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Murre's conjectures for certain product varieties

By

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Abstract

We consider Murre's conjectures on Chow groups for a fourfold which is a product of two curves and a surface. We give a result which concerns Conjecture D:the kernel of a certain projector is equal to the homologically trivial part of the Chow group. We also give a proof of Conjecture B for a product of two surfaces.

1. Introduction

Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Let $\Delta \subset X \times X$ be the diagonal. There is a cohomology class $cl(\Delta) \in H^{2d}(X \times X)$.
In this paper we use Betti cohomology with rational coefficients. There is the Künneth decomposition

$$H^{2d}(X \times X) \simeq \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X).$$

We write $cl(\Delta) = \sum_{i=0}^{2d} \pi_i^{hom}$ according to this decomposition. Here $\pi_i^{hom} \in H^{2d-i}(X) \otimes H^i(X)$. If the Künneth conjecture is true, then each π_i^{hom} is an algebraic cycle.
Murre ([Mu], [Mu2]) formulated the following conjecture. For an abelian group M, we write $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$.

(A) The π_i^{hom} lift to a set of orthogonal projectors π_i in $CH^d(X \times X)_{\mathbb{Q}}$ which satisfy the equality

$$\sum_{i=0}^{2d} \pi_i = \Delta.$$

(B) The correspondences $\pi_0, \cdots, \pi_{j-1}, \pi_{2j+1}, \cdots, \pi_{2d}$ act as zero on $CH^j(X)_{\mathbb{Q}}$.

(C) Let $F^\nu CH^j(X) = Ker\pi_{2j} \cap Ker\pi_{2j-1} \cdots \cap Ker\pi_{2j-\nu+1}$. Then the filtration $F^\cdot$ is independent of the choice of π_i.

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(D) $F^1CH^j(X)_{\mathbb{Q}} = CH^j(X)_{hom,\mathbb{Q}}$.
It is shown by Jannsen ([Ja]) that this conjecture of Murre is equivalent to Beilinson's conjectures on the filtration on Chow groups.
There are not yet many evidences for this conjecture. For a projective smooth curve C and a closed point p on C, set $\pi_0 = p \times C$, $\pi_2 = C \times p$ and $\pi_1 = \Delta - \pi_0 - \pi_2$. Then Conjectures (A), (B) and (D) are true for these projectors. For a projective smooth surface Murre ([Mu]) constructed a set of projectors $\pi_0, \cdots, \pi_4$ for which Conjectures (A), (B) and (D) are true. About Conjecture (C) he proved that the filtration on Chow groups given by these projectors is a natural one in the following sense (Theorem 3 in [Mu]):

- $F^1(CH^1(S)_{\mathbb{Q}}) = Ker(\pi_2) = Pic^0(S)_{\mathbb{Q}}$.
- $F^1(CH^2(S)_{\mathbb{Q}}) = CH^2(S)_{hom,\mathbb{Q}}$. $F^2(CH^2(S)_{\mathbb{Q}}) = Ker(\pi_3) = Ker(alb : CH^2(S)_{hom,\mathbb{Q}} \to Alb(S)_{\mathbb{Q}})$.

Conjecture (A) is also true for abelian varieties (Shermenev [Sh], Deninger-Murre [DM]), hypersurfaces (easy), certain class of threefolds (del Angel-Müller-Stach [deM], [deM2]), and some modular varieties (Gordon-Murre [GM], Gordon-Hanamura-Murre [GHM], [GHM2], Miller-Müller-Stach-Wortmann-Yang-Zuo [Pic]).
Note that if Conjecture (A) is true for varieties X and Y, then it is also true for $X \times Y$. One can put $\pi_{i\,X\times Y} = \sum_{p+q=i} \pi_{p\,X} \times \pi_{q\,Y}$.
In [Mu2] Murre proves that Conjectures (B) and (D) are true for a product of a curve and a surface for this product Chow-Künneth decomposition.

Recently Murre ([KMP]) proved the validity of Conjecture (B) and some part of Conjecture (D) for a product of two surfaces. More precisely, Murre proved that Conjecture (D) is true for a product $S_1 \times S_2$ of two smooth projective surfaces except the following part:

The projector $\pi_{2S_1} \times \pi_{2S_2}$ act as zero on $CH^2(S_1 \times S_2)_{hom,\mathbb{Q}}$.

