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Macroscopic dimension of the ℓ^p-ball with respect to the ℓ^q-norm

By

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Abstract

We give an estimate of the "macroscopic dimension" of the ℓ^p-ball with respect to the ℓ^q-norm.

1. Introduction

1.1. Macroscopic dimension

Let (X, d) be a compact metric space, Y a topological space. For $\varepsilon > 0$, a continuous map $f : X \to Y$ is called an ε-embedding if $\mathrm{Diam} f^{-1}(y) \le \varepsilon$ for all $y \in Y$. Following Gromov [2, p. 332], we define the "width dimension" $\mathrm{Widim}_\varepsilon X$ as the minimum integer n such that there exist an n-dimensional polyhedron P and an ε-embedding $f : X \to P$. When we need to make the used distance d explicit, we use the notation $\mathrm{Widim}_\varepsilon(X, d)$. If we let $\varepsilon \to 0$, then $\mathrm{Widim}_\varepsilon$ gives the usual covering dimension:

$$\lim_{\varepsilon \to 0} \mathrm{Widim}_\varepsilon X = \dim X.$$

$\mathrm{Widim}_\varepsilon X$ is a "macroscopic" dimension of X at the scale $\ge \varepsilon$ (cf. Gromov [2, p. 341]). It discards the information of X "smaller than ε". For example, $[0, 1] \times [0, \varepsilon]$ (with the Euclidean distance) macroscopically looks one-dimensional ($\varepsilon < 1$):

$$\mathrm{Widim}_\varepsilon [0, 1] \times [0, \varepsilon] = 1.$$

Using this notion, Gromov [2] defines "mean dimension". He proposed open problems about this $\mathrm{Widim}_\varepsilon$ (see [2, pp. 333–334]). In this paper we give (partial) answers to some of them.

In [2, p. 334], he asks whether the simplex $\Delta^{n-1} := \{x \in \mathbb{R}^n |\, x_k \ge 0\, (1 \le k \le n),\ \sum_{k=1}^n x_k = 1\}$ satisfies

$$\mathrm{Widim}_\varepsilon \Delta^{n-1} \sim \mathrm{const}_\varepsilon\, n. \tag{1.1}$$

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Our main result below gives the answer: If we consider the standard Euclidean distance on Δ^{n-1}, then (1.1) does *not* hold.

In [2, p. 334], he also asks what is the value of $\mathrm{Widim}_\varepsilon B_{\ell^p}(\mathbb{R}^n)$ with respect to the ℓ^q-norm, where (for $1 \le p$)

$$B_{\ell^p}(\mathbb{R}^n) := \left\{ x \in \mathbb{R}^n \middle| \sum_{k=1}^n |x_k|^p \le 1 \right\}.$$

Our main result concerns this problem. For $1 \le q \le \infty$, let d_{ℓ^q} be the ℓ^q-distance on $\mathbb{R}^n$ given by

$$d_{\ell^q}(x, y) := \left(\sum_{k=1}^n |x_k - y_k|^q \right)^{1/q}.$$

We want to know the value of $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})$. Especially we are interested in the behavior of $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})$ as $n \to \infty$ for small (but fixed) ε. When $q = p$, we have (from "Widim inequality" in [2, p. 333])

$$\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^p}) = n \quad \text{for all } \varepsilon < 1. \tag{1.2}$$

(For its proof, see Gromov [2, p. 333], Gournay [1, Lemma 2.5] or Tsukamoto [7, Appendix A].) More generally, if $1 \le q \le p \le \infty$, then $d_{\ell^p} \le d_{\ell^q}$ and hence

$$\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) = n \quad \text{for all } \varepsilon < 1. \tag{1.3}$$

I think this is a satisfactory answer. (For the case of $\varepsilon \ge 1$, there are still some problems; see Gournay [1].) So the problem is the case of $1 \le p < q \le \infty$. Our main result is the following:

