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Energy and Riemannian flows

By

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Abstract

We define and compute the energy of 1-foliations on riemannian manifolds. We then derive the Euler-Lagrange equations associated with the energy. We also prove that Riemannian flows on manifolds of constant curvature are critical if and only if they are isometric. Finally we prove that isometric flows on 3-manifolds are critical if and only if either they are transverse to 2-dimensional foliations or they provide K-contact structures.

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1. Introduction

There have been several studies of the energy of unit vector fields on riemannian manifolds in recent years. The energy (or, up to a multiplicative constant, the total bending) is a measure of deviation of a unit vector field from being parallel; more precisely, the energy of a unit vector field N is the integral over a compact manifold M of the squared norm of the covariant derivative of N. See [3], [4], [11], [21], [22]. Most of these studies focus on the critical points or the Euler-Lagrange equations associated with the energy.

In this paper, we study the variational problem of the energy functional in general and then for Riemannian flows. We also interpret the energy integral in terms of curvature functions associated with the flow. This is motivated by the energy of liquid crystals.

A liquid crystal is an arrangement of molecules whose centers occupy well determined positions in a 3-dimensional domain. While in an ordinary liquid the molecules are in disorder, in a liquid crystal the molecules are ordered along a foliation of dimension 1 (nematic) or dimension 2 (smectic).

To a liquid crystal, we associate a direction D on the region W of $\mathbf{R}^3$ filled by the liquid crystal. When the liquid crystal is nematic, this direction D is

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tangent to the foliation of dimension 1 that orders the molecules. Imagine a nematic is ordered along a foliation $\mathcal{L}$ by parallel lines. If we impose constraints (magnetic field, heating, etc.) the foliation $\mathcal{L}$ is deformed. Physicists associate to this situation a free energy E. Locally, or in the absence of defects which are points or curves in W, the direction D is given by a unit vector field N. We have

$$E = C_1|divN|^2 + C_2|N.curlN|^2 + C_3|N \wedge curlN|^2 \tag{1.1}$$

where C_1, C_2, and C_3 are constants. Compare with Section 2.

A special case that has been frequently studied is the one constant approximation. See [1], [14].

To state the main results of this paper we recall some definitions and we fix notations. Unless otherwise stated, (M^{n+1}, g) is a smooth, closed, connected riemannian manifold of dimension $n+1$, $n \geq 2$, and $\mathcal{L}$ a 1-foliation on M given by a nonsingular unit vector field N. We suppose that M and $\mathcal{L}$ are oriented, and we let μ be the volume form on M coming from the metric g. Let ∇ be the Levi-Civita connection on TM, the tangent bundle to M, associated with the metric g. We will call $e(\mathcal{L})p = \dfrac{1}{2}|\nabla N|^2(p)$ the energy density of $\mathcal{L}$ at the point p. We define the energy of the foliation $\mathcal{L}$ by

$$E(\mathcal{L}) = \int_M e(\mathcal{L})\ \mu \tag{1.2}$$

We say that the foliation $\mathcal{L}$ is harmonic if it is a critical foliation for this energy functional under variations of $\mathcal{L}$ through foliations $\mathcal{L}_t$, $|t| < \epsilon$. This is motivated by the harmonic map theory of Eells and Sampson [9].

In a previous paper [11] we studied harmonic foliations on compact Riemann surfaces and their associated energies. Basically, they are given by the real parts of meromorphic (holomorphic if possible) vector fields. Moreover, their energy integral diverges (except on 2-tori T^2 the energy integral need not diverge) and the finite part of the energy is given by the Green's function associated with the Laplace operator. See [11].

A very important class of foliations are the so-called "measured foliations" or Riemannian foliations, for which the layers (or rather the leaves) are all equidistant. See Section 2. One can think of a Riemannian foliation as being a very rough mathematical model of a smectic liquid crystal. See [15].

We begin by recalling some notions related to foliations in general. Let $\mathcal{L}$ be a $p-$dimensional oriented foliation on a smooth oriented manifold M (no metric yet to be involved) of dimension $n = p + q$. A vector field Y on M is projectable or an infinitesimal automorphism of $\mathcal{L}$, if

$$[X, Y] \in L \text{ for all } X \in L$$

where L is the tangent bundle to $\mathcal{L}$. This means that the local flow (global if M is compact) of Y preserves the foliation, i.e. maps leaves into leaves. In distinguished coordinates $(x;y) = (x_1, \ldots, x_p; y_1, \ldots, y_q)$, such a vector field is of the form

$$Y = \sum_{i=1}^{p} a_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} b_\alpha \frac{\partial}{\partial y_\alpha} \tag{1.3}$$

with $a_i = a_i(x,y)$ and $\dfrac{\partial b_\alpha}{\partial x_i} = 0$, i.e. $b_\alpha = b_\alpha(y)$.

A differential form ω of degree r is basic, if

$$i_X \omega = 0, \ \ \theta(X)\omega = 0 \ \text{ for } \ X \in L. \tag{1.4}$$

Here i_X and $\theta(X)$ are the interior product and the Lie derivative in the direction X. By Cartan's formula we have

$$\theta(X)\omega = d i_X \omega + i_X d\omega \tag{1.5}$$

where d is the exterior derivative.

In distinguished coordinates $(x;y) = (x_1, \ldots, x_p; y_1, \ldots, y_q)$ of $\mathcal{L}$ a basic form of degree r is of the form

$$\omega = \sum_{\alpha_1 < \ldots < \alpha_r} \omega_{\alpha_1 \ldots \alpha_r} dy_{\alpha_1} \wedge \ldots \wedge dy_{\alpha_r} \tag{1.6}$$

where the functions $\omega_{\alpha_1 \ldots \alpha_r}(y)$ are independent of x, i.e. $\dfrac{\partial \omega_{\alpha_1 \ldots \alpha_r}}{\partial x_i} = 0$.

It is clear that projectable vector fields and basic differential forms descend to the local quotient $U/\mathcal{L}$ where U is an open distinguished set.

