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A computation of universal weight function for quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$

By

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Abstract

We compute weight functions (off-shell Bethe vectors) in any representation with a weight singular vector of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ applying the method of projections of Drinfeld currents developed in [EKP].

1. Introduction

In [EKP] a new method for the construction of weight functions, also known as off-shell Bethe vectors, was suggested. In this paper we apply the constructions of [EKP] and [KPT] to the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ and compute explicitly off-shell Bethe vectors in representations with a weight singular vector.

Off-shell Bethe vectors in integrable models associated with the Lie algebra $\mathfrak{gl}_N$ have appeared in [KR] in the framework of the algebraic nested Bethe ansatz. Except for $N = 2$, off-shell Bethe vectors are defined inductively. They are functions of several complex variables. If the variables satisfy the Bethe ansatz equations, the Bethe vectors are eigenvectors of the transfer matrix of the system. The construction of [KR] was developed in [TV1], where weight functions were defined as certain matrix elements of monodromy operators. Recently, the construction of [TV1] was used in [TV2] for the calculation of nested Bethe vectors in evaluation modules and their tensor products.

The approach of [EKP] is based on the "new realization" of quantum affine algebras [D] and its connection to fundamental coalgebraic properties of weight functions, found in [TV1]. The key role is played by certain projections to the intersection of Borel subalgebras of different kind of the quantum affine algebra, introduced in [ER]. It is shown in [KP], [EKP] that acting by a projection of a product of Drinfeld currents on highest weight vectors of irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{gl}}_N)$ one obtains a collection of rational functions with the required comultiplication properties, that is, a weight function.

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In this paper we perform a calculation of those projections and present explicit expressions for weight functions of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ in modules with a weight singular vector. We present the result in two different ways. First, we write the weight function as a polynomial over Gauss coordinates of defining L-operators with rational dependence of parameters, see eq (5.3); this is equivalent to certain Cauchy type integral over Drinfeld currents, see Theorem 1; eq. (4.38) and eq. (4.39). Second, we express the weight function as a polynomial with rational functional coefficients over matrix elements of the same L-operators (Theorem 2). As an application we compute off-shell Bethe vectors in evaluation modules.

We compare our results with calculation of [TV2] and verify a variant of the conjecture of [KPT] about the coincidence of the two constructions of weight functions. For such a verification we need to adjust our construction to the data of [TV2], were the different R-matrix is used.

The paper is organized as follows. In Section 2 we recall the two descriptions of $U_q(\widehat{\mathfrak{gl}}_N)$: by means of the fundamental L-operators and the so called 'new realization' of Drinfeld. In Section 3 we describe, following [KPT], the construction of a weight function as a certain projection of products of Drinfeld currents. Section 4 contains the main calculations with the constructions of [EKP] and [KPT]. For this we extend the projection operators of [ER] to a natural completion of the algebra, where we use so called composed currents of [DK] and strings. Their projections are then expressed in Gauss coordinates of the fundamental L-operator.

In Section 5 we translate the results to the L-operator's language. The resulting formula, see Theorem 2, is more general then in [TV2]: it does not refer to any evaluation map. Its structure is rather curious: the order of L-operators's entries does not correspond to any normal ordering of the root system. A specialization to the evaluation map is given by eq. (5.24). In Section 6 we compare our results with [TV2]. The R-matrix in [TV2] differs from ours by a finite-dimensional twist. We modify accordingly our construction and observe the resulting literal coincidence with [TV2]. This allows us to verify the conjecture of [KPT] for the defining R-matrix as in [TV2]: the two constructions of weight functions give the same result in any irreducible $U_q(\widehat{\mathfrak{gl}}_N)$-module with a weight singular vector. This result can be formulated on a formal level, see Theorem 3. The paper contains three appendices with important technical results, including various properties of strings and of their projections.

2. Quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$

2.1. L-operator description

Let $\mathrm{E}_{ij} \in \mathrm{End}(\mathbb{C}^N)$ be a matrix with the only nonzero entry equal to 1 at the intersection of the i-th row and j-th column. Let q be a complex number, which is neither 0 nor a root of unity. Let $\mathrm{R}(u,v) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}[[v/u]]$,

$$
\begin{aligned}
\mathrm{R}(u,v) = \sum_{1\le i\le N} &\mathrm{E}_{ii} \otimes \mathrm{E}_{ii} \\
&+ \frac{u-v}{qu-q^{-1}v} \sum_{1\le i<j\le N} (\mathrm{E}_{ii}\otimes \mathrm{E}_{jj} + \mathrm{E}_{jj}\otimes \mathrm{E}_{ii}) \\
&+ \frac{q-q^{-1}}{qu-q^{-1}v} \sum_{1\le i<j\le N} (v\mathrm{E}_{ij}\otimes \mathrm{E}_{ji} + u\mathrm{E}_{ji}\otimes \mathrm{E}_{ij}),
\end{aligned} \tag{2.1}
$$

be a trigonometric R-matrix associated with the vector representation of $\mathfrak{gl}_N$. It satisfies the Yang-Baxter equation

$$
\mathrm{R}_{12}(u_1,u_2)\mathrm{R}_{13}(u_1,u_3)\mathrm{R}_{23}(u_2,u_3) = \mathrm{R}_{23}(u_2,u_3)\mathrm{R}_{13}(u_1,u_3)\mathrm{R}_{12}(u_1,u_2), \tag{2.2}
$$

and the inversion relation

$$
\mathrm{R}_{12}(u_1,u_2)\mathrm{R}_{21}(u_2,u_1) = 1. \tag{2.3}
$$

The algebra $U_q(\widehat{\mathfrak{gl}}_N)$ (with the zero central charge and the gradation operator dropped out) is a unital associative algebra generated by the modes $\mathrm{L}^{\pm}_{ij}[\pm k]$, $k\ge 0$, $1\le i,j\le N$, of the L-operators $\mathrm{L}^{\pm}(z) = \sum_{k=0}^{\infty}\sum_{i,j=1}^{N} \mathrm{E}_{ij}\otimes \mathrm{L}^{\pm}_{ij}[\pm k]z^{\mp k}$, subject to relations

$$
\begin{gathered}
\mathrm{R}_{12}(u,v)\cdot \mathrm{L}^{\pm}_1(u)\cdot \mathrm{L}^{\pm}_2(v) = \mathrm{L}^{\pm}_2(v)\cdot \mathrm{L}^{\pm}_1(u)\cdot \mathrm{R}_{12}(u,v), \\
\mathrm{R}_{12}(u,v)\cdot \mathrm{L}^{+}_1(u)\cdot \mathrm{L}^{-}_2(v) = \mathrm{L}^{-}_2(v)\cdot \mathrm{L}^{+}_1(u)\cdot \mathrm{R}_{12}(u,v), \\
\mathrm{L}^{+}_{ij}[0] = \mathrm{L}^{-}_{ji}[0] = 0, \quad \mathrm{L}^{+}_{kk}[0]\mathrm{L}^{-}_{kk}[0] = 1,
\end{gathered} \tag{2.4}
$$

where $1\le i<j\le N$, $1\le k\le N$ and $\mathrm{L}^{\pm}_1(u) = \mathrm{L}^{\pm}(u)\otimes \mathbf{1}$, $\mathrm{L}^{\pm}_2(u) = \mathbf{1}\otimes \mathrm{L}^{\pm}(u)$.

The coalgebraic structure of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ is defined by the rule

$$
\Delta\left(\mathrm{L}^{\pm}_{ij}(u)\right) = \sum_{k=1}^{N} \mathrm{L}^{\pm}_{kj}(u)\otimes \mathrm{L}^{\pm}_{ik}(u). \tag{2.5}
$$

2.2. The current realization of $U_q(\widehat{\mathfrak{gl}}_N)$

The algebra $U_q(\widehat{\mathfrak{gl}}_N)$ in the current realization (with the zero central charge and the gradation operator dropped out) is generated by the modes of the Cartan currents

$$
k^{\pm}_i(z) = \sum_{m\ge 0} k^{\pm}_i[\pm m]z^{\mp m}, \qquad k^{+}_i[0]k^{-}_i[0] = 1, \tag{2.6}
$$

$i = 1,\dots,N$, and by the modes of the generating functions (called 'Drinfeld currents')

$$
E_i(z) = \sum_{n\in\mathbb{Z}} E_i[n]z^{-n}, \qquad F_i(z) = \sum_{n\in\mathbb{Z}} F_i[n]z^{-n}, \tag{2.7}
$$

$i=1,\dots,N-1$, subject to the commutation relations

(2.8)
$$
\begin{aligned}
&(q^{-1}z-qw)E_i(z)E_i(w)=E_i(w)E_i(z)(qz-q^{-1}w),\\
&(z-w)E_i(z)E_{i+1}(w)=E_{i+1}(w)E_i(z)(q^{-1}z-qw),\\
&(qz-q^{-1}w)F_i(z)F_i(w)=F_i(w)F_i(z)(q^{-1}z-qw),\\
&(q^{-1}z-qw)F_i(z)F_{i+1}(w)=F_{i+1}(w)F_i(z)(z-w),\\
&k_i^{\pm}(z)F_i(w)\left(k_i^{\pm}(z)\right)^{-1}=\frac{q^{-1}z-qw}{z-w}F_i(w),\\
&k_{i+1}^{\pm}(z)F_i(w)\left(k_{i+1}^{\pm}(z)\right)^{-1}=\frac{qz-q^{-1}w}{z-w}F_i(w),\\
&k_i^{\pm}(z)F_j(w)\left(k_i^{\pm}(z)\right)^{-1}=F_j(w)\qquad\text{if}\quad i\neq j,j+1,\\
&k_i^{\pm}(z)E_i(w)\left(k_i^{\pm}(z)\right)^{-1}=\frac{z-w}{q^{-1}z-qw}E_i(w),\\
&k_{i+1}^{\pm}(z)E_i(w)\left(k_{i+1}^{\pm}(z)\right)^{-1}=\frac{z-w}{qz-q^{-1}w}E_i(w),\\
&k_i^{\pm}(z)E_j(w)\left(k_i^{\pm}(z)\right)^{-1}=E_j(w)\qquad\text{if}\quad i\neq j,j+1,\\
&[E_i(z),F_j(w)]=\delta_{i,j}\ \delta(z/w)\ (q-q^{-1})\left(k_i^{+}(z)/k_{i+1}^{+}(z)-k_i^{-}(w)/k_{i+1}^{-}(w)\right),
\end{aligned}
$$

where $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$ and to the Serre relations

$$
\begin{aligned}
&\mathrm{Sym}_{z_1,z_2}(E_i(z_1)E_i(z_2)E_{i\pm1}(w)-(q+q^{-1})E_i(z_1)E_{i\pm1}(w)E_i(z_2)+\\
&\qquad\qquad+E_{i\pm1}(w)E_i(z_1)E_i(z_2))=0,\\
&\mathrm{Sym}_{z_1,z_2}(F_i(z_1)F_i(z_2)F_{i\pm1}(w)-(q+q^{-1})F_i(z_1)F_{i\pm1}(w)F_i(z_2)+\\
&\qquad\qquad+F_{i\pm1}(w)F_i(z_1)F_i(z_2))=0.
\end{aligned}
\tag{2.9}
$$

The isomorphism of the two realization is established with a help of the Gauss decomposition of L-operators. We define Gauss coordinates $\mathrm{F}^{\pm}_{j,i}(z)$, $\mathrm{E}^{\pm}_{i,j}(z)$ $1\le i<j\le N$ and $k_i^{\pm}(z)$, $i=1,\dots,N$ by the relations

$$
\begin{aligned}
\mathrm{L}^{\pm}(z)=&\left(\sum_{i=1}^{N}\mathrm{E}_{ii}+\sum_{i<j}^{N}\mathrm{F}^{\pm}_{j,i}(z)\mathrm{E}_{ij}\right)\\
&\times\left(\sum_{i=1}^{N}k_i^{\pm}(z)\mathrm{E}_{ii}\right)\cdot\left(\sum_{i=1}^{N}\mathrm{E}_{ii}+\sum_{i<j}^{N}\mathrm{E}^{\pm}_{i,j}(z)\mathrm{E}_{ji}\right).
\end{aligned}
\tag{2.10}
$$

We identify $k_i^{\pm}(z)$ with the Cartan currents (2.6) and set [DF]

$$
E_i(z)=\mathrm{E}^{+}_{i,i+1}(z)-\mathrm{E}^{-}_{i,i+1}(z),\quad F_i(z)=\mathrm{F}^{+}_{i+1,i}(z)-\mathrm{F}^{-}_{i+1,i}(z).
\tag{2.11}
$$

For any series $G(t)=\sum_{m\in\mathbb{Z}}G[m]t^{-m}$ we denote $G(t)^{(+)}=\sum_{m>0}G[m]\,t^{-m}$, and $G(t)^{(-)}=-\sum_{m\le0}G[m]\,t^{-m}$. The initial conditions (2.4) imply the relations

$$
\mathrm{F}^{\pm}_{i+1,i}(z)=F_i(z)^{(\pm)},\qquad \mathrm{E}^{\pm}_{i,i+1}(z)=z\left(z^{-1}E_i(z)\right)^{(\pm)}.
\tag{2.12}
$$

In [D] the *current* Hopf structure for the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ has been defined,

$$\begin{aligned}\Delta^{(D)}\left(E_i(z)\right) &= E_i(z)\otimes 1 + k_i^-(z)\left(k_{i+1}^-(z)\right)^{-1}\otimes E_i(z),\\ \Delta^{(D)}\left(F_i(z)\right) &= 1\otimes F_i(z) + F_i(z)\otimes k_i^+(z)\left(k_{i+1}^+(z)\right)^{-1},\\ \Delta^{(D)}\left(k_i^\pm(z)\right) &= k_i^\pm(z)\otimes k_i^\pm(z).\end{aligned} \tag{2.13}$$

We consider two types of Borel subalgebras of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$. Borel subalgebras $U_q(\mathfrak{b}^\pm)\subset U_q(\widehat{\mathfrak{gl}}_N)$ are generated by the modes of the L-operators $\mathrm{L}^{(\pm)}(z)$, respectively. Another type of Borel subalgebras is related to the current realization of $U_q(\widehat{\mathfrak{gl}}_N)$. The Borel subalgebra $U_F\subset U_q(\widehat{\mathfrak{gl}}_N)$ is generated by the modes $F_i[n]$, $k_j^+[m]$, $i=1,\dots,N-1$, $j=1,\dots,N$, $n\in\mathbb{Z}$ and $m\geq 0$. The Borel subalgebra $U_E\subset U_q(\widehat{\mathfrak{gl}}_N)$ is generated by the modes $E_i[n]$, $k_j^-[-m]$, $i=1,\dots,N-1$, $j=1,\dots,N$, $n\in\mathbb{Z}$ and $m\geq 0$. We also consider a subalgebra $U_F'\subset U_F$, generated by the elements $F_i[n]$, $k_j^+[m]$, $i=1,\dots,N-1$, $j=1,\dots,N$, $n\in\mathbb{Z}$ and $m>0$, and a subalgebra $U_E'\subset U_E$ generated by the elements $E_i[n]$, $k_j^-[-m]$, $i=1,\dots,N-1$, $j=1,\dots,N$, $n\in\mathbb{Z}$ and $m>0$. Further, we will be interested in the intersections,

$$U_f^- = U_F'\cap U_q(\mathfrak{b}^-),\qquad U_F^+ = U_F\cap U_q(\mathfrak{b}^+) \tag{2.14}$$

and will describe properties of projections to these intersections.

It was proved in [KPT] that the subalgebras U_f^- and U_F^+ are coideals with respect to Drinfeld coproduct (2.13)

$$\Delta^{(D)}(U_F^+)\subset U_q(\widehat{\mathfrak{gl}}_N)\otimes U_F^+,\qquad \Delta^{(D)}(U_f^-)\subset U_f^-\otimes U_q(\widehat{\mathfrak{gl}}_N),$$

and the multiplication m in $U_q(\widehat{\mathfrak{gl}}_N)$ induces an isomorphism of vector spaces

$$m: U_f^-\otimes U_F^+\to U_F.$$

According to the general theory presented in [EKP] we define projection operators $P^+: U_F\subset U_q(\widehat{\mathfrak{gl}}_N)\to U_F^+$ and $P^-: U_F\subset U_q(\widehat{\mathfrak{gl}}_N)\to U_f^-$ by the prescriptions

$$\begin{gathered}P^+(f_-\ f_+) = \varepsilon(f_-)\ f_+,\qquad P^-(f_-\ f_+) = f_-\ \varepsilon(f_+),\\ \text{for any}\qquad f_-\in U_f^-,\quad f_+\in U_F^+.\\ \text{Here } \varepsilon: U_q(\widehat{\mathfrak{gl}}_N)\to\mathbb{C} \text{ is the counit map.}\end{gathered} \tag{2.15}$$

Denote by $\overline{U}_F$ an extension of the algebra U_F formed by linear combinations of series, given as infinite sums of monomials $a_{i_1}[n_1]\cdots a_{i_k}[n_k]$ with $n_1\leq\cdots\leq n_k$, and $n_1+\cdots+n_k$ fixed, where $a_{i_l}[n_l]$ is either $F_{i_l}[n_l]$ or $k_{i_l}^+[n_l]$. It was proved in [EKP] that

(1) the action of the projections (2.15) can be extended to the algebra $\overline{U}_F$;

(2) for any $f\in\overline{U}_F$ with $\Delta^{(D)}(f)=\sum_i f_i'\otimes f_i''$ we have

$$f=\sum_i P^-(f_i'')\cdot P^+(f_i'). \tag{2.16}$$

3. Weight functions

3.1. Definitions

This section is based on the paper [KPT][*1].

We call a vector v *a weight singular vector* if it is annihilated by any non-negative mode $E_i[n]$, $i=1,\dots,N-1$, $n\geq 0$ and is an eigenvector for $k_i^+(z)$, $i=1,\dots,N$

$$\mathrm{E}^+_{i,i+1}(z)\cdot v=0\ ,\qquad k_i^+(z)\cdot v=\lambda_i(z)\,v, \tag{3.1}$$

where $\lambda_i(z)$ is a meromorphic function, decomposed as a power series in z^{-1}. The L-operator (2.10), acting on a weight singular vector v, becomes upper-triangular

$$\mathrm{L}^+_{ij}(z)\ v=0,\quad i>j,\quad \mathrm{L}^+_{ii}(z)\ v=\lambda_i(z)\ v,\quad i=1,\dots,N. \tag{3.2}$$

We define a weight function by its comultiplication properties.

Let Π be the set $\{1,\dots,N-1\}$ of indices of the simple positive roots of $\mathfrak{gl}_N$. A finite collection $I=\{i_1,\dots,i_n\}$ with a linear ordering $i_i\prec\cdots\prec i_n$ and a 'coloring' map $\iota:I\to\Pi$ is called an *ordered* Π-*multiset*. Sometimes, we denote the map ι by ι_I. A morphism between two ordered Π-multisets I and J is a map $m:I\to J$ that respects the orderings in I and J and intertwines the maps ι_I and ι_J: $\iota_J m=m\iota_I$. In particular, any subset $I'\subset I$ of a Π-ordered multiset has a unique structure of Π-ordered multiset, such that the inclusion map is a morphism of Π-ordered multisets.

To each Π-ordered multiset $I=\{i_1,\dots,i_n\}$ we attach an ordered set of variables $\{t_i|i\in I\}=\{t_{i_1},\dots,t_{i_n}\}$. Each element $i_k\in I$ and each variable t_{i_k} has its own 'type': $\iota(i_k)\in\Pi$.

Let i and j be elements of an ordered Π-multiset. Define a rational function

$$\gamma(t_i,t_j)=\begin{cases}\dfrac{t_i-t_j}{qt_i-q^{-1}t_j}\ , & \text{if}\quad \iota(i)=\iota(j)+1\ ,\\[2mm] \dfrac{q^{-1}t_i-qt_j}{t_i-t_j}\ , & \text{if}\quad \iota(j)=\iota(i)+1\ ,\\[2mm] \dfrac{qt_i-q^{-1}t_j}{q^{-1}t_i-qt_j}\ , & \text{if}\quad \iota(i)=\iota(j)\ ,\\[2mm] 1\ , & \text{otherwise.}\end{cases} \tag{3.3}$$

Assume that for any representation V of $U_q(\widehat{\mathfrak{gl}}_N)$ with a weight singular vector v, and any ordered Π-multiset $I=\{i_1,\dots i_n\}$, there is a V-valued rational function $w_{V,I}(t_{i_1},\dots,t_{i_n})\in V$ depending on the variables $\{t_i|i\in I\}$. We call such a collection of rational functions a *weight function* w, if:

[*1] The definition of the weight function used in this paper differs from that of [KPT] by the reverse order of the variables. The definition of the modified weight function is the same.

(a) The rational function, corresponding to the empty set, is equal to v: $w_{V,\emptyset} \equiv v$.