Theorem 1.1. *Let $1 \le p < q \le \infty$ (q may be ∞). We define $r\,(\ge p)$ by $1/r = 1/p - 1/q$. For any $\varepsilon > 0$ and $n \ge 1$, we have*

$$\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \le \min(n, \lceil (2/\varepsilon)^r \rceil - 1), \tag{1.4}$$

where $\lceil (2/\varepsilon)^r \rceil$ denotes the smallest integer $\ge (2/\varepsilon)^r$. Note that the right-hand-side of (1.4) *is constant for large n (and fixed ε). Therefore the value of* $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})$ *becomes stable as $n \to \infty$.*

This result makes a sharp contrast with the above (1.3). For the simplex $\Delta^{n-1} \subset \mathbb{R}^n$ we have

$$\mathrm{Widim}_\varepsilon \Delta^{n-1} \le \mathrm{Widim}_\varepsilon(B_{\ell^1}(\mathbb{R}^n), d_{\ell^2}) \le \lceil (2/\varepsilon)^2 \rceil - 1.$$

Therefore (1.1) does not hold. This result means that the "macroscopic dimension" of Δ^{n-1} is constant for large n.

When $q = \infty$, we can prove that the inequality (1.4) actually becomes an equality:

Corollary 1.1. *For $1 \le p < \infty$,*

$$\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) = \min(n, \lceil (2/\varepsilon)^p \rceil - 1).$$

This result was already obtained by A. Gournay [1, Proposition 1.3]; see Remark 2 at the end of the introduction. For general $q > p$, I don't have an exact formula. However we can prove the following asymptotic result as a corollary of Theorem 1.1.

Corollary 1.2. *For $1 \le p < q \le \infty$,*

$$\lim_{\varepsilon \to 0} \left(\lim_{n \to \infty} \log \mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) / |\log \varepsilon| \right) = r = \frac{pq}{q-p}.$$

Note that there exists the limit of $\log \mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})$ *as* $n \to \infty$ *because* $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})$ *is monotone non-decreasing in* n *and has an upper bound independent of* n

Remark 1. Gournay [1, Example 3.1] shows $\mathrm{Widim}_\varepsilon(B_{\ell^1}(\mathbb{R}^2), d_{\ell^p}) = 2$ for $\varepsilon < 2^{1/p}$.

1.2. Mean dimension theory

Theorem 1.1 has an application to Gromov's mean dimension theory. Let Γ be an infinite countable group. For $1 \le p \le \infty$, let $\ell^p(\Gamma) \subset \mathbb{R}^\Gamma$ be the ℓ^p-space, $B(\ell^p(\Gamma)) \subset \ell^p(\Gamma)$ the unit ball (in the ℓ^p-norm). We consider the natural right action of Γ on $\ell^p(\Gamma)$ (and $B(\ell^p(\Gamma))$):

$$(x \cdot \delta)_\gamma := x_{\delta\gamma} \quad \text{for } x = (x_\gamma)_{\gamma \in \Gamma} \in \ell^p(\Gamma) \text{ and } \delta \in \Gamma.$$

We give the standard product topology on $\mathbb{R}^\Gamma$, and consider the restriction of this topology to $B(\ell^p(\Gamma)) \subset \mathbb{R}^\Gamma$. (This topology coincides with the restriction of the weak topology of $\ell^p(\Gamma)$ for $p > 1$.) Then $B(\ell^p(\Gamma))$ becomes compact and metrizable. (The Γ-action on $B(\ell^p(\Gamma))$ is continuous.) Let d be the distance on $B(\ell^p(\Gamma))$ compatible with the topology. For a finite subset $\Omega \subset \Gamma$ we define a distance d_Ω on $B(\ell^p(\Gamma))$ by

$$d_\Omega(x, y) := \max_{\gamma \in \Omega} d(x \cdot \gamma, y \cdot \gamma).$$