Finally, a foliation $\mathcal{L}$ is Riemannian if one can equip each transversal submanifold with a metric invariant by the holonomy pseudogroup of $\mathcal{L}$. A metric g on M is bundle-like, if it induces an invariant metric on each transversal submanifold. In distinguished coordinates $(x;y) = (x_1, \ldots, x_p; y_1, \ldots, y_q)$, of $\mathcal{L}$ such a metric g is of the form
(1.7)

$$g = \sum_{i,j=1}^{p} g_{ij}(x;y) dx_i \otimes dx_j + \sum_{i=1,\alpha=1}^{p,q} g_{i\alpha}(x,y) dx_i \otimes dy_\alpha + \sum_{\alpha,\beta=1}^{q} g_{\alpha\beta}(y) dy_\alpha \otimes dy_\beta$$

where the functions $g_{\alpha\beta}(y)$ are independent of x. For more details about Riemannian foliations see [20].

We now suppose that the foliation $\mathcal{L}$ is one dimensional. Let $Q \cong L^\perp$ (via g) be the normal bundle of $\mathcal{L}$, and S be the shape operator on Q. See Section 2.

We prove the following theorem.

Theorem 1.1. *Let $\mathcal{L}$ be an oriented Riemannian flow on a closed oriented riemannian manifold (M^{n+1}, g) of dimension $n+1$, $n \geq 2$. We assume that g is bundle-like with respect to $\mathcal{L}$ and that the mean curvature form of $\mathcal{L}$ is basic with respect to g. Then the flow $\mathcal{L}$ is harmonic if and only if*

$$g(Ric(N) - 2S(\nabla_N N), E) = 0 \quad \text{for all} \quad E \in Q \tag{1.8}$$

where $Ric(N)$ is the Ricci curvature operator in the direction N.

Moreover, if (M, g) has constant sectional curvature C then the following holds.

The flow $\mathcal{L}$ is harmonic if and only if it is isometric, and in this case, $E(\mathcal{L}) = \dfrac{nC}{2} Vol(M)$ if n is even, and $E(\mathcal{L}) = 0$ otherwise, where $Vol(M)$ is the volume of M with respect to the metric g.

Remarks 1.2. (i) A result of Dominguez [8] insures that any bundle-like metric can be modified to another bundle-like for which the mean curvature of $\mathcal{L}$ is basic.

(ii) If the curvature C satisfies $C < 0$, then the flow $\mathcal{L}$ can't be Riemannian. See [17]. See also [10].

(iii) Every Riemannian flow on a closed manifold is locally isometric. See [19]; we recall the key ideas in his proof. Let τ be the metric dual of the vector field $\nabla_N N$. Then $d\tau = 0$ [20]; it follows that, locally $\tau = df$ for some C^∞ function f. The Lie derivative of the metric g in the direction $e^{-f}N$ is zero; i.e., $\theta(e^{-f}N)g = 0$. In particular, the flow $\mathcal{L}$ is locally geodesible with respect to the metric g_1 defined by

$$g_1 = e^{2f} g|_L \oplus g|_{L^\perp} \tag{1.9}$$

However, in general $\mathcal{L}$ is not locally geodesible with respect to the original metric g. Also note that if M is simply connected then $\mathcal{L}$ is globally isometric.

(iv) If (M, g) is an Einstein manifold and $\mathcal{L}$ Riemannian, then $\mathcal{L}$ is harmonic if and only if the vector field $\nabla_N N$ is an asymptotic direction at all points of M.

Corollary 1.3. *Let $\mathcal{L}$ be a Riemannian flow on a closed oriented riemannian manifold M. If $\mathcal{L}$ is transverse to a foliation $\mathcal{F}$, then $\mathcal{L}$ is harmonic if and only if the Ricci curvature in the direction N is proportional to N.*

Now we recall some definitions and facts about contact manifolds. A $(2n+1)-$dimensional manifold M has an almost contact structure if its structural group is reducible to $U(n) \times 1$ or equivalently if it admits a nonsingular vector field ξ (the so-called characteristic vector field), a one-form η and a $(1,1)-$tensor ϕ satisfying

$$\eta(\xi) = 1, \ \phi^2 = -I + \eta \otimes \xi \tag{1.10}$$

where I denotes the field of identity transformations of the tangent spaces at all points. In particular the conditions above imply that $\phi(\xi) = 0$, $\eta \circ \phi = 0$, and the endomorphism ϕ has rank $2n$ at every point in M . A riemannian metric g on M satisfying $g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y)$, for all vector fields X and Y, is called compatible with the almost contact structure, and (ξ, η, ϕ, g) is called an almost contact metric structure, and (M, ξ, η, ϕ, g) is called an almost contact metric manifold. We note that there is always at least a metric g compatible with a given almost contact structure. See [2]. Moreover, we define the fundamental $2-$form Φ by $\Phi(X, Y) = g(X, \phi(Y))$. If in addition, $d\eta = \Phi$ holds, then (M, η) is called a contact metric manifold, that is η satisfies $\eta \wedge (d\eta)^n \neq 0$, and (M, ξ, η, ϕ, g) is called a contact metric manifold. Finally, if M is a contact metric manifold and the characteristic vector field ξ is Killing with respect to the metric g, i.e. $\theta(\xi)g = 0$ then (M, ξ, η, ϕ, g) is called a $K-$contact manifold. It is important to remark that the fact that M is a $K-$contact manifold implies some very interesting restrictions on the curvature of M; for instance the sectional curvature of every plane section containing ξ is equal to 1, i.e. $R_{\xi X \xi X} = g(X, X)$ for all vector fields X orthogonal to ξ. See [2]. Here R is the Riemann curvature tensor. As a consequence, the Ricci curvature $Ric(\xi) = 2n\xi$. It is also remarkable that, when M carries a contact metric structure, the condition $Ric(\xi) = 2n\xi$ is also sufficient. See [2].

