J. Math. Kyoto Univ. (JMKYAZ)
47-3 (2007), 643–655

Exact controllability of a Timoshenko beam with dynamical boundary*

By

Chun-Guo Zhang and Hong-Xiang Hu

Abstract

We consider the exact controllability of a Timoshenko beam system, clamped at one end and attached at the other end to a rigid antenna. Such a system is governed by two partial differential equations and two ordinary differential equations. Using the HUM method, we show that the system is exactly controllable in the usual energy space.

1. Introduction

In recent years, the boundary feedback stabilization of a Timoshenko beam or the exact boundary controllability of a hybrid system of elasticity has been studied extensively (see [1]–[7]), but little attention has been paid to the exact controllability for a Timoshenko beam with dynamical boundary. In this work we consider a Timoshenko beam system clamped at one end and attached at the other end to a rigid antenna, whereon are applied the dynamical controls u_1,u_2. More precisely, we consider the following control problem:

$$(1.1)\quad \begin{cases} w_{tt}(x,t) - k_1 w_{xx}(x,t) + k_1 \varphi_x(x,t) = 0, \quad 0 < x < L, \quad t > 0, \\ \varphi_{tt}(x,t) - k_2 \varphi_{xx}(x,t) - k_1 w_x(x,t) + k_1 \varphi(x,t) = 0, \\ \quad 0 < x < L, \quad t > 0, \\ w(0,t) = \varphi(0,t) = 0, \quad t > 0, \\ M w_{tt}(L,t) + w_x(L,t) - \varphi(L,t) = u_1(t), \quad t > 0, \\ J \varphi_{tt}(L,t) + \varphi_x(L,t) = u_2(t), \quad t > 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \varphi(x,0) = \varphi_0(x), \\ \varphi_t(x,0) = \varphi_1(x), \quad 0 < x < L, \end{cases}$$

here a uniform beam of length L moves in $w - x$ plane, $w(x,t)$ is the displacement of the center line of the beam, $\varphi(x,t)$ is the rotation angle of the cross-section area at the location $x \in [0, L]$, $k_j > 0$ $(j = 1, 2)$ are two wave speeds, M is a mass, and J is rotatory inertia. For more details concerning

Received April 13, 2007

*This work is supported by the National Natural Science Foundation of China grants (no.10501039) and by the scientific research project (no.20050481) of Educational Bureau of Zhejiang province.

the descriptions of the physical structure of the system, we refer to Kim and Renardy [1] or Morgul [2].

For the usual initial data $(w_0, \varphi_0, w_1, \varphi_1) \in H^1(0,L) \times H^1(0,L) \times L^2(0,L) \times L^2(0,L)$, the regularity of weak solution is insufficient to define the traces $w_{tt}(L,t)$ and $\varphi_{tt}(L,t)$. In that case following an idea of Slemrod [8], the boundary conditions involving the dynamical terms $w_{tt}(L,t)$ and $\varphi_{tt}(L,t)$ can be treated as two ordinary differential equations (with respect to the time variable t). More specifically, denoting by $y(x,t) = (w(x,t), \varphi(x,t), w(L,t), \varphi(L,t))$ the state of the system (1.1) and by $u = (0, 0, u_1, u_2)$ the control, we transform the system (1.1) into an abstract system

$$y_{tt} + Ay = u, \quad y(0) = y_0, \quad y_t(0) = y_1, \tag{1.2}$$

where A is a self-adjoint and positive definite operator in the product space $L^2(0,L) \times L^2(0,L) \times \mathbf{R} \times \mathbf{R}$. We obtain a weak formulation of the original system (1.1).

In this paper, we will adapt the Hilbert uniqueness method to the exact controllability of the abstract problem (1.2). The main difficulty in this approach consists in establishing an inverse energy inequality, the direct energy inequality is easy to establish for this kind of problem. Inspired from the uniform stability results, we look for the estimates of the traces of higher order of the solution of the associated homogeneous problem. This allows us to establish the exact controllability of the abstract system (1.2) for usual initial data by means of two controllers u_1,u_2.

The plan of this paper is as follows. In Section 2, we consider the associated homogeneous problem. The direct and inverse energy inequalities are established with the usual norm. In Section 3, we show that the system is exactly controllable in the usual energy space.

2. Homogeneous system

In this section we consider the homogeneous problem:

$$v_{tt} - v_{xx} + \phi_x = 0, \quad 0 < x < L, \quad t > 0, \tag{2.1}$$

$$\phi_{tt} - \phi_{xx} - v_x + \phi = 0, \quad 0 < x < L, \quad t > 0, \tag{2.2}$$

$$v(0,t) = \phi(0,t) = 0, \quad t > 0, \tag{2.3}$$

$$v_{tt}(L,t) + v_x(L,t) - \phi(L,t) = 0, \quad t > 0, \tag{2.4}$$

$$\phi_{tt}(L,t) + \phi_x(L,t) = 0, \quad t > 0. \tag{2.5}$$

Since the physical constants k_1, k_2, M, J are strictly positive, without loss of generality, we will take $k_1 = k_2 = M = J = 1$ throughout this paper. We first write formally the system (2.1)–(2.5) into

$$\begin{aligned} &(v(x,t), \phi(x,t), v(L,t), \phi(L,t))_{tt} \\ &\quad + (-v_{xx}(x,t) + \phi_x(x,t), -\phi_{xx}(x,t) - v_x(x,t) \\ &\quad + \phi(x,t), v_x(L,t) - \phi(L,t), \phi_x(L,t)) = 0. \end{aligned} \tag{2.6}$$