We are interested in the growth behavior of $\mathrm{Widim}_\varepsilon(B(\ell^p(\Gamma)), d_\Omega)$ as $|\Omega| \to \infty$. In particular, if Γ is finitely generated and has an amenable sequence $\{\Omega_i\}_{i \ge 1}$ (in the sense of [2, p. 335]), the mean dimension is defined by (see [2, pp. 335-339])

$$\dim(B(\ell^p(\Gamma)) : \Gamma) = \lim_{\varepsilon \to 0} \left(\lim_{i \to \infty} \mathrm{Widim}_\varepsilon(B(\ell^p(\Gamma)), d_{\Omega_i}) / |\Omega_i| \right).$$

As a corollary of Theorem 1.1, we get the following:

Corollary 1.3. *For $1 \leq p < \infty$ and any $\varepsilon > 0$, there is a positive constant $C(d, p, \varepsilon) < \infty$ (independent of Ω) such that*

$$\mathrm{Widim}_\varepsilon(B(\ell^p(\Gamma)), d_\Omega) \leq C(d, p, \varepsilon) \quad \text{for any finite set } \Omega \subset \Gamma. \tag{1.5}$$

In particular, for a finitely generated infinite amenable group Γ

$$\dim(B(\ell^p(\Gamma)) : \Gamma) = 0. \tag{1.6}$$

(1.6) is the answer to a question of Gromov in [2, p. 340]. Actually the above (1.5) is much stronger than (1.6).

Remark 2. This paper is a revised version of the preprint [5]. A referee of [5] pointed out that the above (1.6) can be derived from the theorem of Lindenstrauss-Weiss [4, Theorem 4.2]. This theorem tells us that if the topological entropy is finite then the mean dimension is 0. We can see that the topological entropy of $B(\ell^p(\Gamma))$ (under the Γ-action) is 0. Hence the mean dimension is also 0. I am most grateful to the referee of [5] for pointing out this argument. The essential part of the proof of Theorem 1.1 (and Corollary 1.1 and Corollary 1.2) is the construction of the continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ described in Section 2. This construction was already given in the preprint [5]. When I was writing this revised version of [5], I found the paper of A. Gournay [1]. [1] proves Corollary 1.1 ([1, Proposition 1.3]) by using essentially the same continuous map as mentioned above. I submitted the paper [5] to a certain journal in June of 2007 before [1] appeared on the arXiv in November of 2007. [5] is quoted as one of the references in [1].

2. Proof of Theorem 1.1

Let S_n be the n-th symmetric group. We define the group G by

$$G := \{\pm 1\}^n \rtimes S_n.$$

The multiplication in G is given by

$$((\varepsilon_1, \cdots, \varepsilon_n), \sigma) \cdot ((\varepsilon'_1, \cdots, \varepsilon'_n), \sigma') := ((\varepsilon_1 \varepsilon'_{\sigma^{-1}(1)}, \cdots, \varepsilon_n \varepsilon'_{\sigma^{-1}(n)}), \sigma\sigma')$$

where $\varepsilon_1, \cdots, \varepsilon_n, \varepsilon'_1 \cdots, \varepsilon'_n \in \{\pm 1\}$ and $\sigma, \sigma' \in S_n$. G acts on $\mathbb{R}^n$ by

$$((\varepsilon_1, \cdots, \varepsilon_n), \sigma) \cdot (x_1, \cdots, x_n) := (\varepsilon_1 x_{\sigma^{-1}(1)}, \cdots, \varepsilon_n x_{\sigma^{-1}(n)})$$

where $((\varepsilon_1, \cdots, \varepsilon_n), \sigma) \in G$ and $(x_1, \cdots, x_n) \in \mathbb{R}^n$. The action of G on $\mathbb{R}^n$ preserves the ℓ^p-ball $B_{\ell^p}(\mathbb{R}^n)$ and the ℓ^q-distance $d_{\ell^q}(\cdot, \cdot)$.