Theorem 1.1 has also the following Corollary.

Corollary 1.4. *Let M^{2n+1} be a $K-$contact manifold with structure tensors (ξ, η, ϕ, g). Then the foliation $\mathcal{L}$ defined by the characteristic vector field ξ is harmonic. Moreover, $E(\mathcal{L}) = Vol(M)$.*

We now restrict the dimension of the manifold M to 3. We have:

Theorem 1.5. *Let $\mathcal{L}$ be an oriented isometric flow on a closed oriented riemannian manifold (M^3, g) of dimension 3. We assume that the metric g is bundle-like with respect to $\mathcal{L}$. Then the following properties are equivalent:*

(i) *$\mathcal{L}$ is harmonic*;

(ii) *$\mathcal{L}$ is either transverse to a 2-dimensional foliation $\mathcal{F}$ and $E(\mathcal{L}) = 0$, or $\mathcal{L}$ is transverse to a contact structure on M and $E(\mathcal{L}) = Vol(M, g)$.*

This paper is organized as follows: In Section 2, we give a geometric expression of the energy using the symmetric functions of the curvature of the orthogonal distribution $L^{\perp}$; we also give a criteria for a flow to be harmonic by deriving the Euler-Lagrange equations associated with the energy. In Section 3 we prove Theorem 1.1 by using the results of Section 2. In Section 4, we prove Theorem 1.5 by showing that the integrability tensor of the orthogonal distribution $L^{\perp}$ is constant. Finally, Section 5 will be devoted to comments.

2. Energy of flows

Let the foliation $\mathcal{L}$ be $1-$dimensional and the dimension of M be $n + 1$. Let also $L = T\mathcal{L}$ be the tangent bundle to $\mathcal{L}$ and $Q \cong L^{\perp}$ (via g) the normal bundle of $\mathcal{L}$.

The second fundamental form B of the plane field Q is defined in terms of the unit vector field N by

$$B(X,Y) = g(\nabla_X N, Y) \quad \text{for} \quad X, Y \in Q \tag{2.1}$$

Note that B is not necessarily symmetric. Actually the symmetry of B is equivalent to the integrability of the distribution Q. To B we associate the shape operator $S : Q \longrightarrow Q$ defined by $g(S(X), Y) = B(X,Y)$ for $X, Y \in Q$.

Recall that the symmetric functions of the curvature η_k of Q are defined at any point $x \in M$ by

$$\det(I + tB_x) = \Sigma_{k=0}^{n} \, \eta_k(x) \, t^k \tag{2.2}$$

where I is the identity endomorphism of Q, and B_x is viewed as the shape operator in the direction N. Observe that $\eta_1(x) = \mathit{trace}\ B_x$ is the mean curvature of Q, and $\eta_n(x) = \det(B_x)$.

Finally, let A be the integrability tensor of Q defined by

$$A(X,Y) = g([X,Y], N) \quad for \quad X, Y \in Q \tag{2.3}$$

where $[X,Y]$ denotes the Lie bracket of X and Y. Observe that $A(X,Y) = g(\nabla_Y N, X) - g(\nabla_X N, Y)$ measures the deviation of the operator B from being symmetric.

We prove the following proposition.

Proposition 2.1. *Let $\mathcal{L}$ be a 1-foliation on a riemannian manifold (M^{n+1}, g) given by a nonsingular unit vector field N. Then the energy density of $\mathcal{L}$ is given by*

$$e(\mathcal{L}) = \frac{1}{2}k^2 + \frac{1}{2}\eta_1^2 - \eta_2 + \frac{1}{4}|A|^2 \tag{2.4}$$

where $k = |\nabla_N N|$ is the geodesic curvature of the leaves, η_1 and η_2 are the symmetric functions of Q, and $|A|^2$ is the Hilbert-Schmidt norm of the tensor A.

Proof. We compute $|\nabla N|^2(p)$ for $p \in M$. To do that, consider a local orthonormal frame $(E_0, E_1, \ldots, E_n)$ defined in a neighborhood of p such that

$E_0(p) = N(p)$ and $E_1(p) = \dfrac{\nabla_N N}{|\nabla_N N|}(p)$ (if $(\nabla_N N)(p) = 0$, then any E_1 will be convenient).

The matrix of ∇N at the point p relative to the frame above is given by

$g(\nabla_{E_a} N, E_b)$ for $a, b = 0, 1, \ldots, n+1$, or

$$\begin{pmatrix} 0 & k & 0 & . & . & . & 0 \\ 0 & a_{11} & a_{12} & . & . & . & a_{1n} \\ 0 & a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}$$

Note that $\{a_{ij}\}$ is the matrix of the second fundamental form B of the hyperplane field Q. Therefore, the energy density of $\mathcal{L}$ at p is

$$e(\mathcal{L}) = \frac{1}{2}k^2 + \frac{1}{2}|B|^2.$$

The proof of the proposition follows immediately from the following linear algebra lemma whose proof is elementary. □

Lemma 2.2. *For any $n \times n$ matrix $A = \{a_{ij}\}$, we have*

$$|A|^2 = \Sigma_{i,j=1,2,\ldots,n} \ a_{ij}^2 = \eta_1^2 - 2\eta_2 + \Sigma_{i<j} \ |a_{ij} - a_{ji}|^2 \tag{2.5}$$

where η_1 and η_2 are the symmetric functions of the matrix A.

Remark 2.3. Let M be a region of the Euclidean space $\mathbf{R}^3$. Then

$$e(\mathcal{L}) = \frac{1}{2}|N \wedge curlN|^2 + \frac{1}{2}|divN|^2 + \frac{1}{2}|N.curlN|^2 - \eta_2 \tag{2.6}$$

This follows from the following interpretations:

$k = |N \wedge curlN|$ is the geodesic curvature of the leaves of $\mathcal{L}$,

$\eta_1 = trace\ B = divN$ is the mean curvature of the distribution Q, and

$|A| = |N.curlN|$ is the nonintegrability term of Q. See [13], [14], [18].