According to this formulation, we introduce the product space

$$H = L^2(0,L) \times L^2(0,L) \times \mathbf{R} \times \mathbf{R}, \tag{2.7}$$

endowed with the usual norm

$$\|\Phi\|_H^2 = \int_0^L (v^2 + \phi^2)dx + \xi^2 + \eta^2, \quad \forall \Phi = (v, \phi, \xi, \eta) \in H. \tag{2.8}$$

We next define the linear operator A as follows

$$D(A) = \begin{pmatrix} \Phi = (v,\phi,\xi,\eta): & v,\phi \in H^2(0,L), \\ v(0) = \phi(0) = 0; & \xi = v(L), \eta = \phi(L) \end{pmatrix}, \tag{2.9}$$

$$A\Phi = (-v_{xx} + \phi_x, -\phi_{xx} - v_x + \phi, v_x(L) - \phi(L), \phi_x(L)), \tag{2.10}$$
$$\forall \Phi = (v,\phi,\xi,\eta) \in D(A).$$

Then setting

$$\xi(t) = v(L,t), \quad \eta(t) = \phi(L,t), \quad \Phi(x,t) = (v(x,t), \phi(x,t), \xi(t), \eta(t)). \tag{2.11}$$

We write the equation (2.6) into evolutionary equation

$$\Phi_{tt} + A\Phi = 0, \quad \Phi(0) = \Phi_0, \quad \Phi_t(0) = \Phi_1. \tag{2.12}$$

Proposition 2.1. *The operator A defined in* (2.9)–(2.10) *is self-adjoint and definite positive. Moreover A^{-1} is compact in H.*

Proof. We first prove that A is a symmetric operator in H. A straightforward computation gives that

$$(A\Phi, \widetilde{\Phi})_H = \int_0^L [(v_x\widetilde{v}_x + \phi_x\widetilde{\phi}_x + \phi\widetilde{\phi}) - (v_x\widetilde{\phi} + \phi\widetilde{v}_x)]dx = (\Phi, A\widetilde{\Phi})_H,$$

for all $\Phi = (v,\phi,\xi,\eta)$, $\widetilde{\Phi} = (\widetilde{v}, \widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}) \in D(A)$. In particular, using Poincaré's inequality we get

$$(A\Phi, \Phi)_H = \int_0^L [(v_x^2 + \phi_x^2 + \phi^2) - 2v_x\phi]dx = \int_0^L [(v_x - \phi)^2 + \phi_x^2]dx \geq C\|\Phi\|_H^2.$$

Now let $\widetilde{\Phi} = (\widetilde{v}, \widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}) \in D(A^*)$, then there exists $\Phi_0 = (v_0, \phi_0, \xi_0, \eta_0) \in H$ such that $(\widetilde{\Phi}, A\Phi) = (\Phi_0, \Phi)$ for all $\Phi \in D(A)$. This means that

$$\int_0^L [\widetilde{v}(-v_{xx} + \phi_x) + \widetilde{\phi}(-\phi_{xx} - v_x + \phi)]dx + \widetilde{\xi}(v_x(L) - \phi(L)) + \widetilde{\eta}\phi_x(L)$$
$$= \int_0^L (v_0 v + \phi_0\phi)dx + \xi_0 v(L) + \eta_0\phi(L)$$

for all $v, \phi \in H^2(0,L)$ satisfying $v(0) = \phi(0) = 0$. Then indeed a straightforward computation shows that

$$\widetilde{v}, \widetilde{\phi} \in H^2(0,L), \quad \widetilde{v}(0) = \widetilde{\phi}(0) = 0, \quad \widetilde{\xi} = \widetilde{v}(L), \quad \widetilde{\eta} = \widetilde{\phi}(L).$$

It follows that $\widetilde{\Phi} \in D(A)$, therefore we obtain $A^* = A$.

Finally let $\Phi = (v, \phi, v(L), \phi(L)) \in D(A)$ solve the equation $A\Phi = \Phi_0$ for $\Phi_0 = (v_0, \phi_0, \xi_0, \eta_0) \in H$. Then we have

$$\begin{cases} v_{xx} - \phi_x = -v_0, \quad 0 < x < L, \\ \phi_{xx} + v_x - \phi = -\phi_0, \quad 0 < x < L, \\ v(0) = \phi(0) = 0, \\ v_x(L) - \phi(L) = \xi_0, \quad \phi_x(L) = \eta_0. \end{cases} \tag{2.13}$$