If this is true for the case of a self-product $S_1 = S_2$ of a surface, then Bloch's conjecture ($p_g = 0 \Rightarrow$ albanese map is injective) for S_1 is true. If one assumes that the Chow group of S_1 is finite dimensional in the sense of Kimura ([Ki]), then for an element $z \in CH^2(S_1 \times S_1)_{hom,\mathbb{Q}}$ one has the equality

$$(\pi_2 \times \pi_2(z))^n = 0$$

where ^{n} means the power as a correspondence and n is determined by the second Betti number of S_1.

In this paper we consider Conjecture (D) for the case where X is a product of two curves and a surface $C_1 \times C_2 \times S$. In this case the most crucial part is to show that $\pi_{1C_1} \times \pi_{1C_2} \times \pi_{2S}$ act as zero on $CH^2(X)_{hom,\mathbb{Q}}$. Here the projectors π_{1C_i} for $i = 1$ and 2 are defined as above and we refer the reader to [Mu] for the definition of the projector π_{2S}. Our original aim was to show that if the cohomology $H^1(C_1) \otimes H^1(C_2) \otimes H^2(S)$ has no non-zero Hodge cycle, then $\pi_{1C_1} \times \pi_{1C_2} \times \pi_{2S}$ kills all the codimension 2 cycles on X. We could not completely solve the problem, so instead we studied what kind of cycles are killed by $\pi_{1C_1} \times \pi_{1C_2} \times \pi_{2S}$. It seems that under certain assumptions on X, "generic" cycles are killed by this projector (Theorem 2.1). This is the main result of this paper.

We also give a proof of the essential part of Conjecture (B) for a product of two surfaces. Our proof is similar to that of Murre in that we make essential use of the properties of the Chow-Künneth projectors for surfaces constructed by Murre. However there are still some differences, so we decided to include our proof here. The basic ideas of the proof come from [Mu2].

This paper is organized as follows. In Section two we prove our main result about Conjecture (D). Section three is devoted to a proof of Conjecture (B) for a product of two surfaces.

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2. The main result

Theorem 2.1. *Let C_1 and C_2 be a projective smooth curves over $\mathbb{C}$ and let S be a projective smooth surface over $\mathbb{C}$.*
Let $X = C_1 \times C_2 \times S$. Assume that these varieties satisfy the following conditions:

- *$NS(S) \otimes \mathbb{Q} = \mathbb{Q}H$ where H is a hyperplane section of S.*
- *The cohomology groups $H^1(C_2) \otimes H^1(S)$ and $H^1(C_1) \otimes H^1(S)$ have no non-zero Hodge cycle.*

Let $Z = \sum a_s Z_s$ be a codimension 2 cycle of X which is homologically trivial. Assume that each component Z_s satisfies one of the following conditions.

1. *$pr_{12}(Z_s) \subset C_1 \times C_2$ has dimension ≤ 1.*
2. *$pr_3(Z_s) \subset S$ has dimension ≤ 1.*
3. *$pr_{12} : Z_s \to C_1 \times C_2$ and $pr_3 : Z_s \to S$ are surjective and for $i = 1$ or 2, $pr_{i3}(Z_s) \subset C_i \times S$ satisfies the following:*

There exists a resolution of singularity $f : \mathbf{S} \to pr_{i3}(Z_s)$ for which there exists a divisor Z' on $C_{3-i} \times \mathbf{S}$ with the following property:

- *$(id_{C_{3-i}} \times f)_*(Z') = Z_s$.*
- *Let $cl(Z') \in H^2(C_{3-i} \times \mathbf{S})$ be the cohomology class of Z'. When we write $cl(Z') = c_1 + c_2 + c_3$ according to the decomposition*

$$H^2(C_{3-i} \times \mathbf{S}) \simeq H^2(C_{3-i}) \oplus H^1(C_{3-i}) \otimes H^1(\mathbf{S}) \oplus H^2(\mathbf{S})$$

then c_2 is contained in $H^1(C_{3-i}) \otimes f^* H^1(pr_{i3}(Z_s))$.

Then the Chow-Künneth projector $\pi_{1C_1} \times \pi_{1C_2} \times \pi_{2S}$ kills Z in $CH^2(X)_{\mathbb{Q}}$.