Remark 2.4. For any vector field X on a closed riemannian manifold (M, g) we have the following integral formula

$$\int_M |\nabla X|^2 \mu = \int_M \left\{ \frac{1}{2}|\theta(X)g|^2 - |divX|^2 + Ric(X) \right\} \mu \tag{2.7}$$

For a proof see [16, 5.9, 5.10]. This will be used to prove Proposition 2.6.

Assume for now that the flow $\mathcal{L}$ is Riemannian. The expression of the energy will simplify considerably. Recall that $\mathcal{L}$ is Riemannian if one can equip each transversal submanifold with a metric invariant by the holonomy pseudogroup of $\mathcal{L}$. We will suppose that the metric g on M is bundle-like; i.e., it induces a holonomy invariant metric on the normal bundle Q. With these assumptions, the orthogonal distribution Q is totally geodesic, the operator B is antisymmetric, and the mean curvature η_1 vanishes.

We prove the following:

Proposition 2.5. *Let $\mathcal{L}$ be a Riemannian flow on a closed riemannian manifold (M^{n+1}, g). Then*

$$E(\mathcal{L}) = \int_M \left(\frac{1}{2}k^2 + \eta_2\right) \ \mu = \int_M \left(\frac{1}{2}k^2 + \frac{1}{2}Ric(N)\right) \ \mu \tag{2.8}$$

If moreover, (M, g) has constant sectional curvature C (necessarily nonnegative), then

$$E(\mathcal{L}) = \int_M \frac{1}{2}k^2 \ \mu + \frac{nC}{2}Vol(M) \tag{2.9}$$

Proof. It is easy to see that the second symmetric function $\eta_2 = \Sigma_{i<j}|a_{ij}|^2 = -\frac{1}{2}trace(B^2)$. Thus by an integral formula due to Ranjan [17], we have

$$\int_M \eta_2 \ \mu = \frac{1}{2}\int_M Ric(N) \ \mu \tag{2.10}$$

where $Ric(N) = g(Ric(N), N)$ is the Ricci curvature in the direction N. The statements follow immediately from Proposition 2.1 and the fact that $Ric(N) = nC$ in the constant curvature case. □

To state the next proposition we introduce the following differential operators. Let $S^2(M)$ be the bundle of smooth symmetric $(0, 2)-$tensors on M and $\chi(M)$ the Lie algebra of C^∞ vector fields. Define

$\delta : S^2(M) \longrightarrow \chi(M)$ and $\delta^* : \chi(M) \longrightarrow S^2(M)$

by

$\delta h = -tr_{12}\nabla h = -\sum_{i=1}^{n+1}(\nabla_{e_i}h)(e_i, -)$ where $e_1, \ldots, e_{n+1}$ is a local orthonormal frame, and $\delta^* X = \frac{1}{2}\theta(X)g$; here $\theta(X)g$ is the Lie derivative of the metric g in the direction X. δ^* is the adjoint of δ with respect to the global scalar product $\langle , \rangle$ on M that is $\langle \delta h, X\rangle = \langle h, \delta^* X\rangle$. See [5].

We have:

Proposition 2.6. *Let $\mathcal{L}$ be an oriented flow defined by a unit vector field N on a closed oriented riemannian manifold (M^{n+1}, g) of dimension $n+1$. Then $\mathcal{L}$ is harmonic if and only if the "vertical tension field"*

$$\tau(N) = 2\delta\delta^* N + \nabla H + Ric(N) \tag{2.11}$$

is parallel to N, where H is the mean curvature of the orthogonal distribution to $\mathcal{L}$ and $Ric(N)$ is the Ricci curvature in the direction N.

Proof. For any vector field Y perpendicular to N we consider variations of the flow $\mathcal{L}$ by foliations $\mathcal{L}_t$ given by vector fields of the form

$N_t = N + tY$. By Remark 2.4 the energy of $\mathcal{L}_t$

(2.12)

$$E(\mathcal{L}_t) = \int_M \frac{1}{2}|\nabla N_t|^2\mu = \int_M \left\{\frac{1}{4}|\theta(N_t)g|^2 - \frac{1}{2}|divN_t|^2 + \frac{1}{2}Ric(N_t)\right\}\mu$$

Write $\langle,\rangle = \int_M g(,)\mu$ and $\omega_t = \omega + t\psi$ where ω and ψ are the dual forms of N and Y respectively. Also observe that if d^* is the adjoint of the exterior derivative d, we have $divN = -d^*\omega = H$ the mean curvature of the orthogonal distribution $Q \approx L^\perp$.

We compute

$$\begin{aligned}\frac{d}{dt}&E(\mathcal{L}_t)|_{t=0}\\ &= 2\left\langle \frac{d}{dt}|_{t=0}\delta^* N_t, \delta^* N\right\rangle - \left\langle \frac{d}{dt}|_{t=0}d^*\omega_t, d^*\omega\right\rangle + \left\langle \frac{d}{dt}|_{t=0}Ric(\omega_t), \omega\right\rangle\end{aligned}$$

since *Ricci* is symmetric. Thus

$$\begin{aligned}\frac{d}{dt}E(\mathcal{L}_t)|_{t=0} &= 2\langle\delta^*Y, \delta^*N\rangle - \langle d^*\psi, d^*\omega\rangle + \langle Ric(\psi), \omega\rangle\\ &= 2\langle\delta\delta^*N, Y\rangle - \langle dd^*\omega, \psi\rangle + \langle Ric(\omega), \psi\rangle\\ &= \langle 2\delta\delta^*N + \nabla H + Ric(N), Y\rangle.\end{aligned}$$

Since the vector field Y is arbitrary perpendicular to N, the Proposition follows immediately. □

Remarks 2.7. (i) One could also use variations of $\mathcal{L}$ through foliations $\mathcal{L}_t$ given by vector fields of the form $N_t = N + t\xi Y$ defined on a smooth compact domain D with smooth boundary, and ξ is a C^∞ function on D vanishing on the boundary ∂D.