The uniqueness theorem of ODEs shows that there exists a unique solution $(v, \phi) \in H^2(0,L) \times H^2(0,L)$ such that $\|(v,\phi)\|_{H^2(0,L)\times H^2(0,L)} \le C\|\Phi_0\|_H$. We obtain thus the compactness of A^{-1}. The proof is complete. $\square$

Since A is self-adjoint and definite positive, and A^{-1} is compact, using the spectral decomposition theorem, we can define the powers $A^{\frac{1}{2}} \in \mathcal{L}(D(A^{\frac{1}{2}}), D(A^{-\frac{1}{2}}))$, we have

$$D(A^{\frac{1}{2}}) = \begin{pmatrix} \Phi = (v, \phi, \xi, \eta) \in H : \quad (v, \phi) \in H^1(0,L) \times H^1(0,L), \\ v(0) = \phi(0) = 0; \quad \xi = v(L), \eta = \phi(L) \end{pmatrix}, \tag{2.14}$$

$$\|\Phi\|_V^2 = \int_0^L [(v_x - \phi)^2 + \phi_x^2]dx, \quad \forall \Phi = (v, \phi, \xi, \eta) \in V, \tag{2.15}$$

where $V = D(A^{\frac{1}{2}})$.

The following result is classic (see [9]).

Proposition 2.2. *Assume that* $(\Phi_0, \Phi_1) \in V \times H$*, then the equation* (2.12) *admits a unique solution* $\Phi(t)$ *such that*

$$\Phi(t) \in C([0,+\infty); V) \cap C^1([0,+\infty); H). \tag{2.16}$$

Moreover, for $t > 0$*, we have*

$$\|\Phi(t)\|_V^2 + \|\Phi_t(t)\|_H^2 = \|\Phi_0\|_V^2 + \|\Phi_1\|_H^2. \tag{2.17}$$

Theorem 2.1. *Let* $\Phi(x,t) = (v(x,t), \phi(x,t), v(L,t), \phi(L,t))$ *be the solution of the equation* (2.12)*. Then for any* $T > 0$*, there exist constants* $C_1 > 0$ *and* $C_2 > 0$ *depending only on* T*, such that the following estimates hold*

$$C_1 \int_0^T (v_t^2(L,t) + \phi_t^2(L,t))dt \le \|\Phi_0\|_V^2 + \|\Phi_1\|_H^2, \tag{2.18}$$

$$\|\Phi_0\|_V^2 + \|\Phi_1\|_H^2 \le C_2 \int_0^T (v_t^2(L,t) + \phi_t^2(L,t))dt. \tag{2.19}$$

Proof. Since $D(A^\infty)\times D(A^\infty)$ is dense in $V\times H$, we can assume that (v,ϕ) is sufficiently smooth. Then we have

$$\|\Phi_t(t)\|_H^2=\int_0^L (v_t^2(x,t)+\phi_t^2(x,t))dx+v_t^2(L,t)+\phi_t^2(L,t). \tag{2.20}$$

Therefore using (2.17) and (2.20), it is easy to verify that the direct inequality (2.18).

Now, we use the multipliers xv_x and $x\phi_x$ to the equations (2.1) and (2.2). Integrating by parts, we obtain that

(2.21)
$$\begin{aligned}0=&\int_0^L [xv_tv_x]_0^T dx-\int_0^T\int_0^L xv_tv_{xt}dxdt-\int_0^T L(v_x(L,t)-\phi(L,t))v_x(L,t)dt\\&+\int_0^T\int_0^L (v_x-\phi)v_xdxdt+\int_0^T\int_0^L (v_x-\phi)xv_{xx}dxdt,\end{aligned}$$

and

$$\begin{aligned}0=&\int_0^L [x\phi_t\phi_x]_0^T dx-\int_0^T\int_0^L x\phi_t\phi_{xt}dxdt\\&-\int_0^T\int_0^L \phi_x x\phi_{xx}dxdt-\int_0^T\int_0^L (v_x-\phi)x\phi_xdxdt.\end{aligned} \tag{2.22}$$

Inserting

$$\begin{aligned}\int_0^T\int_0^L xv_tv_{xt}dxdt&=\frac{L}{2}\int_0^T v_t^2(L,t)dt-\frac12\int_0^T\int_0^L v_t^2(x,t)dxdt,\\ \int_0^T\int_0^L x\phi_t\phi_{xt}dxdt&=\frac{L}{2}\int_0^T \phi_t^2(L,t)dt-\frac12\int_0^T\int_0^L \phi_t^2(x,t)dxdt,\end{aligned}$$

and

$$\int_0^T\int_0^L x\phi_{xx}\phi_xdxdt=\frac{L}{2}\int_0^T \phi_x^2(L,t)dt-\frac12\int_0^T\int_0^L \phi_x^2(x,t)dxdt,$$

into (2.21) and (2.22), therefore we have

$$\begin{aligned}0=&\int_0^L [xv_tv_x]_0^T dx-\frac{L}{2}\int_0^T v_t^2(L,t)dt+\frac12\int_0^T\int_0^L v_t^2(x,t)dxdt\\&-\int_0^T L(v_x(L,t)-\phi(L,t))v_x(L,t)dt+\int_0^T\int_0^L (v_x-\phi)^2dxdt\\&+\int_0^T\int_0^L (v_x-\phi)\phi dxdt+\int_0^T\int_0^L (v_x-\phi)xv_{xx}dxdt,\end{aligned} \tag{2.23}$$