The condition 3 is satisfied if Z_s is a Cartier divisor of $C_{3-i} \times pr_{i3}(Z)$ which is the case if the projection $pr_{i3} : Z_s \to pr_{i3}(Z_s)$ is flat (Lemma 2.3).

Proof. Assume that the condition 1 holds for Z_s. Note that we have a factorization

$$\pi_{1C_1} \times \pi_{1C_2} \times \pi_{2S} = (\pi_{1C_1} \times \pi_{1C_2} \times id_S) \circ (id_{C_1 \times C_2} \times \pi_{2S})$$

and they commute. We write $C = pr_{12}(Z_s) \subset C_1 \times C_2$. Let $\eta_C \overset{j}{\hookrightarrow} C$ be the generic point of C. We apply the projector $id_{C_1 \times C_2} \times (\pi_2)_S$ on Z_s as a cycle on $C \times S$. We have the equality

$$(j \times id_S)^*(id_C \times \pi_{2S})(Z_s) = (\eta_C \times \pi_{2S})((j \times id_S)^* Z_s).$$

We write $(j \times id_S)^* Z_s = Z_{s\eta}$.

Lemma 2.1. *The cycle $Z_{s\eta}$ is algebraically equivalent to a cycle $\eta_C \times E$ on the surface $\eta_C \times S$ where E is a divisor on S defined over the base field $\mathbb{C}$.*

Proof. Consider the cycle $\eta_C \times Z_s$ on $\eta_C \times S \times C = \eta_C \times S \times_{\eta_C} (\eta_C \times C)$. The fiber of $\eta_C \times Z_s$ over $\eta_C \in C(\eta_C) = C_{\eta_C}(\eta_C) = Z_{s\eta}$ and the fiber over a closed point $p \in C(\mathbb{C}) \subset C(\eta_C)$ is of the form $\eta_C \times E$ for a divisor E on S. $\square$

By Lemma 2.1 and ([Mu, Theorem 3]) we have the equality

$$(\eta_C \times \pi_{2S}) Z_{s\eta} = (\eta_C \times \pi_{2S})(\eta_C \times E).$$

Here we use that $\pi_{2S}(Pic^0(S)_{\mathbb{Q}}) = 0$. By taking the closure of this equality in $C \times S$, it follows that

$$(id_C \times \pi_{2S})(Z_s) = C \times \pi_{2S}(E) + \sum_t p_t \times S$$

where for each t, p_t is a closed point on C. Applying $id_C \times \pi_{2S}$ on both sides of the equality kills $p_t \times S$ because by Conjecture (B) for S, $\pi_{2S}(S) = 0$.
By applying $\pi_{1C_1} \times \pi_{1C_2} \times id_S$ on both sides of the equality we see that

$$(\pi_1 \times \pi_1 \times \pi_2)(Z_s) = (\pi_1 \times \pi_1)(C) \times \pi_2(E).$$

If the cycle Z_s satisfies the condition 2, we can see in a similar way that $(\pi_1 \times \pi_1 \times \pi_2)(Z_s)$ is of the form $(\pi_1 \times \pi_1)(C) \times \pi_2(E)$ for a curve E on S and for a divisor C on $C_1 \times C_2$.
Next we assume that the condition 3 holds for Z_s with $i = 2$.

Lemma 2.2. *The subvariety $pr_{23}(Z_s) \subset C_2 \times S$ is an ample divisor.*

Proof. By the assumptions on C_2 and S, we see that

$$NS(C_2 \times S) \otimes \mathbb{Q} = \mathbb{Q}(pt \times S) \oplus \mathbb{Q}(C_2 \times H).$$

We denote $D_1 = pt \times S$ and $D_2 = C_2 \times H$. Write $aD_1 + bD_2$ for the class of $pr_{23}(Z_s)$ in $NS(C_2 \times S) \otimes \mathbb{Q}$. We see that $a = (C_2 \times pt, pr_{23}(Z_s)) > 0$ and $b = \frac{(pt \times H, pr_{23}(Z_s))}{(H,H)} > 0$. Here $(*,*)$ denotes intersection number. So it follows that $pr_{23}(Z_s) - aD_1 - bD_2 \in Pic^0(C_2 \times S) \simeq Pic^0(C_2) \oplus Pic^0(S)$. So there are divisors $d_1 \in Pic(C_2)$ and $d_2 \in Pic(S)$ such that $pr_{23}(Z_s) = pr_2^* d_1 + pr_3^* d_2$ in $Pic(C_2 \times S)$. By Nakai's criterion d_2 is an ample divisor on S and d_1 is ample on C_2. □

