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Locally Stein domains over holomorphically convex manifolds

By

Viorel VÂJÂITU

Abstract

Let $\pi : Y \longrightarrow X$ be a domain over a complex space X. Assume that π is locally Stein. Then we show that Y is Stein provided that X is Stein and either there is an open set W containing X_{sing} with $\pi^{-1}(W)$ Stein or π is locally hyperconvex over any point in X_{sing}. In the same vein we show that, if X is q-complete and X has isolated singularities, then Y results q-complete.

1. Introduction

A well-known theorem due to Oka [Oka] states that, if D is a locally Stein domain over $\mathbb{C}^n$, then D is Stein. This remains true for domains over Stein manifolds as well as over $\mathbb{C}^n$ ([DG]). Recently this has been extended to locally Stein domains over Stein spaces with isolated singularities [CD]; namely the following result is proved: (For definitions see Section 2.)

Theorem 0. *Let $\pi : Y \longrightarrow X$ be a locally Stein domain over a Stein space X with isolated singularities. Then Y is Stein.*

Alternatively, we can view this as an extension of Corollary 1 from [AN] which is recovered when (Y, π) is schlicht over X. However, there Andreotti and Narasimhan have proven a more general result, namely,

Theorem. *Let X be a Stein space and D an open set in X which is locally Stein. If there is a neighborhood W of X_{sing} such that $D \cap W$ is Stein, then D is Stein.*

See [AN], Theorem 4. Note that, as is shown in *loc. cit.* Corollary 2, the existence of W is superfluous if D is strongly pseudoconvex at any of its boundary points lying in X_{sing}.

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The goal of this paper is to prove analogous results in the setting of locally Stein domains over singular Stein spaces, *cf.* Theorems 1.1 and 1.2 and Corollary 1.1 from below.

Theorem 1.1. *Let $\pi : Y \longrightarrow X$ be a locally Stein domain over a Stein space X. If there is an open neighborhood W of X_{sing} in X such that $\pi^{-1}(W)$ is Stein, then Y is Stein.*

Corollary 1.1. *Let $\pi : Y \longrightarrow X$ be a locally Stein domain over a Stein space X. If π is locally hyperconvex over points in X_{sing}, then Y is Stein.*

The existence of W, which is given in the subsequent Corollary 2.2, is related to a generalization of Stehlé's criterion of Steinness [S] on Serre's problem; see Proposition 2.1 from below.

In particular, because the notion of a Stein space is invariant under normalization, we get:

Corollary 1.2. *Any locally Stein domain over a two dimensional Stein space is itself Stein.*

In the same circle of ideas we give another interesting generalization of Theorem 0 which is recovered for $q = 1$.

Theorem 1.2. *Let $\pi : Y \longrightarrow X$ be a locally Stein domain over a q-complete space X with isolated singularities. Then Y is q-complete.*

Theorem 1.1 is a consequence, via the existence of resolution of singularities, of the subsequent "smooth" version.

Theorem 1.3. *Let $\pi : D \longrightarrow \Omega$ be a locally Stein domain over a holomorphically convex manifold Ω. Let A be the analytic subset of Ω given as the union of all positive dimensional compact analytic subsets of Ω.*

Then D is holomorphically convex if, and only if, there is an open neighborhood W of A in Ω such that $\pi^{-1}(W)$ is holomorphically convex.

We remark that the existence of W is crucial as there are examples of even non-holomorphically convex surfaces which cover compact surfaces; see *e.g.*, [Na, pp. 451–452].

2. Preliminaries

Throughout this paper complex spaces are assumed to be reduced and with countable topology. As usual, we abbreviate "plurisubharmonic" and write "psh".

Definition 2.1. Let X be a complex space. Following Stehlé [S] we say that X is *hyperconvex* if X is Stein and there exist a continuous psh proper function $\varphi : X \longrightarrow [-1, 0)$.

The space X is said to be C^0-*pseudoconvex* if there exists a continuous psh exhaustion function $\varphi : X \longrightarrow \mathbb{R}$. We may define similarly spaces that are C^k-*pseudoconvex*, where $k = 1, 2, \ldots$ or C^∞-*pseudoconvex.* For the latter, the standard terminology in the literature is *weakly* 1-*complete.*

As examples we note that any holomorphically convex space is weakly 1-complete. The converse is also true for complex spaces of dimension one. For dimension ≥ 2 there are counterexamples, *e.g.* the total space of a topologically trivial holomorphic line bundle F over a compact Riemann surface of genus one such that there is no integer $k \neq 0$ with F^k analytically trivial.

Definition 2.2. Let $\pi : Y \longrightarrow X$ be a holomorphic map between complex spaces. We say that π is *locally Stein* (resp., *locally hyperconvex*) over a set $S \subset X$ if, for every point $x \in S$ there exists an open neighborhood $V = V_x$ of x in X such that $\pi^{-1}(V)$ is Stein (resp., hyperconvex). The set Y, or more precisely the couple (Y, π), is said to be a *domain over* X if π is locally biholomorphic.

Lemma 2.1. *Let Y be a C^0-*pseudoconvex *space and $\pi : Y \longrightarrow X$ a holomorphic map into a Stein space X. If π has fibres Stein, then Y is Stein.*

Proof. Let φ be the function displaying the C^0-pseudoconvexity of Y and ψ a smooth strictly psh function on X. Granting Runge approximation and Grauert's theorem characterizing Stein spaces, the lemma reduces to show that, for any $\lambda \in \mathbb{R}$, on the set $Y(\lambda) := \{y \in Y\,;\, \varphi(y) < \lambda\}$ there are continuous strictly psh functions.

This is an obvious consequence of the following claim.

Claim. On each relatively compact open subset V of Y there are such functions.

In order to do this, using Siu's theorem [Siu] the following condition is satisfied. Let $L := \overline{V}$. Then there are Stein open sets V_j in Y and D_j in X, $j = 1, \ldots, m$, such that:

1) For each index j one has $\pi^{-1}(D_j) \cap L \subset V_j$;

2) $\{\pi^{-1}(D_j)\}_j$ is a covering of L.