and

(2.24)

$$0 = \int_0^L [x\phi_t\phi_x]_0^T dx - \frac{L}{2}\int_0^T \phi_t^2(L,t)dt + \frac{1}{2}\int_0^T\int_0^L \phi_t^2(x,t)dxdt - \frac{L}{2}\int_0^T \phi_x^2(L,t)dt + \frac{1}{2}\int_0^T\int_0^L \phi_x^2(x,t)dxdt - \int_0^T\int_0^L (v_x-\phi)x\phi_x dxdt.$$

Taking the sum of (2.23) and (2.24), we obtain

(2.25)

$$\begin{aligned} 0 = &\int_0^L [xv_tv_x + x\phi_t\phi_x]_0^T dx - \frac{L}{2}\int_0^T (v_t^2(L,t)+\phi_t^2(L,t))dt \\ &+ \frac{1}{2}\int_0^T\int_0^L (v_t^2+\phi_t^2)dxdt - \int_0^T L(v_x(L,t)-\phi(L,t))v_x(L,t)dt \\ &+ \int_0^T\int_0^L (v_x-\phi)^2 dxdt + \int_0^T\int_0^L (v_x-\phi)\phi dxdt - \frac{L}{2}\int_0^T \phi_x^2(L,t)dt \\ &+ \frac{1}{2}\int_0^T\int_0^L \phi_x^2(x,t)dxdt + \int_0^T\int_0^L (v_x-\phi)x((v_x-\phi)_x)dxdt. \end{aligned}$$

Because of

$$\int_0^T\int_0^L (v_x-\phi)x((v_x-\phi)_x)dxdt = \frac{L}{2}\int_0^T (v_x(L,t)-\phi(L,t))^2dt - \frac{1}{2}\int_0^T\int_0^L (v_x-\phi)^2dxdt,$$

therefore we have

(2.26)

$$\begin{aligned} &\int_0^T\int_0^L ((v_x-\phi)^2+\phi_x^2)dxdt + \int_0^T\left(\int_0^L (v_t^2+\phi_t^2)dx + v_t^2(L,t)+\phi_t^2(L,t)\right)dt \\ &= (L+1)\int_0^T (v_t^2(L,t)+\phi_t^2(L,t))dt - 2\int_0^L [xv_tv_x + x\phi_t\phi_x]_0^T dx \\ &\quad + L\int_0^T \phi_x^2(L,t)dt + L\int_0^T (v_x(L,t)-\phi(L,t))v_x(L,t)dt \\ &\quad + L\int_0^T (v_x(L,t)-\phi(L,t))\phi(L,t)dt - 2\int_0^T\int_0^L (v_x-\phi)\phi dxdt. \end{aligned}$$

Then

(2.27)

$$
\begin{aligned}
T(\|\Phi(t)\|_V^2 + \|\Phi_t(t)\|_H^2) &= \int_0^T (\|\Phi(t)\|_V^2 + \|\Phi_t(t)\|_H^2)dt \\
&= (L+1)\int_0^T (v_t^2(L,t) + \phi_t^2(L,t))dt - 2\int_0^L [xv_tv_x + x\phi_t\phi_x]_0^T dx \\
&\quad + L\int_0^T (v_x(L,t) - \phi(L,t))v_x(L,t)dt + L\int_0^T (v_x(L,t) - \phi(L,t))\phi(L,t)dt \\
&\quad + L\int_0^T \phi_x^2(L,t)dt - 2\int_0^T\int_0^L (v_x - \phi)\phi dxdt.
\end{aligned}
$$

Now applying Cauchy-Schwartz inequality, we obtain that

$$
\begin{aligned}
&\left|2\int_0^L (xv_tv_x + x\phi_t\phi_x)dx\right| \\
&\qquad \le \varepsilon(\|\Phi(t)\|_V^2 + \|\Phi(t)_t\|_H^2) + C_\varepsilon \|(v,\phi)\|^2_{L^\infty(0,T;H^s(0,L)\times H^s(0,L))},
\end{aligned}
\tag{2.28}
$$

(2.29)

$$
\begin{aligned}
&|L(v_x(L,t) - \phi(L,t))v_x(L,t) + L(v_x(L,t) - \phi(L,t))\phi(L,t) + L\phi_x^2(L,t)| \\
&\qquad \le \varepsilon(\|\Phi(t)\|_V^2 + \|\Phi(t)_t\|_H^2) + C_\varepsilon \|(v,\phi)\|^2_{L^\infty(0,T;H^s(0,L)\times H^s(0,L))},
\end{aligned}
$$

$$
\begin{aligned}
&\left|2\int_0^L (v_x - \phi)\phi dx\right| \\
&\qquad \le \varepsilon(\|\Phi(t)\|_V^2 + \|\Phi(t)_t\|_H^2) + C_\varepsilon \|(v,\phi)\|^2_{L^\infty(0,T;H^s(0,L)\times H^s(0,L))}
\end{aligned}
\tag{2.30}
$$

for any $\varepsilon > 0$, provided $\frac{1}{2} < s < 1$.