By Lemma 2.2 the open subscheme $C_2 \times S - pr_{23}(Z_s)$ is affine. It follows that $H^1(pr_{23}(Z_s)) \simeq H^1(C_2 \times S) \simeq H^1(C_2) \oplus H^1(S)$.
Let $f : \mathbf{S} \to pr_{23}(Z_s)$ be a resolution of singularity which satisfies the condition 3. By the assumption $H^1(C_1) \otimes H^1(S)$ has no Hodge cycle. So there is a divisor D on $C_1 \times C_2$ such that $(pr_1 \times (pr_2 \circ f))^* cl(D) = c_2$.
So there are divisors $d_1 \in Pic(C_1)$ and $d_2 \in Pic(\mathbf{S})$ such that in $Pic(C_1 \times \mathbf{S})$, there is an equality

$$(id_{C_1} \times f)^* Z_s - (pr_1 \times (pr_2 \circ f))^* D = d_1 \times \mathbf{S} + C_1 \times d_2.$$

Pushing down to $C_1 \times pr_{23}(Z_s)$ by the map $id_{C_1} \times f$ we have an equality

$$Z_s = d_1 \times pr_{23}(Z_s) + C_1 \times f_*(d_2) + pr_{12}^* D \cap pr_{23}(Z)$$

in $CH_2(C_1 \times pr_{23}(Z_s))$.
One can see that Chow-Künneth projector $\pi_{1C_1} \times \pi_{1C_2} \times \pi_{2S}$ kills $d_1 \times pr_{23}(Z_s) + C_1 \times f_*(d_2)$ in $CH^2(X)$ because by Conjecture (B) for $C_2 \times S$ ([Mu2]), $\pi_{1C_2} \times \pi_{2S}$ kills $pr_{23}(Z_s)$ and π_{1C_1} kills C_1. Each component of $pr_{12} * (pr_{12}^* D \cap pr_{23}(Z))$ has dimension $\leqq 1$. So by a similar argument to the one above it follows that $(\pi_1 \times \pi_1 \times \pi_2)(Z_s)$ is a sum of the cycles of the form $(\pi_1 \times \pi_1)(C) \times \pi_2(E)$ where C is a curve on $C_1 \times C_2$ and E is a curve on S.
So we can assume that each component $Z_s$of Z is of the form $C \times E$ where C is a curve on $C_1 \times C_2$ and E is a curve on S. Since $(\pi_1 \times \pi_1)(Pic^0(C_1 \times C_2)) = 0$ and $\pi_2(Pic^0(S)) = 0$ and Z is homologically trivial, it follows that $(\pi_1 \times \pi_1 \times \pi_2)(Z) = 0$. □

Lemma 2.3. *If the projection $pr_{23} : Z_s \to pr_{23}(Z_s)$ is flat, then Z_s is a Cartier divisor on $C_1 \times pr_{23}(Z_s)$.*

Proof. Let I_{Z_s} be the ideal sheaf of Z_s in $C_1 \times pr_{23}(Z_s)$. For any point $x \in pr_{23}(Z_s)$, Let $\{z_i\}_i$ be the set of closed points on the fiber $Z_s \times_{pr_{23}(Z_s)} \mathrm{Spec}\kappa(x)$. The image of I_{Z_s} in the local ring $\mathcal{O}_{C_1 \times_{\mathbb{C}} \mathrm{Spec}\kappa(x),\, z_i}$ is a principal ideal (f_i). For each i take a local section $\tilde{f}_i \in I_{Z_s}$ which has the image f_i in $\mathcal{O}_{C_1 \times_{\mathbb{C}} \mathrm{Spec}\kappa(x),\, z_i}$. For a sufficiently small neighborhood U of x in $pr_{23}(Z_s)$ we can consider a Cartier divisor D on $C_1 \times U$ which is defined by the equation $\tilde{f}_i$ in a neighborhood of z_i. Let K be the kernel of natural surjection $\mathcal{O}_D \to \mathcal{O}_{Z_s}$.
Let ϕ_D be the function on the set of points on U defined by

$$\phi_D(y) = \dim_{\kappa(y)} \mathcal{O}_D \otimes_{\mathcal{O}_U} \kappa(y).$$