(b) The function $w_{V,I}(t_{i_1},\dots,t_{i_n})$ depends only on an isomorphism class of an ordered Π-multiset, that is, for any isomorphism $f : I \to J$ of ordered Π-multisets

$$w_{V,I}(t_{f(i)}|_{i\in I}) = w_{V,J}(t_j|_{j\in J}). \tag{3.4}$$

(c) The functions $w_{V,I}$ satisfy the following comultiplication property. Let $V = V_1 \otimes V_2$ be a tensor product of two representations with singular vectors v_1, v_2 and weight series $\{\lambda_b^{(1)}(u)\}$ and $\{\lambda_b^{(2)}(u)\}$, $b = 1,\dots,N$. Then for any ordered Π-multiset I we have

$$\begin{aligned} w_{V,I}(t_i|_{i\in I}) = &\sum_{I=I_1\coprod I_2} w_{V_1,I_1}(t_i|_{i\in I_1}) \otimes w_{V_2,I_2}(t_i|_{i\in I_2}) \\ &\times \Phi_{I_1,I_2}(t_i|_{i\in I}) \prod_{j\in I_1} \frac{\lambda^{(2)}_{\iota(j)}(t_j)}{\lambda^{(2)}_{\iota(j)+1}(t_j)}, \end{aligned} \tag{3.5}$$

where

$$\Phi_{I_1,I_2}(t_i|_{i\in I}) = \prod_{i\in I_2,\ j\in I_1,\ i\prec j} \gamma(t_i,t_j).$$

The summation in (3.5) runs over all possible decompositions of the set I into a disjoint union of two non-intersecting subsets I_1 and I_2. The structure of ordered Π-multiset on each subset is induced from that of I.

Given elements i, j of an ordered multiset define two functions $\tilde\gamma(t_i,t_j)$ and $\beta(t_i,t_j)$ by the formulae

$$\tilde\gamma(t_i,t_j) = \begin{cases} \dfrac{t_i - t_j}{qt_i - q^{-1}t_j}\ , & \text{if} \quad \iota(i) = \iota(j)+1\ , \\[2ex] \dfrac{q^{-1}t_i - qt_j}{t_i - t_j}\ , & \text{if} \quad \iota(j) = \iota(i)+1\ , \\[2ex] 1\ , & \text{otherwise} \end{cases}$$

and

$$\beta(t_i,t_j) = \begin{cases} \dfrac{q^{-1}t_i - qt_j}{t_i - t_j}\ , & \text{if} \quad \iota(i) = \iota(j)\ , \\[2ex] 1\ , & \text{otherwise.} \end{cases} \tag{3.6}$$

A collection of rational V-valued functions $\mathbf{w}_{V,I}(t_i|_{i\in I})$, depending on a representation V of $U_q(\widehat{\mathfrak{gl}}_N)$ with a weight singular vector v, and an ordered Π-multiset I, is called a *modified weight function* $\mathbf{w}$, if it satisfies conditions (a), (b) above and the condition (c$'$):

(c′) Let $V = V_1 \otimes V_2$ be a tensor product of two representations with singular vectors v_1, v_2 and weight series $\{\lambda_b^{(1)}(u)\}$ and $\{\lambda_b^{(2)}(u)\}$, $b = 1, \ldots, N$. Then for any multiset I we have

$$\begin{aligned} \mathbf{w}_{V,I}(t_i|_{i\in I}) = & \sum_{I=I_1 \coprod I_2} \mathbf{w}_{V_1,I_1}(t_i|_{i\in I_1}) \otimes \mathbf{w}_{V_2,I_2}(t_i|_{i\in I_2}) \\ & \times \tilde{\Phi}_{I_1,I_2}(t_i|_{i\in I}) \cdot \prod_{j\in I_1} \lambda^{(2)}_{\iota(j)}(t_j) \prod_{j\in I_2} \lambda^{(1)}_{\iota(j)+1}(t_j), \end{aligned} \tag{3.7}$$

where

$$\tilde{\Phi}_{I_1,I_2}(t_i|_{i\in I}) = \prod_{i\in I_1,\ j\in I_2} \beta(t_i,t_j) \prod_{i\in I_2,\ j\in I_1,\ i\prec j} \tilde{\gamma}(t_i,t_j).$$

There is a bijection between weight functions and modified weight functions given by the following relations. Let w be a weight function. Then the collection $\mathbf{w}_{V,I}(t_i|_{i\in I})$, where

$$\mathbf{w}_{V,I}(t_i|_{i\in I}) = w_{V,I}(t_i|_{i\in I}) \prod_{i\prec j} \beta(t_i,t_j) \prod_{i\in I} \lambda_{\iota(i)+1}(t_i) \tag{3.8}$$

is a modified weight function.

3.2. Weight function and projections

Let $I = \{i_1, \ldots, i_n\}$ and $J = \{j_1, \ldots, j_n\}$ be two ordered Π-multisets. Let $\sigma : I \to J$ be an invertible map, which intertwines the coloring maps ι_I and ι_J: $\iota_J \sigma = \sigma \iota_I$, but does not necessarily respect the orderings in I and J (that is, σ is a 'permutation' on classes of isomorphisms of ordered Π-multisets).

Let $w(t_j|_{j\in J})$ be a function of the variables $t_j|_{j\in J}$. Define a *pullback* ${}^{\sigma,\gamma}w(t_i|_{i\in I})$ by the rule

$${}^{\sigma,\gamma}w(t_i|_{i\in I}) = w(t_{\sigma(i)}|_{i\in I}) \prod_{i,j\in I,\, i\prec j,\ \sigma(j)\prec\sigma(i)} \gamma(t_i,t_j). \tag{3.9}$$

One may check [KPT] that the pullback operation (3.9) is compatible with the comultiplication rule (3.5). We call a weight function $w_{V,I}(t_i|_{i\in I})$ *q-symmetric*, if for any ordered Π-multisets I and J and an invertible map $\sigma : I \to J$, intertwining the colouring maps, we have

$${}^{\sigma,\gamma}w_{V,J}(t_i|_{i\in I}) = w_{V,I}(t_i|_{i\in I}). \tag{3.10}$$

For any multiset I we define a U_F^+-valued series $\mathcal{W}_I(t_i|_{i\in I})$ as the projection

$$\mathcal{W}_I(t_i|_{i\in I}) = P^+\left(F_{\iota(i_n)}(t_{i_n}) F_{\iota(i_{n-1})}(t_{i_{n-1}}) \cdots F_{\iota(i_2)}(t_{i_2}) F_{\iota(i_1)}(t_{i_1})\right) \tag{3.11}$$

We call the series $\mathcal{W}_I(t_i|_{i\in I})$ *universal weight function.*

Theorem 1 (KPT). *A collection of V-valued functions*

$$w_{V,I}(t_i|_{i\in I}) = \mathcal{W}_I(t_i|_{i\in I})\ v,$$

where V *is a* $U_q(\widehat{\mathfrak{gl}}_N)$*-module with a singular weight vector* v*, form a* q*-symmetric weight function.*

Let $\bar{n} = \{n_1, n_2, \ldots, n_{N-2}, n_{N-1}\}$ be a set of non-negative integers. Let $I_{\bar{n}}$, be an ordered Π-multiset, such that its first n_1 elements have the type 1, the next n_2 elements have the type 2 and the last n_{N-1} elements have the type $N-1$. Denote by $\bar{t}_{[\bar{n}]}$ the set of related variables:

$$\bar{t}_{[\bar{n}]} = \left\{ t^1_1, \ldots, t^1_{n_1}; t^2_1, \ldots, t^2_{n_2};\ \ldots\ ; t^{N-1}_1, \ldots, t^{N-1}_{n_{N-1}} \right\}. \tag{3.12}$$

The variable t^a_k is of type a. If $n_a = 0$ for some a, then the variables of the type a are absent in the set (3.12). Denote by $\mathcal{W}^{N-1}(\bar{t}_{[\bar{n}]})$ the universal weight function associated with the set of variables (3.12).

$$\begin{aligned} \mathcal{W}^{N-1}(\bar{t}_{[\bar{n}]}) = P^+ \Big(& F_{N-1}(t^{N-1}_{n_{N-1}}) \cdots F_{N-1}(t^{N-1}_1) \quad \cdots \\ & \cdots \quad F_2(t^2_{n_2}) \cdots F_2(t^2_1) F_1(t^1_{n_1}) \cdots F_1(t^1_1) \Big). \end{aligned} \tag{3.13}$$

For any weight singular vector v let $w^{N-1}_V(\bar{t}_{[\bar{n}]}) = \mathcal{W}^{N-1}(\bar{t}_{[\bar{n}]})\ v$ be the related weight function and

$$\mathbf{w}^{N-1}_V(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]}) \prod_{a=2}^{N} \prod_{\ell=1}^{n_{a-1}} \lambda_a(t^{a-1}_\ell)\ w^{N-1}(\bar{t}_{[\bar{n}]}) \tag{3.14}$$

the corresponding modified weight function. Here

$$\beta(\bar{t}_{[\bar{n}]}) = \prod_{a=1}^{N-1} \prod_{1 \le \ell < \ell' \le n_a} \frac{q - q^{-1} t^a_\ell / t^a_{\ell'}}{1 - t^a_\ell / t^a_{\ell'}}.$$

We call the modified weight function (3.14) *off-shell Bethe vector*.

One can see that any Π-ordered multiset is isomorphic to a permutation of some $I_{\bar{n}}$. The q-symmetric property then implies that the series (3.13) completely describe the universal weight function (3.11), and of-shell Bethe vectors (3.14) completely describe the corresponding modified weight function.

4. A computation of the universal weight function

In this section we express the universal weight function $\mathcal{W}^{N-1}(\bar{t}_{\bar{n}})$ in generators of $U_q(\widehat{\mathfrak{gl}}_N)$, using the definition of the projection operator P^+. Our strategy is as follows. Under projection operator in (3.13) we separate all factors $F_a(t^a_i)$ with $a < N-1$ and apply to this product the ordering procedure of Proposition 4.1, based on the property (4.9). We get under total projection a symmetrization of a sum of terms $x_i P^-(y_i) P^+(z_i)$ with rational functional coefficients; here each x_i is a monomial on the modes of $F_{N-1}(t)$, and y_i, z_i are monomials on the modes of $F_a(t)$ with $a < N-1$. Then we reorder x_i

and $P_-(y_i)$. At this stage composed currents, collected in so called strings, appear. The calculation of the projection of strings is a separate problem, which is solved by analytical tools.

4.1. Basic notations

Let $\bar{l}$ and $\bar{r}$ be two collections of nonnegative integers satisfying a set of inequalities

$$l_a \le r_a, \quad a = 1, \dots, N-1. \tag{4.1}$$

Denote by $[\bar{l}, \bar{r}]$ a collection of segments which are sets of positive increasing integers $\{l_a+1, \dots, r_a-1, r_a\}$ including r_a and excluding l_a. The length of each segment is equal to $r_a - l_a$.

For a given set $[\bar{l}, \bar{r}]$ of segments we denote by $\bar{t}_{[\bar{l},\bar{r}]}$ the set of variables

$$\bar{t}_{[\bar{l},\bar{r}]} \; = \{t^1_{l_1+1}, \dots, t^1_{r_1}; \; \dots \; ; t^{N-2}_{l_{N-2}+1}, \dots, t^{N-2}_{r_{N-2}}; \, t^{N-1}_{l_{N-1}+1}, \dots, t^{N-1}_{r_{N-1}}\}. \tag{4.2}$$

The number of the variables of the type a is equal to $r_a - l_a$. In this notation, the set of the variables (3.12) is $\bar{t}_{[\bar{n}]} \equiv \bar{t}_{[\bar{0},\bar{n}]}$. One can consider (4.2) as a list of variables, corresponding to the ordered multiset, naturally related to $[\bar{l}, \bar{r}]$.

For any $a = 1, \dots, N-1$ we consider the segment $[l_a, r_a] = \{l_a+1, \dots, r_a-1, r_a\}$ as an ordered multiset $\{l_a+1 \prec \cdots \prec r_a-1 \prec r_a\}$, in which all the elements are of the type a. The related set of variables is denoted as

$$\bar{t}^a_{[l_a,r_a]} \; = \{t^a_{l_a+1}, \dots, t^a_{r_a}\}. \tag{4.3}$$

All the variables in (4.3) have the type a. For the segment $[l_a, r_a] = [0, n_a]$ we use the shorten notation $\bar{t}^a_{[0,n_a]} \equiv \bar{t}^a_{[n_a]}$.

Our basic calculations are performed on a level of formal series attached to certain ordered multisets. To save space we often write some series as rational homogeneous functions with the following prescription. Let $\{t_i | i \in I\} = \{t_{i_1}, \dots, t_{i_n}\}$ be the ordered set of variables attached to an ordered set $I = \{i_1 \prec i_2 \prec \cdots \prec i_n\}$ and $g(t_i | i \in I)$ be a rational function. Then we associate to $g(t_i | i \in I)$ a Laurent series which is the expansion of $g(t_i | i \in I)$ in the region $|t_{i_1}| \ll |t_{i_2}| \ll \cdots \ll |t_{i_n}|$. If, for instance, $1 \prec 2$, then we associate to a rational function $\dfrac{1}{t_1 - t_2}$ a series $-\sum_{k\ge 0} t_1^k t_2^{-k-1}$. With this convention to a rational function of the variables $\bar{t}_{[\bar{n}]}$ we associate a Taylor series on t^b_k / t^c_l with $b < c$ and on t^a_i / t^a_j with $i < j$.

For a collection of variables $\bar{t}_{[\bar{l},\bar{r}]}$ we consider an ordered product

$$\begin{aligned} \mathcal{F}(\bar{t}_{[\bar{l},\bar{r}]}) &= \prod_{N-1\ge a\ge 1}^{\longleftarrow} \left(\prod_{r_a \ge \ell > l_a}^{\longleftarrow} F_a(t^a_\ell) \right) \\ &= F_{N-1}(t^{N-1}_{r_{N-1}}) \cdots F_{N-1}(t^{N-1}_{l_{N-1}+1}) \quad \cdots \quad F_1(t^1_{r_1}) \cdots F_1(t^1_{l_1+1}), \end{aligned} \tag{4.4}$$

where the series $F_a(t) \equiv F_{a+1,a}(t)$ is defined by (2.7). As a particular case, we have $\mathcal{F}(\bar{t}^a_{[\bar{l}_a,\bar{r}_a]}) = F_a(t^a_{r_a}) \cdots F_a(t^a_{l_a+2}) F_a(t^a_{l_a+1})$.

Symbols $\overleftarrow{\prod\limits_a} A_a$ and $\overrightarrow{\prod\limits_a} A_a$ mean ordered products of noncommutative entries A_a, such that A_a is on the right (resp., on the left) from A_b for $b > a$:

$$\overleftarrow{\prod_{j\geq a\geq i}} A_a = A_j\, A_{j-1}\, \cdots\, A_{i+1}\, A_i, \qquad \overrightarrow{\prod_{i\leq a\leq j}} A_a = A_i\, A_{i+1}\, \cdots\, A_{j-1}\, A_j.$$

For collections of positive integers $\bar r$ and $\bar l$ which satisfy inequalities (4.1) we denote by $S_{\bar l,\bar r} = S_{r_{N-1}-l_{N-1}} \times \cdots \times S_{r_1-l_1}$ a direct product of the symmetric groups. The group $S_{\bar l,\bar r}$ naturally acts on functions of the variables $\bar t_{[\bar l,\bar r]}$ by permutations of variables of the same type. If $\sigma = \sigma^{N-1} \times \cdots \times \sigma^1 \in S_{\bar l,\bar r}$, set

$$(4.5)\qquad {}^{\sigma}\bar t_{[\bar l,\bar r]} = \{t^{N-1}_{\sigma^{N-1}(l_{N-1}+1)}, \ldots, t^{N-1}_{\sigma^{N-1}(r_{N-1})}; \ldots; t^1_{\sigma^1(l_1+1)}, \ldots, t^1_{\sigma^1(r_1)}\}.$$

For a formal series or a function $G(\bar t_{\bar r,\bar l})$ the q-symmetrization means

$$(4.6)\qquad \begin{aligned} &\overline{\mathrm{Sym}}_{\;\bar t_{[\bar l,\bar r]}}\; G(\bar t_{[\bar l,\bar r]}) = \\ &= \sum_{\sigma\in S_{\bar l,\bar r}} \prod_{1\leq a\leq N-1} \prod_{\substack{\ell>\ell' \\ \sigma^a(\ell)<\sigma^a(\ell')}} \frac{q - q^{-1} t^a_{\sigma^a(\ell)}/t^a_{\sigma^a(\ell')}}{q^{-1} - q t^a_{\sigma^a(\ell)}/t^a_{\sigma^a(\ell')}}\; G({}^{\sigma}\bar t_{[\bar l,\bar r]}). \end{aligned}$$

The operation (4.6) is well-defined throughout the paper, see [KP] for details. According to (3.10) we call a formal series $G(\bar t_{[\bar l,\bar r]})$ q-symmetric if

$$(4.7)\qquad \overline{\mathrm{Sym}}_{\;\bar t_{[\bar l,\bar r]}} \Big(G(\bar t_{[\bar l,\bar r]})\Big) = \prod_{a=1}^{N-1} (r_a - l_a)!\; G(\bar t_{[\bar l,\bar r]}).$$

The q-symmetrization of any series is q-symmetric.

For a set of segments $[\bar l,\bar r]$ we introduce a third collection of nonnegative integers $\bar s$ such that $0 \leq s_a \leq r_a - l_a$, $a = 1, \ldots, N-1$. Each integer s_a divides the segment $[\bar l,\bar r]$ into two nonintersecting segments $[\bar l,\bar r - \bar s]$ and $[\bar r - \bar s,\bar r]$. Recall that we include into segment its left edge and exclude the right one. It means that $[l_a, r_a - s_a] = \{l_a + 1, \ldots, r_a - s_a - 1, r_a - s_a\}$ and $[r_a - s_a, r_a] = \{r_a - s_a + 1, \ldots, r_a - 1, r_a\}$.

For a set of the variables $\bar t_{[\bar l,\bar r]}$ and a collection of integers $\bar s$ which divide the set of segments $[\bar l,\bar r]$ we define a series

$$(4.8)\qquad Z_{\bar s}(\bar t_{[\bar l,\bar r]}) = \prod_{a=1}^{N-2} \prod_{\substack{r_a-s_a<\ell\leq r_a \\ l_{a+1}<\ell'\leq r_{a+1}-s_{a+1}}} \frac{q - q^{-1}\; t^a_\ell \,/\, t^{a+1}_{\ell'}}{1 - t^a_\ell \,/\, t^{a+1}_{\ell'}}.$$

Note that this series does not depend on the variables $t^{N-1}_{r_{N-1}-s_{N-1}+1}, \ldots, t^{N-1}_{r_{N-1}}$ and $t^1_{l_1+1}, \ldots, t^1_{r_1-s_1}$. If $s_{N-1} = r_{N-1}$ then this series does not depend on all variables of the type $N-1$ from the set $t_{[\bar l,\bar r]}$. Also if $r_a = l_a$ for all a except one value $a = j$ then the collection $[\bar l,\bar r]$ of segments contains only one segment of the type j and we set the series (4.8) equal to 1.

We call any expression $\sum_i f_-^{(i)} \cdot f_+^{(i)}$, where $f_-^{(i)} \in U_f^-$ and $f_+^{(i)} \in U_f^+$ (*normal*) *ordered*. Using the property (2.16) of the projections we can present any product (4.4) in a normal ordered form.

Proposition 4.1. *We have an equality*

$$
\begin{aligned}
&\mathcal{F}(\bar t_{[\bar l,\bar r]}) = \\
&= \sum_{0\le s_{N-1}\le r_{N-1}-l_{N-1}} \cdots \sum_{0\le s_1\le r_1-l_1} \prod_{1\le a\le N-1} \frac{1}{(s_a)!(r_a-l_a-s_a)!} \\
&\quad\times\ \overline{\mathrm{Sym}}_{\ \bar t_{[\bar l,\bar r]}} \Big(Z_{\bar s}(\bar t_{[\bar l,\bar r]})\ P^-\left(\mathcal{F}(\bar t_{[\bar r-s,\bar r]})\right) \cdot P^+ \left(\mathcal{F}(\bar t_{[\bar l,\bar r-\bar s]})\right)\Big) .
\end{aligned}
\tag{4.9}
$$

Proof. We use the coproduct (2.13). The q-symmetrization appears in (4.9) due to the commuting of the Cartan and simple root currents corresponding to the same type, and the series $Z_{\bar s}(\bar t_{[\bar l,\bar r]})$ appears due to the relation

$$
\begin{aligned}
&k^+_{a+1}(t^{a+1}_{r_{a+1}-s_{a+1}})\cdots k^+_{a+1}(t^{a+1}_{l_{a+1}+1})\cdot F_a(t^a_{r_a})\cdots F_a(t^a_{r_a-s_a+1}) = \\
&= \prod_{\substack{r_a-s_a<\ell\le r_a\\ l_{a+1}<\ell'\le r_{a+1}-s_{a+1}}} \frac{q-q^{-1}\ t^a_\ell\,/\,t^{a+1}_{\ell'}}{1-t^a_\ell\,/\,t^{a+1}_{\ell'}}\ F_a(t^a_{r_a})\cdots F_a(t^a_{r_a-s_a+1}) \\
&\qquad\qquad \times k^+_{a+1}(t^{a+1}_{r_{a+1}-s_{a+1}})\cdots k^+_{a+1}(t^{a+1}_{l_{a+1}+1})
\end{aligned}
$$

which has to be used in order to apply the operator P^- in the right hand side of (4.9). □

4.2. Composed currents and strings

Following [DK], [KP], we introduce *composed currents* $F_{j,i}(t)$ for $i<j$. The composed currents for nontwisted quantum affine algebras were defined in [DK]. The series $F_{i+1,i}(t)$, $i=1,\ldots,N-1$, coincides with $F_i(t)$, cf. (2.7). According to [DK], the coefficients of the series $F_{j,i}(t)$ belong to the completion $\overline{U}_F$ of the algebra U_F, see Section 2.2.