Finally inserting (2.28)–(2.30) into (2.27) gives that

$$
\begin{aligned}
&\|\Phi(t)\|_V^2 + \|\Phi_t(t)\|_H^2 \\
&\quad \le C\left(\int_0^T (v_t^2(L,t) + \phi_t^2(L,t))dt + \|(v,\phi)\|^2_{L^\infty(0,T;H^s(0,L)\times H^s(0,L))}\right)
\end{aligned}
\tag{2.31}
$$

provided $0 < 3\varepsilon < T$ and $\frac{1}{2} < s < 1$.

We will use a compactness- uniqueness argument of Zuazua [10] to prove that the term of lower order $\|(v,\phi)\|^2_{L^\infty(0,T;H^s(0,L)\times H^s(0,L))}$ can be absorbed. If (2.19) is false, then there exists a sequence $\{(\Phi_0^n, \Phi_1^n)\} \subset D(A^\infty) \times D(A^\infty)$ such that

$$
\|\Phi^n(t)\|_V^2 + \|\Phi_t^n(t)\|_H^2 = 1, \quad \forall t \in \mathbf{R}^+,
\tag{2.32}
$$

$$
\int_0^T (|v_t^n(L,t)|^2 + |\phi_t^n(L,t)|^2)dt \to 0, \quad n \to \infty,
\tag{2.33}
$$

where $\Phi^n(x,t) = (v^n(x,t), \phi^n(x,t), v^n(L,t), \phi^n(L,t))$ is the solution of the following equation

$$
\Phi_{tt}^n + A\Phi^n = 0, \quad \Phi^n(0) = \Phi_0^n, \quad \Phi_t^n(0) = \Phi_1^n.
\tag{2.34}
$$

On the other hand, from (2.32) it follows that

$$\begin{aligned}\|(v^n,\phi^n)\|^2_{L^\infty(0,T;H^1(0,L)\times H^1(0,L))}\\ +\|(v^n_t,\phi^n_t)\|^2_{L^\infty(0,T;L^2(0,L)\times L^2(0,L))}\le 1.\end{aligned}\tag{2.35}$$

By the compact embedding (see Simon [11]), there exists a sequence $\{(v^n,\phi^n)\}$, still indexed by n for convenience such that

$$(v^n(t),\phi^n(t))\longrightarrow (v(t),\phi(t))\tag{2.36}$$

strongly in $L^\infty(0,T;H^s(0,L)\times H^s(0,L))$ for $\frac{1}{2}<s<1$. Then using (2.33) and (2.36) in (2.31), we see that $\{(\Phi^n(t),\Phi^n_t(t))\}$ is a Cauchy sequence in the space $L^\infty(0,T;V\times H)$. This implies that

$$(\Phi^n(t),\Phi^n_t(t))\longrightarrow (\Phi(t),\Phi_t(t))$$

strongly in $L^\infty(0,T;V\times H)$.

Let $n\longrightarrow\infty$ in (2.32)–(2.33), we obtain $\Phi(x,t)=(v(x,t),\phi(x,t),v(L,t),\phi(L,t))$ solves the equation

$$\Phi_{tt}(t)+A\Phi(t)=0,\quad \Phi(0)=\Phi_0,\quad \Phi_t(0)=\Phi_1,\tag{2.37}$$

and satisfies the following conditions

$$v_t(L,t)=0,\quad \phi_t(L,t)=0,\tag{2.38}$$

$$\|\Phi(t)\|^2_V+\|\Phi_t(t)\|^2_H=1,\quad \forall t\in\mathbf{R}^+.\tag{2.39}$$

From (2.37)–(2.38), a straightforward computation gives

$$\begin{cases} v_{tt}(x,t)-v_{xx}(x,t)+\phi_x(x,t)=0,\quad 0<x<L,\quad t>0,\\ \phi_{tt}(x,t)-\phi_{xx}(x,t)-v_x(x,t)+\phi(x,t)=0,\\ \quad 0<x<L,\quad t>0,\\ v(0,t)=\phi(0,t)=0,\quad t>0,\\ v_{tt}(L,t)=v_x(L,t)-\phi(L,t)=0,\quad t>0,\\ \phi_{tt}(L,t)=\phi_x(L,t)=0,\quad t>0.\end{cases}\tag{2.40}$$

Setting $\widetilde{v}=v_{tt},\widetilde{\phi}=\phi_{tt}$ in (2.40), we find that $(\widetilde{v},\widetilde{\phi})$ solves, in the sense of distributions, the following equation:

$$\begin{cases} \widetilde{v}(x,t)-\widetilde{v}_{xx}(x,t)+\widetilde{\phi}_x(x,t)=0,\quad 0<x<L,\quad t>0,\\ \widetilde{\phi}(x,t)-\widetilde{\phi}_{xx}(x,t)-\widetilde{v}_x(x,t)+\widetilde{\phi}(x,t)=0,\quad 0<x<L,\quad t>0,\\ \widetilde{v}(0,t)=\widetilde{\phi}(0,t)=0,\quad t>0,\\ \widetilde{v}_x(L,t)-\widetilde{\phi}(L,t)=\widetilde{\phi}_x(L,t)=0,\quad t>0.\end{cases}$$