It is an upper semicontinuous function on U. So there is an neighborhood $U' \subset U$ of x such that for any $y \in U'$, one has

$$\phi_D(y) \leq \phi_D(x).$$

On the other hand, $\dim_{\kappa(y)}\mathcal{O}_{Z_s} \otimes_{\mathcal{O}_U} \kappa(y)$ is a constant function since Z_s is flat over $pr_{23}(Z_s)$. Also note that $\phi_D(y) \geq \dim_{\kappa(y)}\mathcal{O}_{Z_s} \otimes_{\mathcal{O}_U} \kappa(y)$ on U'. Since $\phi_D(x) = \dim_{\kappa(x)}\mathcal{O}_{Z_s} \otimes_{\mathcal{O}_U} \kappa(x)$ it follows that

$$\phi_D(y) = \dim_{\kappa(y)}\mathcal{O}_{Z_s} \otimes_{\mathcal{O}_U} \kappa(y)$$

on U'. As $\mathcal{O}_{Z_s}$ is a flat $\mathcal{O}_U$ module, it follows that $K \otimes_{\mathcal{O}_U} \kappa(y) = 0$ for any point $y \in U'$. Hence $K = 0$. □

3. A proof of Conjecture (B) for a product of two surfaces

In this section we give a proof of the essential part of Conjecture (B) for a product of two surfaces.
Let S_1 and S_2 be projective smooth surfaces over $\mathbb{C}$ and let $X = S_1 \times S_2$. For each S_i there is a Chow-Künneth decomposition $\pi_{0S_i}, \cdots, \pi_{4S_i}$ of the diagonal constructed by Murre ([Mu]). They have the following properties:

π_4, π_3 and π_0 act as 0 on $CH^1(S_i)_{\mathbb{Q}}$.
$F^1CH^1(S_i)_{\mathbb{Q}} = Ker(\pi_2) = CH^1(S_i)_{hom,\mathbb{Q}}$.
$F^2CH^1(S_i)_{\mathbb{Q}} = Ker(\pi_1|_{F^1}) = 0$.

π_0 and π_1 act as 0 on $CH^2(S_i)_{\mathbb{Q}}$.
$F^1CH^2(S_i)_{\mathbb{Q}} = Ker(\pi_4) = CH^2(S_i)_{hom,\mathbb{Q}}$.
$F^2CH^2(S_i)_{\mathbb{Q}} = Ker(\pi_3|_{F^1}) = Ker(alb : CH^2(S_i)_{hom,\mathbb{Q}} \to Alb(S_i) \otimes \mathbb{Q})$.
$F^3CH^2(S_i)_{\mathbb{Q}} = Ker(\pi_2|_{F^2}) = 0$.

There is a Chow-Künneth decomposition for X given by the product of those for S_i.
Murre has proven Conjecture (B) for X. Here we give another proof of the essential part of his result.

Theorem 3.1. *The Chow-Künneth projectors $\pi_{3S_1} \times \pi_{3S_2}$ and $\pi_{3S_1} \times \pi_{2S_2}$ act as zero on $CH^2(X)_{\mathbb{Q}}$.*