The completion $\overline{U}_F$ determines analyticity properties of products of currents (and coincide with analytical properties of their matrix coefficients for highest weight representations). One can show that for $|i-j|>1$, the product $F_i(t)F_j(w)$ is an expansion of a function analytic at $t\ne 0$, $w\ne 0$. The situation is more delicate for $j=i,i\pm 1$. The products $F_i(t)F_i(w)$ and $F_i(t)F_{i+1}(w)$ are expansions of analytic functions at $|w|<|q^2t|$, while the product $F_i(t)F_{i-1}(w)$ is an expansion of an analytic function at $|w|<|t|$. Moreover, the only singularity of the corresponding functions in the whole region $t\ne 0$, $w\ne 0$, are simple poles at the respective hyperplanes, $w=q^2t$ for $j=i,i+1$, and $w=t$ for $j=i-1$. Recall, that the deformation parameter q is a generic complex number, which is neither 0 nor a root of unity.

The definition of the composed currents may be written in analytical form

$$
F_{j,i}(t) \;=\; -\operatorname*{res}_{w=t} F_{j,a}(t)F_{a,i}(w)\,\frac{dw}{w} \;=\; \operatorname*{res}_{w=t} F_{j,a}(w)F_{a,i}(t)\,\frac{dw}{w}
\tag{4.10}
$$

for any $a = i+1, \dots, j-1$. It is equivalent to the relation

$$F_{j,i}(t) = \oint F_{j,a}(t)F_{a,i}(w)\,\frac{dw}{w} - \oint \frac{q^{-1}-qt/w}{1-t/w}\,F_{a,i}(w)F_{j,a}(t)\,\frac{dw}{w},$$
$$F_{j,i}(t) = \oint F_{j,a}(w)F_{a,i}(t)\,\frac{dw}{w} - \oint \frac{q^{-1}-qw/t}{1-w/t}\,F_{a,i}(t)F_{j,a}(w)\,\frac{dw}{w}. \tag{4.11}$$

In (4.11) $\oint \frac{dw}{w} g(w) = g_0$ for any formal series $g(w) = \sum_{n\in\mathbb{Z}} g_n z^{-n}$.

Using the relations (2.8) on $F_i(t)$ we can calculate the residues in (4.10) and obtain the following expressions for $F_{j,i}(t)$, $i<j$:

$$F_{j,i}(t) = (q-q^{-1})^{j-i-1}F_i(t)F_{i+1}(t)\cdots F_{j-1}(t). \tag{4.12}$$

For example, $F_{i+1,i}(t) = F_i(t)$, and $F_{i+2,i}(t) = (q-q^{-1})F_i(t)F_{i+1}(t)$. The last product is well-defined according to the analyticity properties of the product $F_i(t)F_{i+1}(w)$, described above. In a similar way, one can show inductively that the product in the right hand side of (4.12) makes sense for any $i<j$. Formulas (4.12) prove that the defining relations for the composed currents (4.10) or (4.11) yields the same answers for all possible values $i<a<j$.

Calculating formal integrals in (4.11) we obtain the following presentations for the composed currents:

$$F_{j,i}(t) = F_{j,a}(t)F_{a,i}[0] - qF_{a,i}[0]F_{j,a}(t) + (q-q^{-1})\sum_{k\le 0} F_{a,i}[k]\,F_{j,a}(t)\,t^{-k}, \tag{4.13}$$

$$F_{j,i}(t) = F_{j,a}[0]F_{a,i}(t) - q^{-1}F_{a,i}(t)F_{j,a}[0] + (q-q^{-1})\sum_{k>0} F_{a,i}(t)\,F_{j,a}[k]\,t^{-k}, \tag{4.14}$$

which are useful for the calculation of their projections.

The analytical properties of the products of the composed currents, used in the paper, are presented in Appendix A.

For two sets of variables $\{u_1,\dots,u_k\}$ and $\{v_1,\dots,v_k\}$ we introduce the series

$$Y(u_k,\dots,u_1;v_k,\dots,v_1) = \prod_{m=1}^{k}\frac{1}{1-v_m/u_m}\prod_{m'=m+1}^{k}\frac{q-q^{-1}v_{m'}/u_m}{1-v_{m'}/u_m}$$
$$= \prod_{m=1}^{k}\frac{1}{1-v_m/u_m}\prod_{m'=1}^{m-1}\frac{q-q^{-1}v_m/u_{m'}}{1-v_m/u_{m'}}. \tag{4.15}$$

Consider again a collection of segments $[\bar{l},\bar{r}]$ and associated set of variables $\bar{t}_{[\bar{l},\bar{r}]}$. Let $j=\max(a)$ such that $r_b = l_b$ for $b=a+1,\dots,N-1$. Let $\bar{s}$ be a set of nonnegative integers, which besides the inequalities

$$0\le s_a\le r_a-l_a,\quad a=1,\dots,j \tag{4.16}$$

satisfies the following admissibility conditions

$$0 = s_0 \le s_1 \le s_2 \ldots \le s_{j-1} \le s_j = r_j - l_j. \tag{4.17}$$

Having the set $\bar{s}$ which satisfy both restrictions (4.16) and (4.17) we define a series depending on the set of the variables $\bar{t}_{[\bar{r}-\bar{s},\bar{r}]}$:

$$X(\bar{t}_{[\bar{r}-\bar{s},\bar{r}]}) = \prod_{a=1}^{j-1} Y(t^{a+1}_{r_{a+1}-s_{a+1}+s_a},\ldots,t^{a+1}_{r_{a+1}-s_{a+1}+1};t^a_{r_a},\ldots,t^a_{r_a-s_a+1}). \tag{4.18}$$

When $j = 1$ we set $X(\cdot) = 1$.

Define a special ordered product of the composed currents, which we call *a string*:

$$\mathcal{F}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]}) = \prod_{j\ge a\ge 1}^{\longleftarrow} \left(\prod_{l_j+s_a\ge \ell > l_j+s_{a-1}}^{\longleftarrow} F_{j+1,a}(t^j_\ell) \right). \tag{4.19}$$

The string (4.19) depends only on the variables $\{t^j_{l_j+1},\ldots,t^j_{r_j}\}$ of the type j, corresponding to the segment $[l_j, r_j]$. The set of nonnegative integers $\bar{s}$ satisfying the admissibility condition (4.17) divides the segment $[l_j, r_j]$ into j subsegments $[l_j{+}s_{a-1}, l_j{+}s_a]$ for $a = 1,\ldots,j$. This division defines the product of the composed currents in the string (4.19).

Besides the product (4.19) which we called the string we consider the inverse ordered product of the same composed currents which we call the *inverse string*

$$\tilde{\mathcal{F}}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]}) = \prod_{1\le a\le j}^{\longrightarrow} \left(\prod_{l_j+s_{a-1}<\ell\le l_j+s_a}^{\longrightarrow} F_{j+1,a}(t^j_\ell) \right). \tag{4.20}$$

The inverse string satisfies analytical properties formulated in Appendix A which allows to calculate the projection of the direct and inverse string.

4.3. Recurrence relation

Let $\bar{n}$ be the set of nonnegative integers $\bar{n} = \{n_1,\ldots,n_{N-1}\}$. We claim that the projection (3.13) satisfies the recurrence relation given by the following

Proposition 4.2.

$$\begin{aligned} \mathcal{W}^{N-1}(\bar{t}_{[\bar{n}]}) &= \sum_{\bar{s}} \prod_{a=1}^{N-1} \frac{1}{(s_a-s_{a-1})!(n_a-s_a)!} \\ &\times \overline{\mathrm{Sym}}_{\ \bar{t}_{[\bar{n}]}} \left(Z_{\bar{s}}(\bar{t}_{[\bar{n}]}) X(\bar{t}_{[\bar{n}-\bar{s},\bar{n}]}) P^+ \left(\mathcal{F}^{N-1}_{\bar{s}}(\bar{t}^{N-1}_{[n_{N-1}]}) \right) \mathcal{W}^{N-2}(\bar{t}_{[\bar{n}-\bar{s}]}) \right), \end{aligned} \tag{4.21}$$

where the sum is taken over all collections $\bar{s} = \{s_1, s_2,\ldots,s_{N-1}\}$, *such that* $0 = s_0 \le s_1 \le s_2 \ldots \le s_{N-2} \le s_{N-1} = n_{N-1}$ *and* $0 \le s_a \le n_a$, $a = 1,\ldots,N-2$.

Due to the restriction $s_{N-1} = n_{N-1}$ and definition (4.8), the series $Z_{\bar{s}}(\bar{t}_{[\bar{n}]})$ depends only on the variables with type $N-1$ missing. The same restriction implies that the set $\bar{t}_{[\bar{n}-\bar{s}]}$ does not contain variables of the type $N-1$ and an element $\mathcal{W}^{N-2}(\bar{t}_{[\bar{n}-\bar{s}]})$ is a universal weight function for the algebra $U_q(\widehat{\mathfrak{gl}}_{N-1})$. This fact allows to iterate the recurrence relation (4.21) and reduce the universal weight function (3.13) to the q-symmetrization of the product of projection of strings. This will be done in the next section.

Proof of Proposition 4.2. The proof of the recurrence relation (4.21) follows the strategy of calculation of the universal weight function which was described in the introductory part to the Section 4.

Set $\bar{n}' = \{n_1, \dots, n_{N-2}, 0\}$. We have a decomposition $\mathcal{F}(\bar{t}_{[\bar{n}]}) = \mathcal{F}(\bar{t}^{N-1}_{[n_{N-1}]})$ $\mathcal{F}(\bar{t}_{[\bar{n}']})$ where the first term only contains the variables of the type $N-1$, and the last term does not contain the variables of the type $N-1$

$$\begin{aligned}\mathcal{F}(\bar{t}^{N-1}_{[n_{N-1}]}) &= F_{N-1}(t^{N-1}_{n_{N-1}}) \cdots F_{N-1}(t^{N-1}_1),\\ \mathcal{F}(\bar{t}_{[\bar{n}']}) &= F_{N-2}(t^{N-2}_{n_{N-2}}) \cdots F_{N-2}(t^{N-2}_1) \cdots F_1(t^1_{n_1}) \cdots F_1(t^1_1).\end{aligned}$$

We apply to the product $\mathcal{F}(\bar{t}_{[\bar{n}']})$ the ordering procedure of Proposition 4.1 and substitute the result into (3.13):

$$\begin{aligned}\mathcal{W}^{N-1}(\bar{t}_{[\bar{n}]}) = &\sum_{s_{N-2},\dots,s_1} \prod_{a=1}^{N-2} \frac{1}{s_a!(n_a-s_a)!}\, \overline{\mathrm{Sym}}_{\;\bar{t}_{[\bar{n}']}} \big(Z_{\bar{s}'}(\bar{t}_{[\bar{n}']})\\ &\times P^+\Big(\mathcal{F}(\bar{t}^{N-1}_{[n_{N-1}]}) P^-\big(\mathcal{F}(\bar{t}_{[\bar{n}'-\bar{s}',\bar{n}']})\big)\Big) \cdot \mathcal{W}^{N-2}(\bar{t}_{[\bar{n}'-\bar{s}']})\Big).\end{aligned} \tag{4.22}$$

The sum is taken over all nonnegative integers $\{s_1, \dots, s_{N-2}\}$ such that $s_a \le n_a$, $a = 1, \dots, N-2$; $\bar{s}'$ means the collection $\{s_1, \dots, s_{N-2}, 0\}$ and q-symmetrization is performed over the variables $\bar{t}_{[\bar{n}']}$.

Lemma 4.1. *For any* $\bar{n} = \{n_1, \dots, n_{N-1}\}$, $\bar{s} = \{s_1, \dots, s_{N-1}\}$ *such that all* $s_a \le n_a$ *for* $a = 1, \dots, N-2$ *and* $s_{N-1} = n_{N-1}$ *we have*

$$\begin{aligned}&P^+\Big(\mathcal{F}(\bar{t}^{N-1}_{[n_{N-1}]}) P^-\big(\mathcal{F}(\bar{t}_{[\bar{n}'-\bar{s}',\bar{n}']})\big)\Big) =\\ &= \frac{1}{s_1!} \prod_{a=2}^{N-1} \frac{1}{(s_a - s_{a-1})!}\\ &\times \overline{\mathrm{Sym}}_{\;\bar{t}_{[\bar{n}-\bar{s},\bar{n}]}} \Big(X(\bar{t}_{[\bar{n}-\bar{s},\bar{n}]}) \cdot P^+\Big(\mathcal{F}^{N-1}_{\bar{s}}(\bar{t}^{N-1}_{[n_{N-1}]})\Big)\Big)\end{aligned} \tag{4.23}$$

if $s_1 \le s_2 \le \dots \le s_{N-1}$; *otherwise the projection in* (4.23) *is equal to zero.*

In (4.23) we follow the above notations $\bar{n}' = \{n_1, \dots, n_{N-2}, 0\}$, $\bar{s}' = \{s_1, \dots, s_{N-2}, 0\}$. The series $X(\bar{t}_{[\bar{n}-\bar{s},\bar{n}]})$ is defined in (4.18). The proof of Lemma is given in the end of this section.

Substitute (4.23) into (4.22). By definition (4.8) the series $Z_{\bar{s}'}(\bar{t}_{[\bar{n}']}) = Z_{\bar{s}}(\bar{t}_{[\bar{n}]})$ is symmetric with respect to permutations of the variables $t_{[\bar{n}-\bar{s},\bar{n}]}$ of the same type, and the universal weight function $\mathcal{W}^{N-2}(\bar t_{[\bar n'-\bar s']})=\mathcal{W}^{N-2}(\bar t_{[\bar n-\bar s]})$ does now depend on the variables $t_{[\bar n-\bar s,\bar n]}$. We can include the series $Z_{\bar s}(\bar t_{[\bar n]})$ and $\mathcal{W}^{N-2}(\bar t_{[\bar n-\bar s]})$ inside the q-symmetrization $\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n-\bar s,\bar n]}}(\cdot)$ and replace the double symmetrization by a single one, using (4.7):

$$\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n']}}\ \overline{\mathrm{Sym}}_{\ \bar t_{[\bar n-\bar s,\bar n]}}(\ \cdot\ )=\prod_{a=1}^{N-2}s_a!\ \overline{\mathrm{Sym}}_{\ \bar t_{[\bar n]}}(\ \cdot\ )$$

Proposition 4.2 is proved. □

Proof of Lemma 4.1. For any $j=1,\dots,N-1$ denote by U_j the subalgebra of $\overline{U}_f$ formed by the modes of $F_1(t),\dots,F_j(t)$. Let $U_j^\varepsilon=U_j\cap \mathrm{Ker}\,\varepsilon$ be the corresponding augmentation ideal.

We claim first that the projection $P^-\left(\mathcal{F}(\bar t_{[\bar n'-\bar s',\bar n']})\right)$ can be presented as

$$\begin{aligned}&\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n'-\bar s',\bar n']}}\left(\frac{1}{s_1!}\frac{X(\bar t_{[\bar n'-\bar s',\bar n']})}{\prod_{a=1}^{N-3}(s_{a+1}-s_a)!}\right.\\ &\qquad\left.\times P^-\left(\mathcal{F}^{N-2}_{\bar s}(\bar t^{N-2}_{[n_{N-2}-s_{N-2},n_{N-2}]})\right)\right)\qquad \mathrm{mod}\ \ P^-\left(U^\varepsilon_{N-3}\right)\cdot U_{N-1}\end{aligned}\tag{4.24}$$

if admissibility conditions $0\le s_1\le s_2\le\dots\le s_{N-3}\le s_{N-2}$ are satisfied and is zero modulo $P^-\left(U^\varepsilon_{N-3}\right)\cdot U_{N-1}$ otherwise.

This can be shown by iteratively using Proposition C.2, proved in Appendix C. Due to (2.15) and (2.16) we have $P^-(f_1\cdot f_2)=P^-(f_1\cdot P^-(f_2))$ for any elements $f_1,f_2\in\overline{U}_f$. Thus we can present the projection $P^-\left(\mathcal{F}(\bar t_{[\bar n'-\bar s',\bar n']})\right)$ as

$$\begin{aligned}&\frac{1}{s_1!}P^-\left(\prod^{\longleftarrow}_{N-2\ge a\ge 3}\mathcal{F}(\bar t^a_{[n_a-s_a,n_a]})\right.\\ &\qquad\qquad\left.\times\mathcal{F}(\bar t^2_{[n_2-s_2,n_2]})\ \overline{\mathrm{Sym}}_{\ \bar t^1_{[n_1-s_1,n_1]}}P^-\left(\mathcal{F}(\bar t^1_{[n_1-s_1,n_1]})\right)\right).\end{aligned}\tag{4.25}$$

We now apply (C.10) with $j=2$ to the last two terms of (4.25), and under condition $s_1\le s_2$ replace them by

$$\begin{aligned}&\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n^{(2)}-\bar s^{(2)},\bar n^{(2)}]}}\left(X(\bar t_{[\bar n^{(2)}-\bar s^{(2)},\bar n^{(2)}]})\mathcal{F}^2_{\bar s^{(2)}}(\bar t^2_{[n_2-s_2,n_2]})\right)=\\ &=\overline{\mathrm{Sym}}_{\ \bar t^2_{[n_2-s_2,n_2]},\bar t^1_{[n_1-s_1,n_1]}}\left(X(t^2_{n_2-s_2+s_1},\dots,t^2_{n_2-s_2+1};t^1_{n_1},\dots,t^1_{n_1-s_1+1})\right.\\ &\qquad\left.\times F_{32}(t^2_{n_2})\cdots F_{32}(t^2_{n_2-s_2+s_1+1})F_{31}(t^2_{n_2-s_2+s_1})\cdots F_{31}(t^2_{n_2-s_2+1})\right).\end{aligned}$$

modulo $P^-\left(U_1^\varepsilon\right)\cdot U_2$. Here we use the notation $\bar n^{(2)}=\{n_1,n_2,0,\dots,0\}$ and $\bar s^{(2)}=\{s_1,s_2,0,\dots,0\}$. If $s_2<s_1$, the last two terms in (4.25) are zero

modulo $P^-(U_1^\varepsilon)\cdot U_2$. Due to the commutativity $[F_a(t),F_1(t')]=0$ for $a\ge 3$ we can move the elements of $P^-(U_1^\varepsilon)$ to the left through the first product $\prod_{a\ge 3}\mathcal{F}(\bar t^a_{[n_a-s_a,n_a]})$ and then out of the projection P^-, since $P^-(P^-(f')\cdot f)=P^-(f')\cdot P^-(f)$. These terms are absorbed into $P^-\left(U^\varepsilon_{N-3}\right)\cdot U_{N-1}$ in (4.24).

Going further, we replace the appearing string $\mathcal{F}^2_{\bar s^{(2)}}(\bar t^2_{[n_2-s_2,n_2]})$ by its projection P^- and apply Proposition C.2 to the product

$$\mathcal{F}(\bar t^3_{[n_3-s_3,n_3]})\cdot\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n^{(2)}-\bar s^{(2)},\bar n^{(2)}]}}\left(X(\bar t_{[\bar n^{(2)}-\bar s^{(2)},\bar n^{(2)}]})P^-\left(\mathcal{F}^2_{\bar s^{(2)}}(\bar t^2_{[n_2-s_2,n_2]})\right)\right).$$

Using the notations $\bar n^{(3)}=\{n_1,n_2,n_3,0,\dots,0\}$, $\bar s^{(3)}=\{s_1,s_2,s_3,0,\dots,0\}$ and the assumption $s_2\le s_3$ we can replace this product by

$$\frac{1}{(s_3-s_2)!}\,\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n^{(3)}-\bar s^{(3)},\bar n^{(3)}]}}\left(X(\bar t_{[\bar n^{(3)}-\bar s^{(3)},\bar n^{(3)}]})\mathcal{F}^3_{\bar s^{(3)}}(\bar t^3_{[n_3-s_3,n_3]})\right)$$

modulo elements of $P^-(U_2^\varepsilon)\cdot U_3$, which are again moved to the left out of the projection and are absorbed into $P^-\left(U^\varepsilon_{N-3}\right)\cdot U_{N-1}$ in (4.24). Finally we get (4.24).