(ii) The energy of $\mathcal{L}$ is given by

$$E(\mathcal{L}) = \frac{1}{2}\int_M g(\tau(N), N)\mu \tag{2.13}$$

Corollary 2.8 ([21]). *Let $\mathcal{L}$ be an isometric flow on a closed oriented riemannian manifold (M, g). The $\mathcal{L}$ is harmonic if and only if the Ricci curvature $Ric(N)$ is parallel to N.*

Proof. If $\mathcal{L}$ is isometric then $\delta^*N = 0$; moreover, isometric flows are Riemannian because their flows preserve the orthogonal distribution. Hence $H = 0$ and the corollary is proved. □

Example 2.9. The Hopf flow $\mathcal{L}$ tangent to the fibration $\pi : S^{2n+1} \longrightarrow \mathbf{CP}^n$ is harmonic because it is isometric and S^{2n+1}, the unit sphere in R^{2n+2} with the induced metric, has constant curvature 1; here $\mathbf{CP}^n$ is the complex projective space of complex dimension n. Note that the energy of $\mathcal{L}$ is given by $\frac{2n}{2}Vol(S^{2n+1}) = n\frac{2\pi^{n+1}}{n!} = \frac{2\pi^{n+1}}{(n-1)!}$. Use Proposition 2.5.

We finish this section by recalling the following theorem which plays a crucial role in the proof of Theorem 1.1.

Theorem 2.10 ([6]). *Let (M^{n+1}, g) be a closed riemannian manifold of constant curvature C, and $\mathcal{L}$ an oriented 1-foliation on M. We assume M is oriented. Then*

$$\int_M \mu_k \ \mu = \begin{cases} C^{\frac{k}{2}} \begin{pmatrix} n/2 \\ k/2 \end{pmatrix} Vol(M), & \text{if } \ n \text{ and } k \text{ are even,} \\ 0 \qquad\qquad\qquad , & \text{otherwise.} \end{cases}$$

See also [13].

Remark 2.11. Observe that when the dimension of M is even, then C=0. This follows from the fact that M admits a nonsingular flow $\mathcal{L}$, which implies that the characteristic Euler−Poincaré $\chi(M)$ of M is zero.

3. Proof of Theorem 1.1

Proof. We begin with the proof of the first statement; since $\mathcal{L}$ is Riemannian we have $H = 0$; whence by Proposition 2.6 it suffices to prove that $\delta\delta^* N = -S(\nabla_N N)$ when we restrict the operator $\delta\delta^* N$ to the normal bundle $L^\perp$.

Let $e_1, e_2, \ldots, e_n, e_{n+1}$ be a local orthonormal frame defined on an open set U such that $e_{n+1} = N$ and $e_1, e_2, \ldots, e_n$ are projectables; this is possible because the flow is Riemannian; moreover, we assume that this local frame is consistent with the orientation of M. We will write $\nabla_N N = \sum_{i=1}^{n} k_i e_i$ for some local functions k_i and $h = 2\delta^* N = \theta(N)g$. The matrix $g(\nabla_{e_A} N, e_B), A, B = 1, 2, \ldots, n+1$ of ∇N with respect to the frame above may be written

$$\begin{pmatrix} 0 & a_{12} & a_{13} & . & . & . & a_{1n} & 0 \\ -a_{12} & 0 & a_{23} & . & . & . & a_{2n} & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & a_{n-1,n} & 0 \\ -a_{1n} & -a_{2n} & . & . & . & -a_{n-1,n} & 0 & 0 \\ k_1 & k_2 & k_3 & . & . & . & k_n & 0 \end{pmatrix}$$

where $\{a_{i,j}\}_{,i,j=1,2,\ldots,n}$ is the matrix of the second fundamental form of $L^\perp$ or also the matrix of the operator S; it is antisymmetric because $\mathcal{L}$ is Riemannian. Therefore, since $h = 2\delta^* N = \{(\nabla N) + (\nabla N)^t\}$ $((\nabla N)^t$ is the transpose of

$(\nabla N))$ we have

$$h = \begin{pmatrix} 0 & 0 & . & . & . & 0 & k_1 \\ 0 & 0 & . & . & . & 0 & k_2 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & k_n \\ k_1 & k_2 & . & . & . & k_n & 0 \end{pmatrix}$$

In order to save writing we agree to use the following range of indices

$$1 \leq i, j, k \leq n; \quad 1 \leq A, B, C \leq n+1.$$

The matrix h is of course symmetric, and satisfies $h(e_i, e_j) = 0$, $h(e_{n+1}, e_{n+1}) = 0$, and $h(e_{n+1}, e_i) = k_i$.

We compute next $\delta h(e_i)$. We have

$$\begin{aligned} \delta h(e_i) &= -\sum_A (\nabla_{e_A} h)(e_A, e_i) \\ &= -\sum_A \{e_A(h(e_A, e_i)) - h(\nabla_{e_A} e_A, e_i) - h(\nabla_{e_A} e_i, e_A)\} \\ &= -N(k_i) + \sum_A \{h(\nabla_{e_A} e_A, e_i) + h(\nabla_{e_A} e_i, e_A)\} \end{aligned}$$

Note that the assumption on the curvature form of $\mathcal{L}$ being basic implies that $N(k_i) = 0$.