Select smooth functions ρ_j on X with compact support S_j contained in D_j, $0 \leq \rho_j \leq 1$, and such that $\{\pi^{-1}(S_j)\}_j$ is still a covering of L. Let ψ_j be a smooth strictly psh function on V_j, $j = 1, \ldots, m$.

Now, for every smooth strictly psh function ψ on X and for every constant $M > 0$ we define a smooth function Φ on V by setting:

$$\Phi(y) = \sum \psi_j(y)\rho_j(\pi(y)) + M\psi(\pi(y)),\ y \in V.$$

Straightforward computations show that for M sufficiently large, this Φ becomes strictly psh on V, whence the proof of the lemma. $\square$

Corollary 2.1. *Let X be a holomorphically convex space and D an open set of X such that D is C^0-pseudoconvex. Then D is Stein if, and only if, the trace of D on any compact irreducible analytic subset of X is Stein.*

Proof. This follows easily from the above lemma using Remmert's reduction for X. □

Proposition 2.1. *Let $\pi : Y \longrightarrow X$ be a holomorphic map of complex spaces such that X is Stein. If π is locally hyperconvex over X, then Y is Stein.*

Corollary 2.2. *Let $\pi : Y \longrightarrow X$ be a holomorphic map of complex spaces and A a Stein analytic subset of X. If π is locally hyperconvex over A, then there is an open set W in X containing A such that $\pi^{-1}(W)$ is Stein.*

Proof of Corollary 2.2. Using Siu's theorem, we may assume that X is Stein. Now we let $V := \cup_{x\in A} V_x$, where V_x is an open neighborhood of x such that $\pi^{-1}(V_x)$ is hyperconvex. By [Na3] there is a continuous psh function ψ on X which is negative when restricted to A and if $W := \{\psi < 0\}$, then $\overline{W} \subset V$. Noticing that $\pi|_{\pi^{-1}(W)} : \pi^{-1}(W) \longrightarrow W$ is locally hyperconvex (over W) and W is Stein, the proof results by Proposition 2.1. □

Proof of Proposition 2.1*, begining.* In order to proceed, we need the subsequent Lemma 2.2 (due essentially to Kerzman and Rosay [KR] for two functions). Before stating it, for practical purposes, we introduce the set $\mathcal{H}$ consisting of all functions $f : [-1, 0) \longrightarrow (-\infty, 0)$ satisfying the following properties:

- $\lim_{t\to 0} f(t) = 0$;
- For any compact interval K in $[-1, 0)$ one has $\sup_K f < 0$.

Obviously the set $\mathcal{H}$ contains all functions $h : [-1, 0) \longrightarrow (-\infty, 0)$ such that $\lim_{t\to 0} h(t) = 0$ and are either continuous or non-decreasing. It is worth to observe that to any $f \in \mathcal{H}$ we may associate in a natural way two non-decreasing functions $f^+, f^- \in \mathcal{H}$ by setting for $t < 0$:

$$f^+(t) = \sup\{f(s)\,;\, -1 \le s \le t\}$$

and

$$f^-(t) = \inf\{f(s)\,;\, t \le s < 0\}.$$

Notice also that $f^- \le f \le f^+$.

Lemma 2.2. *Let $\{f_\nu\}_\nu$ be a sequence of functions in $\mathcal{H}$. Then there is a non-decreasing convex function $\chi \in \mathcal{H}$ such that, for any $\nu, \mu \in \mathbb{N}$,*

$$(\star) \qquad \lim_{t\to 0} \frac{\chi(f_\nu(t))}{\chi(f_\mu(t))} = 1.$$

Proof of Lemma 2.2. We follow the idea due to Kerzman and Rosay [KR] with some modifications due to the presence of infinitely many functions.

For a sequence of points $-1 = r_0 < r_1 < \cdots < r_n < \cdots < 0$ we define a function $\chi \in \mathcal{H}$ as follows: $\chi = -1$ on $[-1, r_1]$, $\chi(r_n) = -1/n$ for $n > 1$ and affine between r_n and r_{n+1}. This gives that χ is non-decreasing. To get convexity it suffices to impose $4|r_n| < |r_{n-1}|$ for $n > 1$.

Now we precise the choice of r_n in order that $(\star)$ is fulfilled. By hypothesis there is a sequence $\{\varepsilon_n\}_n$ of negative numbers, strictly increasing to 0, such that $\varepsilon_1 = -1$ and for all n it holds:

$$\max\{f_1^+(\varepsilon_n), \ldots, f_n^+(\varepsilon_n)\} < 4\min\{f_1^-(\varepsilon_{n+1}), \ldots, f_{n+1}^-(\varepsilon_{n+1})\}.$$

Then put

$$r_n := \begin{cases} \max\{f_1^+(\varepsilon_n), \ldots, f_n^+(\varepsilon_n)\}, & \text{if n is even;} \\ \min\{f_1^-(\varepsilon_n), \ldots, f_n^-(\varepsilon_n)\}, & \text{if n is odd.} \end{cases}$$

Now we verify that the function χ has the desired properties. Granting the above discussion, one has to check only the assertion $(\star)$.

Fix ν and μ. Let n large; *e.g.* $n \geq \max(\nu/2, \mu/2) + 1$. For $\varepsilon_{2n-1} \leq t < \varepsilon_{2n}$ it follows that $r_{2n-1} \leq f_\nu(t) \leq r_{2n}$ and similarly for $f_\mu(t)$. Thus

$$\frac{2n-1}{2n} \leq \frac{\chi(f_\nu(t))}{\chi(f_\mu(t))} \leq \frac{2n}{2n-1}.$$

In the same way, if $\varepsilon_{2n} \leq t < \varepsilon_{2n+1}$ then

$$\frac{2n-1}{2n+2} \leq \frac{\chi(f_\nu(t))}{\chi(f_\mu(t))} \leq \frac{2n+2}{2n-1}.$$

Then $(\star)$ follows immediately, whence the lemma. □

Proof of Proposition 2.1, *ending.* A first idea is to carefully examine the technique developped by Stehlé [S] where the case of locally analytically trivial holomorphic fiber bundle with hyperconvex fiber is treated.