Proof. Let Z be an element of $CH^2(X)$. Let $\eta_i \overset{j_i}{\hookrightarrow} S_i$ be the generic point of S_i for $i = 1, 2$ and Z_{η_i} be the generic fiber of Z.
The case of $\pi_{3S_1} \times \pi_{3S_2}$. $(id_{S_1} \times j_2)^*(\pi_{3S_1} \times id_{S_2})(Z) = \pi_3 \times \eta_2((id_{S_1} \times j_2)^*Z)$. We write $\pi_3 \times \eta_2 = \pi_{3\eta_2}$. For $p = 1$ and 2 let $C_p \overset{i_p}{\hookrightarrow} S_p$ be a smooth hyperplane section defined over the base field $\mathbb{C}$. Then by Lemma 2.3 of [Mu], $i_{p*} : Jac(C_p) \to Alb(S_p)$ is a surjection. So it follows that $i_{1*} : Jac(C_1)(\eta_2)_{\mathbb{Q}} \to Alb(S_1)(\eta_2)_{\mathbb{Q}}$ is also surjective. Let d be the degree of Z_{η_2} and let e_1 be a closed point on S_1 which is rational over the base field $\mathbb{C}$. Then $Z_{\eta_2} - d(e_1) \in CH^2(S_{1\eta_2})_{hom,\mathbb{Q}}$ and so there is a cycle $D \in Pic^0C_1(\eta_2)_{\mathbb{Q}}$ such that $alb(Z_{\eta_2} - d(e_1)) = i_{1*}(D)$. Let $\bar{D}$ be the closure of D in X. Since D is supported on $C_1 \times \eta_2$, $\bar{D}$ is supported on $\overline{C_1 \times \eta_2} = C_1 \times S_2$.
Since $Ker\pi_3 = Ker(alb)$, we have the equality

$$(id_{S_1} \times j_2)^*(\pi_{3S_1} \times id_{S_2})(Z - d(e_1) \times S_2 - \bar{D}) = \pi_{3S_{1\eta_2}}(Z_{\eta_2} - d(e_1) - i_{1*}D) = 0.$$

So it follows that

$$(\pi_{3S_1} \times id_{S_2})(Z) = (\pi_{3S_1} \times id_{S_2})(\bar{D}) + d\pi_{3S_1}(e_1) \times S_2 + \sum_k D_k$$

where for each k D_k is supported on $S_1 \times Y_k$ for an irreducible curve Y_k. We apply the projector $\pi_{3S_1} \times id_{S_2}$ again on both sides of the equality. We apply $\pi_{3S_1} \times id_{S_2}$ on each D_k as a cycle on $S_1 \times Y_k$. Let $\eta_Y \overset{j_Y}{\hookrightarrow} Y_k$ be the generic point of Y_k. We have the equality

$$(id_{S_1} \times j_Y)^*(\pi_{3S_1} \times id_{Y_k})(D_k) = (\pi_{3S_1} \times \eta_Y)((id_{S_1} \times j_Y)^* D_k).$$

Since $(id_{S_1} \times j_Y)^* D_k$ is a divisor on the surface $S_1 \times \eta_Y$, from Conjecture (B) for S_1 it follows that

$$(\pi_{3S_1} \times \eta_Y)((id_{S_1} \times j_Y)^* D_k) = 0.$$

By taking closure of this equality in $S_1 \times Y_k$ we have the equality

$$(\pi_{3S_1} \times id_{Y_k})(D_k) = \sum_i S_1 \times p_i$$

where for each i p_i is a closed point on Y_k. Applying $\pi_{3S_1} \times id_{S_2}$ again on both sides of the equality it follows that $(\pi_{3S_1} \times id_{S_2})(S_1 \times p_i) = 0$ since by Conjecture (B) for S_1 $\pi_{3S_1}(S_1) = 0$.
Next we apply $id_{S_1} \times \pi_{3S_2}$ on both sides of the equality. By Conjecture (B) for S_2 we see that

$$(id_{S_1} \times \pi_{3S_2})(d\pi_{3S_1}(e_1) \times S_2) = d\pi_{3S_1}(e_1) \times \pi_{3S_2}(S_2) = 0.$$

Let $\eta_{C_1} \overset{j_{C_1}}{\hookrightarrow} C_1$ be the generic point of C_1. We apply $id_{S_1} \times \pi_{3S_2}$ on $\bar{D}$ as a cycle on $C_1 \times S_2$. By Conjecture (B) for $\eta_{C_1} \times \pi_3$, $(\eta_{C_1} \times \pi_3)(j_{C_1} \times id_{S_2})^*(\bar{D}) = 0$. Since

$$(\pi_{3S_1} \times id_{S_2})(id_{S_1} \times \pi_{3S_2}) = (id_{S_1} \times \pi_{3S_2})(\pi_{3S_1} \times id_{S_2})$$

it follows that

$$\begin{aligned}(id_{S_1} \times \pi_{3S_2})(\pi_{3S_1} \times id_{S_2})(\bar{D}) &= (\pi_{3S_1} \times id_{S_2})(id_{S_1} \times \pi_{3S_2})(\bar{D}) \\ &= (\pi_{3S_1} \times id_{S_2})\left(\sum_l p_l \times S_2\right)\end{aligned}$$

for a set of closed points p_l on S_1. So we are reduced to the case where each component of Z is of the form $pt \times S_2$ for a closed point pt. We can see that the projector $\pi_{3S_1} \times \pi_{3S_2}$ kills $pt \times S_2$, because by Conjecture (B) for surfaces $\pi_{3S_i}(S_i) = 0$ for $i = 1$ and 2.