For the calculation of the projection $P^+\left(\mathcal{F}(\bar t^{N-1}_{[n_{N-1}]})P^-\left(\mathcal{F}(\bar t_{[\bar n'-\bar s',\bar n']})\right)\right)$ we replace the second factor by (4.24). Elements of $P^-\left(U^\varepsilon_{N-3}\right)\cdot U_{N-1}$ do not contribute, since any element of U_{N-3} commutes with modes of $F_{N-1}(t)$: $P^+\left(\mathcal{F}(\bar t^{N-1}_{[n_{N-1}]})\cdot P^-\left(U^\varepsilon_{N-3}\right)\cdot U_{N-1}\right)=0$. Thus we are rest to calculate (we set $s_0\equiv 0$)

$$\prod_{a=1}^{N-2}\frac{1}{(s_a-s_{a-1})!}\,\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n'-\bar s',\bar n']}}\,P^+\Big(X(\bar t_{[\bar n'-\bar s',\bar n']})\times\mathcal{F}(\bar t^{N-1}_{[n_{N-1}]})P^-\left(\mathcal{F}^{N-2}_{\bar s}(\bar t^{N-2}_{[n_{N-2}-s_{N-2},n_{N-2}]})\right)\Big).$$

Due to Proposition C.2 the latter expression is non-zero only iff $s_{N-2}\le n_{N-1}$ and is equal to the right hand side of (4.23). □

4.4. Iteration of the recurrence relation

Let $[[\bar s]]=\{s_i^j,\ 1\le i\le j\}$ be a triangular matrix with nonnegative integer coefficients. We say that the matrix $[[\bar s]]$ is $\bar n$-admissible, and denote this by the symbol $[[\bar s]]\prec\bar n$, if it does not increase in the lines, see equation (4.27), and its sum over the columns is $\bar n$:

$$n_a=\sum_{b=a}^{N-1}s_a^b,\qquad a=1,\dots,N-1. \tag{4.26}$$

We also follow the convention $s_0^j=0$ for $j=1,...,N-1$.
(4.27)

$$[[\bar s]]=\begin{pmatrix} s_1^1 & & & & \\ s_1^2 & s_2^2 & & 0 & \\ \vdots & \vdots & \ddots & & \\ s_1^{N-2} & s_2^{N-2} & \dots & s_{N-2}^{N-2} & \\ s_1^{N-1} & s_2^{N-1} & \dots & s_{N-2}^{N-1} & s_{N-1}^{N-1} \end{pmatrix} \begin{array}{l} 0=s_0^1\le s_1^1 \\ 0=s_0^2\le s_1^2\le s_2^2 \\ \vdots \\ 0=s_0^{N-2}\le s_1^{N-2}\le\ldots\le s_{N-2}^{N-2} \\ 0=s_0^{N-1}\le s_1^{N-1}\le\ldots\le s_{N-1}^{N-1} \end{array}$$

Let $\bar s^j$, $j=1,\dots,N-1$ be the j-th line of the admissible matrix $[[\bar s]]$. Define a collection of vectors

$$\bar p(\bar s)^j=\bar s^j+\bar s^{j+1}+\dots+\bar s^{N-2}+\bar s^{N-1},\quad j=1,\dots,N-1 \tag{4.28}$$

with non-negative integer components. Set $\bar p(\bar s)^N=\bar 0$. Note that according to admissibility condition (4.26) $\bar p(\bar s)^1=\bar n$.

The iteration of the recurrence relations (4.21) gives the following

Theorem 4.1. *The weight function* (3.13) *can be presented as a total q-symmetrization of the sum over all $\bar n$ admissible matrices $[[\bar s]]$ of the ordered products of the projections of strings with rational coefficients*:

(4.29)

$$\mathcal{W}^{N-1}(\bar t_{[\bar n]})=\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n]}}\sum_{[[\bar s]]\prec\bar n}\left(\prod_{b=1}^{N-1}\prod_{a=1}^{b}\frac{1}{(s_a^b-s_{a-1}^b)!}\right.$$
$$\left.\times\prod_{j=3}^{N-1}Z_{\bar s^j}(\bar t_{[\bar n-\bar p(\bar s)^{j+1}]})\prod_{j=2}^{N-1}X(\bar t_{[\bar n-\bar p(\bar s)^j,\bar n-\bar p(\bar s)^{j+1}]})\prod_{N-1\ge j\ge 1}^{\longleftarrow}P^+\left(\mathcal{F}^j_{\bar s^j}(\bar t^j_{[s^j_j]})\right)\right).$$

The rational series in (4.29) may be gathered into a single multi-variable series. Set

$$\begin{aligned}\mathcal{Z}_{[[\bar s]]}(\bar t_{[\bar n]})&=\prod_{j=3}^{N-1}Z_{\bar s^j}(\bar t_{[\bar n-\bar p(\bar s)^{j+1}]})\prod_{j=2}^{N-1}X(\bar t_{[\bar n-\bar p(\bar s)^j,\bar n-\bar p(\bar s)^{j+1}]})\\ &=\prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s_a^b}\frac{1}{1-t^a_{\ell+n_a-p_a^b}/t^{a+1}_{\ell+n_{a+1}-p^b_{a+1}}}\\ &\qquad\times\prod_{\ell'=1}^{\ell+n_{a+1}-p^b_{a+1}-1}\frac{q-q^{-1}t^a_{\ell+n_a-p^b_a}/t^{a+1}_{\ell'}}{1-t^a_{\ell+n_a-p^b_a}/t^{a+1}_{\ell'}},\end{aligned} \tag{4.30}$$

where p_a^b are components of the vector $\bar p(\bar s)^b$. Using (4.30) the formula for the universal weight function (4.29) can be written in a compact form:

(4.31)

$$\mathcal{W}^{N-1}(\bar t_{[\bar n]})=\overline{\mathrm{Sym}}_{\ \bar t_{[\bar n]}}\sum_{[[\bar s]]\prec\bar n}\left(\frac{\mathcal{Z}_{[[\bar s]]}(\bar t_{[\bar n]})}{\prod_{a\le b}(s_a^b-s_{a-1}^b)!}\prod_{N-1\ge j\ge 1}^{\longleftarrow}P^+\left(\mathcal{F}^j_{\bar s^j}(\bar t^j_{[s^j_j]})\right)\right).$$

Theorem 4.1 reduces the calculation of the universal weight function to the calculation of the projections of the strings.

4.5. Projection of composed currents and strings

Let Δ be the standard comultiplication in $U_q(\widehat{\mathfrak{gl}}_N)$ defined by (2.5). For any elements $x, y \in U_q(\widehat{\mathfrak{gl}}_N)$ we define the adjoint action by the relation

$$\mathrm{ad}_x\, y = \sum\nolimits_l a(x'_l)\cdot y\cdot x''_l\ , \quad \text{where} \quad \Delta x = \sum\nolimits_l x'_l \otimes x''_l \tag{4.32}$$

and $a(x)$ is an antipode map related to the comultiplication (2.5). By *screening operators* we understand the operators of the adjoint actions of zero modes $F_i[0]$ of $F_i(t)$:

$$S_i\,(y) = \mathrm{ad}_{F_i[0]}\,(y) = y\ F_i[0] - F_i[0]\ k_{i+1}^{-1}k_i\ y\ k_i^{-1}k_{i+1}, \tag{4.33}$$

where k_i are zero modes of $k_i^+(t)$: $k_i = k_i^+[0] = k_i^-[0]^{-1}$.

Proposition 4.3. *For any i, j, $1 \le i < j \le N$ we have the equalities*

$$\begin{aligned} P^+\left(F_{j+1,i}(u)\right) &= S_iS_{i+1}\cdots S_{j-1}\left(P^+(F_j(u))\right)\\ &= S_iS_{i+1}\cdots S_{j-1}\left(F_j(u)^{(+)}\right) = \left(S_iS_{i+1}\cdots S_{j-1}\left(F_j(u)\right)\right)^{(+)}\end{aligned} \tag{4.34}$$

Proof. Taking (4.13) for $a = i+1$ and using the definition of the projection P^+ we obtain

$$P^+\left(F_{j,i}(t)\right)\ =\ S_i\big(P^+(F_{j,i+1}(t))\big), \qquad i < j-1. \tag{4.35}$$

The first equality of Proposition follows from (4.35) by induction. One should take into account the commutativity of the projections and screening operators proved in [KP]. Then we apply (2.12). □

An analog of Proposition 4.3 for $P^-\left(F_{j+1,i}(u)\right)$ is given in Appendix B.

For a set $\{u_1, \dots, u_n\}$ of formal variables we introduce a set of the rational functions

$$\varphi_{u_m}(u; u_1, \dots, u_n) = \prod_{k=1,\ k\neq m}^{n} \frac{u-u_k}{u_m-u_k}\prod_{k=1}^{n}\frac{q^{-1}u_m - qu_k}{q^{-1}u - qu_k} \tag{4.36}$$

satisfying the normalization conditions $\varphi_{u_m}(u_s; u_1, \dots, u_n) = \delta_{ms}$. We set

$$F_{j+1,i}(u; u_1, \dots, u_n) = F_{j+1,i}(u) - \sum_{m=1}^{n}\varphi_{u_m}(u; u_1, \dots, u_n)F_{j+1,i}(u_m) \tag{4.37}$$

for $1 \le i \le j < N$. Here we regard $\varphi_{u_m}(u; u_1, \dots, u_n)$ as a formal power series in a region $|u_1| \gg \cdots \gg |u_m| \gg |u|$. By (4.34) we have

$$\begin{aligned} P^+\left(F_{i,j+1}(u; u_1, \dots, u_n)\right) &= S_iS_{i+1}\cdots S_{j-1}\left(F_j(u)^{(+)}\right) -\\ &- \sum_{m=1}^{n}\varphi_{u_m}(u; u_1, \dots, u_n)S_iS_{i+1}\cdots S_{j-1}\left(F_j(u_m)^{(+)}\right).\end{aligned} \tag{4.38}$$

Proposition 4.4 below and the relation (4.38) suggest a factorized form for the projection of the inverse string $\tilde{\mathcal{F}}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]})$.

Proposition 4.4.

$$P^+\left(\tilde{\mathcal{F}}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]})\right) = \tag{4.39}$$
$$= \prod_{1\le a\le j}^{\longrightarrow}\left(\prod_{l_j+s_{a-1}<\ell\le l_j+s_a}^{\longrightarrow} P^+\left(F_{j+1,a}(t^j_\ell;t^j_{l_j+1},\ldots,t^j_{\ell-1})\right)\right).$$

Proof of Proposition 4.4 is shifted to the Appendix B.

Projections of the string (4.19) and of the inverse string (4.20) are related, namely

$$P^+\left(\mathcal{F}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]})\right) = P^+\left(\tilde{\mathcal{F}}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]})\right)\prod_{l_j<\ell<\ell'\le r_j}\frac{q^{-1}-qt^j_\ell/t^j_{\ell'}}{1-t^j_\ell/t^j_{\ell'}} \tag{4.40}$$
$$\times\prod_{1\le a\le j}\left(\prod_{l_j+s_{a-1}<\ell<\ell'\le l_j+s_a}\frac{1-t^j_\ell/t^j_{\ell'}}{q-q^{-1}t^j_\ell/t^j_{\ell'}}\right).$$

This statement is the direct consequence of the relations between composed currents given by Proposition A.1. For the details see [KP].

The particular case of the basic relations (2.8),

$$k^+_j(t')F_{j,j-1}(t)k^+_j(t')^{-1} = \frac{q-q^{-1}t/t'}{1-t/t'}F_{j,j-1}(t)$$

implies the following relation for the projections:

$$k^+_j(t')P^+\left(F_{j,j-1}(t)\right)k^+_j(t')^{-1} = \frac{q-q^{-1}t/t'}{1-t/t'}P^+\left(F_{j,j-1}(t;t')\right), \tag{4.41}$$

and in general

$$\prod_{a=1}^n k^+_j(t_a)P^+\left(F_{j,j-1}(t)\right)\prod_{a=1}^n k^+_j(t_a)^{-1} \tag{4.42}$$
$$=\prod_{a=1}^n\frac{q-q^{-1}t/t_a}{1-t/t_a}\;P^+\left(F_{j,j-1}(t;t_1,\ldots,t_n)\right).$$

Applying a sequence of the screening operators $S_i\cdots S_{j-2}$ to (4.42) we obtain

$$\prod_{a=1}^n k^+_j(t_a)P^+\left(F_{j,i}(t)\right)\prod_{a=1}^n k^+_j(t_a)^{-1} \tag{4.43}$$
$$=\prod_{a=1}^n\frac{q-q^{-1}t/t_a}{1-t/t_a}P^+\left(F_{j,i}(t;t_1,\ldots,t_n)\right)$$

due to the commutativity of any of zero modes $F_i[0],\dots,F_{j-1}[0]$ with $k^+_{j+1}(t)$. We rewrite the relation (4.43) in the form

$$
\begin{aligned}
&P^+\left(F_{j,i}(t;t_1,\dots,t_n)\right)\prod_{a=1}^{n}k^+_j(t_a)\\
&\quad=\prod_{a=1}^{n}\frac{1-t/t_a}{q-q^{-1}t/t_a}\prod_{a=1}^{n}k^+_j(t_a)\ P^+\left(F_{j,i}(t)\right).
\end{aligned}
\tag{4.44}
$$

Using (4.44) and (4.40) we rewrite the relation (4.39) and its analog for the string (4.19) in the following form:

$$
\begin{aligned}
&P^+\left(\tilde{\mathcal{F}}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]})\right)\prod_{\ell=l_j+1}^{r_j}k^+_{j+1}(t^j_\ell)=\prod_{l_j<\ell<\ell'\le r_j}\frac{1-t^j_{\ell'}/t^j_\ell}{q-q^{-1}t^j_{\ell'}/t^j_\ell}\\
&\quad\times\prod_{1\le a\le j}^{\longrightarrow}\left(\prod_{l_j+s_{a-1}<\ell\le l_j+s_a}^{\longrightarrow}P^+\left(F_{j+1,a}(t^j_\ell)\right)k^+_{j+1}(t^j_\ell)\right),
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
&P^+\left(\mathcal{F}^j_{\bar{s}}(\bar{t}^j_{[l_j,r_j]})\right)\prod_{\ell=l_j+1}^{r_j}k^+_{j+1}(t^j_\ell)=\\
&\quad=\prod_{1\le a\le j}\left(\prod_{l_j+s_{a-1}<\ell<\ell'\le l_j+s_a}\frac{1-t^j_\ell/t^j_{\ell'}}{q-q^{-1}t^j_\ell/t^j_{\ell'}}\right)\\
&\quad\times\prod_{1\le a\le j}^{\longrightarrow}\left(\prod_{l_j+s_{a-1}<\ell\le l_j+s_a}^{\longrightarrow}P^+\left(F_{j+1,a}(t^j_\ell)\right)k^+_{j+1}(t^j_\ell)\right).
\end{aligned}
\tag{4.46}
$$

5. Weight function and L-operators

5.1. From Gauss coordinates to L-operator's entries

The results of the previous section show that the projection of the string multiplied by certain number of $k^+_{j+1}(t^j_\ell)$ can be factorized in such a way that each factor is a product of projection of the composed currents and of some $k^+_{j+1}(t^j_\ell)$. In this section we use this observation and express the weight functions via matrix elements of L-operators (2.10). First, we relate projection of composed currents with Gauss coordinates of L-operators. This is given by the following

Proposition 5.1. *We have for any* $i<j-1$

$$
P^+\left(F_{j,i}(t)\right)\ =\ (q-q^{-1})^{j-i-1}\mathrm{F}^+_{j,i}(t).
\tag{5.1}
$$

Proof. We use the equality

$$
(q-q^{-1})\mathrm{F}^+_{j,i}(t)\ =\ S_i\left(\mathrm{F}^+_{j,i+1}(t)\right),\qquad i<j-1.
\tag{5.2}
$$

It is proved in [KPT] and is a direct consequence of the relations (2.4) taken for the modes of L-operators. The claim of Proposition now follows by induction from Proposition 4.3 and relation (2.12). $\square$

Let V be $U_q(\widehat{\mathfrak{gl}}_N)$-module with a weight singular vector v. The relation (4.46) and Proposition 5.1 allows to rewrite the corollary (4.31) to Theorem 4.1 in the following form:

$$
\begin{aligned}
\mathbf{w}_V^{N-1}(\bar t_{[\bar n]}) &= \beta(\bar t_{[\bar n]})\ \overline{\mathrm{Sym}}\ _{\bar t_{[\bar n]}} \sum_{[[\bar s]]\prec \bar n} \left(\frac{(q-q^{-1})^{\sum_{b=1}^{N-1}(n_b-s^b_b)}}{\prod_{a\le b}^{N-1}(s^b_a-s^b_{a-1})!} \right.\\
&\times \mathcal{Z}_{[[\bar s]]}(\bar t_{[\bar n]}) \prod_{N-1\ge b\ge 1}^{\longleftarrow} \left(\prod_{1\le a\le b}^{\longrightarrow} \prod_{s^b_{a-1}<\ell\le s^b_a}^{\longrightarrow} \right.\\
&\times \left.\left.\left(\mathrm{F}^+_{b+1,a}(t^b_\ell)k^+_{b+1}(t^b_\ell) \prod_{\ell'=\ell+1}^{s^b_a} \frac{t^b_\ell-t^b_{\ell'}}{q^{-1}t^b_\ell-qt^b_{\ell'}} \right)\right) \prod_{\ell=s^b_b+1}^{n_b} k^+_{b+1}(t^b_\ell) \right) v
\end{aligned}
\tag{5.3}
$$

where the series $\mathcal{Z}_{[[\bar s]]}(\bar t_{[\bar n]})$ is given by (4.30).

For any $c=1,\dots,N$ denote by I_c the left ideal of $U_q(\widehat{\mathfrak{b}}^+)$, generated by the modes of $\mathrm{E}^+_{j,i}(u)$ with $i>j\ge c$. We have inclusions $0=I_N\subset I_{N-1}\subset\cdots\subset I_1$.

Lemma 5.1. *Fix any $c=1,\dots,N-1$. Then*
(i) *the left ideal I_c is generated by modes of $\mathrm{L}^+_{i,j}(u)$ with $i>j\ge c$;*
(ii) *for any a and b with $a<b$ and $b\ge c$ we have equalities*

$$
\mathrm{L}^+_{a,b}(t) \equiv \mathrm{F}^+_{b,a}(t)k^+_b(t) \mod I_c, \qquad \mathrm{L}^+_{b,b}(t) \equiv k^+_b(t) \mod I_c,
\tag{5.4}
$$

(iii) *for any $a\le c$ and $b\ge c$ the modes of $\mathrm{L}^+_{a,c}(t)$ and of $\mathrm{L}^+_{b,b}(t)$ normalize the ideal I_c:*

$$
I_c\cdot \mathrm{L}^+_{a,c}(t)\subset I_c \qquad I_c\cdot \mathrm{L}^+_{b,b}(t)\subset I_c.
\tag{5.5}
$$

Proof. We have three types of relations (5.1):

$$
\mathrm{L}^+_{a,b}(t) = \mathrm{F}^+_{b,a}(t)k^+_b(t) + \sum_{b<m\le N} \mathrm{F}^+_{m,a}(t)k^+_m(t)\mathrm{E}^+_{b,m}(t), \qquad a<b,
\tag{5.6}
$$

$$
\mathrm{L}^+_{b,b}(t) = k^+_b(t) \quad + \sum_{b<m\le N} \mathrm{F}^+_{m,b}(t)k^+_m(t)\mathrm{E}^+_{b,m}(t),
\tag{5.7}
$$

$$
\mathrm{L}^+_{a,b}(t) = k^+_a(t)\mathrm{E}^+_{b,a}(t) + \sum_{a<m\le N} \mathrm{F}^+_{m,a}(t)k^+_m(t)\mathrm{E}^+_{b,m}(t), \qquad a>b.
\tag{5.8}
$$

Denote $\bar{\mathrm{E}}^+_{j,i}(u)=k^+_i(t)\mathrm{E}^+_{j,i}(t)$ and $\bar{\mathrm{F}}^+_{i,j}(u)=\mathrm{F}^+_{i,j}(t)k^+_i(t)$ for $i>j$. In these notations the relation (5.8) looks as

$$
\mathrm{L}^+_{a,b}(t) = \bar{\mathrm{E}}^+_{b,a}(t) + \sum_{a<m\le N} \mathrm{F}^+_{m,a}(t)\bar{\mathrm{E}}^+_{b,m}(t), \qquad a>b.
\tag{5.9}
$$

Since $k_b^+(t)$ is invertible, the ideal I_c is generated by the modes of $\bar{\mathrm{E}}_{j,i}^+(u)$ with $i > j \geq c$ as well. Now we inverse the relations (5.9), that is we write $\bar{\mathrm{E}}_{j,N}^+(t) = \mathrm{L}_{N,j}^+(t)$ for $j < N$; then $\bar{\mathrm{E}}_{j,N-1}^+(t) = \mathrm{L}_{N-1,j}^+(t) - \mathrm{F}_{N,N-1}^+(t)\mathrm{L}_{N,j}^+(t)$ for $j < N-1$ and so on by induction. This proves (i). In the same manner we rewrite first (5.6) and (5.7) in $\bar{\mathrm{F}}_{i,j}^+(u)$, $\mathrm{E}_{j,i}^+(u)$ and $k_i^+(t)$ and prove by induction that $\mathrm{L}_{a,b}^+(t) \equiv \bar{\mathrm{F}}_{b,a}^+(t) \mod I_c$ and $\mathrm{L}_{b,b}^+(t) \equiv k_b^+(t) \mod I_c$ for $b \geq c$ and $a < b$, which means (ii).