For simplicity we will use the Christoffel symbols $\nabla_{e_A} e_B = \sum_C \Gamma_{AB}^C e_C$. Thus

$$\delta h(e_i) = \sum_A \sum_B \Gamma_{AA}^B h(e_B, e_i) + \sum_A \sum_B \Gamma_{Ai}^B h(e_B, e_A)$$

Now $\sum_A \sum_B \Gamma_{AA}^B h(e_B, e_i) = \sum_A \Gamma_{AA}^{n+1} k_i = 0$ because $\Gamma_{AA}^{n+1} = g(\nabla_{e_A} e_A, N) = 0$ and

$$\begin{aligned} \sum_A \sum_B \Gamma_{Ai}^B h(e_B, e_A) &= \sum_A \Gamma_{Ai}^{n+1} h(e_{n+1}, e_A) + \sum_A \sum_j \Gamma_{Ai}^j h(e_j, e_A) \\ &= \sum_j \Gamma_{ji}^{n+1} k_j + \sum_j \Gamma_{n+1,i}^j k_j. \end{aligned}$$

Lemma 3.1. $\Gamma_{ji}^{n+1} = \Gamma_{n+1,i}^j$

Proof. $\Gamma_{ji}^{n+1} = g(\nabla_{e_j} e_i, N) = -g(\nabla_{e_j} N, e_i) = -g([e_j, N] + \nabla_N e_j, e_i) = -g(\nabla_N e_j, e_i)$ (because e_j is projectable) $= g(\nabla_N e_i, e_j) = \Gamma_{n+1,i}^j$. The lemma is proved. □

Thus $\delta h(e_i) = 2\sum_j \Gamma_{ji}^{n+1} k_j$ and the first statement of the Theorem follows immediately because the Γ_{ji}^{n+1}'s are the negatives of the coefficients of the second fundamental form.

To prove the rest of the statements in Theorem 1.1, we suppose that (M, g) has constant sectional curvature C, in particular $Ric(N) = nCN$.

If the flow $\mathcal{L}$ is isometric then it is geodesible and since N has unit length we have $\nabla_N N = 0$. Therefore $\mathcal{L}$ is harmonic. We now prove the converse that is we suppose that $\mathcal{L}$ is harmonic.

If $C = 0$ then $\eta_2 = 0$ by Theorem 2.10, but then $\mathcal{L}$ is transverse to a totally geodesic n-dimensional foliation $\mathcal{F}$; now the foliation $\mathcal{F}$ lifts to a foliation by hyperplanes of the universal cover $\mathbf{R}^{n+1}$ of M, and this implies that $\mathcal{L}$ is a foliation by geodesics. It is easy to see that the flow $\mathcal{L}$ is isometric with respect to the metric g; i.e. $\theta(N)g = 0$ ($\theta(N)g$ is the Lie derivative of the metric g in the direction N). Note that in this case $\mathcal{L}$ is the projection of a linear foliation on the torus T^{n+1}. Also observe that if $S \equiv 0$ or if $n+1$ is even then $C = 0$ by Remark 2.11 , and $\mathcal{L}$ is a foliation by geodesics by the same argument.

If $C > 0$, then S is not identically 0 and $n+1$ is odd. Let $\Sigma = \{p \in M : (\nabla_N N)(p) = 0\}$. Note that the assumption on the mean curvature of $\mathcal{L}$ being basic implies that the set Σ is saturated by $\mathcal{L}$ that is if $p \in \Sigma$ then the leaf $\mathcal{L}_p$ through p is inside Σ.

Now since $\mathcal{L}$ is harmonic, we have $\nabla_N N \in KerS$; this clearly implies that the function η_n is zero on $M \backslash \Sigma$. We claim that $\Sigma \neq \emptyset$. Actually $\mu(\Sigma) > 0$.
If $\mu(\Sigma) = 0$ then since the function η_n is bounded on M, we have:

$$\int_M \eta_n \ \mu = \int_{M\backslash\Sigma} \eta_n \ \mu + \int_\Sigma \eta_n \ \mu = 0.$$

But this contradicts Theorem 2.10. Our claim is sustained. We will prove that $\Sigma = M$. We will follow the exact notations in [6]. For any point p in Σ consider a local orthonormal frame $e_1, e_2, \ldots, e_{n+1}$ defined on an open set $U \subset M$ containing p such that $e_1, e_2, \ldots, e_n$ are projectables, $e_{n+1} = N$, and the frame is consistent with the orientation of M. We also let $\theta_1, \theta_2, \ldots, \theta_{n+1}$ be the dual coframe. Since $\mathcal{L}$ is Riemannian and since $e_1, e_2, \ldots, e_n$ are projectables, the forms $\theta_1, \theta_2, \ldots, \theta_n$ are basic. Recall that the connection forms associated with the frame $e_1, e_2, \ldots, e_{n+1}$ are defined by

$$\omega_{i,j}(u) = g(\nabla_u e_i, e_j) \tag{3.1}$$

Define the differential $n-$forms ψ_k on U using the polynomials in t by

$$\Sigma_{k=0}^n \psi_k t^k = (t\theta_1 + \omega_{1,n+1}) \wedge \ldots \wedge (t\theta_n + \omega_{n,n+1}). \tag{3.2}$$

By [6, pages 22, 23] the forms ψ_k are well defined and satisfy

$$\psi_k \wedge \theta_{n+1} = \eta_{n-k}\ \mu. \tag{3.3}$$

Moreover there are $n-$forms τ_k such that

(3.4)

$$d\tau_k = \psi_k \wedge \theta_{n+1} - C^{\frac{n-k}{2}} \begin{pmatrix} n/2 \\ (n-k)/2 \end{pmatrix} \mu = \left\{ \eta_{n-k} - C^{\frac{n-k}{2}} \begin{pmatrix} n/2 \\ (n-k)/2 \end{pmatrix} \right\} \mu$$

The forms τ_k are defined by

(3.5)

$$\tau_k = (-1)^{n+1} \\ \times \left(\tfrac{1}{n-k} \psi_{k+1} + \tfrac{C(k+2)}{(n-k)(n-k-2)} \psi_{k+3} + \ldots + C^{\frac{n-k-2}{2}}\ \tfrac{(k+2)(k+4)\ldots(n-2)}{2.4.6\ldots(n-2)} \psi_{n-1} \right)$$

See [6, page 28]. We will prove that the forms τ_k are basic.

