(e) = 1$. In the case where T is not discrete or equivalently not finite, the continuity at $g = e$ is thus guaranteed by (5.4).

5.2. Remarks on the case where T is a finite group

Assume T be finite. Let $\mathbf{1}_T$ be the trivial representation of T, and put $\widehat{T}^* := \widehat{T}\setminus\{\mathbf{1}_T\}$, $T^* = T\setminus\{e_T\}$. Then, as functions on T, we have

$$(5.6)\qquad 0 = \sum_{\zeta\in\widehat{T}}(\dim\zeta)\,\chi_\zeta\,,\quad 1 = \chi_{\mathbf{1}_T} = -\sum_{\zeta\in\widehat{T}^*}(\dim\zeta)\,\chi_\zeta\,,\quad \text{on } T^*.$$

Therefore, in the parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$ of f_A, putting $\mu'_\zeta = \mu_\zeta - \nu(\dim\zeta)^2$ $(\zeta\in\widehat{T})$ with a $\nu\in\boldsymbol{R}$, we get for $t\in T^*$ or $t\neq e_T$,

$$\sum_{\zeta\in\widehat{T}}\frac{\mu_\zeta}{\dim\zeta}\chi_\zeta(t) = \sum_{\zeta\in\widehat{T}}\frac{\mu'_\zeta}{\dim\zeta}\chi_\zeta(t).$$

By this reason, we can accept the parameter A for f_A not necessarily under the equality condition (5.4) but also under the weaker inequality condition

$$(5.7)\qquad \sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \le 1\,,$$

loosing the validity of the formula of f_A for $t_{q_k} = e_T$ and accordingly for $g = e$.

Under the above condition (5.7), the uniqueness of the part $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}}$, $\mu_\zeta\in\boldsymbol{R}$, is lost. To recover the uniqueness of the parameter A for f_A, we can put, in place of the original condition (5.4) which may be called as (MAX) condition, one of the following conditions (cf. [HH1]–[HH3]):

(MIN) $\qquad \mu_\zeta \ge 0\ (\zeta\in\widehat{T})$, and $\min\{\mu_\zeta\,;\,\zeta\in\widehat{T}\} = 0\,;$

(ZERO) $$\sum_{\zeta\in\widehat{T}}\frac{\mu_\zeta}{(\dim\zeta)^2} = 0\,.$$

5.3. Remarks and examples

5.3.1. In the case of finite groups T, we admit, under the contition (5.7), "$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}} = \mathbf{0}$ for all $(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}$ and $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}} = \mathbf{0}$". Then we have $f_A = \delta_e$, the delta function on G supported by $\{e\}$, which is the character of the regular representation λ_G of G. This also corresponds to another parameter given as

$$\begin{cases} \alpha_{\zeta,\varepsilon} = \mathbf{0} \ \text{ for all } (\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}\ , \\ \mu = (\mu_\zeta)_{\zeta\in\widehat{T}} \text{ with } \mu_\zeta = |T|^{-1}(\dim\zeta)^2\ , \end{cases}$$

under the condition (5.4).

In general, in the case of non-discrete compact groups T, there appear non-continuous positive definite class functions as pointwise limits of centralizations of matrix elements of unitary representations of G. Simple examples are given as sums of continuous ones and constant multiples of the delta function δ_e.

5.3.2. The case of the infinite symmetric group $\mathfrak{S}_\infty$ itself is considered as an extreme case of the wreath product groups $\mathfrak{S}_\infty(T)$ with a trivial group $T = \{\, e_T \,\}$. For $\mathfrak{S}_\infty$, we have originally only the so-called Thoma parameters $\alpha = (\alpha_p)_{p\in\boldsymbol{N}}, \beta = (\beta_p)_{p\in\boldsymbol{N}}$ in [Tho2] satisfying the inequality condition $\|\alpha\| + \|\beta\| \le 1$.

Then, for the parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{\,0,1\,\}}\,;\,\mu\big)$ of the character f_A, we put for the trivial representation $\mathbf{1}_T$ of $T = \{e_T\}$

$$\alpha_{0,\mathbf{1}_T} = \alpha, \quad \alpha_{1,\mathbf{1}_T} = \beta, \tag{5.8}$$

and introduce a fake parameter $\mu = (\mu_{\mathbf{1}_T})$ for the trivial representation $\mathbf{1}_T$ of $T = \{e_T\}$ by putting $\mu_{\mathbf{1}_T} := 1 - (\|\alpha_{0,\mathbf{1}_T}\| + \|\alpha_{1,\mathbf{1}_T}\|)$. Then the equality condition (5.4) is established.

5.3.3. In the case where $\mu_\zeta = 1$ for some $\zeta \in \widehat{T}$ and all other parameters in A are zero,

$$f_A(g) = \prod_{1\le k\le r} \frac{1}{\dim\zeta}\,\chi_\zeta(t_{q_k}) \qquad \text{for } \ g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}, \tag{5.9}$$

and $f_A(g) = 0$ if g has components $g_j = (d_j, \sigma_j)$ with $\ell(\sigma_j) \ge 2$. This case is related to a kind of 'ζ-twisted' regular representation of the group $\mathfrak{S}_\infty \cong G/D$, $D = D_\infty(T)$. Note that this case does not exist for the symmetric group $\mathfrak{S}_\infty$ itself.

Taking an IUR ρ_ζ of D given by tensor product as in **5.3.4** below, we obtain the induced representation $R_\zeta := \mathrm{Ind}_D^G \rho_\zeta$. In the case where $\zeta = \mathbf{1}_T$ the trivial representation of T, R_ζ is essentially the regular representation of $\mathfrak{S}_\infty \cong G/D$. Take appropriately a positive definite matrix element F of R_ζ, and an increasing sequence G_N, we get f_A as a limit of centralizations F^{G_N} of F.

5.3.4. In the case where $\alpha_{\zeta,\varepsilon,1} = 1$ for a fixed $(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}$ and all other parameters in A are zero, we have $\alpha_{\zeta,\varepsilon} = (1,0,0,\dots)$ and

$$f_A(g) = \prod_{1\le k\le r} \frac{1}{\dim\zeta}\,\chi_\zeta(t_{q_k}) \times \prod_{1\le j\le m} \left(\frac{1}{\dim\zeta}\right)^{\ell(\sigma_j)} \chi_\zeta\big(P_{\sigma_j}(d_j)\big)\chi_\varepsilon(\sigma_j).$$

This corresponds to an IUR $\rho_{\zeta,\varepsilon}$ of G constructed as follows. For $D_\infty(T) = \prod'_{i\in\boldsymbol{N}} T_i\,,\ T_i = T$, consider an infinite tensor product of $\zeta_i = \zeta \in \widehat{T}$ with respect to a reference vector $a = (a_i)_{i\in\boldsymbol{N}}$, where $a_i \in V_i, \|a_i\| = 1$. Its representation space is the so-called incomplete tensor product $\otimes^a_{i\in\boldsymbol{N}} V_i$ of V_i in the sense of von Neumann, which is defined as a completion of a linear span of vectors of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes \cdots$ with $v_i \in V_i, \|v_i\| = 1$, such that $v_i = a_i$ $(i \gg 1)$. For $d = (t_i)_{i\in\boldsymbol{N}} \in D_\infty(T)$, and $\sigma \in \mathfrak{S}_\infty$, their operations are given respectively as

$$\begin{aligned}\rho_\zeta(d)\big(\otimes_{i\in\boldsymbol{N}} v_i\big) &:= \otimes_{i\in\boldsymbol{N}}\big(\zeta_i(t_i)v_i\big)\\ \rho_\zeta(\sigma)\big(\otimes_{i\in\boldsymbol{N}} v_i\big) &:= \otimes_{i\in\boldsymbol{N}} v'_i \quad \text{with} \quad v'_i = v_{\sigma^{-1}(i)}.\end{aligned}$$

Then, for $g = (d, \sigma)$, we put $\rho_{\zeta,\varepsilon}(g) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon\, \rho_\zeta(d)\rho_\zeta(\sigma)$. Take a matrix element $F(g) = \langle \rho_{\zeta,\varepsilon}(g)v, v\rangle$ for a $v = \otimes_{i\in\boldsymbol{N}} v_i$. Then a limit of centralizations F^{G_N} as $N \to \infty$ gives the character f_A in question, as will be seen later.

5.3.5. More generally than the case **5.3.1**, assume all the parameters $\alpha_{\zeta,\varepsilon}$ in A are zero, and $\|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta = 1$. Then, the formula (5.5) gives

$$f_A(g) = \prod_{1\le k\le r} \left\{ \sum_{\zeta\in\widehat{T}} \left(\frac{\mu_\zeta}{\dim\zeta}\right) \chi_\zeta(t_{q_k}) \right\}. \tag{5.10}$$

This case relates to an induced representation $\rho = \mathrm{Ind}^G_H \pi$ of G from a subgroup H given below, in the sense that f_A is obtained from a matrix element F of ρ by taking a limit of its centralizations F^{G_N} with respect to $G_N := \mathfrak{S}_N(T)$ as $N \to \infty$ (cf. §12).

To give such a subgroup H and its unitary representation π, take first a partition of $\boldsymbol{N}$ as $\boldsymbol{N} = \bigsqcup_{\zeta\in\widehat{T}_0} I_\zeta$, where $\widehat{T}_0 := \{\,\zeta \in \widehat{T}\,;\, \mu_\zeta \neq 0\,\}$, and each subsets I_ζ are all infinite. Corresponding to this partition, we define a subgroup

$$H = \prod_{\zeta\in\widehat{T}_0} H_\zeta \quad \text{with} \quad H_\zeta = D_{I_\zeta}(T), \tag{5.11}$$

and a representation $\pi = \otimes^b_{\zeta\in\widehat{T}_0} \pi_\zeta$ of H, with a reference vector $b = (b_\zeta)_{\zeta\in\widehat{T}_0}$, such that

$$\pi_\zeta(d) = \otimes^{a_\zeta}_{i\in I_\zeta} \zeta_i(t_i) \quad \text{for} \quad d = (t_i)_{i\in I_\zeta} \in H_\zeta = D_{I_\zeta}(T), \tag{5.12}$$

where $\zeta_i = \zeta$ $(i \in I_\zeta)$, and $a_\zeta = (a_i)_{i\in I_\zeta}$ is a reference vector with $a_i \in V(\zeta_i), \|a_i\| = 1$.

6. Characters of wreath product group $\mathfrak{S}_\infty(T)$ with T abelian

When T is abelian, the general character formula (5.5) for $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with a compact group T has a simplified form.

In this abelian case, $\widehat{T}$ is nothing but the dual group consisting of all one-dimensional characters of T, and for each $\zeta \in \widehat{T}$, its character χ_ζ is identified with ζ itself.

For a $g \in G = \mathfrak{S}_\infty(T)$, let its standard decomposition be $g = \xi_{q_1}\xi_{q_2}\cdots \xi_{q_r} g_1 g_2 \cdots g_m$, as in (5.1) with $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, for $1 \le k \le r$, and $g_j = (d_j, \sigma_j)$ for $1 \le j \le m$. Put $K_j = \mathrm{supp}(\sigma_j)$, and for $d_j = (t_i)_{i \in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put

$$P_{K_j}(d_j) = \prod_{i \in K_j} t_i\,, \quad \zeta(d_j) := \zeta(P_{K_j}(d_j)) = \prod_{i \in K_j} \zeta(t_i)\,. \tag{6.1}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T},\ \varepsilon \in \{\,0,1\,\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta \in \widehat{T}}\,, \tag{6.2}$$

of decreasing sequences of non-negative real numbers $\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i \in \boldsymbol{N}}$, and a set of non-negative $\mu_\zeta \ge 0\ (\zeta \in \widehat{T})$, which satisfies the condition

$$\sum_{\zeta \in \widehat{T}} \sum_{\varepsilon \in \{\,0,1\,\}} \|\alpha_{\zeta,\varepsilon}\| \;+\; \|\mu\| \;=\; 1\,. \tag{6.3}$$

Theorem 6.1. *Let $G = \mathfrak{S}_\infty(T)$ with a compact abelian group T. For a parameter $A := \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{\,0,1\,\}}\,;\, \mu\big)$ in (6.2)–(6.3), the following formula gives an element in $F(G) = E(G)$: for a $g \in G$, let its standard decomposition be as above, then put*

$$\begin{aligned} f_A(g) = &\prod_{1 \le k \le r} \left\{ \sum_{\zeta \in \widehat{T}} \left(\sum_{\varepsilon \in \{\,0,1\,\}} \sum_{i \in \boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} \;+\; \mu_\zeta \right) \zeta(t_{q_k}) \right\} \\ &\times \prod_{1 \le j \le m} \left\{ \sum_{\zeta \in \widehat{T}} \left(\sum_{\varepsilon \in \{\,0,1\,\}} \sum_{i \in \boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right) \zeta(d_j) \right\}, \end{aligned} \tag{6.4}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, and $\zeta(d_j)$ as in (6.1).

Converdely any element in $F(G) = E(G)$ is given in the form of f_A.

Example 6.1. The case where $\alpha_{\zeta,\varepsilon,1} = 1$ for a fixed $(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}$ and all other parameters in A are zero, whence $\alpha_{\zeta,\varepsilon} = (1,0,0,\dots)$, corresponds to one-dimensional character $\pi_{\zeta,\varepsilon}$ of G in Lemma 3.4. In fact,

$$f_A(g) = \prod_{1 \le k \le r} \zeta(t_{q_k}) \times \prod_{1 \le j \le m} \chi_\varepsilon(\sigma_j)\zeta(d_j) \;=\; \pi_{\zeta,\varepsilon}(g).$$

Except these cases of one-dimensional representations of G, a character f_A given above corresponds to a factor representation of G of type II_1.

Example 6.2. Consider the case where $\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| + \mu_\zeta = 1$ for a fixed $\zeta \in \widehat{T}$ and all other parameters in A are zero. Put $\alpha = \alpha_{\zeta,0}, \beta = \alpha_{\zeta,1}$, and let $f_{\alpha,\beta}$ be Thoma's character for $\mathfrak{S}_\infty$. Denote by Ψ the natural homomorphism from G onto $\mathfrak{S}_\infty \cong G/D$ with normal subgroup $D = D_\infty(T)$, and put $f^{\#}_{\alpha,\beta} :=$

$f_{\alpha,\beta}\circ\Psi$. Then the character $f_A(g)$ in this case is equal to $\pi_{\zeta,0}(g)\cdot f^{\#}_{\alpha,\beta}(g)$ with a one-dimensional character $\pi_{\zeta,0}$ of G with $\varepsilon=0$. In fact,

$$f_A(g)=\prod_{1\le k\le r}\zeta(t_{q_k})\times\prod_{1\le j\le m}\zeta(d_j)\times\prod_{1\le j\le m}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}(\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)}\cdot\chi_\varepsilon(\sigma_j)\right).$$

In particular, the case where $\mu_\zeta=1$ for a fixed $\zeta\in\widehat{T}$, corresponds to the induced representation $\mathrm{Ind}_D^G\zeta_D$, where $\zeta_D(d):=\zeta(P(d)), d\in D$, is a one-dimensional character of $D=D_\infty(T)$. The character f_A is equal to ζ_D on $D\hookrightarrow G$, and zero outside of D. In the case $\zeta=\mathbf{1}_T$, this induced representation is nothing but the regular representation of $G/D\cong\mathfrak{S}_\infty$.