Remark 1. Murre pointed out that there is a simpler argument than the one above. We use the equality

$$\pi_{3S_1} \times id_{S_2}(Z) = Z \circ {}^t\pi_{3S_1} = Z \circ \pi_{1S_1}$$

where $\circ$ is composition as correspondences and ^{t} is transpose. By construction of π_1 there is a curve C on S_1 such that π_{1S_1} is supported on $C \times S_1$(cf. (ii) of Proposition 2.1 in [KMP]). So one can immediately conclude that $\pi_{3S_1} \times id_{S_2}(Z)$ is supported on $C \times S_2$.

The case of $\pi_{3S_1} \times \pi_{2S_2}$. We use the factorization $\pi_{3S_1} \times \pi_{2S_2} = (id_{S_1} \times \pi_{2S_2})(\pi_{3S_1} \times id_{S_2})$. We have the equality

$$(\pi_{3S_1} \times id_{S_2})(Z) = (\pi_{3S_1} \times id_{S_2})(\bar{D}) + d\pi_{3S_1}(e_1) \times S_2 + \sum_k D_k$$

where for each k D_k is supported on $S_1 \times Y_k$ for an irreducible curve Y_k and $\bar{D}$ is supported on $C_1 \times S_2$. The D_k part can be treated as above. Then we apply $id_{S_1} \times \pi_{2S_2}$ on both sides of the equality. By Conjecture (B) for S_2 it follows that

$$(id_{S_1} \times \pi_{2S_2})(d\pi_{3S_1}(e_1) \times S_2) = d\pi_{3S_1}(e_1) \times \pi_{2S_2}(S_2) = 0.$$

By using the equality

$$(\pi_{3S_1} \times id_{S_2})(id_{S_1} \times \pi_{2S_2}) = (id_{S_1} \times \pi_{2S_2})(\pi_{3S_1} \times id_{S_2})$$

we have

$$(id_{S_1} \times \pi_{2S_2})(\pi_{3S_1} \times id_{S_2})(\bar{D}) = (\pi_{3S_1} \times id_{S_2})(id_{S_1} \times \pi_{2S_2})(\bar{D}).$$

Let $\eta_{C_1} \overset{j_{C_1}}{\hookrightarrow} C_1$ be the generic point of C_1. We apply $id_{S_1} \times \pi_{2S_2}$ on $\bar{D}$ as a cycle on $C_1 \times S_2$. Since the divisor $(j_{C_1} \times id_{S_2})^*(\bar{D})$ on $\eta_{C_1} \times S_2$ is algebraically equivalent to a divisor $\eta_{C_1} \times E$ on $\eta_{C_1} \times S_2$ where E is a divisor on S_2 defined over the base field $\mathbb{C}$, it follows that

$$\begin{aligned}&(j_{C_1} \times id_{S_2})^*(id_{S_1} \times \pi_{2S_2})(\bar{D} - C_1 \times E)\\ &\quad = (\eta_{C_1} \times \pi_{2S_2})((j_{C_1} \times id_{S_2})^*(\bar{D}) - \eta_{C_1} \times E) = 0.\end{aligned}$$

So by taking the closure of equality in $C_1 \times S_2$ it follows that

$$(id_{S_1} \times \pi_{2S_2})(\bar{D}) = (id_{S_1} \times \pi_{2S_2})(C_1 \times E) + \sum_k p_k \times S_2$$

for a set $\{p_k\}$ of closed points on S_1. In this way we are reduced to the case where each component of Z is a product of two curves or is of the form $pt \times S_2$ or $S_1 \times pt$. By Conjecture (B) for surfaces one can see that the projector $\pi_{3S_1} \times \pi_{2S_2}$ kills the cycles of this form in $CH^2(X)_{\mathbb{Q}}$. □

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