Lemma 3.2. *The coefficients $\{a_{ij}\}$ of the second fundamental form of $L^\perp$ are basic inside Σ.*

Proof. The flow $\mathcal{L}$ is isometric inside Σ because it is Riemannian and its leaves are geodesics. Therefore the form $d\theta_{n+1}$ is basic. Now since the frame $e_1, e_2, \ldots, e_n$ is projectable the functions $d\theta_{n+1}(e_i, e_j)$ for $i, j = 1, 2, \ldots, n$, are basic. But

$$\begin{aligned} d\theta_{n+1}(e_i, e_j) &= e_i(\theta_{n+1}(e_j)) - e_j(\theta_{n+1}(e_i)) - \theta_{n+1}[e_i, e_j] \\ &= -g(\nabla_{e_i} e_j - \nabla_{e_j} e_i, e_{n+1}) = g(\nabla_{e_i} N, e_j) - g(\nabla_{e_j} N, e_i) = 2a_{ij}. \end{aligned}$$

The lemma is proved. □

Lemma 3.3. *The forms ψ_k are basic inside Σ.*

Proof. From the definition of ψ_k and from the fact that the forms $\theta_1, \ldots, \theta_n$ are basic, it suffices to prove that the connection forms $\omega_{i,n+1}, i = 1, 2, \ldots, n$ are basic. We have

$$i_N \omega_{i,n+1} = \omega_{i,n+1}(N) = g(\nabla_N e_i, N) = -g(\nabla_N N, e_i) = 0$$

and

$$(\theta(N)\omega_{i,n+1})(e_j) = N(\omega_{i,n+1}(e_j)) - \omega_{i,n+1}[N, e_j] = N(g(\nabla_{e_j} e_i, N)) - 0.$$

(because $[N, e_j]$ is proportional to N)

Therefore $(\theta(N)\omega_{i,n+1})(e_j) = -N(a_{ji})$. But $N(a_{ji}) = 0$ by the previous lemma. □

We continue the proof of the Theorem. By the previous lemma the forms τ_k are basic. Since the forms τ_k are of degree n, $d\tau_k = 0$ for all k. In particular, $d\tau_0 = 0$. But

$d\tau_0 = (\eta_n - C^{\frac{n}{2}})\mu$. This clearly implies $\eta_n = C^{\frac{n}{2}}$ inside Σ.

Now, since M is connected, and since $\eta_n = 0$ on $M \backslash \Sigma$ and because $\Sigma \neq \emptyset$, we have $M \backslash \Sigma = \emptyset$. Therefore $\Sigma = M$ and the foliation $\mathcal{L}$ is isometric and harmonic. □

Proof of Corollary 1.3

Proof. If $\mathcal{L}$ is transverse to a foliation $\mathcal{F}$, then S vanishes identically. Therefore $\mathcal{L}$ is harmonic if and only if $Ric(N)$ is proportional to N. □

Proof of Corollary 1.4

Proof. It suffices to observe that for a $K-$contact structure on M, the characteristic vector field ξ is isometric and has unit length; moreover $Ric(N) = 2N$. See [2]. □

4. Proof of Theorem 1.5

Proof. Let the flow $\mathcal{L}$ be isometric and given by a unit vector field N; we also write λ the metric dual of N. Note that since the flow $\mathcal{L}$ is isometric with respect to the given metric g, we have $\nabla_N N = 0$ and therefore

$$\mathcal{L} \text{ harmonic} \iff Ric(N) \text{ is parallel to } N$$

by Theorem 1.1; however, to keep the computations as minimal as possible we will derive the Euler-Lagrange equations for the energy.

The matrix of the second fundamental form B is of the form

$$\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

where β is a C^∞ function that measures the nonintegrablity of the distribution $L^\perp$; by Proposition 2.1 the energy of $\mathcal{L}$ is given by

$$E(\mathcal{L}) = \int_M \beta^2 \ \mu = \frac{1}{2}\int_M |*(\lambda \wedge d\lambda)|^2 \ \mu, \tag{4.1}$$

where $*$ is the Hodge star operator.

For any 1-form ψ perpendicular to λ we consider variations of the form

$$\lambda_t = \lambda + t\psi \text{ and write } \beta(t) = *(\lambda_t \wedge d\lambda_t).$$

We have

$$E(\mathcal{L}_t) = \int_M |\beta(t)|^2 \ \mu = \frac{1}{2}\int_M |*(\lambda_t \wedge d\lambda_t)|^2 \ \mu \tag{4.2}$$

We compute

$$\frac{d}{dt}E(\mathcal{L}_t)|_{t=0} = \int_M \beta(\lambda \wedge d\psi + \psi \wedge d\lambda).$$

Since $d(\psi \wedge \beta\lambda) = d\psi \wedge (\beta\lambda) - \psi \wedge d(\beta\lambda)$, we have by Stokes theorem

$$\frac{d}{dt}E(\mathcal{L}_t)|_{t=0} = \int_M \psi \wedge d\lambda - \psi \wedge d(\beta\lambda) = \int_M (d\lambda - d(\beta\lambda)) \wedge \psi$$

Lemma 4.1. $d\lambda \wedge \psi = 0$

Proof. Since the orbits of N are geodesics we have $i_N d\lambda = 0$ (actually the form $d\lambda$ is basic; i.e., $\theta(N)d\lambda = 0$); this implies that $d\lambda = \psi \wedge \alpha$ for some local 1-form α perpendicular to λ. Therefore $d\lambda \wedge \psi = 0$. □

Thus

$$\frac{d}{dt}E(\mathcal{L}_t)|_{t=0} = -\int_M d\beta \wedge \lambda \wedge \psi.$$

Hence $\mathcal{L}$ is harmonic if and only if $d\beta \wedge \lambda \wedge \psi = 0$ or equivalently $d\beta|_{L^\perp} = 0$.