The statement (iii) is a corollary of the Yang-Baxter relations (2.2). Let $\mathrm{R}_{ij;kl}(u,v)$ be the matrix elements of the R-matrix (2.1). We have for $a < c < k$ and for $a < c < j < i$:

$$\begin{aligned}
\mathrm{L}_{k,c}^+(u)\mathrm{L}_{a,c}^+(t) &= \frac{\mathrm{R}_{cc;cc}(u,t)}{\mathrm{R}_{kk;aa}(u,t)}\mathrm{L}_{a,c}^+(t)\mathrm{L}_{k,c}^+(u) - \frac{\mathrm{R}_{ka;ak}(u,t)}{\mathrm{R}_{kk;aa}(u,t)}\mathrm{L}_{a,c}^+(u)\mathrm{L}_{k,c}^+(t),\\
\mathrm{L}_{i,j}^+(u)\mathrm{L}_{a,c}^+(t) &= \frac{\mathrm{R}_{jj;cc}(u,t)}{\mathrm{R}_{ii;aa}(u,t)}\mathrm{L}_{a,c}^+(t)\mathrm{L}_{i,j}^+(u) - \frac{\mathrm{R}_{ia;ai}(u,t)}{\mathrm{R}_{ii;aa}(u,t)}\mathrm{L}_{a,c}^+(u)\mathrm{L}_{i,c}^+(t)\\
&\qquad + \frac{\mathrm{R}_{cj;jc}(u,t)}{\mathrm{R}_{ii;aa}(u,t)}\mathrm{L}_{a,c}^+(t)\mathrm{L}_{i,c}^+(u).
\end{aligned}$$

These relations precisely mean the inclusion $I_c \cdot \mathrm{L}_{a,c}^+(t) \subset I_c$. The second part of (iii) is proved in an analogous manner and is actually well known. □

Theorem 5.1. *For any* $U_q(\widehat{\mathfrak{gl}}_N)$ *module* V *with a weight singular vector* v *we have*

$$\begin{aligned}
\mathbf{w}_V^{N-1}(\bar{t}_{[\bar{n}]}) &= \beta(\bar{t}_{[\bar{n}]})\ \overline{\mathrm{Sym}}_{\ \bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]\prec\bar{n}} \left(\frac{(q-q^{-1})^{\sum_{b=1}^{N-1}(n_b-s_b^b)}}{\prod_{a\leq b}^{N-1}(s_a^b - s_{a-1}^b)!} \right.\\
&\quad \times\ \mathcal{Z}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]}) \prod_{N-1\geq b\geq 1}^{\longleftarrow} \left(\prod_{1\leq a\leq b}^{\longrightarrow} \prod_{\ell=s_{a-1}^b+1}^{s_a^b} \right.\\
&\quad \times \left.\left.\left(\mathrm{L}_{a,b+1}^+(t_\ell^b) \prod_{\ell'=\ell+1}^{s_a^b} \frac{t_\ell^b - t_{\ell'}^b}{q^{-1}t_\ell^b - qt_{\ell'}^b} \right)\right) \prod_{\ell=s_b^b+1}^{n_b} \mathrm{L}_{b+1,b+1}^+(t_\ell^b) \right) v.
\end{aligned}\tag{5.10}$$

Observe that non-commutative products over b and a run in the opposite directions and the ordering in the product over ℓ is not important now, because of commutativity of the matrix elements of L-operators with the same matrix indices.

Proof. The theorem states that in (5.3) we can replace each entry of $\bar{\mathrm{F}}_{b,a}^+(t) = \mathrm{F}_{b,a}^+(t)k_b^+(t)$ by $\mathrm{L}_{a,b}^+(t)$ and each entry of $k_b^+(t)$ in the last product of (5.3) by $\mathrm{L}_{b,b}^+(t)$. This is done with a help of Lemma 5.1. Indeed, we can present the right hand side of (5.3) as a linear combination of terms

$$A_i^N A_i^{N-1} \cdots A_i^2 B_i v,$$

where each A_i^c is an ordered product of some $\bar{\mathrm{F}}_{c,a}^+(t_j^{c-1})$, and B_i is a product of some $k_b(t_j^{b-1})$. We start from A_i^N. Here we have $\bar{\mathrm{F}}_{N,a}^+(t) = \mathrm{L}_{a,N}^+(t)$ by (5.6).

By the statement (ii) of Lemma 5.1 each multiplier of A_i^{N-1} can be written as $\mathrm{L}^+_{a,N-1}(t_j^{N-2})+x_j$ for some $a<N-1$, parameter t_j^{N-2} and $x_j\in I_{N-1}$. Due to the part (iii) of Lemma 5.1, we can rewrite the whole product in A_i^{N-1} as $\prod_j \mathrm{L}^+_{a,N-1}(t_j^{N-2})+y_{N-1}$ with a single $y_{N-1}\in I_{N-1}$. Moving further, we replace the product $A_i^N A_i^{N-1}\cdots A_i^2 B_i$ by

$$\left(\bar A_i^N \bar A_i^{N-1}\cdots \bar A_i^2+y_2\right)B_i v, \tag{5.11}$$

where each $\bar A_i^c$ equals to A_i^c with all $\bar{\mathrm{F}}^+_{c,a}(t)$ replaced by $\mathrm{L}^+_{a,c}(t)$, and $y_2\in I_2\subset I_1$. We apply finally the last part of Lemma 5.1, (iii) and replace (5.11) by the product $\bar A_i^N \bar A_i^{N-1}\cdots \bar A_i^2 \bar B_i v$, where $\bar B_i$ consists of $\mathrm{L}^+_{b,b}(t_k^{b-1})$ only. This proves the theorem. □

We can slightly simplify the formula (5.10) by a renormalization of q-symmetrization (see [TV2]). We set for a series or function $G(\bar t_{[\bar n]})$:

$$\overline{\mathrm{Sym}}\,^{(q)}_{\bar t_{[\bar n]}}\ G(\bar t_{[\bar n]})=\beta(\bar t_{[\bar n]})\ \overline{\mathrm{Sym}}\,_{\bar t_{[\bar n]}}\left(G(\bar t_{[\bar n]})\right). \tag{5.12}$$

Let $G^{\rm sym}(u_1,\dots,u_n)$ be any symmetric function of n variables u_k, that is $G^{\rm sym}({}^\sigma\bar u)=G^{\rm sym}(\bar u)$ for any element σ from the symmetric group S_n. Let $\beta(\bar u)=\prod_{k<k'}\frac{q-q^{-1}u_k/u_{k'}}{1-u_k/u_{k'}}$. One can check the following property of renormalized q-symmetrization:

$$\frac{1}{n!}\ \overline{\mathrm{Sym}}\,^{(q)}_{\bar u}\left(\beta(\bar u)^{-1}G^{\rm sym}(\bar u)\right)=\frac{1}{[n]_q!}\ \overline{\mathrm{Sym}}\,^{(q)}_{\bar u}\left(G^{\rm sym}(\bar u)\right), \tag{5.13}$$

where $[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$, and $[n]_q!=[n]_q[n-1]_q\cdots[2]_q[1]_q$.

We can apply the relation (5.13) to the right hand side of (5.10) because in this formula inside the total q-symmetrization there is a symmetric series in the sets of variables $\{t^b_\ell\}$ for $s^b_{a-1}+1\le\ell\le s^b_a$, $a=1,\dots,b$ and $b=1,\dots,N-1$. This follows from the commutativity of matrix elements of L-operators $[\mathrm{L}^+_{a,b}(t),\mathrm{L}^+_{a,b}(t')]=0$, and from the explicit form of the series (4.30). Restoring this series and denoting $\tilde p^b_a=n_a-p^b_a=s^a_a+\cdots+s^{b-1}_a$ we formulate the following corollary of Theorem 5.1

Corollary 5.1. *The off-shell Bethe vectors for quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ can be written as*

(5.14)

$$\begin{aligned}\mathbf{w}_V^{N-1}(\bar t_{[\bar n]})=\overline{\mathrm{Sym}}\,^{(q)}_{\bar t_{[\bar n]}}\sum_{[[\bar s]]\prec\bar n}\Bigg(&(q-q^{-1})^{\sum_{b=1}^{N-1}(n_b-s^b_b)}\prod_{a\le b}^{N-1}\frac{1}{[s^b_a-s^b_{a-1}]_q!}\\ &\times\prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s^b_a}\frac{1}{1-t^a_{\ell+\tilde p^b_a}/t^{a+1}_{\ell+\tilde p^b_{a+1}}}\prod_{\ell'=1}^{\ell+\tilde p^b_{a+1}-1}\frac{q-q^{-1}t^a_{\ell+\tilde p^b_a}/t^{a+1}_{\ell'}}{1-t^a_{\ell+\tilde p^b_a}/t^{a+1}_{\ell'}}\\ &\times\overleftarrow{\prod_{N-1\ge b\ge1}}\left(\overrightarrow{\prod_{1\le a\le b}}\left(\prod_{\ell=s^b_{a-1}+1}^{s^b_a}\mathrm{L}^+_{a,b+1}(t^b_\ell)\right)\prod_{\ell=s^b_b+1}^{n_b}\mathrm{L}^+_{b+1,b+1}(t^b_\ell)\right)\Bigg)v.\end{aligned}$$

Expression (5.14) for the off-shell Bethe vectors given in terms of matrix elements of L-operators can be written as the recurrence relation

(5.15)
$$\mathbf{w}_V^{N-1}(\bar t_{[\bar n]}) = \sum_{\bar s^{N-1}} \frac{1}{[s_1^{N-1}]_q!} \prod_{a=1}^{N-2} \frac{(q-q^{-1})^{s_a^{N-1}}}{[s_{a+1}^{N-1}-s_a^{N-1}]_q![n_a-s_a^{N-1}]_q!}$$
$$\times\, \overline{\mathrm{Sym}}\;{}^{(q)}_{\bar t_{\bar n}} \left(X(\bar t_{[\bar n-\bar s^{N-1},\bar n]})\; Z_{\bar s^{N-1}}(\bar t_{[\bar n]}) \prod_{a=1}^{N-2}\; \prod_{\ell\,=\,n_a-s_a^{N-1}+1}^{n_a} \lambda_{a+1}(t_\ell^a) \right.$$
$$\left. \times \prod_{1\le a\le N-1}^{\longrightarrow} \left(\prod_{\ell=s_{a-1}^{N-1}+1}^{s_a^{N-1}} \mathrm{L}^+_{a,N}(t_\ell^{N-1}) \right) \mathbf{w}_V^{N-2}(\bar t_{[\bar n-\bar s^{N-1}]}) \right),$$

where summation in (5.15) runs over first row of the admissible matrix $[[\bar s]]$. The vector valued function $\mathbf{w}_V^{N-2}(\bar t_{[\bar n-\bar s^{N-1}]})$ is the off-shell Bethe vector for the algebra $U_q(\widehat{\mathfrak{gl}}_{N-1})$ embedded into $U_q(\widehat{\mathfrak{gl}}_N)$. This embedding $\phi : U_q(\widehat{\mathfrak{gl}}_{N-1}) \hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$ can be described on the level of Gauss coordinates of the corresponding L-operators:

$$(5.16)\qquad \begin{gathered} \phi\left(\mathrm{F}^\pm_{b,a}(t)^{\langle N-1\rangle}\right) = \mathrm{F}^\pm_{b,a}(t)^{\langle N\rangle}, \quad \phi\left(\mathrm{E}^\pm_{a,b}(t)^{\langle N-1\rangle}\right) = \mathrm{E}^\pm_{a,b}(t)^{\langle N\rangle}, \\ \phi\left(k^\pm_a(t)^{\langle N-1\rangle}\right) = k^\pm_a(t)^{\langle N\rangle}, \quad 1\le a<b\le N-1. \end{gathered}$$

Here $\mathrm{F}^\pm_{b,a}(t)^{\langle N-1\rangle}$ etc., denote Gauss coordinates of the source $U_q(\widehat{\mathfrak{gl}}_{N-1})$ while $\mathrm{F}^\pm_{b,a}(t)^{\langle N\rangle}$ etc., the Gauss coordinate of the target $U_q(\widehat{\mathfrak{gl}}_N)$.

5.2. Evaluation homomorphism

The quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ contains a quantum group $U_q(\mathfrak{gl}_N)$. It is generated by the zero modes of L-operators (2.10). Using them, we introduce Cartan-Weyl generators E_{ab}, $1\le a,b\le N$ of $U_q(\mathfrak{gl}_N)$. We set $\mathrm{L}^\pm \equiv \mathrm{L}^\pm[0]$, and

(5.17)
$$\mathrm{L}^+ = \mathrm{K}\mathrm{L}^+_0 = \begin{pmatrix} \mathsf{E}_{11} & & & \\ & \mathsf{E}_{22} & 0 & \\ & 0 & \ddots & \\ & & & \mathsf{E}_{NN} \end{pmatrix} \begin{pmatrix} 1 & & & \\ \nu\mathsf{E}_{12} & 1 & 0 & \\ \vdots & \ddots & \ddots & \\ \nu\mathsf{E}_{1N} & \cdots & \nu\mathsf{E}_{N-1,N} & 1 \end{pmatrix},$$

(5.18)
$$\mathrm{L}^- = \mathrm{L}^-_0\mathrm{K}^{-1} = \begin{pmatrix} 1 & -\nu\mathsf{E}_{21} & \cdots & -\nu\mathsf{E}_{N1} \\ & \ddots & \ddots & \vdots \\ & & 1 & -\nu\mathsf{E}_{N,N-1} \\ & & & 1 \end{pmatrix} \begin{pmatrix} \mathsf{E}_{11}^{-1} & & & \\ & \mathsf{E}_{22}^{-1} & 0 & \\ & 0 & \ddots & \\ & & & \mathsf{E}_{NN}^{-1} \end{pmatrix},$$

where $\nu = q - q^{-1}$. Generators E_{aa}, $\mathsf{E}_{a,a+1}$ and $\mathsf{E}_{a+1,a}$ may be considered as Chevalley generators of $U_q(\mathfrak{gl}_N)$ with commutation relations

$$\begin{aligned} \mathsf{E}_{aa}\mathsf{E}_{bc}\mathsf{E}_{aa}^{-1} &= q^{\delta_{ab}-\delta_{ac}}\mathsf{E}_{bc}, \\ [\mathsf{E}_{a,a+1},\mathsf{E}_{b+1,b}] &= \delta_{ab}\frac{\mathsf{E}_{aa}\mathsf{E}_{a+1,a+1}^{-1} - \mathsf{E}_{aa}^{-1}\mathsf{E}_{a+1,a+1}}{q-q^{-1}}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} &\mathsf{E}_{a\pm1,a}^2\mathsf{E}_{a,a\mp1} - (q+q^{-1})\mathsf{E}_{a\pm1,a}\mathsf{E}_{a,a\mp1}\mathsf{E}_{a\pm1,a} + \mathsf{E}_{a,a-1}\mathsf{E}_{a\pm1,a}^2 = 0, \\ &\mathsf{E}_{a,a\pm1}^2\mathsf{E}_{a\mp1,a} - (q+q^{-1})\mathsf{E}_{a,a\pm1}\mathsf{E}_{a\mp1,a}\mathsf{E}_{a,a\pm1} + \mathsf{E}_{a\mp1,a}\mathsf{E}_{a,a\pm1}^2 = 0. \end{aligned} \tag{5.20}$$

The rest of E_{ab} may be constructed from these Chevalley generators as follows

$$\begin{aligned} \mathsf{E}_{c,a} &= \mathsf{E}_{c,b}\mathsf{E}_{b,a} - q\mathsf{E}_{b,a}\mathsf{E}_{c,b}, \\ \mathsf{E}_{a,c} &= \mathsf{E}_{a,b}\mathsf{E}_{b,c} - q^{-1}\mathsf{E}_{b,c}\mathsf{E}_{a,b}, \qquad a<b<c. \end{aligned} \tag{5.21}$$

Using generators E_{ab} we define an evaluation homomorphism:

$$\mathcal{E}v_z\left(\mathrm{L}^+(u)\right) = \mathrm{L}^+ - \frac{z}{u}\,\mathrm{L}^-, \quad \mathcal{E}v_z\left(\mathrm{L}^-(u)\right) = \mathrm{L}^- - \frac{u}{z}\,\mathrm{L}^+. \tag{5.22}$$

One can check that the relations (5.19) and (5.21) follow from (2.4).

Let M_Λ be a $U_q(\mathfrak{gl}_N)$-module generated by a vector v, satisfying the conditions $\mathsf{E}_{a,a}\,v = q^{\Lambda_a}\,v$ and $\mathsf{E}_{a,b}\,v = 0$ for $a<b$. Then v is a singular weight vector of the evaluation $U_q(\widehat{\mathfrak{gl}}_N)$ module $M_\lambda(z)$. Taking into account reordering of the factors

$$\prod_{b=1}^{N-2}\prod_{\ell=s_b^b+1}^{n_b}\lambda_{b+1}(t_\ell^b) = \prod_{b=2}^{N-2}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s_a^b}\lambda_{a+1}(t^a_{\ell+\tilde p^b_a}) \tag{5.23}$$

we can present the off-shell Bethe vector in $M_\lambda(z)$ as

(5.24)

$$\begin{aligned} \mathbf{w}^{N-1}_{M_\Lambda(z)}(\bar t_{[\bar n]}) &= \frac{(q-q^{-1})^{\sum_{a=1}^{N-1}n_a}}{\prod_{a=1}^{N-1}\prod_{\ell=1}^{n_a}t^a_\ell}\sum_{[[\bar s]]\prec\bar n}\left(\prod_{a\le b}^{N-1}\frac{1}{[s^b_a-s^b_{a-1}]_q!}\right. \\ &\times\left(\prod_{N-1\ge b\ge1}^{\longleftarrow}\left(\prod_{1\le a\le b}^{\longrightarrow}\left(z\mathsf{E}_{b+1,a}\mathsf{E}^{-1}_{b+1,b+1}\right)^{s^b_a-s^b_{a-1}}\right)\right)v \\ &\times\overline{\mathrm{Sym}}^{\,(q)}_{\,\bar t_{[\bar n]}}\left(\prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s^b_a}\frac{q^{\Lambda_{a+1}}t^a_{\ell+\tilde p^b_a}-q^{-\Lambda_{a+1}}z}{1-t^a_{\ell+\tilde p^b_a}/t^{a+1}_{\ell+\tilde p^b_{a+1}}}\prod_{\ell'=1}^{\ell+\tilde p^b_{a+1}-1}\frac{q-q^{-1}t^a_{\ell+\tilde p^b_a}/t^{a+1}_{\ell'}}{1-t^a_{\ell+\tilde p^b_a}/t^{a+1}_{\ell'}}\right)\Bigg), \end{aligned}$$

where $\tilde p^b_a = s^a_a+\cdots+s^{b-1}_a$.

6. Relation to the Tarasov-Varchenko construction

The original inductive construction of nested Bethe vectors of [KR] was then developed in [TV1], where these vectors were defined as certain matrix elements of monodromy operators (see below). In [KPT] a conjecture about the coincidence of the construction from [TV1] and those used in the present paper was stated. Recently, Tarasov and Varchenko managed to calculate nested Bethe vectors in evaluation modules [TV2]. In this section we use the results of [TV2] to prove a variant of the conjecture of [KPT].