7. Characters of the subgroup $\mathfrak{S}_\infty^S(T)\subset\mathfrak{S}_\infty(T)$, $S\subset T$ abelian

Let T be abelian and S its subgroup. Then, $G^S=\mathfrak{S}_\infty^S(T)=D_\infty^S(T)\rtimes\mathfrak{S}_\infty$ with

$$D_\infty^S(T)=\{\,d=(t_i)_{i\in\boldsymbol{N}}\ ;\ P(d)\in S\,\}\qquad\text{with}\quad P(d):=\prod_{i\in\boldsymbol{N}}t_i, \tag{7.1}$$

is a natural subgroup of G. We can deduce a general character formula for this normal subgroup $N=G^S$ from the one for $G=\mathfrak{S}_\infty(T)$, especially when S is open in T.

Take an element $g\in N$ and let its standard decomposition in $G\supset N$ be $g=\xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}\,g_1g_2\cdots g_m$ with $\xi_{q_k}=(t_{q_k},(q_k))$ and $g_j=(d_j,\sigma_j), d_j=(t_i)_{i\in K_j}, K_j=\mathrm{supp}(\sigma_j)$. Note that each component ξ_{q_k} does not necessarily belong to N, and that the component $g_j=(d_j,\sigma_j)$ belongs to N if and only if $P(d_j)=\prod_{i\in K_j}t_i\in S$. However, after careful discussions in Section 14 on the relation between N and G, we obtain the following result for the normal subgroup $N=G^S$ from the result for G.

Theorem 7.1. *Let T be abelian and S a subgroup of T.*

(i) *Let $N=G^S=\mathfrak{S}_\infty^S(T)$ be the normal subgroup of $G=\mathfrak{S}_\infty(T)$ given in* (7.1)*. Then, for any character $f\in E(G)$ of a factor representaion of G of finite type, the restriction $f^S=f|_N$ on N is again such a character of N or $f^S\in E(N)$.*

(ii) *For $f_A\in E(G)$ with a parameter $A=\big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$ in* (6.2)–(6.3)*, the following formula for $f_A|_N$ gives an element in $E(N)$: for a $g\in N$, let its standard decomposition in G be as above, then put*

$$\begin{aligned}f_A^S(g)=&\prod_{1\le k\le r}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\alpha_{\zeta,\varepsilon,i}\ +\ \mu_\zeta\right)\zeta(t_{q_k})\right\}\\&\times\prod_{1\le j\le m}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}(\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)}\cdot\chi_\varepsilon(\sigma_j)\right)\zeta(d_j)\right\},\end{aligned}\tag{7.2}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, *and* $\zeta(d_j)$ *as in* (6.1).

(iii) *Assume that* S *is open in* T, *or especially* T *is finite. Then any character of* $N = G^S$ *is given in the form of* f_A^S, *that is,* $E(G^S) = \{f_A^S;\ A$ *in* (6.2)–(6.3)$\}$.

The parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\mu\big)$ for f_A^S is not unique even under the normalization condition (6.3). To describe the correspondence of parameters, we introduce a translation $R(\zeta_0)$ on A by an element $\zeta_0 \in \widehat{T}$ as follows:

$$R(\zeta_0)A := \big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,; R(\zeta_0)\mu\big) \tag{7.3}$$

with $\alpha'_{\zeta,\varepsilon} = \alpha_{\zeta{\zeta_0}^{-1},\varepsilon}\,,\ \big((\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}\big);\ R(\zeta_0)\mu = (\mu'_\zeta)_{\zeta\in\widehat{T}}\,,\ \mu'_\zeta = \mu_{\zeta{\zeta_0}^{-1}}$.

Proposition 7.2. *Assume that two parameters of characters of* G

$$A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\mu\big) \quad \text{and} \quad A' = \big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\mu'\big)$$

both satisfy the normalization condition (6.3). *Then, they determine the same function on* $N = G^S$, *that is,* $f_A^S = f_{A'}^S$, *if and only if* $A' = R(\zeta_S)A$ *for some* $\zeta_S \in \widehat{T}$ *which is trivial on* S. *In this case, as elements in* $E(G)$ *for the bigger group* G, *we have*

$$f_{A'}(g) = \pi_{\zeta_S,0}(g)\cdot f_A(g) \qquad (g\in G).$$

8. Method of proving Theorem 5.1

Put $G = \mathfrak{S}_\infty(T)$ with a compact group T. Let $E(G)$ and $F(G)$ be respectively the sets of all characters and of all factorizable continuous positive definite class functions on G, then $E(G) = F(G)$ as is proved in Section 4.

The first part of our proof is to prepare seemingly sufficiently big family $E'(G)$ of factorizable continuous positive definite class functions f_A's on G: $E'(G) \subset F(G)$.

The second part is to prove that $E'(G)$ covers $F(G)$: $E'(G) \supset F(G)$.

Then $E'(G)$ is actually equal to $F(G)$, and we get $E'(G) = F(G) = E(G)$.

8.1. The first part of the proof

The first part has two important ingredients. The one is a method of *taking limits of centralizations* of positive definite functions. The other is *inducing up positive definite functions* from subgroups.

8.1.1. Taking limits of centralizations of positive definite functions

For a continuous positive definite function F on a topological group G and a compact subgroup $G' \subset G$, we define a *centralization* of F with respect to G' as

$$F^{G'}(g) := \int_{g'\in G'} F(g'g{g'}^{-1})\, d\mu_{G'}(g'), \tag{8.1}$$

where $\mu_{G'}$ denotes the normalized Haar measure on G'. Then $F^{G'}$ is automatically invariant under G'.

Assume that we have an increasing sequence of compact subgroups $G_N \nearrow G$. Then we can examine if the series of continuous positive definite functions F^{G_N} converges pointwise to a continuous function $F_\infty = \lim_{N\to\infty} F^{G_N}$. If it does, then F_∞ is necessarily a positive definite *class* function.

Choosing starting functions F as simple as possible, we check what we get as the limit functions F_∞ which also depend heavily on the choice of the series $G_N \nearrow G$. This is a kind of 'trial and error' method. In the case of $G = \mathfrak{S}_\infty(T)$, we can thus get as a result the total set $F(G)$ of factorizable continuous positive functions on G.

However, for the group $G = GL(\infty, \boldsymbol{F})$ studied in [Sku], where $\boldsymbol{F}$ is a finite field, we obtain none until now except the delta function δ_e on G supported by the identity element $e \in G$.

8.1.2. Inducing up of positive definite functions

We choose appropriate subgroups H and their URs π and use their diagonal matrix elements f_π as positive definite functions on H to be induced up to G as $F = \mathrm{Ind}_H^G f_\pi$ (see Lemma 8.1 below). Then we centralize F along with some increasing sequences $G_N \nearrow G$ as F^{G_N} and check their limits $F_\infty = \lim_{N\to\infty} F^{G_N}$.

We have constructed in [Hir1]–[Hir2] a huge family of IURs of a wreath product group $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with any finite group T, by taking so-called wreath product type subgroups H in a 'saturated fashion', and their IURs π of a certain form to get IURs of G as induced representations $\rho = \mathrm{Ind}_H^G \pi$.

For our present purpose of getting a seemingly big enough set $E'(G)$ of f_A's on G, actually it is sufficient to choose simpler subgroups of degenerate wreath product type and their URs π. Here induced representations $\rho = \mathrm{Ind}_H^G \pi$ are very far from to be irreducible, but sufficient for our purpose to get positive definite class functions on G.

In a general setting, we have the following fact.

Lemma 8.1. *Let G be a group and H its subgroup. Take a positive definite function f on H, and extend it trivially onto G by putting zero outside of H, which is denoted by $F = \mathrm{Ind}_H^G f$. Then F is again positive definite on G.*

As an example of positive definite functions f on H, we can take a matrix element of a UR π of H on a Hilbert space $V(\pi)$ as

$$f_\pi(h) = \langle \pi(h)v, v\rangle \ \ (h \in H) \quad \text{with} \quad v \in V(\pi), \|v\| = 1.$$

In the case where H is open in G, or in particular G is discrete, the trivial inducing up $F = \mathrm{Ind}_H^G f_\pi$ is a matrix element for the induced representation $\rho = \mathrm{Ind}_H^G \pi$.

Let G' be a compact subgroup of G and take a centralization $F^{G'}$ of $F = \mathrm{Ind}_H^G f$. Since F is zero outside of H, the value of centralization $F^{G'}(g)$

is $\neq 0$ only for elements g which are conjugate under G' to some $h \in H$, and moreover, for $h \in H$,

$$F^{G'}(h) = \int_{G'} f(g'hg'^{\,-1})\, d\mu_{G'}(g'), \tag{8.2}$$

where, $f(g'hg'^{\,-1}) = 0$ if $g'hg'^{\,-1} \notin H$, by definition, whence the integrand $\neq 0$ only if $g'hg'^{-1} \in H$.

A pointwise limit of F^{G_N} for an increasing sequence $G_N \nearrow G$ of compact subgroups of G, which is certainly positive definite and invariant, may be *continuous* or may be *not*, with respect to the inductive limit topology τ_{ind}. We study cases where H is open in G, f is continuous on H and the limit function F_∞ is continuous.

In calculation, the condition $g'hg'^{-1} \in H$ for $g' \in G_N$, is translated into certain combinatorial conditions, and to get the limit F_∞ of F^{G_N} as $N \to \infty$, we have to calculate asymtotic behavior of several ratios of combinatorial numbers. In the discrete case or the case of a finite group T, the above integral turns out to be a sum which can be calculated by some combinatorics [HH3].

8.2. The second part of the proof

The second part is to guarantee that actually all factorizable continuous positive definite class functions have been already obtained in the first part, that is, $E'(G) = F(G)$, or the *completeness* of $E'(G)$.

As seen from the explicit form of f_A's, the inclusion $E'(G) \subset F(G)$ is trivially clear. To prove the converse inclusion $E'(G) \supset F(G)$, we proceed in Section 13 as follows. Take an $f \in F(G)$. Then, it is written as

$$f(g) = \prod_{1\le k\le r} f(\xi_{q_k}) \prod_{1\le j\le m} f(g_j)$$

for $g = \xi_{q_1} \cdots \xi_{q_r}\, g_1 \cdots g_m \in G$, and we see that f is written in the form of f_s in (4.15) with $s \in S'$ in (4.12).

We can take a kind of *partial Fourier transform* of $f = f_s$ on $G = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with respect to subgroups $D_n(T) \subset D_\infty(T)$, and get a series of positive definite class functions $\mathcal{F}_{\zeta,\varepsilon,n}(f)$ on $\mathfrak{S}_n, n \ge 1$, where $\zeta \in \widehat{T}, \varepsilon = 0, 1$.

For every fixed (ζ, ε), we appeal to Korollar 1 to Satz 2 in [Tho2] for the series of $\mathcal{F}_{\zeta,\varepsilon,n}(f)$ on $\mathfrak{S}_n, n \ge 1$. Then we can specify the range of the parameter $s = (s_\omega)_{\omega\in\Omega}$, and find that f_s is expressed in the form of f_A in Theorem 5.1 with a parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}},\ \mu\big)$ in the range of A given by (5.3)–(5.4).

9. Subgroups $H \subset G = \mathfrak{S}_\infty(T)$ and their representations π

In place of the purpose in [Hir1]–[Hir2] of getting IURs, our present purpose is to get all the characters of $G = \mathfrak{S}_\infty(T)$. In the papers [Hir3]–[Hir4] and [HH1]–[HH3], we apply the method of taking limits of centralizations of matrix elements $F = \mathrm{Ind}_H^G f_\pi$ of $\rho = \mathrm{Ind}_H^G \pi$, where f_π is a diagonal (hence positive definite) matrix element of a UR π of H. Now in the case of continuous (or non-discrete) compact groups T, we apply the similar method using the trivial inducing up $F = \mathrm{Ind}_H^G f_\pi$ of f_π, irrespective of that F is a matrix element of $\rho = \mathrm{Ind}_H^G \pi$ or not. To our present purpose, we look for the best choice of pairs of H and π, following principally the previous papers [HH1]–[HH3], but simplifying the situation without paying attention on the *irreducibility* of the induced representations ρ.

To give such subgroups H, we take first a partition of $\boldsymbol{N}$ as

$$\boldsymbol{N} = \left(\bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}_0\times\{0,1\}} \left(\bigsqcup_{p\in P_{\zeta,\varepsilon}} I_p \right) \right) \bigsqcup \left(\bigsqcup_{\zeta\in\widehat{T}_0} I_\zeta \right), \tag{9.1}$$

where $\widehat{T}_0$ is a countable subset of $\widehat{T}$, and each $P_{\zeta,\varepsilon}$ is a countably infinite index set, and the subsets I_p, I_ζ are all infinite. Corresponding to this partition, we define a subgroup

$$H = \left(\prod_{(\zeta,\varepsilon)\in\widehat{T}_0\times\{0,1\}} \left({\prod_{p\in P_{\zeta,\varepsilon}}}' H_p \right) \right) \times \left({\prod_{\zeta\in\widehat{T}_0}}' H_\zeta \right), \tag{9.2}$$

$$\text{with} \qquad H_p = \mathfrak{S}_{I_p}(T), \quad H_\zeta = D_{I_\zeta}(T) \subset \mathfrak{S}_{I_\zeta}(T).$$

As a unitary representation π of H, we take

$$\pi = \left(\otimes_{(\zeta,\varepsilon)\in\widehat{T}_0\times\{0,1\}} \left(\otimes_{p\in P_{\zeta,\varepsilon}}^{b_{\zeta,\varepsilon}} \pi_p \right) \right) \otimes \left(\otimes_{\zeta\in\widehat{T}_0}^{b} \pi_\zeta \right). \tag{9.3}$$

Here $b_{\zeta,\varepsilon} = (b_p)_{p\in P_{\zeta,\varepsilon}}$ is a reference vector with $b_p \in V(\pi_p), \|b_p\| = 1$ $(p \in P_{\zeta,\varepsilon})$, and π_p for $H_p = \mathfrak{S}_{I_p}(T)$ is given as

$$\pi_p\big((d,\sigma)\big) = \left(\otimes_{i\in I_p}^{a_p} \zeta_i(t_i) \right) I(\sigma)\,\mathrm{sgn}_{\mathfrak{S}}(\sigma)^{\varepsilon} \quad \text{for} \quad d = (t_i)_{i\in I_p},\ \sigma \in \mathfrak{S}_{I_p}, \tag{9.4}$$

where $a_p = (a_i)_{i\in I_p}$ is a reference vector with $a_i \in V(\zeta_i), \|a_i\| = 1$, and $\zeta_i = \zeta$ as a representation of $T_i = T$ $(i \in I_p)$, and $I(\sigma)$ is defined as

$$I(\sigma):\ v = \otimes_{i\in I_p} v_i \longmapsto \otimes_{i\in I_p} v_i',\quad v_i' = v_{\sigma^{-1}(i)} \qquad (v_i \in V(\zeta_{p,i}), i \in I_p).$$

Moreover $b = (b_\zeta)_{\zeta\in\widehat{T}_0}$ is a reference vector, and for $\zeta \in \widehat{T}_0$, π_ζ of H_ζ is given as

$$\pi_\zeta(d) = \otimes_{i\in I_\zeta}^{a_\zeta} \zeta_i(t_i) \quad \text{for} \ \ d = (t_i)_{i\in I_\zeta} \in H_\zeta = D_{I_\zeta}(T), \tag{9.5}$$

where $a_\zeta = (a_i)_{i\in I_\zeta}$ with $a_i \in V(\zeta_i), \|a_i\| = 1$, and $\zeta_i = \zeta$ for $T_i = T$ $(i \in I_\zeta)$.