However, we give a proof which avoids Stehlé's m-plurisubharmonicity and use instead the approaches in [KR] and [P1].

We start by choosing open coverings $\{V_i'\}_i$, $\{V_i\}_i$ and $\{V_i''\}_i$ of X such that $V_i' \Subset V_i \Subset V_i''$ and, for all indices i (running through an at most countable set I): $U_i'' := \pi^{-1}(V_i'')$ are hyperconvex. Put $U_i' = \pi^{-1}(V_i')$ and $U_i = \pi^{-1}(V_i)$.

Let $\theta_i : U_i'' \longrightarrow [-1, 0)$ be the function displaying the hyperconvexity of U_i''. Let Λ be the set of all pairs of indices (i, j) with $V_i \cap V_j \neq \emptyset$. For any $(i, j) \in \Lambda$ define the function $f_{ij} : (-\infty, 0) \longrightarrow (-\infty, 0)$ by setting for $t < 0$:

$$f_{ij}(t) = \inf\{\theta_j(x)\,;\, x \in U_i \cap U_j \text{ such that } t \leq \theta_i(x)\}.$$

It is readily seen that each f_{ij} is non-decreasing and $\lim_{t\to 0} f_{ij}(t) = 0$. Add to the set $\{f_{ij}\,;\,(i, j) \in \Lambda\}$ the identity function of $[-1, 0)$ and apply Lemma

2.2. We get the corresponding function χ. Then set $\sigma : [-1, 0) \longrightarrow (0, \infty)$ by $\sigma = \log(-1/\chi)$. Obviously σ is increasing, convex and $\lim_{t\to 0} \sigma(t) = \infty$. Besides, it is easily seen that, for any $(i, j) \in \Lambda$ the function $\sigma(\theta_i) - \sigma(\theta_j)$ is a bounded function over $\pi^{-1}(V_i \cap V_j)$. Therefore, putting $\varphi_i := \sigma(\theta_i)$, $i \in I$, we get psh exhaustion functions $\varphi_i : U_i'' \longrightarrow (0, \infty)$ such that $\varphi_i - \varphi_j$ is bounded on $U_i \cap U_j$ for $(i, j) \in \Lambda$.

Now the patching procedure goes as follows: Select smooth functions with compact support ρ_i on X, $0 \le \rho_i \le 1$, $\rho_i \equiv 1$ on V_i' and supp $\rho_i \subset V_i$. It is easily seen that there are positive constants C_i such that:

$$\varphi_i + C_i\rho_i \circ \pi > \varphi_j + C_j\rho_j \circ \pi, \text{ over } V_i' \cap \partial V_j.$$

Then we let ψ be a strictly psh exhaustive function on X such that $\psi + C_i\rho_i$ is psh for all i.

Finally, we define $\Phi : Y \longrightarrow \mathbb{R}$ by setting for $y \in Y$:

$$\Phi(y) = \max\{\varphi_i(y) + C_i\rho_i(\pi(y)) + \psi(\pi(y))\,;\, i \text{ such that } \pi(y) \in V_i\}.$$

It is straightforward to see that Φ is continuous, psh and exhausts Y. Thus Y is C^0-pseudoconvex so that Y results Stein by Lemma 2.1. $\square$

3. Domains over complex manifolds

Let (D, π) be a connected domain over a complex manifold X; endow D with a riemannian metric g coming from a complete riemannian metric on X. For $x \in X$ and $r > 0$ denote $B(x; r) := \{y \in X\,;\, \text{dist}(y, x) < r\}$. (This "ball" is a relatively compact open subset of X.)

Definition 3.1. Let $\zeta \in D$ and put $x := \pi(\zeta)$. Define the boundary distance from ζ by setting $\delta(\zeta) :=$ the supremum of all $r > 0$ for which there is an open subset $U(\zeta; r)$ in D which contains ζ and is biholomorphic to $B(x; r)$ via π.

Observe that if D is not schlicht, then $\delta < \infty$. Note also that if $U(\zeta; r)$ exists for some $r > 0$, then $U(\zeta; s)$ is relatively compact in D for every s with $0 < s < r$.

For $\varepsilon > 0$ set $D_\varepsilon := \{\zeta \in D\,;\, \delta(\zeta) > \varepsilon\}$.

Lemma 3.1. *Assume that $\pi(D)$ is relatively compact in X. Then, for each $\varepsilon > 0$ there exists a C^∞-smooth function $h : D_\varepsilon \longrightarrow [0, \infty)$ such that:*

a) *For every $c \in \mathbb{R}$ the set $\{\zeta \in D_\varepsilon\,;\, h(\zeta) < c\}$ is relatively compact in D.*
b) *There is $C > 0$ such that: $\forall\, \zeta \in D_\varepsilon$, $\|\partial h(\zeta)\| \le C$ and $\|L(h, \zeta)\| \le C$.*

Remark 1. For X a Kähler manifold this lemma is stated by Takeuchi [T]; he approximates the function $\varphi(\zeta)$ given on D as the distance from ζ to some fixed point $\zeta_0 \in D$ by averaging the integral over the "ball" of (fixed small) radius r of center ζ through its volume. However, the volume in question is only Lipschitz and, in general not differentiable as is the case when $X = \mathbb{C}^n$. Therefore, *a priori* the first approximate $A_r(\varphi)$ does not follow of class C^1. Then he itterates this process once more in order to get the desired "C^2-smooth approximation". When M. Colţoiu sent me a preliminary version of [CD] where he used Takeuchi's lemma, I pointed to him my concern from above. Then he reproved it using an approximation device as in Hörmander's book [Ho] and with a weaker conclusion, namely statement b) from above asserts only the boundedness from below of the corresponding Leviform. Afterwards I discovered that Takeuchi's proof is eventually correct as can be seen using the notion of "injectivity radius" (well-known to geometers).

Below we give another proof of Takeuchi's approximation lemma, more elementary one, based on ideas from [LeB] and [V2].