We define $\mathbb{R}^n_{\geq 0}$ and Λ_n by

$$\mathbb{R}^n_{\geq 0} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n | \, x_i \geq 0 \ (1 \leq i \leq n)\},$$
$$\Lambda_n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n | \, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

The following can be easily checked:

Lemma 2.1. *For $\varepsilon \in \{\pm 1\}^n$ and $x \in \mathbb{R}^n_{\geq 0}$, if $\varepsilon x \in \mathbb{R}^n_{\geq 0}$, then $\varepsilon x = x$. For $\sigma \in S_n$ and $x \in \Lambda_n$, if $\sigma x \in \Lambda_n$, then $\sigma x = x$. For $g = (\varepsilon, \sigma) \in G$ and $x \in \Lambda_n$, if $gx \in \Lambda_n$, then $gx = \varepsilon(\sigma x) = \sigma x = x$.*

Let m, n be positive integers such that $1 \leq m < n$. We define the continuous map $f_0 : \Lambda_n \to \Lambda_n$ by

$$f_0(x_1, \cdots, x_n) := (x_1 - x_{m+1}, x_2 - x_{m+1}, \cdots, x_m - x_{m+1}, \underbrace{0, 0, \cdots, 0}_{n-m}).$$

The following is the key fact for our construction:

Lemma 2.2. *For $g \in G$ and $x \in \Lambda_n$, if $gx \in \Lambda_n$ ($\Rightarrow gx = x$), then we have*

$$f_0(gx) = g f_0(x).$$

Proof. First we consider the case of $g = \varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \{\pm 1\}^n$. If $x_{m+1} = 0$, then

$$f_0(\varepsilon x) = (\varepsilon_1 x_1, \cdots, \varepsilon_m x_m, 0, \cdots, 0) = \varepsilon f_0(x).$$

If $x_{m+1} > 0$, then $\varepsilon_i = 1$ $(1 \leq i \leq m+1)$ because $\varepsilon_i x_i = x_i \geq x_{m+1} > 0$ $(1 \leq i \leq m+1)$. Hence

$$f_0(\varepsilon x) = (x_1 - x_{m+1}, \cdots, x_m - x_{m+1}, 0, \cdots, 0) = f_0(x) = \varepsilon f_0(x).$$

Next we consider the case of $g = \sigma \in S_n$. $gx \in \Lambda_n$ implies $x_{\sigma(i)} = x_i$ $(1 \leq i \leq n)$. Set $y := f_0(x)$. Let r $(1 \leq r \leq m+1)$ be the integer such that

$$x_{r-1} > x_r = x_{r+1} = \cdots = x_{m+1}.$$

From $x_{\sigma(i)} = x_i$ $(1 \leq i \leq n)$, we have

$$1 \leq i < r \Rightarrow 1 \leq \sigma(i) < r \Rightarrow y_{\sigma(i)} = x_{\sigma(i)} - x_{m+1} = y_i,$$
$$r \leq i \Rightarrow r \leq \sigma(i) \Rightarrow y_{\sigma(i)} = 0 = y_i.$$

Hence we have $f_0(\sigma x) = f_0(x) = \sigma f_0(x)$.

Finally we consider the case of $g = (\varepsilon, \sigma) \in G$. Since $gx \in \Lambda_n$, we have $gx = \varepsilon(\sigma x) = \sigma x = x \in \Lambda_n$ (see Lemma 2.1). Hence

$$f_0(gx) = f_0(\varepsilon(\sigma x)) = \varepsilon f_0(\sigma x) = \varepsilon\sigma f_0(x) = g f_0(x).$$

□

We define a continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ as follows; For any $x \in \mathbb{R}^n$, there is a $g \in G$ such that $gx \in \Lambda_n$. Then we define

$$f(x) := g^{-1} f_0(gx).$$

From Lemma 2.2, this definition is well-defined. Since $\mathbb{R}^n = \bigcup_{g \in G} g\Lambda_n$ and $f|_{g\Lambda_n} = g f_0 g^{-1}$ $(g \in G)$ is continuous on $g\Lambda_n$, f is continuous on $\mathbb{R}^n$. Moreover f is G-equivariant.