10. Increasing sequences of subgroups $G_N \nearrow G = \mathfrak{S}_\infty(T)$

Depending on the choice of increasing sequence $G_N \nearrow G$ of subgroups, we get various positive definite class functions of G as limits of centralizations

F^{G_N} for $F = \mathrm{Ind}_H^G f_\pi$, which turn out to be characters. We choose a series G_N as $G_N = \mathfrak{S}_{J_N}(T)$, $J_N \nearrow \boldsymbol{N}$, and demand an asymptotic condition as

$$\text{(10.1)} \qquad \frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \ \ (p \in P), \qquad \frac{|I_\zeta \cap J_N|}{|J_N|} \to \mu_\zeta \ \ (\zeta \in \widehat{T}),$$

where $P := \sqcup_{(\zeta,\varepsilon)\in \widehat{T}\times\{\,0,1\,\}} P_{\zeta,\varepsilon}$ is the union of index sets, with $P_{\zeta,\varepsilon} = \emptyset$ for $\zeta \in \widehat{T} \setminus \widehat{T}_0$. Then,

$$\text{(10.2)} \qquad \sum_{p\in P} \lambda_p \ + \ \sum_{\zeta\in\widehat{T}} \mu_\zeta \ \le \ 1\,.$$

For each $(\zeta,\varepsilon) \in \widehat{T} \times \{\,0,1\,\}$, reorder the numbers $\{\,\lambda_p\ ;\ p \in P_{\zeta,\varepsilon}\,\}$ in the decreasing order and put it as $\alpha_{\zeta,\varepsilon} := (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}$, and also put $\mu := (\mu_\zeta)_{\zeta\in\widehat{T}}$. Then,

$$\text{(10.3)} \qquad \sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{\,0,1\,\}} \|\alpha_{\zeta,\varepsilon}\| \ + \ \|\mu\| \ \le \ 1\,,$$

which is nothing but the condition (5.7) in the case of a finite group T. In the case of infinite T, if the inequality < 1 holds, the continuity at $g = e$ of $F_\infty(g) = \lim_{N\to\infty} F^{G_N}(g)$ is lost since $F_\infty(e) = 1$, and so we only pick up the cases for which the equality holds here or (5.4) holds. As a pointwise limit F_∞ of the series of centralizations F^{G_N}, we obtain the factorizable positive definite class function f_A with $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{\,0,1\,\}}\,;\,\mu\big)$ in Theorem 5.1.

11. Partial centralization with respect to $D_{J_N}(T) \subset G_N$

As an increasing sequence $G_N \nearrow G = \mathfrak{S}_\infty(T)$ of subgroups, we have chosen $G_N = \mathfrak{S}_{J_N}(T) = D_{J_N}(T) \rtimes \mathfrak{S}_{J_N}$ with $J_N \nearrow \boldsymbol{N}$. Put $D_N = D_{J_N}(T)$ and $S_N = \mathfrak{S}_{J_N}$ for simplicity, then $G_N = D_N \rtimes S_N$, and we identify $d' \in D_N$ and $\sigma' \in S_N$ with their images in G_N respectively. Our task is to calculate centralizations F^{G_N} of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$, and to determine their limits. From the formula (8.2) for F^{G_N} and the explicit form of the subgroup H in (9.1)–(9.2), we see that for $h \in H$

$$\text{(11.1)} \qquad \begin{aligned} F^{G_N}(h) &= \int_{G_N} f_\pi(g'hg'^{-1})\, d\mu_{G_N}(g') \\ &= \frac{1}{|S_N|} \sum_{\sigma'\in S_N\,:\,\sigma'h\sigma'^{-1}\in H} \widetilde{f_\pi}(\sigma'h\sigma'^{-1}), \end{aligned}$$

where $\widetilde{f_\pi}$ is a *partial centralization* of f_π with respect to $D_N \cong T^{J_N}$ defined as

$$\text{(11.2)} \qquad \widetilde{f_\pi}(h') = \int_{D_N} f_\pi(d'h'd'^{-1})\, d\mu_{D_N}(d') \qquad (h' \in H),$$

with the normalized Haar measure μ_{D_N} on D_N.

Note that for a finite number of $h' \in H$, the partial centralization $\widetilde{f_\pi}(h')$ is stable as N is sufficiently large. To calculate it, we apply the explicit form of representation π of H given in (9.3)–(9.5). Then the calculations go on just as in the case of a finite T in [HH3], and we get the following result.

Proposition 11.1. *Take a $g = (d, \sigma)$ from H and let*

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m, \quad \xi_q = \big(t_q, (q)\big), \quad g_j = (d_j, \sigma_j),$$

be its standard decomposition. Then, the partial centralization $\widetilde{f_\pi}(g)$ of matrix element f_π is given as follows. Let $K(\zeta)$ be the set of $k\,(1 \le k \le r)$ such that $\xi_{q_k} \in H_p$ with $p \in \bigsqcup_{\varepsilon\in\{0,1\}} P_{\zeta,\varepsilon}$ or $\xi_{q_k} \in H_\zeta$, and $J(\zeta,\varepsilon)$ be the set of $j\,(1 \le j \le m)$ such that $g_j = (d_j, \sigma_j) \in H_p$ with $p \in P_{\zeta,\varepsilon}$. Then,

$$\widetilde{f_\pi}(g) = \left(\prod_{\zeta\in\widehat{T}} \prod_{k\in K(\zeta)} \frac{\chi_\zeta(t_{q_k})}{\dim\zeta}\right) \times \tag{11.3}$$
$$\times \left(\prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \prod_{j\in J(\zeta,\varepsilon)} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}}\,\mathrm{sgn}(\sigma_j)^{\varepsilon}\right),$$

where, for $\sigma_j = (i_1\ i_2\ \ldots\ i_{\ell_j})$ with $\ell_j = \ell(\sigma_j)$ and $d_j = (t_i)_{i\in K_j}$ with $K_j := \mathrm{supp}(\sigma_j)$,

$$P_{\sigma_j}(d_j) := \big[t'_{\ell_j} t'_{\ell_j-1}\cdots t'_2 t'_1\big] \in T/\!\sim \qquad \text{with} \quad t'_k = t_{i_k}.$$

12. Limits of centralizations of positive definite functions

We are now on the way of calculating centralizations of F^{G_N} of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$ with respect to $G_N = D_{J_N}(T) \rtimes \mathfrak{S}_{J_N}$, and to determine their limits. Recall the formula (11.1) as

$$F^{G_N}(g) = \frac{1}{|S_N|} \sum_{\tau\in S_N\,:\,\tau g\tau^{-1}\in H} \widetilde{f_\pi}(\tau g \tau^{-1}) \qquad (g \in H), \tag{12.1}$$

where $S_N = \mathfrak{S}_{J_N}$, and the partial centralization $\widetilde{f_\pi}$ is calculated as in (11.3).

12.1. Limit of centralizations for a 'monomials' term

For any element in G, there exists an element in H conjugate to it. Therefore it is enough for us to determine the value F^{G_N} on H. Take $g = (d, \sigma) \in H$ and let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$, $\xi_q = \big(t_q, (q)\big)$, $g_j = (d_j, \sigma_j)$, be its standard decomposition. Put $P = \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}_0\times\{0,1\}} P_{\zeta,\varepsilon}$ with a countable subset $\widehat{T}_0 \subset \widehat{T}$, then,

$$H = \left({\prod_{p\in P}}' H_p\right) \times \left({\prod_{\zeta\in\widehat{T}_0}}' H_\zeta\right),$$

and the condition $g \in H$ means that each ξ_{q_k} belongs to one of H_p and H_ζ, and that each g_j belongs to one of H_p. Furthermore, the latter condition can be expressed by means of supports as

$$\begin{cases} \mathrm{supp}(\xi_{q_k}) = \{\, q_k \,\} \subset I_p \ \text{ or } \ \subset I_\zeta, \\ K_j = \mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j) \subset I_p. \end{cases} \tag{12.2}$$

For $\zeta \in \widehat{T}$ and $p \in P_{\zeta,\varepsilon} \subset P$, we put

$$X_\zeta(\xi_q) = \frac{\chi_\zeta(t_q)}{\dim \zeta},\ X_p(\xi_q) = \frac{\chi_\zeta(t_q)}{\dim \zeta},\ X_p(g_j) = \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}}\,\mathrm{sgn}(\sigma_j)^\varepsilon, \tag{12.3}$$

where $\xi_q = (t_q, (q))$ with $t_q \in T$ and $g_j = (d_j, \sigma_j)$. Then the formula (11.3) for $\widetilde{f_\pi}(g)$ is rewritten as

$$\widetilde{f_\pi}(g) = \prod_{\zeta \in \widehat{T}} \left(\prod_{k\,:\,q_k \in I_\zeta} X_\zeta(\xi_{q_k}) \right) \times \prod_{p \in P} \left(\prod_{k\,:\,q_k \in I_p} X_p(\xi_{q_k}) \prod_{j\,:\,K_j \subset I_p} X_p(g_j) \right), \tag{12.4}$$

where $1 \le k \le r, 1 \le j \le m$. The term corresponding to ζ in the first product comes from $\xi_{q_k} \in H_\zeta$, and the term corresponding to $p \in P$ in the second product comes from ξ_{q_k} and g_j in H_p.

Let $Q(g, I_\zeta)$ be the union of supports $q_k = \mathrm{supp}(\xi_{q_k}) \in I_\zeta$, and $QK(g, I_p)$ be the union of supports $q_k = \mathrm{supp}(\xi_{q_k}) \in I_p$ and $K_j = \mathrm{supp}(g_j) \subset I_p$. Since $g \in H$, they give a partition of $\mathrm{supp}(g)$. Let their orders be $n(\zeta)$ and $n(p)$ respectively, then

$$\begin{gathered} \left(\bigsqcup_{\zeta \in \widehat{T}} Q(g, I_\zeta) \right) \bigsqcup \left(\bigsqcup_{p \in P} QK(g, I_p) \right) \ = \ \mathrm{supp}(g), \\ \sum_{\zeta \in \widehat{T}} n(\zeta) \ + \ \sum_{p \in P} n(p) \ = \ |\mathrm{supp}(g)|. \end{gathered} \tag{12.5}$$

Now, for $\tau \in \mathfrak{S}_\infty$, put ${}^\tau g = \tau g \tau^{-1}$, ${}^\tau \xi_q = \tau \xi_q \tau^{-1}$, and ${}^\tau g_j = \tau g_j \tau^{-1}$. Then, the standard decomposition of ${}^\tau g$ into mutually disjoint basic elements is given as

$${}^\tau g = {}^\tau\xi_{q_1} {}^\tau\xi_{q_2} \cdots {}^\tau\xi_{q_r} {}^\tau g_1 {}^\tau g_2 \cdots {}^\tau g_m,\ {}^\tau\xi_q = \big(t_q, (\tau(q))\big),\ {}^\tau g_j = (\tau(d_j), \tau\sigma_j\tau^{-1}).$$

For ξ_q, we have $X_p({}^\tau\xi_q) = X_p(\xi_q)$ if ${}^\tau\xi_q$ is still in H_p, or equivalently if $\tau(q) \in I_p$. For $d_j = (t_i)_{i \in K_j}$, recall that

$$\tau(d_j) = (t_{\tau^{-1}(i')})_{i' \in \tau(K_j)} \quad \text{and} \quad P_{\tau\sigma_j\tau^{-1}}(\tau(d_j)) = P_{\sigma_j}(d_j)$$

and so $X_p({}^\tau g_j) = X_p(g_j)$ if ${}^\tau g_j$ is still in H_p, or equivalently if $\tau(K_j) \subset I_p$.

Let us now consider a partial sum of (12.1), where $\tau \in S_N = \mathfrak{S}_{J_N}$ runs over all such elements that it preserves $Q_\zeta := Q(g, I_\zeta)$ and $QK_p := QK(g, I_p)$

inside of I_ζ and I_p respectively. Suppose that N is sufficiently large so that g is contained in $H \cap G_N$, then this condition on $\tau \in S_N$ is written as

$$\tau(Q_\zeta) \subset I_\zeta \cap J_N, \quad \tau(QK_p) \subset I_p \cap J_N. \tag{12.6}$$

Put $\mathcal{Q} := \{\, Q_\zeta \ (\zeta \in \widehat{T}),\ QK_p \ (p \in P)\,\}$, and denote by $\mathcal{T}(\mathcal{Q}, N)$ the set of $\tau \in S_N = \mathfrak{S}_{J_N}$ satisfying the condition (12.6). Then, for $\tau \in \mathcal{T}(\mathcal{Q}, N)$, we see from the above consideration that $\widetilde{f_\pi}({}^\tau g) = \widetilde{f_\pi}(g)$ for ${}^\tau g = \tau g \tau^{-1}$. Therefore the partial sum over $\tau \in \mathcal{T}(\mathcal{Q}, N)$ is calculated as

$$\frac{1}{|S_N|} \sum_{\tau \in \mathcal{T}(\mathcal{Q},N)} \widetilde{f_\pi}({}^\tau g) = \frac{|\mathcal{T}(\mathcal{Q}, N)|}{|J_N|!}\, \widetilde{f_\pi}(g). \tag{12.7}$$

Under the asymptotic condition (10.1) or

$$\frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \ \ (p \in P), \quad \frac{|I_\zeta \cap J_N|}{|J_N|} \to \mu_\zeta \ \ (\zeta \in \widehat{T}), \tag{12.8}$$

let us calculate the limit as $N \to \infty$. Similarly as in [Hir4] and [HH2]–[HH3], we can calculate the order $|\mathcal{T}(\mathcal{Q}, N)|$ and obtain by (12.5)

$$\frac{|\mathcal{T}(\mathcal{Q}, N)|}{|J_N|!} \longrightarrow \prod_{\zeta \in \widehat{T}} \mu_\zeta^{\,n(\zeta)} \times \prod_{p \in P} \lambda_p^{\,n(p)}. \tag{12.9}$$

Applying the formulas (12.7) and (12.4), we obtain

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{|S_N|} \sum_{\tau \in \mathcal{T}(\mathcal{Q},N)} \widetilde{f_\pi}({}^\tau g) &= \lim_{N\to\infty} \widetilde{f_\pi}(g)\, \frac{|\mathcal{T}(\mathcal{Q}, N)|}{|J_N|!} \\ = \prod_{\zeta \in \widehat{T}} \Bigg(\prod_{k\,:\,q_k \in I_\zeta} \mu_\zeta\, X_\zeta(\xi_{q_k}) \Bigg) &\cdot \prod_{p \in P} \Bigg(\prod_{k\,:\,q_k \in I_p} \lambda_p X_p(\xi_{q_k}) \prod_{j\,:\,K_j \subset I_p} \lambda_p^{\ell(\sigma_j)} X_p(g_j) \Bigg), \end{aligned}$$

where the product over $\zeta \in \widehat{T}_0$ and that over $p \in P$ are actually finite, and for $p \in P_{\zeta,\varepsilon}$, $(\zeta, \varepsilon) \in \widehat{T}_0 \times \{0, 1\}$,

$$\lambda_p X_p(\xi_{q_k}) = \frac{\lambda_p}{\dim \zeta} \chi_\zeta(t_{q_k}), \ \ \lambda_p^{\ell(\sigma_j)} X_p(g_j) = \left(\frac{\lambda_p}{\dim \zeta} \right)^{\ell(\sigma_j)} \chi_\zeta\big(P_{\sigma_j}(d_j)\big) \mathrm{sgn}(\sigma_j)^\varepsilon.$$

The above calculation for a partial sum over $\tau \in \mathcal{T}(\mathcal{Q}, N) \subset \mathfrak{S}_{J_N}$ can be applied to other partial sums. These partial sums come from possible cases of ${}^\tau g$ such that $\mathrm{supp}({}^\tau \xi_{q_k}) = \tau(q_k)$ belongs to which of I_ζ or I_p, and that $\mathrm{supp}({}^\tau g_j) = \tau(K_j)$ is contained in which of I_p. All these cases give us similarly as above limits of centralizations, and they corresponds altogether exactly all the ‘monomial’ terms of the expansion of the right hand side of (5.5) in Theorem 5.1 into ‘monomials’ as explained below.