Proof of Lemma 3.1. Let r and s be positive constants such that $r+2s < \varepsilon$. It is easily seen that for $\zeta \in D_\varepsilon$ and $x \in X$ with $\operatorname{dist}(\pi(\zeta), x) < s$ there exists $\xi \in \pi^{-1}(x)$ such that $U(\xi; r) \subset D_s$ and $U(\xi; r) \ni \zeta$. (In fact, if σ is a section of π over $B(\pi(\zeta); \varepsilon)$ with $\sigma(\pi(\zeta)) = \zeta$, we set $\xi = \sigma(x)$.)

In particular, if $\{x_i\}_{i\in I}$ is a set of points in $\overline{\pi(D_\varepsilon)}$ such that $\{B(x_i; s)\}_i$ cover $\overline{\pi(D_\varepsilon)}$, then $\{U(\zeta_{ij}; r)\}_{ij}$ cover $\overline{D_\varepsilon}$, where $\{\zeta_{ij}\}_{j\in\Lambda_i} := \pi^{-1}(x_i) \cap \overline{D_\varepsilon}$. Moreover, if $j, l \in \Lambda_i$ with $j \neq l$, then $U(\zeta_{ij}; r) \cap U(\zeta_{il}; r) = \emptyset$. Note that as $\pi(D)$ is relatively compact we may choose I be a finite set of indices.

Now fix $r = s = \varepsilon/4$. Let $\{x_i\}_{i\in I}$ and $\{\zeta_{ij}\}_{j\in\Lambda_i}$ be as above. Put $B_i := B(x_i; r)$, $U_{ij} := U(\zeta_{ij}; r)$, $U_i = \cup_{j\in\Lambda_i} U_{ij}$ $(\subset \pi^{-1}(B_i))$ and $B := \cup_{i\in I} B_i$. Let $\{\theta_i\}_i$ be a partition of unity subordinate to the covering $\{B_i\}_i$ of B. Notice that B is an open neighborhood of $\overline{\pi(D_\varepsilon)}$.

For simplicity assume first that D_ε is connected. Fix a pair (i_0, j_0) with $j \in \Lambda_{i_0}$. For each pair of indices (i,j) define ν_{ij} as the length of the shortest chain $U_{i_0j_0}, U_{i_1j_1}, \ldots, U_{i_mj_m}$ with $U_{i_kj_k} \cap U_{i_{k+1}j_{k+1}} \neq \emptyset$ and $(i_m, j_m) = (i,j)$. Observe that if (i,j) and (k,p) are such that $U_{ij} \cap U_{kp} \neq \emptyset$, then $\nu_{ij} - \nu_{kp} \in \{-1, 0, 1\}$.

Consider the locally constant function $h_i : U_i \longrightarrow [0,\infty)$ which equals ν_{ij} on U_{ij}. Let U be the union of all U_{ij}. Then define $h : U \longrightarrow [0,\infty)$ by setting:

$$(\diamond) \qquad h(\zeta) = \sum_i \theta_i(\pi(\zeta)) h_i, \text{ if } \zeta \in U.$$

Here we check the statements of the lemma.

Ad a). Set for $m \in \mathbb{N}$, $T_m := \{(i,j)\,;\, i \in I, j \in \Lambda_i, \nu_{ij} \le m\}$. One shows that T_m is finite for all m by induction since T_1 is a finite set and, if T_m is finite, then T_{m+1} is finite, too.

Now, because every "ball" U_{ij} is relatively compact in D, the assertion follows easily.

Ad b). Let Ω be a small open neighborhood of a given point $\zeta_0 \in D_\varepsilon$ such that $\pi|_\Omega$ is biholomorphic onto its image $\pi(\Omega)$ and, moreover, if $\pi(\Omega) \cap \operatorname{supp}(\theta_i) \neq \emptyset$, then $\pi(\Omega) \subset B_i$. Let $(i_1, j_1), \ldots, (i_r, j_r)$ be all pairs (i_t, j_t) such that $U_{i_t j_t} \supset \Omega$. Thus ($\diamond$) becomes:

$$(\flat) \qquad h(\zeta) = \nu_{i_1 j_1} + \sum_{t=1}^{r} \theta_{i_t}(\pi(\zeta))(\nu_{i_t j_t} - \nu_{i_1 j_1}), \zeta \in \Omega.$$

Granting ($\flat$) and since $\nu_{i_t j_t} - \nu_{i_1 j_1} \in \{-1, 0, 1\}$ (notice also that if K is compact in B, *e.g* $K = \overline{\pi(D_\varepsilon)}$, then $K \cap \mathrm{supp}(\theta_i)$ is compact in B_i) it follows easily that the first and second order derivatives of ρ are bounded.

Finally, if D is not connected, we let $D_\varepsilon^1, D_\varepsilon^2, \ldots, D_\varepsilon^k, \ldots$, be the connected components of D_ε. For each k we have a function $h^{(k)}$ defined as above on D_ε^k. Taking $h : D_\varepsilon \longrightarrow [0, \infty)$, $h|_{D_\varepsilon^k} := k + h^{(k)}$, the properties a) and b) from above follows. $\square$

From [M] we quote:

Lemma 3.2. *Let (D, π) be a domain over a complex manifold X. Let U_1 and U_2 be open subsets of X endowed with riemannian metrics g_1 and g_2, respectively. On each domain $\pi^{-1}(U_i)$ over U_i, $i = 1, 2$, one has the corresponding boundary distance function $\delta_i > 0$. Let K_i be a compact subset in U_i, $i = 1, 2$, such that $\pi^{-1}(K_1 \cap K_2) \neq \emptyset$. Then there is a positive constant C such that on $\pi^{-1}(K_1 \cap K_2)$ one has $C > \delta_1/\delta_2 > 1/C$.*

Let $\mathbb{B}(a; r)$ denotes the open ball in $\mathbb{C}^n$ centered at the point $a \in \mathbb{C}^n$ and of radius $r > 0$.