Proposition 2.1. *Let $1 \le p < q \le \infty$. For any $x \in B_{\ell^p}(\mathbb{R}^n)$, we have*

$$d_{\ell^q}(x, f(x)) \le \left(\frac{1}{m+1}\right)^{\frac{1}{p}-\frac{1}{q}}.$$

Note that the right-hand side does not depend on n.

Proof. Since f is G-equivariant and d_{ℓ^q} is G-invariant, we can suppose $x \in B_{\ell^p}(\mathbb{R}^n) \cap \Lambda_n$, i.e. $x = (x_1, x_2, \cdots, x_n)$ with $x_1 \ge x_2 \ge \cdots \ge x_n \ge 0$. We have

$$f(x) = (x_1 - x_{m+1}, \cdots, x_m - x_{m+1}, 0, \cdots, 0).$$

Hence

$$d_{\ell^q}(x, f(x)) = \|(\underbrace{x_{m+1}, \cdots, x_{m+1}}_{m+1}, x_{m+2}, \cdots, x_n)\|_{\ell^q}.$$

Set $t := x_{m+1}^p$ and $s := q/p > 1$. Since $x_1^p + \cdots + x_n^p \le 1$ and $x_1 \ge x_2 \ge \cdots \ge x_n \ge 0$, we have $t \le 1/(m+1)$, $0 \le x_k^p \le t$ $(m+1 \le k \le n)$ and $x_{m+2}^p + \cdots + x_n^p \le 1 - (m+1)t$. Then

$$\begin{aligned} x_{m+2}^q + \cdots + x_n^q &= (x_{m+2}^p)^{s-1} \cdot x_{m+2}^p + \cdots + (x_n^p)^{s-1} \cdot x_n^p, \\ &\le t^{s-1}(x_{m+2}^p + \cdots + x_n^p), \\ &\le t^{s-1}\{1 - (m+1)t\} = t^{s-1} - (m+1)t^s. \end{aligned}$$

Therefore

$$d_{\ell^q}(x, f(x))^q = (m+1)x_{m+1}^q + x_{m+2}^q + \cdots + x_n^q \le t^{s-1} \le (1/(m+1))^{s-1}.$$

Thus

$$d_{\ell^q}(x, f(x)) \le (1/(m+1))^{1/p-1/q}.$$

□

Proof of Theorem 1.1. Set $m := \min(n, \lceil (2/\varepsilon)^r \rceil - 1)$. We will prove $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \le m$. If $n = m$, then the statement is trivial. Hence we suppose $n > m = \lceil (2/\varepsilon)^r \rceil - 1$. From $m + 1 = \lceil (2/\varepsilon)^r \rceil \ge (2/\varepsilon)^r$ and $1/r = 1/p - 1/q$,

$$2\left(\frac{1}{m+1}\right)^{\frac{1}{p}-\frac{1}{q}} \le \varepsilon.$$

We have

$$f(\mathbb{R}^n) = \bigcup_{g \in G} g f(\Lambda_n).$$

Note that $f(\Lambda_n) \subset \mathbb{R}^m := \{(x_1, \cdots, x_m, 0, \cdots, 0) \in \mathbb{R}^n\}$. Proposition 2.1 implies that

$$f|_{B_{\ell^p}(\mathbb{R}^n)} : (B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \to \bigcup_{g \in G} g \cdot \mathbb{R}^m \text{ is a } 2\left(\frac{1}{m+1}\right)^{\frac{1}{p}-\frac{1}{q}}\text{-embedding.}$$

Therefore we get $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \leq m$. □

3. Proof of Corollaries 1.1 and 1.2

3.1. Proof of Corollary 1.1

We need the following result. (cf. Gromov [2, p. 332]. For its proof, see Lindenstrauss-Weiss [4, Lemma 3.2] or Tsukamoto [6, Example 4.1].)