12.2. Summing up all 'monomial' terms to the whole formula

For $\zeta \in \widehat{T}_0$ and $p \in P$, we see from (12.3) that

$$|X_\zeta(\xi_q)| \le 1\,, \quad |X_p(\xi_q)| \le 1\,, \quad |X_p(g_j)| \le \frac{1}{(\dim\zeta)^{\ell(\sigma_j)-1}} \le 1. \tag{12.10}$$

By the equality in (10.3) or by (5.4) we have

$$\sum_{\zeta\in\widehat{T}_0} \mu_\zeta + \sum_{p\in P} \lambda_p = 1, \tag{12.11}$$

and, letting $\{\alpha_{\zeta,\varepsilon,i}; i \in \boldsymbol{N}\}$ be a reordering of $\{\lambda_p; p \in P_{\zeta,\varepsilon}\}$, we have from the formula (5.5) of $f_A(g)$,

$$\begin{aligned} f_A(g) = \prod_{1\le k\le r} \left(\sum_{p\in P} \lambda_p X_p(\xi_{q_k}) + \sum_{\zeta\in\widehat{T}_0} \mu_\zeta\, X_\zeta(\xi_{q_k}) \right) \times \\ \times \prod_{1\le j\le m} \left(\sum_{p\in P} \lambda_p^{\ell(\sigma_j)} X_p(g_j) \right). \end{aligned} \tag{12.12}$$

Note that by (12.10) each multiplicative factor in (12.12) is evaluated in its absolute value as ≤ 1.

Let $\mathcal{P}_m$ be the set of all partitions $\delta = \{J_p\ (p \in P)\}$ indexed by P of the set of indices $j \in \boldsymbol{I}_m = \{1, 2, \ldots, m\}$ of g_j's, and $\mathcal{Q}_r$ be the set of all partitions $\gamma = \{K_\zeta\ (\zeta \in \widehat{T}_0), K_p\ (p \in P)\}$ indexed by $\widehat{T}_0 \cup P$ of the set of indices $k \in \boldsymbol{I}_r$ of ξ_{q_k}'s. Put $\gamma\cdot\delta := \{\, K_\zeta\ (\zeta \in \widehat{T}_0),\ K_p,\ J_p\ (p \in P)\,\}$, and let $\mathcal{KJ}$ be the set of all these $\gamma\cdot\delta$. Then the expansion of $f_A(g)$ of the right hand side of (12.12) into monomial terms are parametrized by the set $\gamma\cdot\delta \in \mathcal{KJ}$ as

$$f_A(g) = \sum_{\gamma\cdot\delta\in\mathcal{KJ}} \Xi_{\gamma\cdot\delta}(g), \tag{12.13}$$

with 'monomial terms' $\Xi_{\gamma\cdot\delta}(g)$ given as

$$\prod_{\zeta\in\widehat{T}_0} \prod_{k\in K_\zeta} X_\zeta(\xi_{q_k}) \cdot \prod_{p\in P} \left(\prod_{k\in K_p} X_p(\xi_{q_k}) \prod_{j\in J_p} X_p(g_j) \right) \times \prod_{k\in K_\zeta} \mu_\zeta^{n(\zeta)} \prod_{p\in P} \lambda_p^{n(p)}.$$

Here the product over $\zeta \in \widehat{T}_0$ and that over $p \in P$ are actually finite, and

$$n(\zeta) = |K_\zeta|, \quad n(p) = |K_p| + \textstyle\sum_{j\in J_p} \ell(\sigma_j) = |K_p| + \sum_{j\in J_p} |\mathrm{supp}(g_j)|.$$

Now we come back to the centralization F^{G_N} in (12.1). Take $\gamma\cdot\delta := \{\, K_\zeta\ (\zeta \in \widehat{T}_0),\ K_p,\ J_p\ (p \in P)\,\}$, and put

$$Y^N_{\gamma\cdot\delta}(g) := \frac{1}{|S_N|} \sum_{\tau\in\mathcal{T}(\gamma\cdot\delta)} \widetilde{f_\pi}({}^\tau g) \qquad \text{with}$$

$$\mathcal{T}(\gamma\cdot\delta) := \{\tau \in S_N; {}^\tau\xi_{q_k} \in H_\zeta\ (k \in K_\zeta), {}^\tau\xi_{q_k} \in H_p\ (k \in K_p), {}^\tau g_j \in H_p\ (j \in J_p)\}.$$

Then, by a similar calculation as in **12.1**, we have

$$F^{G_N}(g) = \sum_{\gamma\cdot\delta\in\mathcal{KJ}} Y^N_{\gamma\cdot\delta}(g), \tag{12.14}$$

$$Y^N_{\gamma\cdot\delta}(g) = \prod_{\zeta\in\widehat{T}_0}\prod_{k\in K_\zeta} X_\zeta(\xi_{q_k})\cdot\prod_{p\in P}\left(\prod_{k\in K_p} X_p(\xi_{q_k})\prod_{j\in J_p} X_p(g_j)\right)\times\frac{C^N_{\gamma\cdot\delta}}{|J_N|!},$$

where $C^N_{\gamma\cdot\delta} := |\mathcal{T}(\gamma\cdot\delta)|$. Note that $\mathcal{T}(\gamma\cdot\delta)$ is defined by the following condition on $\tau\in S_N$:

$${}^{\tau}q_k \in I_\zeta\cap J_N\ (k\in K_\zeta),\ {}^{\tau}q_k\in I_p\cap J_N\ (k\in K_p),\ {}^{\tau}(\mathrm{supp}(g_j))\subset I_p\cap J_N\ (j\in J_p).$$

Then, similarly as in **12.1**, the order $C^N_{\gamma\cdot\delta} = |\mathcal{T}(\gamma\cdot\delta)|$ is given as $|\mathcal{T}(\mathcal{Q},N)|$, and as in (12.9)

$$\frac{C^N_{\gamma\cdot\delta}}{|J_N|!} \longrightarrow \prod_{\zeta\in\widehat{T}}\lambda_\zeta^{n(\zeta)}\times\prod_{p\in P}\lambda_p^{n(p)}. \tag{12.15}$$

We note that, for $\mathcal{Q} = \{Q_\zeta\,(\zeta\in\widehat{T}),\ QK_p\,(p\in P)\,\}$ in **12.1**, there corresponds a $\gamma\cdot\delta = \{\,K_\zeta\ (\zeta\in\widehat{T}_0),\ K_p,\ J_p\ (p\in P)\,\}$ given by

$$K_\zeta = \{k\in \boldsymbol{I}_r;\ \xi_{q_k}\in H_\zeta\},\ K_p = \{k\in\boldsymbol{I}_r;\ \xi_{q_k}\in H_p\},\ J_p = \{j\in\boldsymbol{I}_m;\ g_j\in H_p\}.$$

Now we can prove the following theorem, a half of Theorem 5.1.

Theorem 12.1. *Let T be a compact group. Let f_A be the class function on $G=\mathfrak{S}_\infty(T)$ given by the formula* (5.5) *in Theorem* 5.1*, with parameter*

$$A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{\,0,1\,\}}\,;\,\mu\big)$$

in (5.3)*. If the parameter A satisfies the equality condition* (5.4)*, then f_A is obtained as a limit of centralizations of a positive definite function $F=\mathrm{Ind}_H^G f_\pi$ with (H,π) given above. The limit is taken according to an increasing sequence of subgroups $G_N=\mathfrak{S}_{J_N}(T)$ with $J_N\nearrow\boldsymbol{N}$ obeying the asymptotic condition* (12.8)*.*

Let $E'(G)$ be the set of all f_A's under the condition (5.4)*, then $E'(G)\subset F(G)$.*

Proof. Note that the condition (5.4) is nothing but (12.11). Under this condition we evaluate $|f_A(g)-F^{G_N}(g)|$ as follows. Let $\varepsilon_n\searrow 0$ $(n\to\infty)$ be a decreasing sequence of positive numbers. Let $\widehat{T}_n\subset\widehat{T}_0$ and $P_n\subset P$ be finite subsets such that

$$\sum_{\zeta\in\widehat{T}_n}\mu_\zeta + \sum_{p\in P_n}\lambda_p > 1-\varepsilon_n. \tag{12.16}$$

(1) Put $\lambda_{\zeta,N} = \dfrac{|I_\zeta \cap J_N|}{|J_N|}$, $\lambda_{p,N} = \dfrac{|I_p \cap J_N|}{|J_N|}$, then

$$\sum_{\zeta \in \widehat{T}_0} \mu_{\zeta,N} + \sum_{p \in P} \lambda_{p,N} = 1.$$

Since $\mu_{\zeta,N} \to \lambda_\zeta, \lambda_{p,N} \to \lambda_p$ $(N \to \infty)$ by assumption, we can take N_n sufficiently large so that for any $N \geq N_n$

$$\sum_{\zeta \in \widehat{T}_n} |\mu_\zeta - \mu_{\zeta,N}| + \sum_{p \in P_n} |\lambda_p - \lambda_{p,N}| \;<\; \varepsilon_n. \tag{12.17}$$

Then we have

$$\sum_{\zeta \notin \widehat{T}_n} \mu_\zeta + \sum_{p \notin P_n} \lambda_p \;<\; \varepsilon_n, \quad \sum_{\zeta \notin \widehat{T}_n} \mu_{\zeta,N} + \sum_{p \notin P_n} \lambda_{p,N} \;<\; 2\varepsilon_n. \tag{12.18}$$

(2) Let $\mathcal{KJ}_n$ be the set of $\gamma \cdot \delta = \{\, K_\zeta \ (\zeta \in \widehat{T}_0),\ K_p,\ J_p \ (p \in P)\,\}$ such that $K_\zeta = \emptyset$ for $\zeta \notin \widehat{T}_n$, and $K_p = J_p = \emptyset$ for $p \notin P_n$. Then $\mathcal{KJ}_n$ is finite. In the formula (12.13) of $f_A(g)$, we divide the sum over $\gamma \cdot \delta \in \mathcal{KJ}$ of $\Xi_{\gamma\cdot\delta}(g)$ into two cases depending on $\gamma \cdot \delta \in \mathcal{KJ}_n$ or not as

$$f_A(g) = f_A^n(g) + f_A^{\sharp n}(g), \tag{12.19}$$

with $$f_A^n(g) := \sum_{\gamma\cdot\delta \in \mathcal{KJ}_n} \Xi_{\gamma\cdot\delta}(g), \quad f_A^{\sharp n}(g) := \sum_{\gamma\cdot\delta \notin \mathcal{KJ}_n} \Xi_{\gamma\cdot\delta}(g).$$

Similarly, in the formula (12.14) of $F^{G_N}(g)$, we divide the sum over $\gamma \cdot \delta \in \mathcal{KJ}$ of $Y^N_{\gamma\cdot\delta}(g)$ into two cases according as $\gamma \cdot \delta \in \mathcal{KJ}_n$ or not as above, and express F^{G_N} as

$$F^{G_N}(g) = F^{G_N,n}(g) + F^{G_N,\sharp n}(g), \tag{12.20}$$

Then we have

$$|f_A(g) - F^{G_N}(g)| \;\leq\; |f_A^n(g) - F^{G_N,n}(g)| \;+\; |f_A^{\sharp n}(g)| \;+\; |F^{G_N,\sharp n}(g)|.$$

(3) We denote by $R_{1,N}, R_{2,N}$ and $R_{3,N}$ the first, the second and the third term in the right hand side respectively. Then $R_{1,N}$ is a finite sum of the terms $\Xi_{\gamma\cdot\delta}(g) - Y^N_{\gamma\cdot\delta}(g)$ each of which tends to 0 as $N \to \infty$. So we can choose $N'_n \geq N_n$ such that, for any $N \geq N'_n$, we have $R_{1,N} < \varepsilon_n$.

For the second term $R_{2,N}$, using the evaluation (12.10) and the note just after (12.12), we get

$$R_{2,N} \leq \sum_{1 \leq k \leq r} \left(\sum_{p \notin P_n} \lambda_p + \sum_{\zeta \notin \widehat{T}_n} \mu_\zeta \right) + \sum_{1 \leq j \leq m} \left(\sum_{p \notin P_n} \lambda_p^{\ell(\sigma_j)} \right) \;<\; (r+m)\varepsilon_n.$$

For the third term $R_{3,N}$, first evaluate each $|Y^N_{\gamma\cdot\delta}(g)|$ as

$$|Y^N_{\gamma\cdot\delta}(g)| \;\le\; C^N_{\gamma\cdot\delta}/|J_N|! \le C \cdot \prod_{\zeta\in\widehat{T}_0} \lambda_{\zeta,N}{}^{n(\zeta)} \cdot \prod_{p\in P} \lambda_{p,N}{}^{n(p)},$$

where C is a constant, for example, we can take $C = 2^{|\mathrm{supp}(g)|}$ if $N \ge 2|\mathrm{supp}(g)|$. Then, a similar evaluation as that for $R_{2,N}$ (using $\lambda_{\zeta,N}, \lambda_{p,N}$ instead of λ_ζ, λ_p respectively) gives us $R_{3,N} < C\,(r+m)\cdot 2\varepsilon_n$.