Lemma 3.3. *Let $\tau : \mathbb{B}^n(0; 2) \longrightarrow \mathbb{C}^n$ be a holomorphic map and f a holomorphic function on $\mathbb{B}^n(0; 2)$, $f \not\equiv 0$, such that $\mathrm{d}\tau$ is of rank n on $\mathbb{B}^n(0; 2) \setminus \{f = 0\}$.*

Let (Ω, π) be a domain over $\mathbb{B}^n(0; 2)$. Let δ' and δ denote boundary distances measured in the domains $(\Omega \setminus \{f \circ \pi = 0\}, \tau \circ \pi)$ and (Ω, π) over $\mathbb{C}^n$ respectively.

Then, there exists positive constants k_0 and C_0 depending only on f and τ, but not on (Ω, π), such that for any $k \geq k_0$ it holds:

$$(3) \qquad -\log \delta' + k \log |f \circ \pi| \leq -\log \delta + kC_0$$

on $\Omega \cap \pi^{-1}\Big(\mathbb{B}^n(0; 1) \setminus \{f = 0\}\Big)$.

Proof. We drop the index n in the notation for a ball. From Lemma 6.5.1 in [FN] we retain the following fact. Consider points $z \in \mathbb{B}(0, 3/2)$. Let $d(z)$ be the distance to $\{f = 0\}$. Then there is an integer N and a neighborhood V of $\{f = 0\}$ in $\mathbb{B}(0, 2)$ such that for $z \in V$ and $0 < \rho < d(z)/4$ we have

$$(\flat) \qquad \tau(\mathbb{B}(z, \rho)) \supset \mathbb{B}(\tau(z), \rho d(z)^N).$$

Moreover, if $\rho > 0$ is small enough, then τ is biholomorphic over $\mathbb{B}(\tau(z), \rho d(z)^N)$; for instance we may take $\rho \leq d(z)^{N+1}$ with $d(z)$ small.

Let M be a constant such that $|f(w)| \leq d(z) \cdot \exp(M)$. It suffices to prove the estimate in the lemma for $\xi \in \Omega$ with $z = \pi(\xi) \in \mathbb{B}(0,1) \cap V \setminus \{f = 0\}$ for a small enough V as above.

We check two alternate cases:

(i) Assume that $\delta(\xi) \geq d(z)^{N+1}$. Therefore there is an open neighborhood W of ξ in Ω that is mapped biholomorphically onto the ball $\mathbb{B}(z, d(z)^{N+1})$.

Then from ($\flat$) and the "moreover" we get that τ is biholomorphic over $\mathbb{B}(\tau(z), d(z)^{2N+1})$. Thus there is an open neighborhood W' of ξ in W that is mapped biholomorphically via $\tau \circ \pi$ onto $\mathbb{B}(\tau(z), d(z)^{2N+1})$ which gives readily that $\delta'(\xi) \geq d(z)^{2N+1}$, hence

$$-\log \delta'(\xi) + k \log |f(z)| \leq (2N + 1 - k) \log(1/d(z)) + kM.$$

Now as $\log(1/d(z)) > 0$ and $-\log \delta(\xi) > 0$, the right hand side of the above inequality is at most $-\log \delta(\xi) + kM$ for $k \geq 2N + 1$.

(ii) Suppose that $\delta(\xi) < d(z)^{N+1}$. Then similarly as above we obtain the inequality $\delta'(\xi) \geq \delta(\xi) d(z)^N$ which in turn gives

$$-\log \delta'(\xi) + k \log |f(z)| \leq -\log \delta(\xi) + (N - k) \log(1/d(z)) + kM.$$

Since $\log(1/d(z)) > 0$, by taking $k \geq N$, the right hand side the above inequality is at most $-\log \delta(\xi) + kM$.

Consequently, the desired inequality in the lemma follows for those points ξ that projects in the set $V \cap \mathbb{B}(0,1) \setminus \{f = 0\}$, with $k_0 = 2N + 1$ and $C_0 = M$. Then we conclude applying Lemma 3.2 for points ξ that maps via π on the compact set $\overline{\mathbb{B}(0,1)} \setminus V$. $\square$

4. Proof of Theorem 1.3

Recall that $\pi : D \longrightarrow \Omega$ is a domain over the holomorphically convex manifold Ω and π is locally Stein. Also for the analytic subset A of Ω given as the union of all compact analytic subsets of positive dimension of Ω, there is an open neighborhood W of A in Ω such that $\pi^{-1}(W)$ is holomorphically convex.

Let $\rho : \Omega \longrightarrow X$ be the Remmert reduction so that X is a normal Stein space and ρ is a proper holomorphic map with connected fibres. Therefore ρ induces a biholomorphic map from $\Omega \setminus A$ onto $X \setminus \rho(A)$.

Let also $\mu' : \pi^{-1}(W) \longrightarrow Y'$ be the Remmert reduction of $\pi^{-1}(W)$; thus Y' is a normal Stein space and μ' is a proper holomorphic map with connected fibers which induces a biholomorphic map from $\pi^{-1}(W) \setminus \pi^{-1}(A)$ onto $Y' \setminus \mu'(\pi^{-1}(A))$.

The standard surgery procedure, via the above biholomorphism, obtained by glueing $D \setminus \pi^{-1}(A)$ and Y', furnishes us a normal complex space Y with a proper holomorphic map $\mu : D \longrightarrow Y$ extending μ'. Moreover there is a canonical holomorphic map $\pi_0 : Y \longrightarrow X$ which is locally Stein over X, *a fortiori* it has fibres Stein. Pictorially we have a natural commutative diagram of holomorphic maps

$$\begin{tikzcd} D \arrow[r, "\pi"] \arrow[d, "\mu"'] & \Omega \arrow[d, "\rho"] \\ Y \arrow[r, "\pi_0"] & X. \end{tikzcd}$$

Notice that, by Runge approximation, there is no loss in generality to suppose that $\pi(D)$ is relatively compact in Ω. On the one hand the statement of the theorem is equivalent to saying that Y is Stein and on the other hand, granting Lemma 2.1 it suffices to prove that Y is C^0-pseudoconvex.