Applying Holmgren's theorem (see Lions [12]), we deduce $\widetilde{v}=v_{tt}=0,\widetilde{\phi}=\phi_{tt}=0$. This implies in turn that

$$\begin{cases} v_{xx}(x,t)-\phi_x(x,t)=0,\quad 0<x<L,\quad t>0,\\ \phi_{xx}(x,t)+v_x(x,t)-\phi(x,t)=0,\quad 0<x<L,\quad t>0,\\ v(0,t)=\phi(0,t)=0,\quad t>0,\\ v_x(L,t)-\phi(L,t)=\phi_x(L,t)=0,\quad t>0.\end{cases}$$

The uniqueness theorem of ODEs shows that there exists a unique solution $(v, \phi) = (0, 0)$, this contradicts (2.39). The proof is complete. $\square$

3. Exact controllability for usual initial data

In this section, we consider the exact controllability of the Timoshenko beam system (1.1) with usual initial data. We first give the following estimates.

Theorem 3.1. *Let $\Phi(x,t) = (v(x,t), \phi(x,t), v(L,t), \phi(L,t))$ be the solution of the homogeneous equation* (2.12). *Then for any $T > 0$, there exist two constants $C_1 > 0$ and $C_2 > 0$ depending only on T such that the following estimates hold*

$$C_1 \int_0^T (v^2(L,t) + \phi^2(L,t))dt \le \|\Phi_0\|_H^2 + \|\Phi_1\|_{V^*}^2, \tag{3.1}$$

$$\|\Phi_0\|_H^2 + \|\Phi_1\|_{V^*}^2 \le C_2 \int_0^T (v^2(L,t) + \phi^2(L,t))dt, \tag{3.2}$$

where V^* is a dual space of V with respect to the pivot space H.

Proof. Because of the density, we consider only $(\Phi_0, \Phi_1) \in D(A^\infty) \times D(A^\infty)$. Defining

$$\widetilde{\Phi}_0 = -A^{-1}\Phi_1, \quad \widetilde{\Phi}_1 = \Phi_0, \tag{3.3}$$

we have

$$\|\widetilde{\Phi}_0\|_V^2 + \|\widetilde{\Phi}_1\|_H^2 = \|\Phi_0\|_H^2 + \|\Phi_1\|_{V^*}^2. \tag{3.4}$$

Now let $\widetilde{\Phi}(x,t) = (\widetilde{v}(x,t), \widetilde{\phi}(x,t), \widetilde{v}(L,t), \widetilde{\phi}(L,t))$ be the solution of the equation

$$\widetilde{\Phi}_{tt}(t) + A\widetilde{\Phi}(t) = 0, \quad \widetilde{\Phi}(0) = \widetilde{\Phi}_0, \quad \widetilde{\Phi}_t(0) = \widetilde{\Phi}_1. \tag{3.5}$$

Then applying the inequalities (2.18)–(2.19) to the solution $\widetilde{\Phi}$ of the equation (3.5), it follows that

$$C_1 \int_0^T (\widetilde{v}_t^2(L,t) + \widetilde{\phi}_t^2(L,t))dt \le \|\widetilde{\Phi}_0\|_V^2 + \|\widetilde{\Phi}_1\|_H^2, \tag{3.6}$$

$$\|\widetilde{\Phi}_0\|_V^2 + \|\widetilde{\Phi}_1\|_H^2 \le C_2 \int_0^T (\widetilde{v}_t^2(L,t) + \widetilde{\phi}_t^2(L,t))dt. \tag{3.7}$$

On the other hand, since $\widetilde{\Phi}_t(0) = \widetilde{\Phi}_1 = \Phi_0$, $\widetilde{\Phi}_{tt}(0) = -A\widetilde{\Phi}(0) = \Phi_1$, it follows that $\widetilde{\Phi}_t = \Phi$. Then replacing $\widetilde{v}_t$ and $\widetilde{\phi}_t$ by v and ϕ in (3.6)–(3.7) and using (3.4) gives the estimates (3.1)–(3.2). Thus the proof is complete. $\square$

Now we consider the following controlled problem

$$(3.8)\qquad \begin{cases} w_{tt} - w_{xx} + \varphi_x = 0, \quad 0 < x < L, \quad t > 0, \\ \varphi_{tt} - \varphi_{xx} - w_x + \varphi = 0, \quad 0 < x < L, \quad t > 0, \\ w(0,t) = \varphi(0,t) = 0, \quad t > 0, \\ w_{tt}(L,t) + w_x(L,t) - \varphi(L,t) = u_1(t), \quad t > 0, \\ \varphi_{tt}(L,t) + \varphi_x(L,t) = u_2(t), \quad t > 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \varphi(x,0) = \varphi_0(x), \\ \varphi_t(x,0) = \varphi_1(x), \quad 0 < x < L. \end{cases}$$