Thus altogether we get for any $N \ge N'_n$,

$$|f_A(g) - F^{G_N}(g)| \;<\; \{1 + (r+m) + 2C(r+m)\}\,\varepsilon_n.$$

This completes the proof of Theorem 12.1. $\square$

13. Determination of the region of parameters for $f \in F(G)$

For each $f \in F(G)$, there corresponds an element $s \in S'$ such that $f = f_s = \Psi_s \circ \Phi$ or $s = s(f)$ as in **4.2.** As the final step of the proof of Theorem 5.1, we specify the range of the parameter $s = (s_\omega)_{\omega\in\Omega}$.

13.1. Values of $s(f) = (s_\omega)_{\omega\in\Omega}$ and the compactness of $F(G) = E(G)$

The functions f_A in Theorem 5.1 with parameter

$$A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$$

in (5.3)–(5.4), are given as limits of centralizations of positive definite function $F = \mathrm{Ind}_H^G f_\pi$ with a matrix element f_π of a UR of a subgroup H of $G = \mathfrak{S}_\infty(T)$. They prepare a big family $E'(G)$ of continuous *factorizable* positive definite class functions on G, or $E'(G) \subset F(G)$, by the results in Sections 9–12.

We see that, for $f = f_A$, the parameter $s(f) = s = (s_\omega)_{\omega\in\Omega}$ in (4.13) is given by absolutely convergent sums as follows: for $\omega = ([t],\ell) \in (T/\!\sim) \times \boldsymbol{N}$,

$$\text{(13.1)}\quad \text{for}\quad \ell = 1,\quad s_\omega = \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} \;+\; \frac{\mu_\zeta}{\dim\zeta} \right) \chi_\zeta(t)\,,$$

$$\text{(13.2)}\quad \text{for}\quad \ell \ge 2,\quad s_\omega = \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \left(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\right)^{\ell} (-1)^{\varepsilon(\ell-1)} \right) \chi_\zeta(t).$$

We put $s_{\omega_0} = 1$ in accordance with (5.4).

Compairing the images of $E'(G)$ and $F(G)$ under the map $f \mapsto s(f)$, we obtain the following result.

Theorem 13.1. *A factorizable positive definite class function $f \in F(G)$ on $G = \mathfrak{S}_\infty(T)$, normalized as $f(e) = 1$, is given in the form of f_A in the formula* (5.5) *in Theorem* 5.1, *with parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$ in* (5.3) *satisfying the condition* (5.4), *or $E'(G) \supset F(G)$. Actually we have $E'(G) = F(G)$.*

As a consequence of the formula (13.1)–(13.2) and the fact that $E'(G) = F(G)$ in the above proposition, and also the fact $E(G) = F(G)$ in Section 4, we obtain the following important theorem on the topology of the space of characters G (cf. [Far]). Let $S' = \{\, s \in S = \prod_{\omega\in\Omega} D_\omega \,;\, \jmath(s) = \overline{s}\}$ be as in (4.12) in **4.2**.

Theorem 13.2. *Let $G = \mathfrak{S}_\infty(T)$ with T a comapct group.*

(i) *Let $S'' \subset S'$ be the image of the set of normalized continuous factorizable positive definite class functions $F(G)$ under the map $f \to s(f) = (s_\omega)_{\omega\in\Omega}$. It is given by* (13.1)–(13.2) *from the parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$ in* (5.3)–(5.4) *of $f_A \in E'(G) = F(G)$. The topology τ_{cu} of compact uniform convergence on $F(G)$ is transformed to the direct product topology τ_{prod} on S'', and $F(G)$ and S'' are compact.*

(ii) *The space of normalized characters $E(G)$ is equal to $F(G)$, and the same statements as in* (i) *for $F(G)$ hold for $E(G)$. In particular, $E(G)$ is compact in $K_{\le 1}(G)$.*

13.2. First step of the proof of Theorem 13.1

The proof of Theorem 13.1 will continue until the end of this section.

We examine a positive definite class function $f = f_s = \Psi_s \circ \Phi \in F(G)$, and study the range of $s(f), f \in F(G)$, in S'. Define a class function on the compact group T by putting

$$X(t) = s_{([t],1)} \quad \text{for} \quad t \in T, \tag{13.3}$$

where $s_{([t],1)} = s_\omega$ for $\omega = ([t],1) \in \Omega$. Then, X is a continuous positive definite class function on T.

Lemma 13.3. *A continuous positive definite class function X on T, normalized as $X(e_T) = 1$, is expressed as an absolutely and uniformly convergent linear combination of $\chi_\zeta, \zeta \in \widehat{T}$, as*

$$X(t) = \sum_{\zeta\in\widehat{T}} a_\zeta\, \chi_\zeta(t) \ \ (t \in T), \quad a_\zeta = \int_T X(t)\, \overline{\chi_\zeta(t)}\, d\nu_T(t) \,\ge\, 0, \tag{13.4}$$

$$\sum_{\zeta\in\widehat{T}} a_\zeta \dim\zeta \;=\; 1\,. \tag{13.5}$$

Proof. Note that in the sum in (13.4), there appear at most countably infinite $\zeta \in \widehat{T}$. The sum is convergent in $L^2(T)$, and so convergent weakly. If the set of finite partial sums evaluated at e_T, $X_F(e_T) = \sum_{\zeta\in F} a_\zeta \chi_\zeta(e_T) = \sum_{\zeta\in F} a_\zeta \dim\zeta$, $F \subset \widehat{T}$ finite, is bounded, then the sum in (13.4) is absolutely and uniformly convergent (and the limit function should coincide with the weak limit $X(t)$).

In fact, $a_\zeta \ge 0$ and the boundedness means the convergence of of the sum of non-negative numbers $\sum_{\zeta\in\widehat{T}} a_\zeta \dim\zeta$, and so the absolute and uniform convergence of $\sum_{\zeta\in F} a_\zeta\, \chi_\zeta(t)$ since $|\chi_\zeta(t)| \le \chi_\zeta(e_T) = \dim\zeta$.

Let us prove that the set of finite partial sums X_F, $F \subset \widehat{T}$ finite, is bounded. We see that $X \succeq X_F$ or X majorizes X_F as positive definite functions. In fact, since X and X_F are both class functions, it is enough to check that, with $f = X - X_F$, $f(\psi^* * \psi) \geq 0$ for any invariant continuous function ψ on T. On the other hand, any invariant $\psi \in C(T)$ can be approximated uniformly by a finite linear combination $\varphi = \sum_{\zeta \in F'} c_\zeta \chi_\zeta$, $F' \subset \widehat{T}$ finite, of irreducible characters χ_ζ's. By calculation, we get $f(\varphi^* * \varphi) = \sum_{\zeta \in F' \setminus F} a_\zeta |c_\zeta|^2 \geq 0$.

Since $X \succeq X_F$, we have $X(e_T) \geq X_F(e_T) = \sum_{\zeta \in F} a_\zeta \dim \zeta$, the boundedness of $X_F(e_T)$, as desired. $\square$

For $\ell \geq 2$, we define also a continuous class function $Y_\ell(t)$ on T by putting

$$Y_\ell(t) = s_{([t],\ell)} \quad (t \in T), \tag{13.6}$$

where $s_{([t],\ell)} = s_\omega$ for $\omega = ([t], \ell) \in \Omega$. Then, similarly as for X, it is expressed as

$$Y_\ell(t) = \sum_{\zeta \in \widehat{T}} b_{\zeta,\ell}\, \chi_\zeta(t) \ \ (t \in T), \quad b_{\zeta,\ell} = \int_T Y_\ell(t)\, \overline{\chi_\zeta(t)}\, d\nu_T(t). \tag{13.7}$$

The uniform convergence of the above some will be guranteed on the way of discussion (see e.g., (13.11) in **13.5**). For $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$, we have from (4.1) and (4.15)

$$f(g) = \prod_{i \leq k \leq r} \left(\sum_{\zeta \in \widehat{T}} a_\zeta\, \chi_\zeta(t_{q_k}) \right) \cdot \prod_{1 \leq j \leq m} \left(\sum_{\zeta \in \widehat{T}} b_{\zeta,\ell(\sigma_j)}\, \chi_\zeta(P_{\sigma_j}(d_j)) \right). \tag{13.8}$$

13.3. Elementary positive definite class functions $F_{\zeta,\varepsilon} \in F(G)$

For a fixed $\zeta \in \widehat{T}$, we extract from f its part relating to ζ. To do so, we utilize an elementary element $F_{\zeta,\varepsilon} \in F(G)$ which is given as follows.

We take an IUR ρ_n of a degenerate form of $G_n = \mathfrak{S}_n(T) = D_{\boldsymbol{I}_n}(T) \rtimes \mathfrak{S}_n$ as follows, where $\boldsymbol{I}_n = \{1, 2, \ldots, n\}$. Define tensor product representation of $D_{\boldsymbol{I}_n}(T)$ as $\otimes_{i \in \boldsymbol{I}_n} \zeta_i$ with $\zeta_i = \zeta$ for $T_i = T$ $(i \in \boldsymbol{I}_n)$, for which the representation space is $V_n = \otimes_{i \in \boldsymbol{I}_n} V(\zeta_i)$. For $\sigma \in \mathfrak{S}_n$, put $I(\sigma)\big(\otimes_{i \in \boldsymbol{I}_n} v_i\big) := \otimes_{i \in \boldsymbol{I}_n} v_{\sigma^{-1}(i)}$ with $v_i \in V(\zeta_i)$. Then, for $(\zeta, \varepsilon) \in \widehat{T} \times \{0, 1\}$, we put for $g = (d, \sigma) \in G_n, d = (t_i)_{i \in \boldsymbol{I}_n}$,

$$\rho_n(g) = \rho_n\big((d,\sigma)\big) := \big(\otimes_{i \in \boldsymbol{I}_n} \zeta_i(t_i)\big) I(\sigma)\, \mathrm{sgn}(\sigma)^\varepsilon. \tag{13.9}$$

Take a $g \in G$. Then, starting from a certain n, g belongs to G_n, and so we can consider the limit of normalized trace characters as $\lim_{n \to \infty} \widetilde{\chi}_{\rho_n}(g)$ with $\widetilde{\chi}_{\rho_n}(g) = \mathrm{tr}\big(\rho_n(g)\big) / \dim \rho_n$.

Lemma 13.4. *Let ρ_n be an IUR of $G_n = \mathfrak{S}_n(T)$ constructed from $(\zeta, \varepsilon) \in \widehat{T} \times \{0, 1\}$ as above. Then, there exists a pointwise limit $F_{\zeta,\varepsilon}$ on $G = \mathfrak{S}_\infty(T)$*

of $\widetilde{\chi}_{\rho_n}$ given as follows. For $g = (d,\sigma) \in G$, let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$, $\xi_q = \big(t_q,(q)\big)$, $g_j = (d_j,\sigma_j)$, be a standard decomposition. Then,

$$\begin{aligned} F_{\zeta,\varepsilon}(g) &:= \lim_{n\to\infty} \frac{\mathrm{trace}\big(\rho_n(g)\big)}{\dim \rho_n} \\ &= \prod_{1\le k\le r} \frac{\chi_\zeta(t_{q_k})}{\dim \zeta} \times \prod_{1\le j\le m} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \mathrm{sgn}(\sigma_j)^\varepsilon. \end{aligned} \tag{13.10}$$

The proof is quite similar as in the case of a finite group T in [HH3, Theorem 9].

The positive definite class function $F_{\zeta,\varepsilon} \in F(G)$ is a special case of f_A in Theorem 5.1, for which $\alpha_{\zeta,\varepsilon} = (1,0,0,\ldots)$ and other parameters $\alpha_{\zeta',\varepsilon'}$ and $\mu_{\zeta'}$ are all zero.

13.4. A general lemma and a partial Fourier transform

We prepare here a lemma which has some generlity. Then we apply it to define a kind of *partial Fourier transform* of such an f on $G = \mathfrak{S}_\infty(T)$ with respect to simple functions $F_{\zeta,\varepsilon} \in F(G)$.

Lemma 13.5. *Let D be a compact normal subgroup of a topological group G. For a continuous positive definite function f on G, put*

$$f^o(g) := \int_D f(gd)\, d\mu_D(d),$$

where μ_D denotes the normalized Haar measure on D. Suppose that for each $g \in G$, the automorphism $D \ni d \mapsto gdg^{-1} \in D$ is measure-preserving. Then, f^o gives a continuous positive definite function on the quotient group G/D, and it is also expressed as $f^o(g) := \int_D f(d'g)\, d\mu_D(d')$.

On the other hand, we also note that the product $(f_1f_2)(g) := f_1(g)f_2(g)$ ($g \in G$) of two positive definite functions f_1 and f_2 on a group G is again positive definite.

Now come back to $G = \mathfrak{S}_\infty(T)$. Fix a $(\zeta_0,\varepsilon) \in \widehat{T}\times\{0,1\}$, and take $F_{\zeta_0,\varepsilon} \in F(G)$ in Lemma 13.4. Then the product $f'(g) := \big(f\,\overline{F_{\zeta_0,\varepsilon}}\big)(g) = f(g)\,\overline{F_{\zeta_0,\varepsilon}}(g)$ is positive definite. *A partial Fourier transform* $\mathcal{F}_{\zeta_0,\varepsilon;n}(f)$ of f with respect to $F_{\zeta_0,\varepsilon}$ is by definition the integral of f' with respect to D_n:

$$\mathcal{F}_{\zeta_0,\varepsilon;n}(f)(g) := \int_{D_n} f(d'g)\,\overline{F_{\zeta_0,\varepsilon}}(d'g)\; d\mu_{D_n}(d').$$

By Lemma 13.5, $\mathcal{F}_{\zeta_0,\varepsilon;n}(f)$ gives a positive definite function on $G_n = \mathfrak{S}_n(T)$, and accordingly on $G_n/D_n \cong \mathfrak{S}_n$. Let us calculate $\mathcal{F}_{\zeta_0,0;n}(f)(g)$. Taking multiplicative factors of $F_{\zeta_0,0}$, we put

$$X_{\zeta_0}(t) \;=\; \frac{\chi_{\zeta_0}(t)}{\dim \zeta_0}, \quad Y_{\ell,\zeta_0}(t) \;=\; \frac{\chi_{\zeta_0}(t)}{(\dim \zeta_0)^\ell} \quad (t \in T).$$

Then, by (13.4)–(13.7), we need the following formulas. Firstly,

$$\int_T X(t)\,\overline{X_{\zeta_0}(t)}\,d\nu_T(t) \;=\; \frac{a_{\zeta_0}}{\dim\zeta_0}\,.$$

Secondly, for a basic element (d',σ') with $d'=(t_1,t_2,\ldots,t_\ell)$, $\sigma'=(1\ 2\ \cdots\ \ell)$, we have $P_{\sigma'}(d')=[t_\ell t_{\ell-1}\cdots t_2t_1]$, and therefore

$$\int_{T^\ell}\big(Y_\ell\,\overline{Y_{\ell,\zeta_0}}\big)\big(t_\ell t_{\ell-1}\cdots t_2t_1\big)\;d\nu_T(t_1)d\nu_T(t_2)\,\cdots\,d\nu_T(t_\ell)=\frac{b_{\zeta_0,\ell}}{(\dim\zeta_0)^\ell}\,.$$