Now, because $\rho(A)$ is an analytic subset of X, which contains the set of singular points of X, by [AN] there are finitely many holomorphic mappings $\tau_k : X \longrightarrow \mathbb{C}^n$ with discrete fibres and holomorphic functions g_k on X, $k = 1, \ldots, m$, such that:

- $\rho(A) = \cap_{k=1}^m \{g_k = 0\}$ and
- the induced maps $\tau_k : X \setminus \{g_k = 0\} \longrightarrow \mathbb{C}^n$ are locally biholomorphic.

Set $\theta_\star := \log(|g_1|^2 + \cdots + |g_m|^2)$. Then $\theta_\star$ is a psh function on X such that $\{\theta_\star = -\infty\} = \rho(A)$. Put $\theta = \theta_\star \circ \rho$ and $f_k = g_k \circ \rho$, $k = 1, \ldots, m$.

Select on X a positive, smooth strictly psh exhaustion function $\sigma_\star$ such that $\sigma_\star + \theta_\star > 0$ on $X \setminus \rho(W)$. Put $\sigma = \sigma_\star \circ \rho$. Then define $V := \{\sigma + \theta < 0\}$ and $U := \{2\sigma + \theta < 0\}$. Obviously U and V are open neighborhoods of A, $\overline{U} \subset V$ and $\overline{V} \subset W$. Then select $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth convex function with $\{\chi = 0\} = (-\infty, 0]$ and $\chi' > 0$ on $(0, \infty)$. Put $\psi_1 := \chi(\sigma + \theta)$ and $\psi_2 := \chi(2\sigma + \theta)$. Then ψ_1 and ψ_2 are smooth psh non-negative functions on D such that $\{\psi_1 = 0\} = V$ and $\{\psi_2 = 0\} = U$. Moreover ψ_1 and ψ_2 are strictly psh at points where they are positive.

Now we produce a psh continuous function $\Phi : D \longrightarrow [0, \infty)$ such that, for any $\lambda \geq 0$, the set $D(\lambda) := \{\zeta \in D\,;\, \Phi(\zeta) < \lambda\}$ is C^0-pseudoconvex.

In order to do this, for each $l = 1, \ldots, m$, let δ_l be the boundary distance function in the domain $Y \setminus \{g_l \circ \pi_0 = 0\}$ over $\mathbb{C}^n$. It is perhaps important to notice that $\Omega \setminus \{f_l = 0\}$ is biholomorphic to $X \setminus \{g_l = 0\}$ and $D \setminus \{f_l \circ \pi\}$ is biholomorphic to $Y \setminus \{g_l \circ \pi_0\}$ (via the naturally induced mappings).

Consider the function $\Phi_l^{(k)} : Y \longrightarrow [0, \infty)$ given by

$$\Phi_l^{(k)} := \begin{cases} \max(-\log \delta_l + k \log |g_l \circ \pi_0|, 0) & \text{on } Y \setminus \{g_l \circ \pi_0 = 0\}, \\ 0 & \text{on } \{g_l \circ \pi_0 = 0\}. \end{cases}$$

By lemmata 3.2 and 3.3 and since $\pi_0(Y)$ is relatively compact in X, it follows that there is a positive integer k_0 such that, for any $k > k_0$, the function $\Phi_l^{(k)}$ becomes continuous on Y. Fix such k. Thus the function $\Phi' : Y \longrightarrow [0, \infty)$ defined by

$$\Phi' := \max(\Phi_1^{(k)}, \cdots, \Phi_l^{(k)})$$

is continuous and psh on Y; it vanishes along the possibly non compact analytic set $\pi_0^{-1}(\rho(A))$.

The desired function is $\Phi := \Phi' \circ \mu$.

In order to check the properties stated above for Φ, let us endow D with a riemannian metric g that comes from a complete riemannian metric on Ω; define the boundary distance $\delta(\zeta)$ for $\zeta \in D$ as in Section 3. For $\varepsilon > 0$ set $D_\varepsilon := \{\zeta \in D\,;\, \delta(\zeta) > \varepsilon\}$.

An important feature of Φ is that, granting lemmata 4 and 5, for each $\lambda \in \mathbb{R}$, there is $\varepsilon = \varepsilon(\lambda) > 0$ such that

$$(\sharp) \qquad D(\lambda) \setminus \pi^{-1}(U) \subset D_\varepsilon.$$

Let $h : D_\varepsilon \longrightarrow [0, \infty)$ be as in Lemma 3.1. Put $\widetilde{\psi}_1 := \psi_1 \circ \pi$ and $\widetilde{\psi}_2 := \psi_2 \circ \pi$. By ($\sharp$), the product function $h\widetilde{\psi}_1$ makes sense as a function on $D(\lambda)$ (one extends it by the value 0 over $D(\lambda) \cap \pi^{-1}(V)$). Now for $M > 0$ define the function $\Psi_\lambda : D(\lambda) \longrightarrow [0, \infty)$ by setting:

$$\Psi_\lambda := h\widetilde{\psi}_1 + M\widetilde{\psi}_2.$$

Observe that Ψ_λ is continuous, it vanishes on $\pi^{-1}(U) \cap D(\lambda)$ and, for every $c \in \mathbb{R}$ the set $\{\Psi_\lambda < c\} \setminus \pi^{-1}(U)$ is relatively compact in D. Moreover Ψ is psh if M is sufficiently large. Clearly it suffices to test the plurisubharmonicity of Ψ_λ only on $D(\lambda) \setminus \pi^{-1}(V)$. Straightforward computations using Lemma 3.1 show that the Levi form of $h\widetilde{\psi}_1$ is bounded from below over the compact set $K := \overline{\pi(D_\varepsilon)} \setminus V$. Then as ψ_2 is strictly psh outside U, *a fortiori* near K, there is $M > 0$ such that Ψ_λ is (even strictly) psh on $D(\lambda) \setminus \pi^{-1}(V)$.