Lemma 3.1. *For $\varepsilon < 1$,*

$$\mathrm{Widim}_\varepsilon([0,1]^n, d_{\ell^\infty}) = n,$$

where d_{ℓ^∞} is the sup-distance given by $d_{\ell^\infty}(x,y) := \max_i |x_i - y_i|$.

From this we get:

Lemma 3.2. *Let $B_{\ell^\infty}(\mathbb{R}^n, \rho)$ be the closed ball of radius ρ centered at the origin in $\ell^\infty(\mathbb{R}^n)$ $(\rho > 0)$. Then for $\varepsilon < 2\rho$*

$$\mathrm{Widim}_\varepsilon(B_{\ell^\infty}(\mathbb{R}^n, \rho), d_{\ell^\infty}) = n.$$

Proof. Consider the bijection

$$[0,1]^n \to B_{\ell^\infty}(\mathbb{R}^n, \rho), \quad (x_1, \cdots, x_n) \mapsto (2\rho x_1 - \rho, \cdots, 2\rho x_n - \rho).$$

Then the statement easily follows from Lemma 3.1. □

Proof of Corollary 1.1. Set $m := \min(n, \lceil (2/\varepsilon)^p \rceil - 1)$. From Theorem 1.1, $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) \leq m$. We want to show $\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) \geq m$. Note that for any real number x we have $\lceil x \rceil - 1 < x$. Hence $m \leq \lceil (2/\varepsilon)^p \rceil - 1 < (2/\varepsilon)^p$. Therefore $m(\varepsilon/2)^p < 1$. Then if we choose $\rho > \varepsilon/2$ sufficiently close to $\varepsilon/2$, then $(m \leq n)$

$$B_{\ell^\infty}(\mathbb{R}^m, \rho) \subset B_{\ell^p}(\mathbb{R}^n).$$

(If $\varepsilon \geq 2$, then $m = 0$ and $B_{\ell^\infty}(\mathbb{R}^m, \rho)$ is $\{0\}$.) From Lemma 3.2,

$$\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) \geq \mathrm{Widim}_\varepsilon(B_{\ell^\infty}(\mathbb{R}^m, \rho), d_{\ell^\infty}) = m.$$

Essentially the same argument is given in Gournay [1, pp. 5-6]. □

3.2. Proof of Corollary 1.2

The following lemma easily follows from (1.2)

Lemma 3.3. *Let $B_{\ell^q}(\mathbb{R}^n, \rho)$ be the closed ball of radius ρ centered at the origin in $\ell^q(\mathbb{R}^n)$ $(1 \leq q \leq \infty$ and $\rho > 0)$. For $\varepsilon < \rho$,*

$$\mathrm{Widim}_\varepsilon(B_{\ell^q}(\mathbb{R}^n, \rho), d_{\ell^q}) = n.$$

Proposition 3.1. *For $1 \le p < q \le \infty$,*

$$\min(n, \lceil \varepsilon^{-r} \rceil - 1) \le \mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}),$$

where r is defined by $1/r = 1/p - 1/q$.

Proof. We can suppose $q < \infty$. Set $m := \min(n, \lceil \varepsilon^{-r} \rceil - 1)$. From Hölder's inequality,

$$(|x_1|^p + \cdots + |x_m|^p)^{1/p} \le m^{1/r}(|x_1|^q + \cdots + |x_m|^q)^{1/q}.$$

As in the proof of Corollary 1.1, we have $m \le \lceil \varepsilon^{-r} \rceil - 1 < \varepsilon^{-r}$, i.e. $m^{1/r}\varepsilon < 1$. Therefore if we choose $\rho > \varepsilon$ sufficiently close to ε, then $(m \le n)$

$$B_{\ell^q}(\mathbb{R}^m, \rho) \subset B_{\ell^p}(\mathbb{R}^n).$$

From Lemma 3.3,

$$\mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \ge \mathrm{Widim}_\varepsilon(B_{\ell^q}(\mathbb{R}^m, \rho), d_{\ell^q}) = m.$$