We will prove that $N(\beta) = 0$. On the one hand we have

$$\theta(N)(d\lambda \wedge \lambda) = (\theta(N)d\lambda) \wedge \lambda + d\lambda \wedge (\theta(N)\lambda) = d\lambda \wedge (i_N d\lambda + d i_N \lambda) = 0,$$

and on the other hand, since $d\lambda \wedge \lambda = 2\beta\mu$ we have

$$\theta(N)(\beta\mu) = 0 \text{ or } d\beta \wedge i_N\mu + \beta\theta(N)\mu = 0;$$

But then $d\beta \wedge i_N\mu = 0$ because $\theta(N)\mu = div(N)\mu = 0$; This relation clearly implies that $N(\beta) = 0$.

Hence $d\beta = 0$ and β is constant (M is connected!).

Without loss of generality $\beta = 0$ or $\beta = 1$.

If $\beta = 0$ then $\mathcal{L}$ is transverse to a codimension 1 foliation $\mathcal{F}$ and $E(\mathcal{L}) = 0$.

If $\beta = 1$ then the $1-$form λ provides a contact structure; moreover since N is isometric we get a $K-$contact structure on M; in addition $E(\mathcal{L}) = vol(M)$. Observe that $Ric(N) = 2N$ in this case. The theorem is proved. □

Example 4.2. (i) Let M be a closed Riemann surface equipped with a smooth metric of constant curvature C. let T^1M be the unit tangent bundle equipped with its canonical metric coming form the metric of M. See [2]. The vertical foliation $\mathcal{L}$ tangent to the fibres is isometric. Moreover, if the genus g of M is not equal to 1, it is well known that this foliation is transverse to a contact structure; thus it is harmonic; in addition, its energy is given by $vol(T^1M) = \dfrac{8\pi^2(1-g)}{C}$. If $g = 1$ then T^1M is trivial, the flow $\mathcal{L}$ is clearly harmonic, and its energy is 0.

(ii) Let $f : M \longrightarrow M$ be an isometry of a closed riemannian surface (M, g). The suspension of f gives a $3-$dimensional manifold M_f which is a fibre bundle over the circle S^1. The fibres are diffeomorphic to M. Let $p : M_f \longrightarrow S^1$ be the projection. The horizontal lift (with respect to a connection) of the canonical vector field $\dfrac{\partial}{\partial\theta}$ on S^1 gives a $1-$dimensional foliation $\mathcal{L}$ on M_f. This flow is transverse to the foliation of M_f by the fibres. Let $G = g \oplus p^*d\theta^2$ be the Kaluza-Klein metric on M_f obtained from g and the pullback metric $d\theta^2$ on S^1. The flow $\mathcal{L}$ is Riemannian because h is an isometry, it is also geodesible. Thus $\mathcal{L}$ is isometric and therefore it is harmonic; moreover, its energy is 0.

Remark 4.3. Isometric flows on $3-$dimensional manifolds are Seifert bundles that is all the leaves are compact and have finite holonomy. The converse is also true. See [7].

5. Comments

The variational problem studied in this paper is contained in the well known Weitzenböck formula (even without any assumptions on the flow $\mathcal{L}$); to see this we suppose that N is a unit vector field tangent to $\mathcal{L}$ and let λ be the metric dual of N. Observe that since $|\nabla N|^2 = |\nabla \lambda|^2$

$$E(\mathcal{L}) = \frac{1}{2}\int_M |\nabla\lambda|^2\mu = \frac{1}{2}\langle\nabla\lambda, \nabla\lambda\rangle \tag{5.1}$$

where $\langle , \rangle$ is the global scalar product on M. The Weitzenböck formula reads

$$\triangle\lambda = -\nabla^*\nabla\lambda + Ric(\lambda) \tag{5.2}$$

where ∇^* is the adjoint connection of ∇ with respect to the global scalar product $\langle , \rangle$, $\triangle$ is the laplace operator; by taking the scalar product of the previous formula with λ and integrating over M we get

$$\langle\triangle\lambda, \lambda\rangle = -\langle\nabla^*\nabla\lambda, \lambda\rangle + \langle Ric(\lambda), \lambda\rangle \tag{5.3}$$

or

$$E(\mathcal{L}) = -\langle\triangle\lambda, \lambda\rangle + \langle Ric(\lambda), \lambda\rangle \tag{5.4}$$

Now since the operators $\triangle$ and *Ricci* are both symmetric, the variational problem leads to $\triangle\lambda - Ric(\lambda)$ is proportional to λ; see also [22].

In this context it is worthwhile to note the following fact.

Proposition 5.1. *Let M be a closed manifold and $\mathcal{F}$ a foliation of codimension one defined by a closed nonsingular 1-form λ. Let also $\mathcal{L}$ be a 1-dimensional foliation given by a nonsingular vector field N and transverse to $\mathcal{F}$. Then there is a metric g on M for which $\mathcal{L}$ is harmonic if and only if $Ric(N)$ is parallel to N.*

Proof. First observe that the foliation $\mathcal{F}$ is Riemannian; a transverse invariant measure is given by $\left|\int_\gamma \lambda\right|$ where γ is any local transverse path. Let g be a bundle-like metric on M for which N is of unit length and the mean curvature of $\mathcal{F}$ is basic with respect to $\mathcal{F}$. Notice that $\triangle\lambda = -dH = -N(H)\lambda$ because H is constant along $\mathcal{F}$.

Therefore, using the Weitzenböck formula we have

$\mathcal{L}$ is harmonic if and only if $Ric(\lambda)$ is parallel to λ. □

Remark 5.2. In the previous proposition, the manifold M cannot be simply connected and the flow $\mathcal{L}$ is not necessarily Riemannian. Moreover, M is a fibre bundle over the circle S^1.

However, in general using the Weitzenböck formula, it is hard to derive geometric information about the flow $\mathcal{L}$ and the computations are unpleasant.

During this work we found that the Ricci curvature tensor is a serious obstacle to a reasonable characterization of harmonic Riemannian flows except in dimension 3; The constant curvature assumption seems to fit within the aspect of harmonic flows.
A forthcoming paper by the author tackles harmonic conformal flows on manifolds of constant curvature.

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