Setting

$$(3.9)\qquad y(x,t) = (w(x,t), \varphi(x,t), w(L,t), \varphi(L,t)), \quad u = (0,0,u_1,u_2),$$

we write the system (3.8) into the following form:

$$(3.10)\qquad y_{tt} + Ay = u, \quad y(0) = y_0, \quad y_t(0) = y_1.$$

Let Φ be a solution of the homogeneous problem (2.12). It follows from the inner product of (3.10) with Φ in H. Integrating by parts, we remark that $(Ay, \Phi)_H = (y, A\Phi)_H = -(y, \Phi_{tt})_H$, we obtain

$$(3.11)\qquad \begin{aligned} &(y_0, \Phi_1)_H - (y_1, \Phi_0)_H + \int_0^t (u_1(s)v(L,s) + u_2(s)\phi(L,s))ds \\ &\quad = (y(t), \Phi_t(t))_H - (y_t(t), \Phi(t))_H. \end{aligned}$$

Next defining the linear form B by setting

$$(3.12)\qquad \begin{aligned} &B(\Phi_0, \Phi_1) \\ &\quad = \langle y_0, \Phi_1 \rangle_{V\times V^*} - \langle y_1, \Phi_0 \rangle_{H\times H} + \int_0^t (u_1(s)v(L,s) + u_2(s)\phi(L,s))ds, \end{aligned}$$

we obtain a weak formulation of equation (3.10)

$$(3.13)\qquad B(\Phi_0, \Phi_1) = \langle (-y_t(t), y(t)), S(t)(\Phi_0, \Phi_1) \rangle_{H\times V, H\times V^*},$$

where $S(t)$ denotes the semigroup of isometries associated to the homogeneous problem (2.12).

Theorem 3.2. *For any* $(y_0, y_1) \in V \times H$ *and* $(u_1, u_2) \in L^2(0,T) \times L^2(0,T)$*, the controlled equation* (3.10) *admits a unique weak solution* y *such that*

$$(3.14)\qquad y \in C(0,T;V) \cap C^1(0,T;H),$$

defined in the sense that the equation (3.13) *is satisfied for all* $(\Phi_0, \Phi_1) \in H \times V^*$ *and all* $0 < t < T$*. Moreover the linear application*

$$(3.15)\qquad (y_0, y_1, u_1, u_2) \longrightarrow (y, y_t)$$

is continuous for the corresponding topologies.

Proof. By Cauchy-Schwartz inequality and theorem 3.1, we have

$$\begin{aligned}\left|\int_0^t (u_1(s)v(L,s)+u_2(s)\phi(L,s))ds\right| \\ &\leq C(\|u_1\|_{L^2(0,T)}+\|u_2\|_{L^2(0,T)})(\|v(L,\cdot)\|_{L^2(0,T)}+\|\phi(L,\cdot)\|_{L^2(0,T)}) \\ &\leq C(\|u_1\|_{L^2(0,T)}+\|u_2\|_{L^2(0,T)})(\|\Phi_0\|_H+\|\Phi_1\|_{V^*}).\end{aligned}$$

This implies that the linear form B is continuous in the space $H\times V^*$. Moreover, we have

$$\|B\| \leq C(\|y_0\|_V+\|y_1\|_H+\|u_1\|_{L^2(0,T)}+\|u_2\|_{L^2(0,T)}). \tag{3.16}$$

From Riesz's representation theorem, there exists a unique $(U(t),U_t(t))\in V\times H$ such that

$$B(\Phi_0,\Phi_1)=\langle(-U_t(t),U(t)),(\Phi_0,\Phi_1)\rangle_{H\times V,H\times V^*},$$

for all $(\Phi_0,\Phi_1)\in H\times V^*$. Then indeed setting

$$(-y_t(t),y(t)):=S(t)(-U_t(t),U(t)),$$

we obtain the equation (3.13). Moreover from (3.16) it follows that

$$\|y(t)\|_V+\|y_t(t)\|_H \leq C(\|y_0\|_V+\|y_1\|_H+\|u_1\|_{L^2(0,T)}+\|u_2\|_{L^2(0,T)}). \tag{3.17}$$

Now let $u_1\in C^\infty([0,T]), u_2\in C^\infty([0,T])$. We know that the equation (3.10) admits a smooth solution y possessing the regularity (3.14). Since $C^\infty([0,T])$ is dense in $L^2(0,T)$, by virtue of (3.17) we see that the weak solution y satisfies also the regularity (3.14). The continuous dependence of the application (3.15) follows also from (3.17). The proof is complete. □

Theorem 3.3. *Let $T>0$, then for all $(y_0,y_1)\in V\times H$, there exists a controller $(u_0,u_1)\in L^2(0,T)\times L^2(0,T)$ such that the weak solution y of controlled problem* (3.10) *satisfies the final conditions*

$$y(T)=y_t(T)=0. \tag{3.18}$$

Proof. Let Φ be the solution of homogeneous system (2.12) with the initial data $(\Phi_0,\Phi_1)\in H\times V^*$. We define the semi-norm

$$\|(\Phi_0,\Phi_1)\|_F^2=\int_0^T (v^2(L,t)+\phi^2(L,t))dt. \tag{3.19}$$

From theorem 3.1, we know that (3.19) defines an equivalent norm in the space $F=H\times V^*$.