□

Proof of Corollary 1.2. From Theorem 1.1 and Proposition 3.1, we have

$$\lceil \varepsilon^{-r} \rceil - 1 \le \lim_{n \to \infty} \mathrm{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \le \lceil (2/\varepsilon)^r \rceil - 1.$$

From this estimate, we can easily get the conclusion. □

4. Proof of Corollary 1.3

Let $1 \le p < \infty$ and $\varepsilon > 0$. Set $X := B(\ell^p(\Gamma))$. To begin with, we want to fix a distance on X (compatible with the topology). Since X is compact, if we prove (1.5) for one fixed distance, then (1.5) becomes valid for any distance on X. Let $w : \Gamma \to \mathbb{R}_{>0}$ be a positive function satisfying

$$\sum_{\gamma \in \Gamma} w(\gamma) \le 1.$$

We define the distance $d(\cdot, \cdot)$ on X by

$$d(x, y) := \sum_{\gamma \in \Gamma} w(\gamma)|x_\gamma - y_\gamma| \quad \text{for } x = (x_\gamma)_{\gamma \in \Gamma} \text{ and } y = (y_\gamma)_{\gamma \in \Gamma} \text{ in } X.$$

As in Subsection 1.2, we define the distance d_Ω on X for a finite subset $\Omega \subset \Gamma$ by

$$d_\Omega(x, y) := \max_{\gamma \in \Omega} d(x \cdot \gamma, y \cdot \gamma).$$

For each $\delta \in \Gamma$, there is a finite set $\Omega_\delta \subset \Gamma$ such that

$$\sum_{\gamma \in \Gamma \setminus \Omega_\delta} w(\delta^{-1}\gamma) \le \varepsilon/4.$$

Set $\Omega' := \bigcup_{\delta\in\Omega}\Omega_\delta$. Ω' is a finite set satisfying

$$\sum_{\gamma\in\Gamma\setminus\Omega'} w(\delta^{-1}\gamma) \le \varepsilon/4 \quad \text{for any } \delta\in\Omega.$$

Set $c := \lceil (4/\varepsilon)^p \rceil - 1$. Let $\pi : X \to B_{\ell^p}(\mathbb{R}^{\Omega'}) = \{x \in \mathbb{R}^{\Omega'} |\ \|x\|_p \le 1\}$ be the natural projection. From Theorem 1.1, there are a polyhedron K of dimension $\le c$ and an $\varepsilon/2$-embedding $f : (B_{\ell^p}(\mathbb{R}^{\Omega'}), d_{\ell^\infty}) \to K$. Then $F := f\circ\pi : (X, d_\Omega) \to K$ is an ε-embedding; If $F(x) = F(y)$, then $d_{\ell^\infty}(\pi(x), \pi(y)) \le \varepsilon/2$ and for each $\delta\in\Omega$

$$\begin{aligned} d(x\cdot\delta, y\cdot\delta) &= \sum_{\gamma\in\Omega'} w(\delta^{-1}\gamma)|x_\gamma - y_\gamma| + \sum_{\gamma\in\Gamma\setminus\Omega'} w(\delta^{-1}\gamma)|x_\gamma - y_\gamma|,\\ &\le \frac{\varepsilon}{2}\sum_{\gamma\in\Omega'} w(\delta^{-1}\gamma) + 2\sum_{\gamma\in\Gamma\setminus\Omega'} w(\delta^{-1}\gamma),\\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $d_\Omega(x, y) \le \varepsilon$. Therefore,

$$\mathrm{Widim}_\varepsilon(X, d_\Omega) \le c.$$

This shows (1.5). If Γ has an amenable sequence $\{\Omega_i\}_{i\ge 1}$, then $|\Omega_i| \to \infty$ and hence

$$\lim_{i\to\infty} \mathrm{Widim}_\varepsilon(X, d_{\Omega_i})/|\Omega_i| = 0.$$

This shows (1.6). □

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