The R-matrix, used in [TV2] differs from (2.1). To achieve a compatibility of the results, we slightly modify our construction. Let $\mathcal{R} \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$ be the universal R-matrix of $U_q(\widehat{\mathfrak{gl}}_N)$ (with droped factor $q^{-(c\otimes d+d\otimes c)/2}$) and $\bar{\mathcal{R}}$ be the universal R-matrix of the Hopf subalgebra $U_q(\mathfrak{gl}_N)$, described in the previous section. Define $\varepsilon_i \in U_q(\mathfrak{gl}_N)$ by the relation $q^{\varepsilon_i} = \mathsf{E}_{ii}$. Define the *reduced* R-matrix $\bar{\mathcal{R}}_0$ of $U_q(\mathfrak{gl}_N)$ by the relation

$$\bar{\mathcal{R}}_0 = \bar{\mathcal{R}} q^{\sum_{i=1}^N \varepsilon_i \otimes \varepsilon_i}.$$

Then $\bar{\mathcal{R}}_0^{21}$ is a two-cocycle with respect to comultiplication Δ. For any $x \in U_q(\widehat{\mathfrak{gl}}_N)$ we set $\tilde{\Delta}(x) = (\bar{\mathcal{R}}_0^{21})^{-1}\Delta(x)\bar{\mathcal{R}}_0^{21}$. The universal R-matrix $\tilde{\mathcal{R}}$ for the comultiplication $\tilde{\Delta}$ is $\tilde{\mathcal{R}} = \bar{\mathcal{R}}_0^{-1}\mathcal{R}\bar{\mathcal{R}}_0^{21}$. It is an element of $U_q(\tilde{\mathfrak{b}}^+) \otimes U_q(\tilde{\mathfrak{b}}^-)$, where the new Borel subalgebras $U_q(\tilde{\mathfrak{b}}^+)$ and $U_q(\tilde{\mathfrak{b}}^-)$ are determined by L-operators $\tilde{\mathrm{L}}^+(z) = (\pi(z)\otimes 1)(\tilde{R}^{21})^{-1})$ and $\tilde{\mathrm{L}}^-(z) = (\pi(z)\otimes 1)(\tilde{R})$. Here $\pi(z)$ is a vector representation of $U_q(\widehat{\mathfrak{gl}}_N)$ evaluated at the point z. We have by the construction

$$\tilde{\mathrm{L}}^{\pm}(z) = \left(\mathrm{L}_0^-\right)^{-1} \mathrm{L}^{\pm}(z) \left(\mathrm{L}_0^+\right)^{-1}, \tag{6.1}$$

where $\mathrm{L}^{\pm}(z)$ are L-operators of Section 2.2, and $\mathrm{L}_0^{\pm}$ are given by the relations (5.17) and (5.18). The L-operators $\tilde{\mathrm{L}}^{\pm}(z)$ satisfy the Yang-Baxter relations with R-matrix (6.2), initial conditions (6.3) and comultiplication rule (6.4):

$$\begin{aligned} \tilde{\mathrm{R}}(u,v) &= \sum_{1\le i\le N} \mathrm{E}_{ii}\otimes\mathrm{E}_{ii} \\ &+ \frac{u-v}{qu-q^{-1}v} \sum_{1\le i<j\le N} (\mathrm{E}_{ii}\otimes\mathrm{E}_{jj} + \mathrm{E}_{jj}\otimes\mathrm{E}_{ii}) \\ &+ \frac{q-q^{-1}}{qu-q^{-1}v} \sum_{1\le i<j\le N} (u\mathrm{E}_{ij}\otimes\mathrm{E}_{ji} + v\mathrm{E}_{ji}\otimes\mathrm{E}_{ij}), \end{aligned} \tag{6.2}$$

$$\tilde{\mathrm{L}}^+_{ji}[0] = \tilde{\mathrm{L}}^-_{ij}[0] = 0, \quad \tilde{\mathrm{L}}^+_{kk}[0]\tilde{\mathrm{L}}^-_{kk}[0] = 1, \quad 1 \le i < j \le N,\ 1 \le k \le N\,, \tag{6.3}$$

$$\tilde{\Delta}\left(\tilde{\mathrm{L}}^{\pm}_{ij}(u)\right) = \sum\nolimits_k \tilde{\mathrm{L}}^{\pm}_{kj}(u) \otimes \tilde{\mathrm{L}}^{\pm}_{ik}(u). \tag{6.4}$$

Let

$$
\tag{6.5}
\begin{aligned}
\tilde{\mathrm{L}}^{\pm}(z) = &\left(\sum_{i=1}^{N} \mathrm{E}_{ii} + \sum_{i<j}^{N} \tilde{\mathrm{F}}^{\pm}_{j,i}(z)\mathrm{E}_{ij}\right)\\
&\times \left(\sum_{i=1}^{N} k_i^{\pm}(z)\mathrm{E}_{ii}\right) \cdot \left(\sum_{i=1}^{N} \mathrm{E}_{ii} + \sum_{i<j}^{N} \tilde{\mathrm{E}}^{\pm}_{i,j}(z)\mathrm{E}_{ji}\right).
\end{aligned}
$$

be the Gauss decomposition of $\tilde{\mathrm{L}}^{\pm}(z)$. We have the following connection to the current realization of $U_q(\widehat{\mathfrak{gl}}_N)$:

$$
\tag{6.6}
E_i(z) = \tilde{\mathrm{E}}^{+}_{i,i+1}(z) - \tilde{\mathrm{E}}^{-}_{i,i+1}(z), \qquad F_i(z) = \tilde{\mathrm{F}}^{+}_{i+1,i}(z) - \tilde{\mathrm{F}}^{-}_{i+1,i}(z),
$$

$$
\tag{6.7}
\tilde{\mathrm{E}}^{\pm}_{i,i+1}(z) = E_i(z)^{(\pm)}, \qquad \tilde{\mathrm{F}}^{\pm}_{i+1,i}(z) = z\left(z^{-1}F_i(z)\right)^{(\pm)}.
$$

Besides, the correspondence (6.1) imply the relations

$$
\tag{6.8}
\tilde{\mathrm{E}}^{\pm}_{i,i+1}(z) = \mathrm{E}^{\pm}_{i,i+1}(z) - E_i[0], \qquad \tilde{\mathrm{F}}^{\pm}_{i,i+1}(z) = \mathrm{F}^{\pm}_{i,i+1}(z) + F_i[0].
$$

We have the new decomposition $U_F = \tilde{U}_f^- \tilde{U}_F^+$, where $\tilde{U}_f^- = U_F' \cap U_q(\tilde{\mathfrak{b}}^-)$, and $\tilde{U}_F^+ = U_F \cap U_q(\tilde{\mathfrak{b}}^+)$, and the new projections $\tilde{P^+} : U_F \to \tilde{U}_F^+$ and $\tilde{P^-} : U_F \to \tilde{U}_f^-$ defined as

$$
\tag{6.9}
\tilde{P^+}(f_-\ f_+) = \varepsilon(f_-)\ f_+, \quad \tilde{P^-}(f_-\ f_+) = f_-\ \varepsilon(f_+),
$$

where $f_- \in \tilde{U}_f^-$ and $f_+ \in \tilde{U}_F^+$. For any Π-ordered multiset I we set

$$
\tag{6.10}
\tilde{\mathcal{W}}_I(t_i|_{i\in I}) = \tilde{P^+}\left(F_{\iota(i_n)}(t_{i_n})F_{\iota(i_{n-1})}(t_{i_{n-1}})\cdots F_{\iota(i_2)}(t_{i_2})F_{\iota(i_1)}(t_{i_1})\right).
$$

A small modification of arguments of [KPT] shows that a collection of V-valued functions $\tilde{w}_{V,I}(t_i|_{i\in I}) = \tilde{\mathcal{W}}_I(t_i|_{i\in I})\ v$, where V is a $U_q(\widehat{\mathfrak{gl}}_N)$-module with a singular weight vector v, form a q-symmetric weight function, that is, satisfy the setting of Section 3.1 with respect to the comultiplication $\tilde{\Delta}$.

Denote by $\tilde{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}]})$ the universal weight function associated with the set of variables (3.12).

$$
\tag{6.11}
\begin{aligned}
\tilde{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}]}) = \tilde{P}^+\Big(&F_{N-1}(t^{N-1}_{n_{N-1}})\cdots F_{N-1}(t^{N-1}_1) \quad \cdots\\
&\cdots \quad F_1(t^1_{n_1})\cdots F_1(t^1_1)\Big),
\end{aligned}
$$

and by $\tilde{\mathbf{w}}_V^{N-1}(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{\bar{n}})\prod_{a=2}^{N}\prod_{\ell=1}^{n_{a-1}}\lambda_a(t^{a-1}_\ell)\ \tilde{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}]})v$ the related modified weight function, associated with a weight singular vector v of a $U_q(\widehat{\mathfrak{gl}}_N)$-module V.

Set $\tilde{Y}(\bar{u};\bar{v}) = \prod_{m=1}^{k}\frac{v_m}{u_m}\,Y(\bar{u};\bar{v})$, where the series $Y(\bar{u};\bar{v})$ is defined by (4.15) and let $\tilde{X}(\bar{t}_{[\bar{r}-\bar{l},\bar{r}]})$ be the series defined by (4.18) with $Y(\cdot)$ replaced by $\tilde{Y}(\cdot)$. The first step of the calculation of recurrence relation for (6.11), as well as the definition of strings is unchanged. The difference appears during the calculation of the projection (4.23). This difference results that the series $X(\bar{t}_{[\bar{n}-\bar{s},\bar{n}]})$ in (4.21) is replaced by $\tilde{X}(\bar{t}_{[\bar{n}-\bar{s},\bar{n}]})$ and the recurrence relation for (6.11) takes the form

$$\tilde{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}]}) = \sum_{\bar{s}} \frac{1}{s_1!} \prod_{a=2}^{N-1} \frac{1}{(s_a - s_{a-1})!} \prod_{a=1}^{N-2} \frac{1}{(n_a - s_a)!}$$
$$\times \ \overline{\mathrm{Sym}}_{\ \bar{t}_{[\bar{n}]}} \left(Z_{\bar{s}}(\bar{t}_{[\bar{n}]}) \cdot \tilde{X}(\bar{t}_{[\bar{n}-\bar{s},\bar{n}]}) \cdot \tilde{P^{+}} \left(\mathcal{F}^{N-1}_{\bar{s}}(\bar{t}^{N-1}_{[n_{N-1}]}) \right) \cdot \tilde{\mathcal{W}}^{N-2}(\bar{t}_{[\bar{n}-\bar{s}]}) \right).$$

The iteration of this relation yields an analog of Theorem 4.1
(6.12)

$$\tilde{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}_{\ \bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]\prec\,\bar{n}} \left(\frac{\tilde{\mathcal{Z}}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]})}{\prod_{a\le b}(s^b_a - s^b_{a-1})!} \overleftarrow{\prod_{N-1\ge j\ge 1}} \tilde{P}^{+} \left(\mathcal{F}^{j}_{\bar{s}^j}(\bar{t}^{j}_{[s^j_j]}) \right) \right),$$

where

$$\tilde{\mathcal{Z}}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]}) = \prod_{b=2}^{N-1} \prod_{a=1}^{b-1} \prod_{\ell=1}^{s^b_a} \frac{t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell+\tilde{p}^b_{a+1}}}{1 - t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell+\tilde{p}^b_{a+1}}} \prod_{\ell'=1}^{\ell+\tilde{p}^b_{a+1}-1} \frac{q - q^{-1}t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell'}}{1 - t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell'}}$$

and $\tilde{p}^b_a = s^a_a + \cdots + s^{b-1}_a$. An analog of Proposition 5.1 reads

$$\tilde{P^{+}}\left(F_{j,i}(t)\right) \ = \ (q - q^{-1})^{j-i-1}\tilde{\mathrm{F}}^{+}_{j,i}(t), \qquad i < j-1.$$

allows to rewrite (6.12) in the Gauss coordinates of L-operator $\tilde{\mathrm{L}}^{+}(t)$:

$$\tilde{\mathbf{w}}^{N-1}_V(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})\ \overline{\mathrm{Sym}}_{\ \bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]\prec\,\bar{n}} \left(\frac{(q-q^{-1})^{\sum_{b=1}^{N-1}(n_b - s^b_b)}}{\prod_{a\le b}^{N-1}(s^b_a - s^b_{a-1})!} \right.$$
$$\times\, \tilde{\mathcal{Z}}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]}) \overleftarrow{\prod_{N-1\ge b\ge 1}} \left(\overrightarrow{\prod_{1\le a\le b}} \prod_{\ell=s^b_{a-1}+1}^{s^b_a} \right.$$
$$\left.\left.\times \left(\tilde{\mathrm{F}}^{+}_{b+1,a}(t^b_\ell) k^{+}_{b+1}(t^b_\ell) \prod_{\ell'=\ell+1}^{s^b_a} \frac{t^b_\ell - t^b_{\ell'}}{q^{-1}t^b_\ell - q t^b_{\ell'}} \right) \right) \prod_{\ell=s^b_b+1}^{n_b} k^{+}_{b+1}(t^b_\ell) \right) v.$$

The arguments for the derivation of an analog of Theorem 5.1 and Corollary 5.1 are unchanged. We get finally the following expression for the modified weight function:

$$
\begin{aligned}
\tilde{\mathbf{w}}_V^{N-1}(\bar{t}_{[\bar{n}]}) &= \overline{\mathrm{Sym}}\;{}^{(q)}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]\prec\,\bar{n}} \Bigg((q-q^{-1})^{\sum_{b=1}^{N-1}(n_b-s^b_b)} \prod_{a\le b}^{N-1} \frac{1}{[s^b_a - s^b_{a-1}]_q!} \\
&\times \prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s^b_a} \frac{t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell+\tilde{p}^b_{a+1}}}{1-t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell+\tilde{p}^b_{a+1}}} \prod_{\ell'=1}^{\ell+\tilde{p}^b_{a+1}-1} \frac{q-q^{-1}t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell'}}{1-t^a_{\ell+\tilde{p}^b_a}/t^{a+1}_{\ell'}} \\
&\times \prod_{N-1\ge b\ge 1}^{\longleftarrow} \left(\prod_{1\le a\le b}^{\longrightarrow} \left(\prod_{\ell=s^b_{a-1}+1}^{s^b_a} \tilde{\mathrm{L}}^+_{a,b+1}(t^b_\ell) \right) \prod_{\ell=s^b_b+1}^{n_b} \tilde{\mathrm{L}}^+_{b+1,b+1}(t^b_\ell) \right) \Bigg) v.
\end{aligned}
\tag{6.13}
$$

The embedded algebra $U_q(\mathfrak{gl}_N)$ is given again by zero modes of L-operators. We set $\tilde{\mathrm{L}}^{\pm} \equiv \tilde{\mathrm{L}}^{\pm}[0]$. Due to (6.1), we have $\tilde{\mathrm{L}}^+ = (\mathrm{L}_0^-)^{-1}\mathrm{K}$ and $\tilde{\mathrm{L}}^- = \mathrm{K}^{-1}(\mathrm{L}_0^+)^{-1}$. Introduce generators $\tilde{\mathsf{E}}_{i,j}$ of $U_q(\mathfrak{gl}_N)$ by the relations

$$
\tilde{\mathrm{L}}^+ = \begin{pmatrix} 1 & \nu\tilde{\mathsf{E}}_{21} & \cdots & \nu\tilde{\mathsf{E}}_{N1} \\ & \ddots & \ddots & \vdots \\ & & 1 & \nu\tilde{\mathsf{E}}_{N,N-1} \\ & & & 1 \end{pmatrix} \begin{pmatrix} \tilde{\mathsf{E}}_{11} & & & \\ & \tilde{\mathsf{E}}_{22} & 0 & \\ & 0 & \ddots & \\ & & & \tilde{\mathsf{E}}_{NN} \end{pmatrix}
$$

$$
\tilde{\mathrm{L}}^- = \begin{pmatrix} \tilde{\mathsf{E}}_{11}^{-1} & & & \\ & \tilde{\mathsf{E}}_{22}^{-1} & 0 & \\ & 0 & \ddots & \\ & & & \tilde{\mathsf{E}}_{NN}^{-1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ -\nu\tilde{\mathsf{E}}_{12} & 1 & 0 & \\ \vdots & \ddots & \ddots & \\ -\nu\tilde{\mathsf{E}}_{1N} & \cdots & -\nu\tilde{\mathsf{E}}_{N-1,N} & 1 \end{pmatrix}
$$

In particular, $\tilde{\mathsf{E}}_{ij} = \mathsf{E}_{ij}$ if $|i-j|\le 1$, so the Chevalley generators $\tilde{\mathsf{E}}_{a,a}$, $\tilde{\mathsf{E}}_{a,a+1}$ and $\tilde{\mathsf{E}}_{a+1,a}$ satisfy the same relations (5.19) and (5.20) with different rules for the composed roots generators

$$
\begin{aligned}
\tilde{\mathsf{E}}_{c,a} &= \tilde{\mathsf{E}}_{c,b}\tilde{\mathsf{E}}_{b,a} - q^{-1}\tilde{\mathsf{E}}_{b,a}\tilde{\mathsf{E}}_{c,b}, \\
\tilde{\mathsf{E}}_{a,c} &= \tilde{\mathsf{E}}_{a,b}\tilde{\mathsf{E}}_{b,c} - q\tilde{\mathsf{E}}_{b,c}\tilde{\mathsf{E}}_{a,b}, \qquad a<b<c.
\end{aligned}
$$

Define the evaluation homomorphism of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ to $U_q(\mathfrak{gl}_N)$ as in (5.22):

$$
\mathcal{E}v_z\left(\tilde{\mathrm{L}}^+(u)\right) = \tilde{\mathrm{L}}^+ - \frac{z}{u}\,\tilde{\mathrm{L}}^-, \quad \mathcal{E}v_z\left(\tilde{\mathrm{L}}^-(u)\right) = \tilde{\mathrm{L}}^- - \frac{u}{z}\,\tilde{\mathrm{L}}^+. \tag{6.14}
$$

The substitution of (6.14) into (6.13) gives

(6.15)
$$\begin{aligned}&\tilde{\mathbf{w}}^{N-1}_{M(z)}(\bar t_{[\bar n]})=(q-q^{-1})^{\sum_{a=1}^{N-1}n_a}\\ &\times\sum_{[[\bar s]]\prec\bar n}\left(\left(\prod^{\longleftarrow}_{N-1\ge b>a\ge1}\frac{q^{s^b_{a-1}(s^b_{a-1}-s^b_a)}}{[s^b_a-s^b_{a-1}]_q!}\,\check{\mathsf{E}}^{s^b_a-s^b_{a-1}}_{b+1,a}\right)v\right.\\ &\times\overline{\mathrm{Sym}}^{\,(q)}_{\bar t_{[\bar n]}}\left.\left(\prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s^b_a}\frac{q^{\Lambda_{a+1}}t^a_{\ell+\tilde p^b_a}-q^{-\Lambda_{a+1}}z}{t^{a+1}_{\ell+\tilde p^b_{a+1}}-t^a_{\ell+\tilde p^b_a}}\prod_{\ell'=1}^{\ell+\tilde p^b_{a+1}-1}\frac{qt^{a+1}_{\ell'}-q^{-1}t^a_{\ell+\tilde p^b_a}}{t^{a+1}_{\ell'}-t^a_{\ell+\tilde p^b_a}}\right)\right),\end{aligned}$$

where $\check{\mathsf{E}}_{b+1,a}=\tilde{\mathsf{E}}_{b+1,a}\tilde{\mathsf{E}}_{b+1,b+1}$. The factor $q^{s^b_{a-1}(s^b_{a-1}-s^b_a)}$ appears after the reordering of the generators $\check{\mathsf{E}}_{b+1,a}$. To obtain (6.15) from (6.13) we use again the reordering (5.23).

Remind the construction of [TV1]. Let L-operator

$$\mathrm{L}(z)=\sum_{k=0}^{\infty}\sum_{i,j=1}^{N}\mathrm{E}_{ij}\otimes\mathrm{L}_{ij}[k]z^{-k}$$

of some Borel subalgebra $U_q(\check{\mathfrak{b}}^+)$ of $U_q(\widehat{\mathfrak{gl}}_N)$ satisfies the Yang-Baxter relation with a R-matrix $\check{\mathrm{R}}(u,v)$. We use the notation $\mathrm{L}^{(k)}(z)\in\left(\mathbb{C}^N\right)^{\otimes M}\otimes U_q(\check{\mathfrak{b}}^+)$ for an L-operator acting nontrivially on k-th tensor factor in the product $\left(\mathbb{C}^N\right)^{\otimes M}$ for $1\le k\le M$. Consider a series on M variables

$$\mathbb{T}(u_1,\dots,u_M)=\mathrm{L}^{(1)}(u_1)\cdots\mathrm{L}^{(M)}(u_M)\cdot\mathbb{R}^{(M,\dots,1)}(u_M,\dots,u_1)\tag{6.16}$$

with coefficients in $\left(\mathrm{End}(\mathbb{C}^N)\right)^{\otimes M}\otimes U_q(\check{\mathfrak{b}}^+)$, where

$$\mathbb{R}^{(M,\dots,1)}(u_M,\dots,u_1)=\prod^{\longleftarrow}_{1\le i<j\le M}\check{\mathrm{R}}^{(ji)}(u_j,u_i).\tag{6.17}$$

In the ordered product of R-matrices (6.17) the factor $\check{\mathrm{R}}^{(ji)}$ is to the left of the factor $\check{\mathrm{R}}^{(ml)}$ if $j>m$, or $j=m$ and $i>l$. Consider the set of variables (3.12). Following [TV1], set

$$\begin{aligned}\mathbb{B}(\bar t_{[\bar n]})=&\mathrm{tr}_{\mathbb{C}^{\otimes|\bar n|}}\otimes\mathrm{id}\ (\mathbb{T}(t^1_1,\dots,t^1_{n_1};\ \dots\ ;t^{N-1}_1,\dots,t^{N-1}_{n_{N-1}})\\ &\times\mathrm{E}^{\otimes n_1}_{21}\otimes\cdots\otimes\mathrm{E}^{\otimes n_{N-1}}_{N,N-1}\otimes1),\end{aligned}\tag{6.18}$$

where $|\bar n|=n_1+\cdots+n_{N-1}$. The element (6.18) is given by (6.16) with the identification: $M=|\bar n|$; for $a=1,\dots,N-1$ and $n_1+\cdots+n_{a-1}<i\le n_1+\cdots+n_a$, $u_i=t^a_{i-n_1-\cdots-n_{a-1}}$. The coefficients of $\mathbb{B}(\bar t_{\bar n})$ are elements of the Borel subalgebra $U_q(\check{\mathfrak{b}}^+)$. For any $U_q(\widehat{\mathfrak{gl}}_N)$-module V with a singular vector v denote

$$\mathbb{B}_V(\bar t_{[\bar n]})=\mathbb{B}(\bar t_{[\bar n]})v.\tag{6.19}$$

We can establish a correspondence of the calculations above and of [TV2] as follows. We identify the R-matrix $\tilde{\mathrm{R}}(u,v)$, see (6.2) and $\check{\mathrm{R}}(u,v)$, the Borel subalgebra $U_q(\tilde{\mathfrak{b}}^+)$ (see the beginning of this section) with $U_q(\check{\mathfrak{b}}^+)$ of [TV2]. Set

$$\eta(\bar{t}_{[\bar{n}]}) = \prod_{1\le a<b\le N-1} \prod_{1\le j\le n_b} \prod_{1\le i\le n_a} \frac{qt^b_j - q^{-1}t^a_i}{t^b_j - t^a_i}.$$

Then we have

$$\tilde{\mathbf{w}}^{N-1}_{M_\Lambda(z)}(\bar{t}_{[\bar{n}]}) = \eta(\bar{t}_{[\bar{n}]})\mathbb{B}_{M_\Lambda(z)}(\bar{t}_{[\bar{n}]}) \tag{6.20}$$

for arbitrary evaluation module $M_\Lambda(z)$. This is just a literal coincidence of eq. (6.15) and of eq. (6.11) in [TV2]. Next, we know that the comultiplication property of the weight functions $\tilde{\mathbf{w}}^{N-1}_V(\bar{t}_{[\bar{n}]})$ and $\mathbb{B}_V(\bar{t}_{[\bar{n}]})$ coincide [KPT]. Thus

$$\begin{aligned}\tilde{\mathbf{w}}^{N-1}_{1_{g(z_0)}\otimes M_{\Lambda_1}(z_1)\otimes\cdots\otimes M_{\Lambda_n}(z_n)}(\bar{t}_{[\bar{n}]}) = \\ = \eta(\bar{t}_{[\bar{n}]})\mathbb{B}_{1_{g(z_0)}\otimes M_{\Lambda_1}(z_1)\otimes\cdots\otimes M_{\Lambda_n}(z_n)}(\bar{t}_{[\bar{n}]})\end{aligned} \tag{6.21}$$

for any tensor product of evaluation modules and a one-dimensional module $1_{g(z_0)}$, in which every $L_{ii}(z)$ acts by multiplication on $g(z_0)$ (both weight functions in consideration are trivial for one-dimensional modules). The identity (6.21) is sufficient to conclude the general coincidence of the two constructions.