Lemma 13.6. *Let $f=f_s$ be a factorizable positive definite class function in* (4.15) *given as $f(g)=\prod_{\omega\in\Omega}{s_\omega}^{n_\omega(g)}$. Let the notations be as in* (13.3)–(13.8)*, then the partial Fourier transform $\mathcal{F}_{\zeta_0,0;n}(f)$ of f with respect to $F_{\zeta_0,0}$ is given as follows: for $\sigma\in\mathfrak{S}_n\cong G_n/D_n$, let $\sigma=\sigma_1\sigma_2\cdots\sigma_m$ be its decomposition into mutually disjoint cycles, then*

$$\mathcal{F}_{\zeta_0,0;n}(f)(\sigma)=\left(\frac{a_{\zeta_0}}{\dim\zeta_0}\right)^{n-|\mathrm{supp}(\sigma)|}\times\prod_{1\le j\le m}\frac{b_{\zeta_0,\ell(\sigma_j)}}{(\dim\zeta_0)^{\ell(\sigma_j)}}\,.$$

13.5. Completion of the proof of Theorem 13.1

For $\sigma\in\mathfrak{S}_\infty$, let $n_\ell(\sigma)$ be the multiplicity of cycles of length ℓ in the standard decomposition of σ into disjoint cycles. For a series of complex numbers $s=(s_1,s_2,\ldots)$, consider a class function α_s^n on each subgroup $\mathfrak{S}_n$ given by

$$\alpha_s^n(\sigma):={s_1}^{n-|\mathrm{supp}(\sigma)|}{s_2}^{n_2(\sigma)}\cdots{s_\ell}^{n_\ell(\sigma)}\quad(\sigma\in\mathfrak{S}_n),$$

where $2\,n_2(\sigma)+3\,n_3(\sigma)+\cdots+\ell\,n_\ell(\sigma)=|\mathrm{supp}(\sigma)|\le n$. Then, Korollar 1 of Satz 2 in [Tho2] says that

($\star$) *The class function α_s^n is positive definite on $\mathfrak{S}_n$ for all $n\ge1$ if and only if there exist series of non-negative real numbers $\alpha=(\alpha_i)_{i\in\boldsymbol{N}}$, $\beta=(\beta_i)_{i\in\boldsymbol{N}}$ with $\|\alpha\|<+\infty$, $\|\beta\|<+\infty$, such that*

$$\|\alpha\|\;+\;\|\beta\|\;\le s_1,\quad s_\ell=\sum_{i\in\boldsymbol{N}}{\alpha_i}^\ell\;+\;(-1)^{\ell-1}\sum_{i\in\boldsymbol{N}}{\beta_i}^\ell\quad(\ell\ge2).$$

In our case, by ($\star$), we have $\alpha=(\alpha_i)_{i\in\boldsymbol{N}}$, $\beta=(\beta_i)_{i\in\boldsymbol{N}}$ such that

$$\|\alpha\|\;+\;\|\beta\|\;\le\;\frac{a_{\zeta_0}}{\dim\zeta_0},$$
$$\sum_{i\in\boldsymbol{N}}{\alpha_i}^\ell\;+\;(-1)^{\ell-1}\sum_{i\in\boldsymbol{N}}{\beta_i}^\ell\;=\;\frac{b_{\zeta_0,\ell}}{(\dim\zeta_0)^\ell}\quad(\ell\ge2).$$

Rearrange α_i's and β_i's in decreasing order and put

$$\begin{aligned}\alpha_{\zeta_0,0,i}&=(\dim\zeta_0)^2\,\alpha_i\;\ge\;0, &\alpha_{\zeta_0,1,i}&\;=\;(\dim\zeta_0)^2\,\beta_i\;\ge\;0,\\ \alpha_{\zeta_0,0}&=(\alpha_{\zeta_0,0,i})_{i\in\boldsymbol{N}}, &\alpha_{\zeta_0,1}&\;=\;(\alpha_{\zeta_0,1,i})_{i\in\boldsymbol{N}},\\ \mu_{\zeta_0}&=(\dim_{\zeta_0})\,a_{\zeta_0}-\|\alpha_{\zeta_0,0}\|-\|\alpha_{\zeta_0,1}\|\;\ge\;0.\end{aligned}$$

Then we have

$$\frac{\|\alpha_{\zeta_0,0}\|}{\dim_{\zeta_0}} + \frac{\|\alpha_{\zeta_0,1}\|}{\dim_{\zeta_0}} + \frac{\mu_{\zeta_0}}{\dim_{\zeta_0}} = a_{\zeta_0},$$

$$\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta_0,0,i}}{\dim\zeta_0}\right)^{\ell} + (-1)^{\ell-1}\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta_0,1,i}}{\dim\zeta_0}\right)^{\ell} = b_{\zeta_0,\ell} \quad (\ell\geq 2). \tag{13.11}$$

Now put $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$ with $\mu = (\mu_\zeta)_{\zeta\in\widehat{T}}$. Then we have from (13.5)

$$\sum_{\zeta\in\widehat{T}}\ \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1,$$

which is nothing but the condition (5.4) guaranteeing the continuity of f_A at $g = e$.

Finally we get the following expression by absolutely convergent sums. For $\omega = ([t],1)$ with $t\in T^*$, the value $s_\omega = f(\xi_q)$ for $\xi_q = (t,(q))$ is given by

$$s_\omega = f(\xi_q) = \sum_{\zeta\in\widehat{T}}\left(\frac{\|\alpha_{\zeta,0}\|}{\dim\zeta} + \frac{\|\alpha_{\zeta,1}\|}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\right)\chi_\zeta(t),$$

and for $\omega = ([t],\ell), \ell\geq 2$, the value $s_\omega = f((d,\sigma))$ for a basic (d,σ), with $P_\sigma(d) = [t]$ and $\ell(\sigma) = \ell$, is given by

$$s_\omega = f((d,\sigma)) = \sum_{\zeta\in\widehat{T}}\left\{\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta,0,i}}{\dim\zeta}\right)^{\ell} + (-1)^{\ell-1}\sum_{i\in\boldsymbol{N}}\left(\frac{\alpha_{\zeta,1,i}}{\dim\zeta}\right)^{\ell}\right\}\chi_\zeta(t).$$

This completes the proof of Theorem 13.1.

Thus we have here $E'(G)\supset F(G)$, and so $E'(G) = F(G)$. Together with the equality $F(G) = E(G)$ in Theorem 4.2, the proof of Theorem 5.1 is now complete.

14. A general theory for reduction of characters to normal subgroups

14.1. General theory

Let G be a topological group and N its normal subgroup. Denote by $K_1(G)$ (resp. $K_1(N,G)$) the set of invariant continuous positive definite functions on G normalized as $f(e) = 1$ (resp. the set of such functions on N which are also *G-invariant*). Further let $E(G) := \mathrm{Extr}(K_1(G))$ (resp. $E(N,G) := \mathrm{Extr}(K_1(N,G))$) be the set of extremal points of the convex set $K_1(G)$ (resp. $K_1(N,G)$).

Theorem 14.1. *Let G be a Hausdorff topological group and N its normal subgroup with the relative topology.*

(i) *For an* $F \in K_1(G)$, *let* $f = F|_N$ *be its restriction on* N. *It belongs to* $K_1(N,G)$, *and if* $f = a_1 f_1 + a_2 f_2$ *with* $a_i > 0, f_i \in K_1(N,G)$, *then there exist extensions* $F_i \in K_1(G)$ *of* f_i *for* $i = 1,2$, *such that* $F = a_1 F_1 + a_2 F_2$.

(ii) *For any* $F \in E(G)$, *its restriction* $f = F|_N$ *belongs to* $E(N,G)$.

Proof. The assertion (ii) follows from the assertion (i).

To prove (i), we borrow certain ideas from [Tho1]. First we prepare some generalities about Gelfand-Raikov representation [GeRa]. As a general setting, for a Hausdorff topological group N, let $\mathfrak{F}(N)$ be the C^*-algebra of functions on N which are zero outside finite number of points, with operations

$$(\varphi * \psi)(\xi) := \sum_{\eta \in N} \varphi(\xi\eta^{-1})\psi(\eta), \quad \varphi^*(\xi) := \overline{\varphi(\xi^{-1})} \qquad (\xi \in N,\, \varphi, \psi \in \mathfrak{F}(N)).$$

Let $\mathcal{P}_1(N)$ be the set of continuous positive definite functions k on N normalized as $k(e) = 1$ at the identity element $e \in N$. Introduce in $\mathfrak{F}(N)$ an inner product as

$$(\varphi, \psi) = \sum_{\xi, \eta \in N} k(\eta^{-1}\xi)\, \varphi(\xi)\, \overline{\psi(\eta)},$$

and let J_k be the kernel of this inner product. Then, we get on $\mathfrak{F}(N)/J_k$ a positive definite inner product $\langle \varphi^k, \psi^k \rangle_k := (\varphi, \psi)$, where $\varphi^k := \varphi + J_k \in \mathfrak{F}(N)/J_k$. By completion we get a Hilbert space $\mathfrak{H}_k(N)$. Since (φ, ψ) is invariant under the left translation on N, it gives a UR of N on $\mathfrak{H}_k(N)$, which we call the *Gelfand-Raikov representaion* associated to k and denote by π_k. Put $v^0 = \delta_e^k$ the image of the delta-function δ_e supported by $\{e\}$, then it is a unit cyclic vector and $k(h) = \langle \pi_k(h)v^0, v^0 \rangle_k$ (cf. [GeRa]).

In the case where k is invariant or $k \in K_1(N)$, the inner product (φ, ψ) is also invariant under the right translation on N and so it induces another UR ρ_k of N. Denote by $\mathfrak{U}_k$ (resp. $\mathfrak{V}_k$) the von-Neumann algebra generated by $\pi_k(N)$ (resp. $\rho_k(N)$), then they are mutually commutant algebras of the other.

Now let N and G be as in the theorem. Take an $F \in K_1(G)$ and put $f = F|_N$. Then $f \in K_1(N,G)$, and we have two Gelfand-Raikov representations π_F of G on $\mathfrak{H}_F(G)$ and π_f of N on $\mathfrak{H}_f(N)$. For $x \in G$, put

$$\Psi_x(\varphi, \psi) := \sum_{\xi, \eta \in N} F(\eta^{-1}x\xi)\, \varphi(\xi)\, \overline{\psi(\eta)} \qquad (\varphi, \psi \in \mathfrak{F}(N)).$$

Then, we have $|\Psi_x(\varphi, \psi)| \leq \|\varphi^f\|\, \|\psi^f\|$. In fact, define $\widetilde{\varphi}, \widetilde{\psi} \in \mathfrak{F}(G)$ by putting

$$\begin{cases} \widetilde{\varphi}(\xi) & := \varphi(\xi) \quad (\xi \in N), \\ \widetilde{\varphi}(g) & := \quad 0 \qquad (g \notin N), \end{cases} \qquad \begin{cases} \widetilde{\psi}(x^{-1}\xi) & := \psi(\xi) \quad (\xi \in N), \\ \widetilde{\psi}(x^{-1}g) & := \quad 0 \qquad (g \notin N). \end{cases}$$

Then, by calculation we obtain

$$\Psi_x(\varphi, \psi) = \sum_{g, g' \in G} F({g'}^{-1}g)\, \widetilde{\varphi}(g)\, \overline{\widetilde{\psi}(g')} = \langle \widetilde{\varphi}^F, \widetilde{\psi}^F \rangle_F,$$

whence $|\Psi_x(\varphi,\psi)| \le |\langle \widetilde{\varphi}^F, \widetilde{\psi}^F\rangle_F| \le \|\widetilde{\varphi}^F\|\,\|\widetilde{\psi}^F\|$ with $\widetilde{\varphi}^F, \widetilde{\psi}^F \in \mathfrak{H}_F(G)$. On the other hand, we see that $\|\widetilde{\varphi}^F\| = \|\varphi^f\|$, $\|\widetilde{\psi}^F\| = \|\psi^f\|$. In fact, for $\|\widetilde{\varphi}^F\| = \|\varphi^f\|$,

$$\begin{aligned}\|\widetilde{\varphi}^F\|^2 &= \sum_{x,y\in G} F(y^{-1}x)\widetilde{\varphi}^F(x)\,\overline{\widetilde{\varphi}^F(y)} \;=\; \sum_{\xi,\eta\in N} F(\eta^{-1}\xi)\varphi(\xi)\,\overline{\varphi(\eta)} \\ &= \sum_{\xi,\eta\in N} f(\eta^{-1}\xi)\varphi(\xi)\,\overline{\varphi(\eta)} \;=\; \|\varphi^f\|^2.\end{aligned}$$

Therefore we get from Ψ_x on $\mathfrak{F}(N)$ an inner product $\widetilde{\Psi}_x$ on $\mathfrak{F}_f(N)$ as

$$\widetilde{\Psi}_x(\varphi^f,\psi^f) := \Psi_x(\varphi,\psi) \qquad (\varphi,\psi\in\mathfrak{F}(N)).$$

Since $|\widetilde{\Psi}_x(\varphi^f,\psi^f)| \le \|\varphi^f\|\,\|\psi^f\|$, we have a bounded linear transformation A_x on $\mathfrak{H}_f(N)$ such that $\|A_x\|\le 1$ and $\Psi_x(\varphi,\psi) = \widetilde{\Psi}_x(\varphi^f,\psi^f) = \langle A_x\varphi^f,\psi^f\rangle_f$. Then, $F(x) = \langle A_x\delta_e^{\,f}, \delta_e^{\,f}\rangle_f$, and we have

$$A_\xi = \pi_f(\xi) \quad (\xi\in N), \qquad A_{zxz^{-1}} = U_zA_xU_{z^{-1}} \quad (z,x\in G), \tag{14.1}$$

where $U_z\varphi^f := (U_z\varphi)^f$ with $(U_z\varphi)(\xi) := \varphi(z^{-1}\xi z)$ $(\xi\in N, \varphi\in\mathfrak{F}(N))$. In fact,

$$\begin{aligned}\langle A_\xi\varphi^f,\psi^f\rangle_f &= \sum_{\eta_1,\eta_2\in N} F({\eta_2}^{-1}\xi\eta_1)\varphi(\eta_1)\overline{\psi(\eta_2)} \\ &= \sum_{\eta_1,\eta_2\in N} f({\eta_2}^{-1}\eta_1)\varphi(\xi^{-1}\eta_1)\overline{\psi(\eta_2)} \;=\; \langle \pi_f(\xi)\varphi^f,\psi^f\rangle_f\,;\end{aligned}$$

$$\begin{aligned}\langle A_{zxz^{-1}}\varphi^f,\psi^f\rangle_f &= \sum_{\xi,\eta\in N} F(\eta^{\,-1}zxz^{-1}\xi)\varphi(\xi)\overline{\psi(\eta)} \\ &= \sum_{\xi,\eta\in N} F(z^{-1}\eta^{\,-1}zxz^{-1}\xi z)\varphi(\xi)\overline{\psi(\eta)} \\ &= \sum_{\xi,\eta\in N} F(\eta^{-1}x\xi)\varphi(z\xi z^{-1})\overline{\psi(z\eta z^{-1})} \\ &= \langle A_xU_{z^{-1}}\varphi^f, U_{z^{-1}}\psi^f\rangle_f \;=\; \langle U_zA_xU_{z^{-1}}\varphi^f,\psi^f\rangle_f.\end{aligned}$$

Furthermore we have $A_x\rho_f(\xi) = \rho_f(\xi)A_x$ $(\xi\in N)$. In fact,

$$\begin{aligned}\langle A_x\rho_f(\xi)\varphi^f,\rho_f(\xi)\psi^f\rangle_f &= \sum_{\eta_1,\eta_2\in N} F({\eta_2}^{-1}x\eta_1)\varphi(\eta_1\xi)\overline{\psi(\eta_2\xi)} \\ &= \sum_{\eta_1,\eta_2\in N} F(\xi{\eta_2}^{-1}x\eta_1\xi^{-1})\varphi(\eta_1)\overline{\psi}(\eta_2) \\ &= \sum_{\eta_1,\eta_2\in N} F({\eta_2}^{-1}x\eta_1)\varphi(\eta_1)\overline{\psi}(\eta_2) \;=\; \langle A_x\varphi^f,\psi^f\rangle_f.\end{aligned}$$

Hence A_x belongs to the commutant $\mathfrak{U}_f = {\mathfrak{V}_f}'$ of $\mathfrak{V}_f$.