We are now in a position to show that Φ has C^0-pseudoconvex sublevel sets. Indeed, the continuous psh function $\alpha : D(\lambda) \longrightarrow \mathbb{R}$ defined by setting:

$$\alpha := \max\Big(-\log(\lambda - \Phi), \Psi_\lambda \Big)$$

exhausts $D(\lambda) \setminus \pi^{-1}(V)$. Let β be a smooth psh exhaustion function on $\pi^{-1}(W)$; then select a smooth rapidly increasing convex function χ such that $\chi(\alpha) > \beta$ on $D(\lambda) \cap \partial\pi^{-1}(V)$. Therefore $\theta : D(\lambda) \longrightarrow \mathbb{R}$ defined by setting:

$$\theta := \begin{cases} \max(\chi(\alpha), \beta) & \text{on } D(\lambda) \cap \pi^{-1}(V), \\ \chi(\alpha) & \text{on } D(\lambda) \setminus \pi^{-1}(V), \end{cases}$$

is a well-defined continuous psh exhaustion function on $D(\lambda)$ so that every $D(\lambda)$ is C^0-pseudoconvex. This Φ descends to a continuous psh function Φ' on Y whose sublevel sets are all C^0-pseudoconvex, thus Stein granting Lemma 2.1. Therefore Y is Stein by Runge approximation, whence the theorem. □

Remark 2. From the given proof we retain the following fact that will be used in the next section, namely, setting

$$Y(\lambda) := \{y \in Y\,;\, \Phi'(y) < \lambda\},$$

the function $\Psi = \Psi_\lambda$, defined *a priori* on $D(\lambda)$, descends to a continuous psh function Ψ'_λ defined on $Y(\lambda)$.

5. Proof of Theorem 1.2

First let us recall a few facts on *q-convexity* [AG] and on *convexity with respect to linear sets* [P2].

Definition 5.1. Let $\widehat{U}$ be an open set in $\mathbb{C}^n$. A function $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbb{R})$ is said to be *q-convex* if its Leviform $L(\varphi, z)$ has at most $q-1$ eigenvalues which are non-positive, for any $z \in \widehat{U}$; equivalently this means that there is a family $\{M_z\}_{z \in \widehat{U}}$ of complex vector spaces, $M_z \subset \mathbb{C}^n = T_z\mathbb{C}^n$, each M_z with codimension $\leq q-1$, and such that the quadratic form $L(\varphi, z)|_{M_z}$ is positively definite, for any $z \in D$.

Now, let us consider X be a complex space. A (local) chart of X at a point $x \in X$ is a holomorphic embedding $\iota : U \longrightarrow \widehat{U}$, where U is an open neighborhood of x in X and $\widehat{U}$ an open subset of some euclidean space $\mathbb{C}^n$, $n = n(x)$. Holomorphic embedding means that $\iota(U)$ is an analytic subset of $\widehat{U}$ and the induced map $\iota : U \longrightarrow \iota(U)$ is biholomorphic.

Definition 5.2. A function $\varphi \in C^2(X, \mathbb{R})$ is said to be *q-convex* if, for any point of X there is a local chart $\iota : U \longrightarrow \widehat{U}$, $U \ni x$, and a q-convex function $\widehat{\varphi} \in C^2(\widehat{U}, \mathbb{R})$ with $\widehat{\varphi} \circ \iota = \varphi|_U$. X is called *q-complete* if there exists a q-convex exhaustion function φ on X. (The normalization is such that "1-complete $\equiv$ Stein".)

Observe that there are simple examples of q-convex functions (for $q > 1$) whose sum fails to be q-convex. In order to remedy this unpleasant feature, M. Peternell [P2] has given the following definition.

Let T_xX denotes the Zariski tangent space of X at $x \in X$.

Put $TX = \cup_{x \in X} T_xX$. (Notice that for a local chart $\iota : U \longrightarrow \widehat{U}$ at x, the differential map $\iota_{\star,x} : T_xX \longrightarrow \mathbb{C}^n$ is an injective homomorphism of complex vector spaces.)

Definition 5.3. A subset $\mathcal{M}$ of TX is said to be *a linear set over* X if, for every point x of X, $\mathcal{M}_x := \mathcal{M} \cap T_xX$ is a complex vector subspace of T_xX. If $\mathcal{M}$ is a linear set over X, we define

$$\operatorname{codim}_X \mathcal{M} := \sup_{x \in X} \operatorname{codim}_{T_xX} \mathcal{M}_x.$$

If Ω is an open subset of X we have an obvious definition for $\mathcal{M}|_\Omega$ as a linear set over Ω. Moreover, if $\pi : Y \longrightarrow X$ is a holomorphic map of complex spaces, we define $\pi^\star\mathcal{M}$ as follows. For every $y \in Y$ we have an induced $\mathbb{C}$-linear map $\pi_{\star,y} : T_yY \longrightarrow T_{\pi(y)}X$. We set

$$\pi^\star\mathcal{M} := \cup_{y \in Y} (\pi_{\star,y})^{-1}(\mathcal{M}_{\pi(y)}).$$

Obviously $\pi^\star\mathcal{M}$ is a linear set over Y and $\operatorname{codim}_Y\pi^\star\mathcal{M} \leq \operatorname{codim}_X\mathcal{M}$. Below we introduce convexity with respect to linear sets following [P2].

Definition 5.4. Let $\varphi \in C^2(X,\mathbb{R})$ and $\mathcal{M}$ a linear set over X.

- Let $x \in X$. We say that φ is *weakly $\mathcal{M}_x$-convex* if there are: a local chart $\iota : U \longrightarrow \widehat{U}$ of x and $\widehat{\varphi} \in C^2(\widehat{U},\mathbb{R})$ with $\widehat{\varphi}\circ\iota = \varphi|_U$ such that $L(\widehat{\varphi},\iota(x))\iota_{\star,x}(\xi) \geq 0$, for every $\xi \in \mathcal{M}_x$.

- The function φ is called *weakly $\mathcal{M}$-convex* if φ is weakly $\mathcal{M}_x$-convex, for any $x \in X$. Then φ is said to be *$\mathcal{M}$-convex* if X can be covered by open sets U such that $\varphi|_U = \theta + \psi$, where θ is smooth and strictly psh on U and ψ is weakly $\mathcal{M}|_U$-covex.