Let J be the left ideal of $U_q(\tilde{\mathfrak{b}}^+)$, generated by all modes of $\tilde{\mathrm{E}}_{b,a}(t)$, $a>b$. Let $\tilde{\mathcal{W}}^{N-1}(\bar{t}_{\bar{n}})$ be the universal weight function given by eq. (6.11).

The following result verifies the conjecture of [KPT] for the universal weight function (6.10), and the weight function of (6.18), related to R-matrix (6.2).

Theorem 6.1.

(i) *The two weight functions are equal for each irreducible finite-dimensional $U_q(\widehat{\mathfrak{gl}}_N)$-module V with a singular vector v:*

$$\tilde{\mathbf{w}}^{N-1}_V(\bar{t}_{[\bar{n}]}) = \eta(\bar{t}_{[\bar{n}]})\mathbb{B}_V(\bar{t}_{[\bar{n}]}).$$

(ii) *Consider $U_q(\tilde{\mathfrak{b}}^+)$ as an algebra over $\mathbb{C}[[(q-1)]]$. Then*

$$\tilde{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}]}) \prod_{a=1}^{N-1}\prod_{\ell=1}^{n_a} k^+_{a+1}(t^a_\ell) \equiv \eta(\bar{t}_{[\bar{n}]})\mathbb{B}(\bar{t}_{[\bar{n}]}) \mod J. \tag{6.22}$$

Proof. For the proof of (i) we apply (6.21) and the classical result [CP]: every irreducible finite-dimensional $U_q(\widehat{\mathfrak{gl}}_N)$-module with a singular vector v is isomorphic to a subquotient of a tensor product of one-dimensional modules and of evaluation modules. More precisely, this subquotient is a quotient of the submodule of that tensor product generated by the tensor product of weight singular vectors. The singular vector corresponds to the image of the tensor product of the singular vectors within this isomorphism.

Due to (6.21), for the proof of (ii) it is sufficient to verify the following statement. Let $x \in U(\tilde{\mathfrak{b}}_+)$ be an element of the universal enveloping algebra of a Borel subalgebra of $\mathfrak{gl}_N[t,t^{-1}]$, which does not belong to J. Then there exists a tensor product of evaluation modules with a weight singular vector v such that $xv \neq 0$. This can be observed as follows. Let $\mathrm{E}_{k,l}[n]$, $i,j = 1,\dots,N$, $n \in \mathbb{Z}$ be the generators of the Lie algebra $\mathfrak{gl}_N[t,t^{-1}]$, such that $[\mathrm{E}_{i,j}[n], \mathrm{E}_{k,l}[m]] = \delta_{j,k}\mathrm{E}_{i,l}[n+m] - \delta_{k,l}\mathrm{E}_{k,j}[n+m]$ and the ideal J is generated by all $\mathrm{E}_{k,l}[m]$, $k<l$, $m>0$. By PBW theorem, we can present x as a sum of ordered monomials on the generators $\mathrm{E}_{j,i}[n] \in \mathfrak{gl}_N[t,t^{-1}]$ with $1 \le i \le j \le N$ and $n \ge 0$. Clearly, x admits a weight decomposition with respect to Cartan subalgebra of $\mathfrak{gl}_N$. Take a maximal weight component x_0. Present x_0 in a form

$$x_0 = \sum_k X_k P_k(\{\mathrm{E}_{a,a}[n_a]\}), \tag{6.23}$$

where each X_k is a monomial $\mathrm{E}_{2,1}[n_1]\mathrm{E}_{2,1}[n_2]\cdots\mathrm{E}_{3,1}[n_l]\cdots\mathrm{E}_{N,N-1}[n_m]$ with all $n_i \ge 0$ of a given weight and $P_k(\{\mathrm{E}_{a,a}[n_a]\})$ is a polynomial over commuting imaginary root vectors $\mathrm{E}_{a,a}[n_a]$, $a = 1,\dots,N$. Let $m_{i,j}$ denotes the number of occurrences of all possible $\mathrm{E}_{j,i}[r]$ in the decomposition of X_1.

Let M_i be a $\mathfrak{gl}_N$ Verma module with a highest vector v, satisfying the relations $\mathrm{E}_{j,i}[0]v = 0$ for $i > j$ and $\mathrm{E}_{a,a}[0]v = \delta_{a,i}v$. Let $M_i(z)$ be the corresponding evaluation $\mathfrak{gl}_N[t,t^{-1}]$ module. For a collection $\bar{n} = \{n_1,\dots,n_N\}$ and a set of variables $z_{\bar{n}} = \{z^1_1,\dots,z^1_{n_1},\dots,\ z^N_{n_N}\}$ define $M(z_{\bar{n}})$ as a tensor product $M(z_{\bar{n}}) = M_1(z^1_1) \otimes \cdots \otimes M_N(z^N_{n_N})$. Its weight singular vector is $\bar{v} = v \otimes \cdots \otimes v$. Denote by $v_{\{m_{ij}\}} \in M(z_{\bar{n}})$ the vector $\mathrm{E}_{2,1}[0]v \otimes \cdots \otimes \mathrm{E}_{2,1}[0]v \otimes \cdots \otimes \mathrm{E}_{N,N-1}[0]v \otimes v \otimes \cdots \otimes v$ where the number of occurrences of each $\mathrm{E}_{i,j}[0]\,v$ is $m_{i,j}$ (we suppose that the numbers n_k are big enough). Equip the dual space $M^*(z_{\bar{n}})$ with a structure of contragredient $\mathfrak{gl}_N$ module ($\mathrm{E}_{j,i}[0]^* = \mathrm{E}_{i,j}[0]$). Then the vector $v^*_{\{m_{ij}\}} \in M^*(z_{\bar{n}})$ is well defined.

Consider the matrix element $\langle v^*_{\{m_{ij}\}}, xv\rangle$. By weight arguments it is equal to $\langle v^*_{\{m_{ij}\}}, x_0 v\rangle$. It is also clear that we have nonzero contribution only for the terms with the same number of occurrences of $\mathrm{E}_{j,i}[r]$ (neglecting degrees r) as in X_1. In this matrix coefficient the imaginary root vector $\mathrm{E}_{a,a}[m]$ contributes as the Newton polynomial $s^a_m = (z^a_1)^m + \cdots + (z^a_{n_a})^m$ of degree m over the variables $z^a_1,\dots,z^a_{n_a}$ and thus each polynomial $P_k(\{\mathrm{E}_{a,a}[n_a]\})$ contributes as this polynomial over symmetric functions s^a_m.

In order to describe the contribution of monomials on $\mathrm{E}_{j,i}[r]$ with $i<j$ we make the following renaming of the variables z^a_i. We write them first in a natural order $z^1_1,\dots,z^N_{n_N}$ and then rename first $m_{1,2}$ of them as $x^{12}_1,\dots,x^{12}_{m_{12}}$, the next $m_{1,3}$ of them as $x^{13}_1,\dots,x^{13}_{m_{13}}$ and so on. The rest of the variables remain unchanged. Then each $\mathrm{E}_{j,i}[r]$ with $i<j$ contributes as the Newton polynomial $s^{ij}_r = (x^{ij}_1)^r + \cdots + (x^{ij}_{m_{ij}})^r$ of degree r over the variables $x^{ij}_1,\dots,x^{ij}_{m_{ij}}$.

Now we use the following properties of symmetric (with respect to a product of symmetric groups) functions: (i) the Newton polynomials of degree less then the number of the variables are algebraically independent; (ii) the products $s_{m_1}s_{m_2}\cdots s_{m_n}$ of the Newton polynomials $s_k = x^k_1 + \ldots + x^k_n$ of n variables taken over all unordered collections $(m_1, ..., m_n)$ of nonnegative integers form a linear basis of the ring of symmetric functions on n variables; (iii) for any finite set of linearly independent polynomials of n variables one can find $N > n$ such that all these polynomials will be linear independent over the ring of symmetric functions of N variables.

These properties of symmetric functions imply that once we take a collection $\bar{n}$ with all n_i^a big enough, the matrix element $\langle v^*_{\{m_{ij}\}}, xv\rangle$ is nonzero polynomial over $\{z_{\bar{n}}\}$. □

Clearly, the formality restriction in (ii) is artificial and is taken for technical simplification of the proof.

A. Analytical properties of composed currents

An analytical reformulation of Serre relations (2.9), see [E], imply the following statements in the completed algebra $\overline{U}_F$:

(i) the products $F_i(z)F_i(w)$ have a simple zero at $z = w$; *2

(ii) the product $(q^{-1}z_1 - qz_2)(z_2 - z_3)(qz_1 - q^{-1}z_3)F_{i-1}(z_1)F_i(z_2)F_{i-1}(z_3)$ vanish on the lines $z_2 = z_1 = q^{-2}z_3$ and $z_2 = q^{-2}z_1 = z_3$;

(iii) the product $(z_1 - z_2)(q^{-1}z_2 - qz_3)(qz_1 - q^{-1}z_3)F_i(z_1)F_{i-1}(z_2)F_i(z_3)$ vanish on the lines $z_2 = z_1 = q^2z_3$ and $z_2 = q^2z_1 = z_3$.

The properties (i), (ii) and (iii) imply the commutativity

$$[F_{i-1}(z)F_i(z)F_{i+1}(z), F_i(w)] = 0. \tag{A.1}$$

Let us prove (A.1). The basic relations (2.8) imply that:

$$(q^{-1}z - qw)(z - w)(qz - q^{-1}w)[F_{i-1}(z)F_i(z)F_{i+1}(z), F_i(w)] = 0. \tag{A.2}$$

Consider the product $F_{i-1}(z)F_i(z)F_{i+1}(z)F_i(w)$. Due to (2.8) this product as a function of w has poles at the points $w = q^{-2}z, z, q^2z$. On the other hand it has zero at $w = z$ due to (i). Due to (ii) the product $F_i(z)F_{i+1}(z)F_i(w)$ has zero at $w = q^{-2}z$ and due to (iii) the product $F_{i-1}(z)F_i(z)F_i(w)$ has zero at $w = q^2z$. This means that the whole product $F_{i-1}(z)F_i(z)F_{i+1}(z)F_i(w)$ has no zeros and no poles. The same is true for the inverse product $F_i(w)F_{i-1}(z)F_i(z)F_{i+1}(z)$. The relations (A.2) implies now the commutativity in (A.1).

We will use (4.12) in order to describe the analytical properties of the product of composed currents.

Proposition A.1. *The following relations hold in $\overline{U}_F$ for any $a < b$ and $c < d$:*

$$F_{b,a}(z)F_{d,c}(w) = F_{d,c}(w)F_{b,a}(z), \quad b < c, \tag{A.3}$$

$$\begin{aligned}(q^{-1}z - qw)F_{b,a}(z)F_{d,c}(w) &= \\ &= (z - w)F_{d,c}(w)F_{b,a}(z), \quad b = c,\end{aligned} \tag{A.4}$$

*2 Let v be a vector of a highest weight module V and $\xi \in V^*$. The analytical property (i) means that the matrix element $\langle \xi, F_i(z)F_i(w)v\rangle$ is a meromorphic function of w and z equal to zero at $w = z$.

$$F_{b,a}(z)F_{d,c}(w) = \frac{q-q^{-1}z/w}{1-z/w}F_{d,c}(w)F_{b,a}(z), \quad a>c,\ b=d, \tag{A.5}$$

$$\begin{aligned} &\frac{q-q^{-1}w/z}{1-w/z}F_{b,a}(z)F_{d,c}(w) = \\ &\qquad = \frac{q-q^{-1}z/w}{1-z/w}F_{d,c}(w)F_{b,a}(z) \quad a=c,\ b=d, \end{aligned} \tag{A.6}$$

$$F_{b,a}(z)F_{d,c}(w) = F_{d,c}(w)F_{b,a}(z), \quad a<c<d<b, \tag{A.7}$$

$$\frac{q-q^{-1}w/z}{1-w/z}F_{b,a}(z)F_{d,c}(w) = F_{d,c}(w)F_{b,a}(z), \quad a=c,\ b<d. \tag{A.8}$$

Proof. We note that in the analytic language, both sides of all relations in (A.3)–(A.8) are analytical functions in $(\mathbb{C}^*)^2$. This means, for instance, that the product $F_{d,a}(w)F_{b,a}(z)$ in (A.8) has no zeroes and no poles for $b<d$, while the product $F_{b,a}(z)F_{d,a}(w)$ has a simple zero at $z=w$ and a simple pole at $z=q^{-2}w$. Let us prove (A.8). Consider the product $F_{b,a}(z)F_{d,a}(w)$. Due to (A.1) and (4.12), analytical properties of this product are the same as of $F_a(z)F_a(w)F_{a+1}(w)$. This product considered as a function of z has poles at the points $z=q^{-2}w$ and $z=q^2w$. On the other hand due to (i) it has a zero at $z=w$ and due to (ii) a zero at $z=q^2w$. It means that the product $F_{b,a}(z)F_{d,a}(w)$ has a simple pole at $z=q^{-2}w$ and a simple zero at $z=w$. Consider now the inverse product $F_{d,a}(w)F_{b,a}(z)$. Its analytical properties are defined again due to (A.1) by the properties of the product $F_a(w)F_{a+1}(w)F_a(z)$. The latter has poles at $z=q^2w$ and at $z=w$. Due to (i) and (ii) it has zeros at the same points. It means that the inverse product $F_{d,a}(w)F_{b,a}(z)$ of composed currents are analytic function in $(\mathbb{C}^*)^2$ without zeros and poles. Now the relations (2.8) imply (A.8). □

Corollary A.1. *The inverse string* (4.20) *have simple zeros at* $t^j_k = t^j_m$ *and simple poles at* $t^j_k = q^{-2}t^j_m$ *for all pairs* (m,k) *such that* $r_j \ge m > k > l_j$. *These poles are the only singularities of the inverse string.*

Proof. Proof follows from the analysis of the ordering of the product of currents in the inverse string. It contains the products of the composed currents $F_{j+1,a}(t^j_k)F_{j+1,c}(t^j_m)$, where always $a\le c$ and $k<m$. Properties (A.5) and (A.6) of the products of the composed currents shows that $F_{j+1,a}(t^j_k)F_{j+1,c}(t^j_m)$ will always have the pole at $t^j_k = q^{-2}t^j_m$ and zero at $t^j_k = t^j_m$. □

The assertion A.1 will be used in the next Appendix in order to calculate the projections of the inverse string (4.20).

B. Projections of the string

In this Appendix we prove Proposition 4.4 describing the projection P^+ of the inverse string. We also describe projection P^-. Up to renaming the variables the inverse string is a product $F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)$ for $1 \le i_1 \le \cdots \le i_n < j$. The region of analyticity of each equality in this Appendix is $|v_1| \gg |v_2| \gg \ldots \gg |v_n|$, where v_1 is the first variable appearing in the equation when we read it from the left, v_2 is the second and so on.

Lemma B.1. *For any $1 \le i_1 \le \cdots \le i_n \le k < j$ we have an equality*

$$\begin{aligned} P^+(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)F_{j,k}(t)) = \\ = P^+(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)) \cdot P^+(F_{j,k}(t)) + \\ + \sum_{b=1}^{n} \frac{V_b(t_1,\ldots,t_n)}{t-q^2t_b}. \end{aligned} \tag{B.1}$$

where $V_b(t_1,\ldots,t_n)$ take value in U_f^+ and do not depend on the variable t.

Proof. We want to move the difference $F_{j,k}(t) - P^+\left(F_{j,k}(t)\right)$ to the left of the product $F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)$ and observe that during this process appear only simple poles at the points $t = q^2t_b$ with (independent on t) operator-valued residues.

A particular case of the formula (4.13) for $a = i+1$ can be written in the form

$$F_{j,k}(t) \;=\; S_k(F_{j,k+1}(t)) + (q^{-1}-q)F_k(t)^{(-)}\,F_{j,k+1}(t). \tag{B.2}$$

Iterating (B.2) we obtain

$$F_{j,k}(t) \;=\; S_k\,S_{k+1}\cdots S_{j-2}(F_{j-1}(t)) + \hat{F}_{j,k}(t), \tag{B.3}$$

where we set $\hat{F}_{k+1,k}(t) = 0$ and

$$\hat{F}_{j,k}(t) = (q^{-1}-q)\sum_{m=k+1}^{j-1}\left(F_{m,k}(t)^{(-)} - \hat{F}_{m,k}(t)^{(-)}\right)F_{j,m}(t), \tag{B.4}$$

if $j-k > 1$. Proposition 4.3 and (B.3) imply that

$$F_{j,k}(t) - P^+\left(F_{j,k}(t)\right) = \hat{F}_{j,k}(t)^{(+)} - F_{j,k}(t)^{(-)}. \tag{B.5}$$

and $P^+\left(\hat{F}_{j,k}(t)\right) = 0$. Applying Proposition 4.3 and (B.3) once again, we get also an equality $P^+\left(F_{j,k}(t)^{(-)}\right) = 0$.

Permuting the difference $F_{j,k}(t) - P^+\left(F_{j,k}(t)\right)$ and the product of the currents $F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)$ we consider the terms $\hat{F}_{j,k}(t)^{(+)}$ and $F_{j,k}(t)^{(-)}$ in the r.h.s. of (B.5) separately.