Now suppose that $f\in K_1(N,G)$ is expressed as $f = a_1f_1 + a_2f_2$ with $a_i>0, f_i\in K_1(N,G)$. Then $a_1+a_2=1$ and f_i is majorized by f or more exactly $\lambda_i f(\varphi*\varphi^*) \ge f_i(\varphi*\varphi^*)$ $(\varphi\in\mathfrak{F}(N))$ with $\lambda_i = {a_i}^{-1}$.

Let $\mathfrak{Z}_f^+$ be the set of positive elements in the commmon center $\mathfrak{Z}_f = \mathfrak{U}_f \cap \mathfrak{V}_f$. Then, corresponding to $f_i \le \lambda_i f$ $(i = 1, 2)$ above, we have positive operators $B_i \in \mathfrak{Z}_f^+$ such that

$$0 \le B_i \le \lambda_i I, \qquad f_i(\xi) = \langle B_i \delta_\xi^f, \delta_e^f \rangle_f \quad (\xi \in N),$$

where I denotes the identity operator on $\mathfrak{H}_f(N)$. In fact, consider the inner product in $\mathfrak{F}(N)$ corresponding to f_i as

$$\Phi_i(\varphi, \psi) := \sum_{\xi, \eta \in N} f_i(\eta^{-1}\xi)\varphi(\xi)\overline{\psi(\eta)},$$

and put as before $\langle \varphi^{f_i}, \psi^{f_i} \rangle_{f_i} := \Phi_i(\varphi, \psi)$ and $\|\varphi^{f_i}\| := \Phi_i(\varphi, \varphi)^{1/2}$. Then, $\|\varphi^{f_i}\| \le \lambda_i^{1/2} \|\varphi^f\|$ and so

$$|\Phi_i(\varphi, \psi)| \le \|\varphi^{f_i}\| \, \|\psi^{f_i}\| \le \lambda_i \, \|\varphi^f\| \, \|\psi^f\|.$$

Therefore Φ_i gives a continuous hermitian form on $\mathfrak{F}(N)/J_f \subset \mathfrak{H}_f(N)$, and it is expressed by a non-negative definite operator B_i as $\Phi_i(\varphi, \psi) = \langle B_i \varphi^f, \psi^f \rangle_f$. Furthermore we have $0 \le B_i \le \lambda_i I$, $\;f_i(\eta^{-1}\xi) = \langle B_i \delta_\xi^f, \delta_\eta^f \rangle_f$, and so B_i commutes with $\pi_f(\xi)$ and also with $\rho_f(\eta)$ (cf. also Lemma 1.4.1 in [HH5]).

We have $U_z B_i = B_i U_z$ $(z \in G)$. In fact, since $\pi_f(\xi) B_i = B_i \pi_f(\xi)$ $(\xi \in N)$,

$$\begin{aligned} \langle U_z B_i U_{z^{-1}} \delta_\xi^f, \delta_\eta^f \rangle_f &= \langle B_i U_{z^{-1}} \delta_\xi^f, U_{z^{-1}} \delta_\eta^f \rangle_f = \langle B_i \delta_{z\xi z^{-1}}^f, \delta_{z\eta z^{-1}}^f \rangle_f \\ &= f_i(z\eta^{-1}z^{-1}z\xi z^{-1}) = f_i(\eta^{-1}\xi) = \langle B_i \delta_\xi^f, \delta_\eta^f \rangle_f. \end{aligned}$$

Put $A_x^i := A_x B_i = B_i A_x$ $(i = 1, 2)$ (recall $A_x \in \mathfrak{U}_f$) and

$$F_i(x) := \langle A_x^i \delta_e^f, \delta_e^f \rangle_f = \langle A_x B_i \delta_e^f, \delta_e^f \rangle_f = \langle B_i A_x \delta_e^f, \delta_e^f \rangle_f \quad (x \in G).$$

Then $a_1 F_1 + a_2 F_2 = F$, because $a_1 B_1 + a_2 B_2 = I$ and $F(x) = \langle A_x \delta_e^f, \delta_e^f \rangle_f$. We shall prove that F_i is a continuous invariant positive definite function on G or $F_i \in K_1(G)$, which extends f_i.

(i) F_i is an extension of f_i. In fact, for $\xi \in N$,

$$F_i(\xi) = \langle A_\xi^i \delta_e^f, \delta_e^f \rangle_f = \langle B_i \pi_f(\xi) \delta_e^f, \delta_e^f \rangle_f = \langle B_i \delta_\xi^f, \delta_e^f \rangle_f = f_i(\xi).$$

(ii) F_i is an invariant function. In fact, for $x, z \in G$,

$$\begin{aligned} F_i(zxz^{-1}) &= \langle B_i A_{zxz^{-1}} \delta_e^f, \delta_e^f \rangle_f \;=\; \langle B_i U_z A_x U_{z^{-1}} \delta_e^f, \delta_e^f \rangle_f \\ &= \langle U_z B_i A_x \delta_e^f, \delta_e^f \rangle_f \;=\; \langle B_i A_x \delta_e^f, U_{z^{-1}} \delta_e^f \rangle_f \\ &= \langle B_i A_x \delta_e^f, \delta_e^f \rangle_f \;=\; F_i(x). \end{aligned}$$

(iii) F_i is continuous. In fact, for any fixed $\varphi, \psi \in \mathfrak{F}(N)$, the map $G \ni x \mapsto \Psi_x(\varphi, \psi) = \langle A_x \varphi^f, \psi^f \rangle_f \in \boldsymbol{C}$ is continuous. Moreover, since $\{\varphi^f ; \varphi \in \mathfrak{F}(N)\}$ is dense in $\mathfrak{H}_f(N)$ and $\|A_x\| \le 1$, the map $G \ni x \mapsto A_x$ is weakly continuous, and so is the map $G \ni x \mapsto B_i A_x$. Hence $F_i(x) = \langle B_i A_x \delta_e^f, \delta_e^f \rangle_f$ is continuous.

(iv) F_i is positive definite. In fact, for $\phi \in \mathfrak{F}(G)$,

$$
\begin{aligned}
0 &\le \sum_{x,y\in G} F(y^{-1}x)\phi(x)\overline{\phi(y)} \\
&= \sum_{x,y\in G/N}\ \sum_{\xi,\eta\in N} F(\eta^{-1}y^{-1}x\xi)\phi(x\xi)\overline{\phi(y\eta)} \\
&= \sum_{x,y\in G/N} \langle A_{y^{-1}x}\phi_x^{\,f}, \phi_y^{\,f}\rangle_f,
\end{aligned} \tag{14.2}
$$

where $\phi_x(\xi) := \phi(x\xi)$ $(\xi \in N)$ and $\phi_x \in \mathfrak{F}(N)$. On the other hand,

$$
\begin{aligned}
\sum_{x,y\in G} F_i(y^{-1}x)\phi(x)\overline{\phi(y)} &= \sum_{x,y\in G/N}\ \sum_{\xi,\eta\in N} F_i(\eta^{-1}y^{-1}x\xi)\phi(x\xi)\overline{\phi(y\eta)} \\
&= \sum_{x,y\in G/N}\ \sum_{\xi,\eta\in N} \langle B_i A_{y^{-1}x}\delta_\xi^{\,f}, \delta_\eta^{\,f}\rangle_f \phi(x\xi)\overline{\phi(y\eta)} \\
&= \sum_{x,y\in G/N} \langle B_i A_{y^{-1}x}\phi_x^{\,f}, \phi_y^{\,f}\rangle_f \\
&= \sum_{x,y\in G/N} \langle A_{y^{-1}x}B_i^{\,1/2}\phi_x^{\,f}, B_i^{\,1/2}\phi_y^{\,f}\rangle_f .
\end{aligned}
$$

Here $x \in G/N$ means that x runs over a fixed complete set $\mathcal{R}$ of representatives of G/N, and $B_i^{\,1/2}\phi_x^{\,f}$ is zero except a finite subset X of $\mathcal{R}$ (since $\phi \in \mathfrak{F}(G)$), and is in $\mathfrak{H}_f(N)$, the completion of $\{\varphi^f = \sum_{\xi\in N}\varphi(\xi)\delta_\xi^{\,f}\,;\,\varphi \in \mathfrak{F}(N)\}$. Therefore we can find a series of functions $Y_n \in \mathfrak{F}(G)$ such that $Y_{n,x} = 0$ except when $x \in X$, and $Y_{n,x}^{\,f} = \sum_{\xi\in N} Y_n(x\xi)\delta_\xi^{\,f}$ converges to $B_i^{\,1/2}\phi_x^{\,f}$ for $x \in X$. Then we get finally from (14.2)

$$
\begin{aligned}
&\sum_{x,y\in G/N} \langle A_{y^{-1}x}B_i^{\,1/2}\phi_x^{\,f}, B_i^{\,1/2}\phi_y^{\,f}\rangle_f = \\
&\qquad = \lim_{n\to\infty} \sum_{x,y\in G/N} \langle A_{y^{-1}x}Y_{n,x}^{\,f}, Y_{n,y}^{\,f}\rangle_f \ \ge\ 0\,.
\end{aligned} \tag{14.3}
$$

The proof of Theorem 14.1 is now complete. □

14.2. Case of a compact group G and its normal subgroup N

To clarify the situation of rather unexpected result Theorem 14.1, we explain a little more in detail by taking the case of a compact group G.

Let N be a closed (but not necessarily open) normal subgroup of G. For an irreducible unitary representation (= IUR) ρ of N, a $\xi \in G$ acts on it as $({}^\xi\rho)(h) := \rho(\xi^{-1}h\xi)$ $(h \in H)$, and the subgroup $Z(\rho) := \{\xi \in G\,;\,{}^\xi\rho \cong \rho\}$ is of finite index in G. Put

$$R(\rho) := \bigoplus_{\xi\in G/Z(\rho)} {}^\xi\rho,$$

where the summation runs over a complete system of representatives of $G/Z(\rho)$, then its chartacter $\chi_{R(\rho)} = \sum_{\xi \in G/Z(\rho)} \chi_{^\xi\rho}$ is determined independent of the choice of the system of representatives. Take the normalized character

$$\widetilde{\chi}_{R(\rho)} := \frac{\chi_{R(\rho)}}{\dim R(\rho)} = \frac{1}{|G/Z(\rho)|} \sum_{\xi \in G/Z(\rho)} \widetilde{\chi}_{^\xi\rho}\,, \qquad \widetilde{\chi}_{^\xi\rho} := \frac{\chi_{^\xi\rho}}{\dim \rho}.$$

Proposition 14.2. *The set $E(N,G)$ of extremal points of $K_1(N,G)$ consists of $\widetilde{\chi}_{R(\rho)}$, where ρ runs over a complete system of representatives of the dual $\widehat{N}$ of N. Moreover, for two IURs ρ and ρ', we have $\widetilde{\chi}_{R(\rho)} = \widetilde{\chi}_{R(\rho')}$ if and only if $\rho' \cong {}^\xi\rho$ for some $\xi \in G$.*

Proposition 14.3. (i) *The set $E(G)$ of extremal points of $K_1(G)$ consists of normalized characters $\widetilde{\chi}_\Pi$ of IURs Π, where Π runs over a complete system of representatives of $\widehat{G}$.*

(ii) *The restriction $\widetilde{\chi}_\Pi|_N$ onto N, as functions, is equal to $\widetilde{\chi}_{R(\rho)}$ for an irreducible component ρ of the restriction $\Pi|_N$. The represention $\Pi|_N$ is a multiple of $R(\rho)$ with multiplicity*

$$m_\Pi := \frac{\dim \Pi}{\dim \rho \cdot |G/Z(\rho)|}.$$

Proposition 14.4. *Fix an IUR ρ of N. Then, for an IUR Π of G, the restriction $\widetilde{\chi}_\Pi|_N$ of its normalized character equals $\widetilde{\chi}_{R(\rho)}$ if and only if Π is an irreducible component of the induced representation $\pi = \mathrm{Ind}_N^G \rho$, and also if and only if $\Pi|_N$ contains $R(\rho)$.*

To prove these Propositions, we utilize the following.

For the induced representation $\pi = \mathrm{Ind}_N^G \rho$, put

$$\chi_\pi(h) := \int_G \chi_\rho(\xi^{-1} h \xi)\, d\xi.$$

Then, for a continuous function φ on G such that the operator $\pi(\varphi) = \int_G \varphi(g)\pi(g)\, dg$ is of finite rank, we have

$$\mathrm{tr}\big(\pi(\varphi)\big) = \int_N \varphi(h)\, \chi_\pi(h)\, dh.$$

Here $d\xi$ and dh denote the normalized Haar measures on G and N respectively.

Note 14.1. The complex measure $\chi_\pi(h)\, dh$ on G supported by N may be called the *character* of π.

15. Restriction of characters from $G = \mathfrak{S}_\infty(T)$ to its canonical subgroups

15.1. Reduction from $G = \mathfrak{S}_\infty(T)$ to its canonical subgroups

Let T be abelian, and assume that S is an open subgroup of T. Let N be one of the subgroups $G' := \mathfrak{A}_\infty(T)$, $G^S = \mathfrak{S}_\infty^S(T)$ and $G'^S = G' \cap G^S$. Then N is an open normal subgroup of G, and in this special case we have the following result.

Theorem 15.1. *Let $G = \mathfrak{S}_\infty(T)$ with T abelian, and N be one of G', G^S and G'^S, where S is an open subgroup of T for G^S and G'^S. Then every character $f \in E(N, G)$ is a restriction of a character $F \in E(G)$.*

Proof. We can apply Theorem 1.3 (ii) as is noted in Remark 1.2. But we give here a proof for this special case different from that for Theorem 1.3 (ii).