- The space X is called *$\mathcal{M}$-complete* if there is an exhaustion function $\varphi : X \longrightarrow \mathbb{R}$ which is $\mathcal{M}$-convex.

Lemma 5.1 ([P2]). *Let X be a complex space and φ a q-convex function on X. Then there is a linear set $\mathcal{M}$ over X of codimension $\leq q-1$ such that φ is $\mathcal{M}$-convex.*

In practice we usually deal with functions which are not $\mathcal{M}$-convex, but they might be written locally on open sets as maximum of finitely many $\mathcal{M}$-convex functions. In order to state a useful approximation result (*viz.* Lemma 5.2 from below), let us introduce the family $C(X;\mathcal{M})$ of continuous functions $\varphi : X \longrightarrow \mathbb{R}$ such that X is covered by open sets U (depending on φ) for which there are finitely many continuous psh functions α_j on U and $\mathcal{M}|_U$-convex functions ψ_j, $1 \leq j \leq k$, such that

$$\varphi|_U = \max(\alpha_1 + \psi_1, \ldots, \alpha_k + \psi_k).$$

Notice that $C(X;\mathcal{M})$ is closed under standard operations as is the case with the set of (strictly) psh functions on X; *e.g.* if $\varphi_1, \varphi_2 \in C(X;\mathcal{M})$ and $\chi \in C^2(\mathbb{R},\mathbb{R})$ is strictly increasing and convex, then $\max(\varphi_1,\varphi_2), \varphi_1+\varphi_2, \chi(\varphi_1) \in C(X;\mathcal{M})$.

The following result is to be deduced from [V1] by a simple perturbation argument.

Lemma 5.2. *Let $\mathcal{M}$ be a linear set over a complex space X and $\varphi \in C(X;\mathcal{M})$. Then, for every $\eta \in C^0(X,\mathbb{R})$, $\eta > 0$, there is $\widetilde{\varphi} \in C^\infty(X,\mathbb{R})$ which is $\mathcal{M}$-convex and such that $|\widetilde{\varphi} - \varphi| < \eta$.*

The counterpart of classical Runge approximation which we need here and is stated below follows immediately from ([V2, Lemma 4, p. 514]).

Proposition 5.1. *Let $\mathcal{M}$ be a linear set over a complex space X and $\varphi \in C(X;\mathcal{M})$. Suppose that there is a sequence $\{c_\nu\}$ of real numbers tending to infinity such that each set $X_\nu := \{x \in X\,;\, \varphi(x) < c_\nu\}$ is $\mathcal{M}|_{X_\nu}$-complete. Then X is $\mathcal{M}$-complete. In particular, if $\operatorname{codim}_X\mathcal{M} \leq q-1$, then X results q-complete.*

Proof of Theorem 1.2. (Sketch!) Recall that $\pi : Y \longrightarrow X$ is a locally Stein domain over a q-complete space X with isolated singularities. Let $\rho : \widehat{X} \longrightarrow X$ be a resolution of singularities and consider $\mu : \widehat{Y} \longrightarrow Y$ obtained from the fibered product of (Y, π) with $(\widehat{X}, \rho)$ over X. Then $(\widehat{Y}, \widehat{\pi})$ becomes a domain over $\widehat{X}$ and one has a canonical commutative diagram:

$$\begin{tikzcd}
\widehat{Y} \arrow[r, "\widehat{\pi}"] \arrow[d, "\mu"'] & \widehat{X} \arrow[d, "\rho"] \\
Y \arrow[r, "\pi"] & X.
\end{tikzcd}$$

We want to prove that Y is q-complete. The idea is to examine carefully the proof of our Theorem 1.3 using weak convexity with certain linear set instead of plurisubharmonicity.

In order to do this, let $\varphi : X \longrightarrow \mathbb{R}$ be q-convex and exhaustive. By Lemma 5.1 there is linear set $\mathcal{M}$ over X of codimension $\leq q-1$ such that φ is $\mathcal{M}$-convex.

Granting Proposition 5.1, we may assume that $\pi(Y)$ is relatively compact in X, and thus that X_{sing} is a finite set (although this assertion is not essential for the proof).

In contrast to the Stein case, here X being only q-complete (with $q > 1$) its singular set X_{sing} cannot be defined as common zero set of globally defined holomorphic functions. Nevertheless, we do this locally and then patch using cut-off functions; the corresponding function $\theta_\star$ will have compact support in X, and $\theta_\star = \log(|g_1|^2 + \cdots + |g_m|^2)$ on a Stein neighborhood W of X_{sing}, where $g_1, \ldots, g_m$ define X_{sing}. Then we construct functions $\Phi_l^{(k)}$ on $\pi^{-1}(W)$. They can be patched as in ([V2, pp. 519–520]; or as in the proof of Proposition 2.1 from above) and yield Φ. We show easily that $\widetilde{\Phi} := \Phi + \varphi \circ \pi \in C(Y; \pi^\star\mathcal{M})$.

Also the function Ψ_λ, which is *a priori* defined on $\widehat{Y}(\lambda)$ (see remark 2 in Section 4), descends to a (smooth) weakly $\pi^\star(\mathcal{M})$-convex function on $Y(\lambda)$. (The role of taking the resolution of singularities is to use Takeuchi's approximation lemma!). Then one shows that each set $Y(\lambda) := \{y \in Y\,;\, \widetilde{\Phi}(y) < \lambda\}$ with $\lambda \in \mathbb{R}$, is $\pi^\star\mathcal{M}$-complete. By Proposition 5.1, Y results $\pi^\star\mathcal{M}$-complete; hence Y is q-complete because $\operatorname{codim}_Y \pi^\star\mathcal{M} \leq q-1$. $\square$

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Institute of Mathematics "Simion Stoilow"
of the Romanian Academy
P.O. Box 1-764, RO-014700 Bucharest, Romania
e-mail: Viorel.Vajaitu@imar.ro

and Université des Sciences et Technologies de Lille 1
Laboratoire Paul Painlevé, Bât
M2, F-59655 Villeneuve d'Ascq Cedex, France
e-mail: Viorel.Vajaitu@math.univ-lille1.fr

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