Choosing the controllers u_1,u_2 as follows

$$u_1=v(L,t), \quad u_2=\phi(L,t). \tag{3.20}$$

According to the inequality (3.1), we have

$$\|(u_1, u_2)\|^2_{L^2(0,T)\times L^2(0,T)} \le C\|(\Phi_0, \Phi_1)\|^2_{H\times V^*}. \tag{3.21}$$

Next we solve the backward problem

$$y_{tt} + Ay = u, \quad y(T) = y_t(T) = 0. \tag{3.22}$$

From theorem 3.2, we see that (3.22) admits a unique weak solution possessing the regularity (3.14). In particular, we have

$$\|(y(t), y_t(t))\|^2_{V\times H} \le C\|(u_1, u_2)\|^2_{L^2(0,T)\times L^2(0,T)}. \tag{3.23}$$

Now defining the operator Λ as

$$\Lambda(\Phi_0, \Phi_1) = (y_t(0), -y(0)), \quad \forall(\Phi_0, \Phi_1) \in H \times V^*, \tag{3.24}$$

by virtue of the inequalities (3.23) and (3.21) we obtain

$$\|\Lambda(\Phi_0, \Phi_1)\|^2_{H\times V} \le C\|(u_1, u_2)\|^2_{L^2(0,T)\times L^2(0,T)} \le C\|(\Phi_0, \Phi_1)\|^2_{H\times V^*}.$$

This proves that Λ is a linear continuous operator from $H \times V^*$ into $H \times V$.

Now it follows from the inner product of (3.22) with Φ in H, integrating by parts, we obtain

$$-(y_0, \Phi_1)_H + (y_1, \Phi_0)_H = \int_0^T (v^2(L, s) + \phi^2(L, s))ds. \tag{3.25}$$

Interpreting (3.25) into the following form

$$\langle \Lambda(\Phi_0, \Phi_1), (\Phi_0, \Phi_1)\rangle_{H\times V, H\times V^*} = \|(\Phi_0, \Phi_1)\|^2_F \tag{3.26}$$

and using Lax-Milgram's theorem, we deduce that Λ is an isomorphism from $H \times V^*$ into $H \times V$. Therefore given any $(y_1, -y_0) \in H \times V$, there exists a unique $(\Phi_0, \Phi_1) \in H \times V^*$ such that

$$\Lambda(\Phi_0, \Phi_1) = (y_t(0), -y(0)) = (y_1, -y_0). \tag{3.27}$$

This means precisely that the weak solution y of the backward problem (3.22), with the right-hand side u given by (3.20), satisfies the initial value conditions

$$y(0) = y_0, \quad y_t(0) = y_1. \tag{3.28}$$

In other words, we have proved that the system (3.10) is driven to rest by the controls u_1, u_2 given in (3.20). The proof is thus complete. □

Acknowledgements. The author is very grateful to the referees for their valuable suggestions concerning the manuscript.

Department of Mathematics
Hangzhou Dianzi University
Hangzhou 310018
People's Republic of China
e-mail: cgzhang0@sina.com

References

[1] J. U. Kim and Y. Renardy, *Boundary control of the Timoshenko beam*, SIAM J. Control Optim. **25** (6) (1987), 1417–1429.

[2] O. Morgul, *Boundary control of a Timoshenko beam attached to a rigid body: planar motion*, Int. J. Control **54** (4) (1991), 763–781.

[3] Q. X. Yan and D. X. Feng, *Stabilization of Timoshenko Beam with Dissipative Boundary Feedback*, Control Theory Appl. **18** (3) (2001), 353–357.

[4] ———, *Feedback stabilization of nonuniform Timoshenko beam with dynamical boundary*, Control Theory Appl. **20** (5) (2003), 673–677.

[5] C. G. Zhang, *Boundary feedback stabilization of the undamped Timoshenko beam with both ends free*, J. Math. Anal. Appl. **326** (2007), 488–499.

[6] W. Littman and L.v Markus, *Exact boundary controllability of a hybrid system of elasticity*, Arch. Rational Mech. Anal. **103** (1988), 193–236.

[7] B. P. Rao, *Exact boundary controllability of a hybrid system of elasticity by the Hilbert uniqueness method*, to appear.

[8] M. Slemrod, *Feedback stabilization of a linear system in Hilbert space with a priori bounded control*, Math. Control Signals Systems, 1989, 265–285.

[9] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[10] E. Zuazua, *Contrôlabilité exacte en un temps arbitrairement petit de quelques modèles de plaques*, in Lions [12], 465–491.

[11] J. Simon, *Compact sets in the space $L^p(0,T;B)$, Annali di mathematica pura ed applicata* (IV), Vol. CXLVI (1987), 65–96.

[12] J. L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Vol. I, Masson, Paris, 1988.