Take an $f \in E(N, G)$. Let π be a factor representation of N associated to f on a Hilbert space $V(\pi)$ in such a way that π has a unit cyclic vector v_0 such that $f(h) = \langle \pi(h)v_0, v_0\rangle$ $(h \in N)$. Then $V(\pi)$ is spanned by the union of strongly compact sets $W_n = \pi(N \cap G_n)v_0, n \geq 1$, where $G_n = \mathfrak{S}_n(T)$. Since each $W_n \subset V(\pi)$ is strongly compact, it has a countable dense subset, and so $V(\pi)$ is separable.

On the other hand, since N is open in G, we have a counting measure as an invariant measure on a finite set G/N, and an induced representation $\rho = \mathrm{Ind}_N^G \pi$ is constructed on the $V(\pi)$-valued ℓ^2-space $V(\rho)$ on G/N, which turns out to be separable. More explicitly, take a $V(\pi)$-valued $\boldsymbol{\psi}$ such that $\boldsymbol{\psi}(hg) = \pi(h)\boldsymbol{\psi}(g)$ $(h \in N, g \in G)$, then $\big(\rho(g_0)\boldsymbol{\psi}\big)(g) := \boldsymbol{\psi}(gg_0)$. We can take another realization, fixing a complete sets of representatives $\{s_i; 1 \leq i \leq M\}$ of G/N with $s_1 = e$ the identity elemet. Then $\boldsymbol{\psi}(g)$ above is represented by a system $\boldsymbol{w} := (v_i)_{1\leq i\leq M}$ of vectors $v_i = \boldsymbol{\psi}(s_i) \in V(\pi)$ with $\|\boldsymbol{w}\|^2 := (1/M)\sum_{1\leq i\leq M} \|v_i\|^2$, and $\rho(g_0)$ is given by $\boldsymbol{w} \to \boldsymbol{w}' := (v'_i)_{1\leq i\leq M}$ with $v'_i = \boldsymbol{\psi}(s_i g_0) = \pi(h'_i)v_{i'}$, where $s_i g_0 = h'_i s_{i'}$ $(1 \leq i \leq M)$. Take a unit vector as

$$\boldsymbol{w}_0 = (v_i)_{1\leq i\leq M} \quad \text{with} \quad v_1 = \sqrt{M}\, v_0, \;\; v_i = \boldsymbol{0}\ (\forall i \neq 1). \tag{15.1}$$

Then, putting $F(g) := \langle \rho(g)\boldsymbol{w}_0, \boldsymbol{w}_0\rangle$, we have $F(h) = f(h)$ $(h \in N)$ and $F(g) = 0$ outside H. In fact, for $g \notin N$, we have $s_1 g = g \notin N$ and so the first component of $\rho(g)\boldsymbol{w}_0$ is $\boldsymbol{0}$ and $\langle \rho(g)\boldsymbol{w}_0, \boldsymbol{w}_0\rangle = 0$. Since f is G-invariant and N is open in G, $F(g_0 g {g_0}^{-1}) = F(g)$ and $F \in K_1(G)$.

Denote by $\mathfrak{B}(\rho)$ the von Neumann algebra generated by ρ, and by Φ the normal trace corresponding to F defined by $\Phi(C) = \langle C\boldsymbol{w}_0, \boldsymbol{w}_0\rangle$ $(C \in \mathfrak{B}(\rho))$.

Let $\mathfrak{Z}$ denotes the center of $\mathfrak{B}(\rho)$. Then, the factorial decomposition of ρ is given as follows. We know that a finite normal trace on $\mathfrak{B}(\rho)$ is determined by its restriction on $\mathfrak{Z}$ (cf. for instance, Theorem V.2.6 in [Tak, V.2]). Let $\widehat{\mathfrak{Z}}$ be the set of one-dimensional representations of $\mathfrak{Z}$. Then, there exists a field of factor representations $(\rho(\chi), \mathcal{H}(\rho(\chi)), \chi \in \widehat{\mathfrak{Z}}$, and a positive measure μ on $\widehat{\mathfrak{Z}}$ such that

$$\rho = \int_{\widehat{\mathfrak{Z}}}^{\oplus} \rho(\chi)\, d\mu(\chi), \qquad V(\rho) = \int_{\widehat{\mathfrak{Z}}}^{\oplus} \mathcal{H}(\rho(\chi))\, d\mu(\chi).$$

In this decomposition, we have, for $\boldsymbol{w}_0 \in V(\rho)$ and $B \in \mathfrak{B}(\rho)$,

$$\boldsymbol{w}_0 = \int_{\widehat{\mathfrak{Z}}}^{\oplus} \boldsymbol{w}_{0,\chi}\, d\mu(\chi), \qquad B = \int_{\widehat{\mathfrak{Z}}}^{\oplus} B(\chi)\, d\mu(\chi),$$

and so $\langle B\boldsymbol{w}_0, \boldsymbol{w}_0\rangle = \int_{\widehat{\mathfrak{Z}}} \langle B(\chi)\boldsymbol{w}_{0,\chi}, \boldsymbol{w}_{0,\chi}\rangle\, d\mu(\chi)$.

Define a linear form $\Phi_\chi(D) = \langle D\boldsymbol{w}_{0,\chi}, \boldsymbol{w}_{0,\chi}\rangle$ on $\mathfrak{B}(\rho(\chi))$, then

$$\Phi(B) = \int_{\widehat{\mathfrak{Z}}} \Phi_\chi(B(\chi))\, d\mu(\chi) \qquad (B \in \mathfrak{B}(\rho)), \tag{15.2}$$

and Φ_χ is a normal trace for almost all χ with central character $\chi \in \widehat{\mathfrak{Z}}$, and is equal to the normalized characer of $\rho(\chi)$. For $B = \rho(g)$ $(g \in G)$, put $\rho(\chi)(g) := B(\chi)$, then $\rho(\chi)$ is a factor representation of G for almost all χ, and Φ_χ corresponds to $F_\chi(g) = \langle \rho(\chi)(g)\boldsymbol{w}_{0,\chi}, \boldsymbol{w}_{0,\chi}\rangle$ $(g \in G)$, which is in $E(G)$ (cf. Theorem 1.1).

Since $F|_N = f$ and f is extremal or $f \in E(N)$, we see from $f = \int_{\widehat{\mathfrak{Z}}} \big(F_\chi|_N\big)\, d\mu(\chi)$ that $F_\chi|N = f$ for almost all χ. This means that $F_\chi \in E(G)$ is an extension of f. □

Lemma 15.2. *Let $G = \mathfrak{S}_\infty(T)$ with T abelian and N be as in Theorem* 15.1. *Then the set of normalized invariant positive definite functions $K_1(N)$ is equal to $K_1(N, G)$, and accordingly $E(N) = E(N, G)$.*

Proof. For any $h \in N, g \in G$, there exists an $h_0 \in N$ such that $ghg^{-1} = h_0hh_0^{-1}$. This gives the assertion in the lemma. □

15.2. Proof of Theorem 7.1

Let T be abelian and S a subgroup of T. Put $G = \mathfrak{S}_\infty(T)$ and $N = G^S = \mathfrak{S}_\infty^S(T)$ be the normal subgroup of G in the theorem. Let $b(t) = (t, e_T, \ldots) \in D_\infty(T)$, and $B = \{b(t); t \in T\}$. Then $G = BN = NB$ and $b(t)N = b(t')N$ if and only if $tS = t'S$, and $G/N \cong T/S$.

Proof for Theorem 7.1. (i) For $G = \mathfrak{S}_\infty(T)$ and its normal subgroup $N = G^S$, we apply Theorem 14.1. Then we see that for any $f \in E(G)$ its restriction $f^S = f|_N$ belongs to $E(N, G) = E(N)$.

(iii) The converse is true if S is open in T, since Theorem 15.1 or Theorem 1.3 (ii) is applicable. □

Remark 15.1. As a general setting, let G be a countable discrete group and N its normal subgroup. Then, Lemma 14 in [Tho1] asserts that, *for an $F \in E(G)$, its restriction $f = F|_N$ on N belongs to $E(N, G)$.*

15.3. Proof of Proposition 7.2

For $g \in G$, let

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m, \quad \xi_{q_k} = (t_{q_k}, (q_k)), \quad g_j = (d_j, \sigma_j),$$

be its standard decomposition. For $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$ in (5.3), we assume the condition

$$\text{(MAX)} \qquad \sum_{\zeta\in\widehat{T}}\ \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| \;+\; \|\mu\| \;=\; 1\,.$$

The formula of f_A in Theorem 7.1(ii) is rewritten as

$$f_A(g) = \prod_{1\le k\le r} Y_1(t_{q_k}) \times \prod_{1\le j\le m} Y_{\ell_j}(P_{K_j}(d_j)), \tag{15.3}$$

where $\ell_j = \ell(\sigma_j)$, $P_{K_j}(d_j) = \prod_{i\in K_j} t_i$ for $d_j = (t_i)_{i\in K_j}$, and

$$Y_1(t) = \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} \;+\; \mu_\zeta \right) \zeta(t),$$

$$Y_\ell(t) = \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^\ell \; (-1)^{\varepsilon(\ell-1)} \right) \zeta(t) \qquad (\ell\ge 2).$$

Since (MAX) is assumed, the formula (15.3) is valid even in the case of $t_{q_k} = e_T$. For another $A' = \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu'\Big)$ satisfying (MAX), $f_{A'}$ is given similarly as

$$f_{A'}(g) = \prod_{1\le k\le r} Y'_1(t_{q_k}) \times \prod_{1\le j\le m} Y'_{\ell_j}(P_{K_j}(d_j)), \tag{15.4}$$

where $Y'_\ell(t)$ $(\ell\ge 1, t\in T)$ are given corresponding to the parameter A' similarly.

Now assume that $f_A|_{G^S} = f_{A'}|_{G^S}$. Put $n = r+m$ the number of basic elements in the standard decomposition of g. Let $\mathcal{NT}$ be the set of pairs $(\ell,t)\in\boldsymbol{N}\times T$ given as $(1,t_{q_k})$, $1\le k\le r$ $(\ell=1)$, and (ℓ_j,t'_j), $1\le j\le m$, with $\ell_j\ge 2, t'_j = P_{K_j}(d_j)\in T$, and we number n elements of $\mathcal{NT}$ as (ℓ_s,t_s) with $1\le s\le n$. Then the equality $f_A(g) = f_{A'}(g)$ for $g\in G^S$ is equivalent to $\prod_{1\leqslant s\leqslant n} Y_{\ell_s}(t_s) \;=\; \prod_{1\leqslant s\leqslant n} Y'_{\ell_s}(t_s)$ under the condition $\prod_{1\leqslant s\leqslant n} t_s\in S$. Here the set $\mathcal{NT}$ of pairs $(\ell_s,t_s)\in\boldsymbol{N}\times T$ $(1\le s\le n)$ are arbitrary except satisfying this last condition.

Consider this equality under more restrictive condition that $\prod_{1\le s\le n} t_s = e_T$. Then, in this case, just as is worked out in [HH3, §17] we have a one-dimensional character ζ_0 of T such that $f_{A'}(g) = \zeta_0(P(d))\cdot f_A(g)$ for $g = (d,\sigma)\in G$. Since $\pi_{\zeta_0,0}(g) = \zeta_0(P(d))$, this is written as $f_{A'} = \pi_{\zeta_0,0}\cdot f_A$. Now consider this equality under the condition $\prod_{1\le s\le n} t_s\in S$, then we conclude that ζ_0 is trivial on S. $\square$

16. Wreath product $\mathfrak{A}_\infty(T)$ of T with the infinite alternating group $\mathfrak{A}_\infty$

16.1. The case of the group $G' = \mathfrak{A}_\infty(T)$

Let us consider a normal subgroup $G' = \mathfrak{A}_\infty(T) = D_\infty(T) \rtimes \mathfrak{A}_\infty$ of $G = \mathfrak{S}_\infty(T)$. The special case where $T = \{ e_T \}$ or the case of $\mathfrak{A}_\infty \subset \mathfrak{S}_\infty$ is treated in [Tho2]. Here we prove the following result.

Theorem 16.1. (i) *Every character of the group $G = \mathfrak{S}_\infty(T)$ gives a character of $G' = \mathfrak{A}_\infty(T)$ by restriction. Conversely any character of G' is a restriction of someone of G.*

(ii) *For two continuous factorizable positive definite functions f_A and $f_{A'}$ on G, with parameters*

$$A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big), \quad A' = \big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu'\big)$$

as in (5.3) *respectively, their restrictions on G' coincide with each other if and only if $f_{A'} = (\mathrm{sgn}_{\mathfrak{S}})^a f_A$ ($a = 0$ or 1), or, under the condition* (MAX) *for both of A and A', if and only if $A' = A$ or $A' = {}^tA$, where ${}^tA$ is defined as*

$$\text{(16.1)} \qquad {}^tA := \big((\alpha''_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu''\big),$$

$$\text{with} \quad \alpha''_{\zeta,0} = \alpha_{\zeta,1}\,,\ \ \alpha''_{\zeta,1} = \alpha_{\zeta,0} \quad (\zeta \in \widehat{T}), \quad \text{and} \quad \mu'' = \mu.$$

The proof for (ii) is quite similar as for the case of T finite in [HH3, Theorem 14].

16.2. The case of the group ${G'}^S = \mathfrak{A}^S_\infty(T)$

Assume T be compact abelian and $S \subset T$ a subgroup of T. Then we have another normal subgroup

$$\text{(16.2)} \qquad \begin{aligned} {G'}^S &:= G' \cap G^S = \mathfrak{A}^S_\infty(T) \\ &= \big\{ g = (d,\sigma) \in \mathfrak{S}_\infty(T)\,;\, \mathrm{sgn}_{\mathfrak{S}}(\sigma) = 1,\ P(d) \in S \big\} \end{aligned}$$

of $G = \mathfrak{S}_\infty(T)$. For this group, we can also prove the analogous result as for $G^S = \mathfrak{S}^S_\infty(T)$ and $G' = \mathfrak{A}_\infty(T)$.

Theorem 16.2. (i) *For the normal subgroup ${G'}^S$ of G, every character of G gives a character of ${G'}^S$ by restriction. Conversely if S is open in T, any characer of ${G'}^S$ is given by restriction of someone of G.*

(ii) *Two functions f_A and $f_{A'}$ in $E(G)$ with parameters A and A' as in Theorem* 16.1 *have the same restriction on ${G'}^S$ if and only if $f_{A'} = \pi_{\zeta_S,a} f_A$ with a one-dimensional character*

$$\pi_{\zeta_S,a}(g) = \zeta_S\big(P(d)\big)\,(\mathrm{sgn}_{\mathfrak{S}})^a(\sigma) \quad \text{for } g = (d,\sigma) \in D_\infty(T) \rtimes \mathfrak{S}_\infty,$$

in Lemma 3.4, *where $a = 0, 1$, and $\zeta_S \in \widehat{T}$ which is trivial on S. This corresponds to the following relation between A and A' both satisfying* (MAX):

$$\begin{aligned} &A' = R(\zeta_S)A \quad \text{in (7.3)} && \text{in case } a = 0, \\ &A' = R(\zeta_S)({}^tA) && \text{in case } a = 1. \end{aligned}$$

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