Lemma B.2. *For* $1 \le i_1 \le \cdots \le i_n \le k < j$

$$P^+\left(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)\cdot \hat{F}_{j,k}(t)\right) = 0. \tag{B.6}$$

Note that the equality (B.6) is equivalent to a system of two equalities, each of them is used further:

$$P^+\left(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)\cdot \hat{F}_{j,k}(t)^{(\pm)}\right) = 0. \tag{B.7}$$

Proof. We use presentation (B.4), the equality $P^+\left(F_{j,k}(t)^{(-)}\right) = 0$ and prove the statement of Lemma by induction over k. For $k = j-2$ the statement (B.6) follows from the relations

$$\begin{aligned} F_{j,s}(t')F_{m,k}(t)^{(-)} &= F_{m,k}(t)^{(-)}F_{j,s}(t'), \quad s<k \\ F_{j,k}(t')F_{m,k}(t)^{(-)} &= \frac{q^{-1}t'-qt}{t'-t}F_{m,k}(t)^{(-)}F_{j,k}(t') \\ &\quad - \frac{(q^{-1}-q)t'}{t'-t}F_{m,k}(t')^{(-)}F_{j,k}(t'), \end{aligned} \tag{B.8}$$

valid for $m < j$. The latter relations are the result of applying the integral transformation $-\oint \frac{dz}{z}\frac{1}{1-t/z}$ to (A.7), (A.8) with renaming $w \to t'$. For $k = j-3$ we use again (B.8) and relations (B.7) for $k = j-2$, which are already proved and so on. □

Lemma B.3. *The following relations hold for* $i < k$

$$\begin{aligned} F_{j,i}(t')F_{j,k}(t)^{(-)} &= \frac{t-t'}{q^{-1}t-qt'}F_{j,k}(t)^{(-)}F_{j,i}(t') + \frac{(q^{-1}-q)t'}{q^{-1}t-qt'}F_{j,i}(t')F_{j,k}(t')^{(-)}, \\ F_{j,k}(t')F_{j,k}(t)^{(-)} &= \frac{qt-q^{-1}t'}{q^{-1}t-qt'}F_{j,k}(t)^{(-)}F_{j,k}(t') + \\ &\quad + \frac{(q^{-1}-q)t'}{q^{-1}t-qt'}\left(F_{j,k}(t')F_{j,k}(t')^{(-)} + F_{j,k}(t')^{(-)}F_{j,k}(t')\right). \end{aligned}$$

Proof. Apply the integral transform $-\oint \frac{dz}{z}\frac{1}{1-t/z}$ to the relations (A.5), (A.6) and rename $w \to t'$. □

Lemmas B.3 and B.2 imply Lemma B.1. Exact form of $V_b(t_1,\ldots,t_n)$ is not important. □

Let $F_{k,j}(t;t_1,\ldots,t_n)$ be the linear combination of currents introduced by the relation (4.37). Lemma B.1 implies now the following

Lemma B.4. *For* $1 \le i_1 \le \cdots \le i_n \le k < j$ *we have the equality of formal series*

$$\begin{aligned} &P^+(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)F_{j,k}(t)) \\ &\quad = P^+(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n))\,P^+\left(F_{j,k}(t;t_1,\ldots,t_n)\right). \end{aligned}$$

Proof. Corollary A.1 to Proposition A.1 states that the product of the currents $F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)F_{j,k}(t)$ has simple zeroes at hyperplanes $t = t_i$, $i = 1,\ldots,n$. Substituting these conditions in (B.1) we get a systems of n linear equations over the field of rational functions $\mathbb{C}(t_1,\ldots,t_n)$ for the operators $V_b(t_1,\ldots,t_n)$:

$$\sum_{b=1}^{n}\frac{V_b(t_1,\ldots,t_n)}{t_a-q^2t_b} = V\cdot P^+\left(F_{k,j}(t_a)\right), \quad i=1,\ldots,n\,, \tag{B.9}$$

where $V = P^+\left(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)\right)$. The determinant of the matrix $B_{a,b} = (t_a-q^2t_b)^{-1}$ of this system is nonzero in $\mathbb{C}(t_1,\ldots,t_n)$,

$$\det(B) = (-q^2)^{\frac{n(n+1)}{2}}\frac{\prod_{a\neq b}(t_a-t_b)^2}{\prod_{a,b}(t_a-q^2t_b)},$$

hence the system has a unique solution over $\mathbb{C}(t_1,...,t_n)$. This implies that the operators V_b are linear combinations over $\mathbb{C}(t_1,...,t_{a-1})$ of the operators $V\cdot P^+\left(F_{j,k}(t_a)\right)$, $a=1,...,n$, and the projection

$$\begin{aligned} P^+\left(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n)F_{j,k}(t)\right) &= V\cdot P^+\left(F_{j,k}(t)\right) \\ &-\sum_{b=1}^{n}\varphi_{t_b}(t;t_1,\ldots,t_n)\ V\cdot P^+\left(F_{j,k}(t_b)\right), \end{aligned} \tag{B.10}$$

where $\varphi_{t_b}(t;t_1,\ldots,t_n) = A_b(t;t_1,\ldots,t_n)/\prod_{m=1}^{n}(t-q^2t_m)$ are rational functions whose numerators are polynomials on t of degree less then n. The system (B.9) is satisfied if the rational functions $\varphi_{t_b}(t;t_1,\ldots,t_n)$ have the property

$$\varphi_{t_b}(t_a;t_1,\ldots,t_n) = \delta_{a,b}, \qquad a,b=1,...,n.$$

This interpolation problem has a unique solution given by formula (4.36). This proves Lemma B.4. □

The iteration of Lemma B.4 proves Proposition 4.4. □

Now we describe the projection P^- of the composed currents and strings. Let $\tilde{\mathrm{ad}}_x$ denotes the adjoint action in $U_q(\widehat{\mathfrak{gl}}_N)$ related to comultiplication Δ^{op}, $\tilde{\mathrm{ad}}_x\ y = \sum_l x''_l\cdot y\cdot a^{-1}(x'_l)$, where $\Delta x = \sum_l x'_l\otimes x''_l$. Define screening operators $\tilde{S}_i$, $i=1,\ldots,N-1$ by the relation

$$\tilde{S}_i\,(y) = \tilde{\mathrm{ad}}_{F_i[0]}\ (y) = F_i[0]\ y - k_i^{-1}k_{i+1}\ y\ k_{i+1}^{-1}k_i\ F_i[0]$$

Inductive use of (4.14) implies an analog of Proposition 4.3 (we do not use it further):

$$P^-\left(F_{j+1,i}(t)\right) = \tilde{S}_j\tilde{S}_{j-1}\cdots\tilde{S}_{i+1}\left(P^-(F_i(t))\right) = -\left(\tilde{S}_j\tilde{S}_{j-1}\cdots\tilde{S}_{i+1}\left(F_i(t)\right)\right)^{(-)}.$$

Set

$$\tilde\varphi_{u_m}(u;u_1,\dots,u_n)=\prod_{a=1,\ a\neq m}^{n}\frac{u-u_a}{u_m-u_a}\prod_{a=1}^{n}\frac{qu_m-q^{-1}u_a}{qu-q^{-1}u_a},\tag{B.11}$$

and

$$\tilde F_{j+1,i}(u;u_1,\dots,u_n)=F_{j+1,i}(u)-\sum_{m=1}^{n}\tilde\varphi_{u_m}(u;u_1,\dots,u_n)F_{j+1,i}(u_m),$$

where $1\le i\le j<N$. The series $\tilde F_{i,j+1}(u;u_1,\dots,u_n)$ admits another presentation, which will be used below. Namely, in the notation $u\equiv u_0$ and

$$\psi_{u_m}(u_0;u_1,\dots,u_n)=\prod_{j=1}^{n}\frac{u_0-u_j}{qu_0-q^{-1}u_j}\cdot\frac{\prod_{j=1}^{n}(qu_k-q^{-1}u_j)}{\prod_{j=0\ j\neq k}^{n}(u_k-u_j)}$$

we have

$$\tilde F_{j+1,i}(u_0;u_1,\dots,u_n)=\sum_{k=0}^{n}\psi_{u_m}(u_0;u_1,\dots,u_n)F_{j+1,i}(u_k)$$

so that

$$\begin{aligned}&P^-\left(\tilde F_{j+1,i}(u_0;u_1,\dots,u_n)\right)\\&\quad=\sum_{k=0}^{n}\psi_{u_m}(u_0;u_1,\dots,u_n)P^-\left(F_{j+1,i}(u_k)\right).\end{aligned}\tag{B.12}$$

We have an analog of Proposition 4.4:

$$\begin{aligned}&P^-(F_{j,i_1}(t_1)\cdots F_{j,i_n}(t_n))\\&\quad=P^-\Big(\tilde F_{j,i_1}(t_1;t_2,\dots,t_n)\Big)\,P^-\Big(\tilde F_{j,i_2}(t_2;t_3,\dots,t_n)\Big)\ \cdots\\&\qquad\cdots\ P^-\Big(\tilde F_{j,i_{n-1}}(t_{n-1};t_n)\Big)\,P^-(F_{j,i_n}(t_n)).\end{aligned}\tag{B.13}$$

Using (B.13) we present the projection of the string $P^-\left(\mathcal F^j_{\bar s'}(\bar t^j_{[s_j]})\right)$ in a factorized form

$$\begin{aligned}P^-\left(\mathcal F^j_{\bar s'}(\bar t^j_{[s_j]})\right)&=\overrightarrow{\prod_{1\le a\le j}}\left(\overrightarrow{\prod_{s_{a-1}<\ell\le s_a}}P^-\left(\tilde F_{j+1,a}(t^j_\ell;t^j_{\ell+1},\dots,t^j_{s_j})\right)\right)\\&\times\prod_{1\le\ell<\ell'\le s_j}\frac{q^{-1}-qt^j_\ell/t^j_{\ell'}}{1-t^j_\ell/t^j_{\ell'}}\prod_{1\le a\le j}\left(\prod_{s_{a-1}<\ell<\ell'\le s_a}\frac{1-t^j_\ell/t^j_{\ell'}}{q-q^{-1}t^j_\ell/t^j_{\ell'}}\right).\end{aligned}\tag{B.14}$$

C. String product expansion

Let $U_{i,j}$ be subalgebra of $\overline{U}_F$ generated by the modes of the currents $F_i(t),\dots,F_j(t)$ and $U^\varepsilon_{i,j}=U_{i,j}\cap\mathrm{Ker}\,\varepsilon$ be the corresponding augmentation ideal.

Lemma C.1. *For any i and j such that $1\le i<j\le N$ we have*

$$P^-\left(F_{j,i}(t)\right)=-F_{j,i}(t)^{(-)}\qquad \mathrm{mod}\quad P^-\left(U^{\varepsilon}_{i,j-2}\right)\cdot U_{i,j-1}. \tag{C.1}$$

Proof. Proof is based on the relation

$$P^-\left(F_{j,i}(t)\right)+F_{j,i}(t)^{(-)}=S_i\left(P^-\left(F_{j,i+1}(t)\right)+F_{j,i+1}(t)^{(-)}\right)$$
$$\mathrm{mod}\ \ P^-\left(U^{\varepsilon}_{i,i}\right)\cdot U_{i,j-1}$$

which is direct consequence of (B.2). Using this relation several times we obtain

$$P^-\left(F_{j,i}(t)\right)+F_{j,i}(t)^{(-)}=S_i\cdots S_{j-2}\left(P^-\left(F_{j,j-1}(t)\right)+F_{j,j-1}(t)^{(-)}\right)$$
$$\mathrm{mod}\ \ P^-\left(U^{\varepsilon}_{i,j-2}\right)\cdot U_{i,j-1}$$

But for the simple roots currents we have $P^-\left(F_{j,j-1}(t)\right)+F_{j,j-1}(t)^{(-)}=0$. Using the commutativity of screening operators and the projections [KP], we prove the Lemma. □

Let $\bar s=\{s_{j+1},s_j,\dots,s_2,s_1\}$ be a collection of non-negative integers satisfying admissibility conditions: $s_{j+1}\ge s_j\ge\cdots\ge s_1\ge s_0=0$. Set $\bar s'=\{0,s_j,\dots,s_1\}$.

Proposition C.1. *For any product $\mathcal{F}(\bar t^{j+1}_{[s_{j+1}]})$ and a string $\mathcal{F}^j_{\bar s'}(\bar t^j_{[s_j]})$ we have an equality*

(C.2)
$$\mathcal{F}(\bar t^{j+1}_{[s_{j+1}]})\cdot P^-\left(\mathcal{F}^j_{\bar s'}(\bar t^j_{[s_j]})\right)=\prod_{i=1}^{j+1}\frac{1}{(s_i-s_{i-1})!}\ \ \overline{\mathrm{Sym}}_{\ t^{j+1}_{[s_{j+1}]}}\left(\mathcal{F}^{j+1}_{\bar s}(\bar t^{j+1}_{[s_{j+1}]})\right.$$
$$\left.\times\ \overline{\mathrm{Sym}}_{\ \bar t^j_{[s_{j-1},s_j]}}\cdots\overline{\mathrm{Sym}}_{\ \bar t^j_{[s_1,s_2]}}\overline{\mathrm{Sym}}_{\ \bar t^j_{[s_1]}}\left(Y(t^{j+1}_{s_j},\dots,t^{j+1}_1;t^j_{s_j},\dots,t^j_1)\right)\right)$$

modulo $P^-\left(U^{\varepsilon}_j\right)\cdot U_{j+1}$.

Proof. Substitute (B.14) instead of the second factor of the product $\mathcal{F}(\bar t^{j+1}_{[s_{j+1}]})\cdot P^-\left(\mathcal{F}^j_{\bar s'}(\bar t^j_{[s_j]})\right)$. Our strategy is to move each factor

$$P^-\Big(\tilde F_{j+1,a}(t^j_\ell;t^j_{\ell+1},\dots,t^j_{s_j})\Big)$$

to the left of the product $\mathcal{F}(\bar t^{j+1}_{[s_{j+1}]})$ and keep the terms modulo $P^-(U^{\varepsilon}_{1,j})\cdot U_{1,j+1}$. We start from the most left factor taken at $a=1$ and $\ell=1$. We replace first $P^-\Big(\tilde F_{j+1,1}(t^j_1;t^j_2,\dots,t^j_{s_j})\Big)$ with a linear combination of $F_{j+1,1}(t^j_k)^{(-)}$ with rational coefficients. This can be done due to Lemma C.1 and (B.14), since the modes of the current $F_{j+1}(t)$ commute with any elements from $P^-(U^{\varepsilon}_{1,j-1})$

and thus can be moved to the left forming modulo terms. Then we use the relations (A.5) and ordering rules

(C.3)
$$\begin{aligned} F_{j+1}(t)F_{j+1,a}(t')^{(-)} &= -\frac{1}{1-t'/t}\ F_{j+2,a}(t)\ + \\ &+\ \left(\frac{q-q^{-1}t'/t}{1-t'/t}\,F_{j+1,a}(t')^{(-)}-\frac{q-q^{-1}}{1-t'/t}F_{j+1,a}(t)^{(-)}\right)F_{j+1}(t), \end{aligned}$$

for $a=1,\ldots,j$, which are consequence of (A.4). They give the equalities

(C.4)
$$\begin{aligned} &F_{j+1}(t^{j+1}_{s_{j+1}})\cdots F_{j+1}(t^{j+1}_1)\cdot F_{j+1,1}(t)^{(-)} = -\sum_{a=1}^{s_{j+1}}\ \prod_{a<\ell\le s_{j+1}}\frac{q^{-1}-qt^{j+1}_a/t^{j+1}_\ell}{1-t^{j+1}_a/t^{j+1}_\ell} \\ &\times\prod_{1\le\ell<a}\frac{q^{-1}-qt^{j+1}_\ell/t^{j+1}_a}{1-t^{j+1}_\ell/t^{j+1}_a}\frac{1}{1-t/t^{j+1}_a}\ \ F_{j+2,1}(t^{j+1}_a)\ \underbrace{F_{j+1}(t^{j+1}_{s_{j+1}})\cdots F_{j+1}(t^{j+1}_1)}_{F_{j+1}(t^{j+1}_a)\ \text{omitted}} \end{aligned}$$

modulo $P^-(U^\varepsilon_{1,j})\cdot U_{1,j+1}$. The iteration of (C.4) using other ordering rules, being consequences of (A.8) and (A.7),

$$\begin{aligned} F_{j+2,a}(t)F_{j+1,a}(t')^{(-)} &= \left(\frac{q^{-1}-qt'/t}{1-t'/t}F_{j+1,a}(t')^{(-)}\right. \\ &\qquad\left.-\ \frac{q^{-1}-q}{1-t'/t}F_{j+1,a}(t)^{(-)}\right)F_{j+2,a}(t), \\ F_{j+2,a}(t)F_{j+1,b}(t')^{(-)} &= F_{j+1,b}(t')^{(-)}F_{j+2,a}(t),\quad a<b. \end{aligned}$$

leads to the following result. Denote

(C.5)
$$\alpha_1(x)=\frac{q^{-1}-qx}{1-x},\quad \alpha_2(x)=\frac{q^{-1}-qx}{q-q^{-1}x},\quad \alpha_3(x)=\frac{q-q^{-1}x}{1-x}.$$

Then the product $\mathcal{F}(\bar t^{j+1}_{[s_{j+1}]})\cdot P^-\left(\mathcal{F}^j_{\bar s'}(\bar t^j_{[s_j]})\right)$ modulo $P^-(U^\varepsilon_{1,j})\cdot U_{1,j+1}$ is equal to the following sum:

(C.6)
$$\begin{aligned} &\prod_{i=1}^{j}\left(\prod_{s_{i-1}<\ell<\ell'\le s_i}\alpha_3(t^j_\ell/t^j_{\ell'})^{-1}\right)\sum_{A_j}\cdots\sum_{A_2}\sum_{A_1}\ Y(t^{j+1}_{a_{s_j}},\ldots,t^{j+1}_{a_1};t^j_{s_j},\ldots,t^j_1) \\ &\times\prod_{i=1}^{j}\prod_{\substack{\ell<\ell'\\ \ell,\ell'\in A_i}}\alpha_1(t^{j+1}_\ell/t^{j+1}_{\ell'})\prod_{\substack{\ell<\ell'\\ \ell\in A^\perp_{i-1}\ \ell'\in A_i}}\alpha_2(t^{j+1}_\ell/t^{j+1}_{\ell'})\prod_{\ell\in A^\perp_{i-1}\ \ell'\in A_i}\alpha_3(t^{j+1}_\ell/t^{j+1}_{\ell'}) \\ &\times\overrightarrow{\prod_{1\le i\le j}}\left(F_{j+2,i}(t^{j+1}_{a_{s_{i-1}+1}})\cdots F_{j+2,i}(t^{j+1}_{a_{s_i}})\right)\ \underbrace{F_{j+1}(t^{j+1}_{s_{j+1}})\cdots F_{j+1}(t^{j+1}_1)}_{F_{j+1}(t^{j+1}_{a_1}),\ldots,F_{j+1}(t^{j+1}_{a_{s_j}})\ \text{omitted}}\ . \end{aligned}$$

The sum in (C.6) goes over all non-ordered subsets $A_1, A_2, \ldots, A_j$ of the set $A = \{1, \ldots, s_{j+1}\}$. These subsets are defined as follows. Let $A_0 = \varnothing$ be an empty set. Denote $A_0^\perp = A$. Define inductively the subsets A_i and $A_i^\perp$ of $A_0^\perp$ for $i = 1, \ldots, j$ by the relations: $A_i \cup A_i^\perp = A_{i-1}^\perp$. Denote the elements of the set A_i by the letters $a_{s_{i-1}+1}, \ldots, a_{s_i-1}, a_{s_i}$. Number of the elements in the subset A_i is equal to $s_i - s_{i-1}$. Note also that because subsets A_k is defined inductively through previous subsets $A_{k-1}, \ldots, A_1$ the summations in (C.6) are not commutative. First, we have to sum over all possible subsets A_1 in $A_0^\perp$, then over all possible A_2 in $A_1^\perp$ and so on.

The last line in (C.6) is the inverse string. We can use the relations (A.5) between the composed currents and the last product of the rational series in the second line of (C.6) to transform it to the string, again modulo $P^-(U^\varepsilon_{1,j})\cdot U_{1,j+1}$:

$$\mathcal{F}(\bar t^{j+1}_{[s_{j+1}]}) \cdot P^-\left(\mathcal{F}^j_{\bar s'}(\bar t^j_{[s_j]})\right) = \prod_{i=1}^{j}\left(\prod_{s_{i-1}<\ell<\ell'\le s_i} \alpha_3(t^j_\ell/t^j_{\ell'})^{-1}\right)\sum_{A_j}\cdots\sum_{A_2}\sum_{A_1}$$
$$\times Y(t^{j+1}_{a_{s_j}}, \ldots, t^{j+1}_{a_{s_1}}; t^j_{s_j}, \ldots, t^j_1)\prod_{i=1}^{j}\prod_{\substack{\ell<\ell'\\ \ell,\ell'\in A_i}}\alpha_1(t^{j+1}_\ell/t^{j+1}_{\ell'})\prod_{\substack{\ell<\ell'\\ \ell\in A^\perp_{i-1}\ \ell'\in A_i}}\alpha_2(t^{j+1}_\ell/t^{j+1}_{\ell'})$$
$$\times \underbrace{F_{j+1}(t^{j+1}_{s_{j+1}})\cdots F_{j+1}(t^{j+1}_1)}_{F_{j+1}(t^{j+1}_{a_1}),\ldots,F_{j+1}(t^{j+1}_{a_{s_j}})\text{ are omitted}}\ \prod_{1\le i\le j}^{\longleftarrow}\left(F_{j+2,i}(t^{j+1}_{a_{s_{i-1}+1}})\cdots F_{j+2,i}(t^{j+1}_{a_{s_i}})\right).$$

Let us also decompose the summation over the non-ordered sets $A_i = \{a_{s_{i-1}+1}, \ldots, a_{s_i}\}$ to the summations over ordered sets $\bar A_i = \{a_{s_{i-1}+1} < \cdots < a_{s_i}\}$ and to the sums over all permutations among fixed $\{a_{s_{i-1}+1}, \ldots, a_{s_i}\}$. Denote the sum over permutation of the fixed elements $\{a_{s_{i-1}+1}, \ldots, a_{s_i}\}$ as $\sum_{\mathrm{per}\bar A_i}$. The previous formula can be written in the form modulo $P^-(U^\varepsilon_{1,j})\cdot U_{1,j+1}$:

(C.7)
$$\mathcal{F}(\bar t^{j+1}_{[s_{j+1}]}) \cdot P^-\left(\mathcal{F}^j_{\bar s'}(\bar t^j_{[s_j]})\right) = \prod_{i=1}^{j}\left(\prod_{s_{i-1}<\ell<\ell'\le s_i} \alpha_3(t^j_\ell/t^j_{\ell'})^{-1}\right)\sum_{\bar A_j}\cdots\sum_{\bar A_1}$$
$$\prod_{i=1}^{j}\prod_{\substack{\ell<\ell'\\ \ell,\ell'\in \bar A_i}}\alpha_1(t^{j+1}_\ell/t^{j+1}_{\ell'})\prod_{\substack{\ell<\ell'\\ \ell\in A^\perp_{i-1}\ \ell'\in \bar A_i}}\alpha_2(t^{j+1}_\ell/t^{j+1}_{\ell'})\sum_{\mathrm{per}\bar A_j}\cdots\sum_{\mathrm{per}\bar A_1} Y(t^{j+1}_{a_{s_j}}, \ldots, t^{j+1}_{a_1}; \bar t^j_{[s_j]})$$
$$\times \underbrace{F_{j+1}(t^{j+1}_{s_{j+1}})\cdots F_{j+1}(t^{j+1}_1)}_{F_{j+1}(t^{j+1}_{a_1}),\ldots,F_{j+1}(t^{j+1}_{a_{s_j}})\text{ omitted}}\ \prod_{1\le i\le j}^{\longleftarrow}\left(F_{j+2,i}(t^{j+1}_{a_{s_{i-1}+1}})\cdots F_{j+2,i}(t^{j+1}_{a_{s_i}})\right).$$

The summations $\sum_{\mathrm{per}\bar A_i}$ can be translated to the q-symmetrization of the series $Y(t^{j+1}_{a_{s_j}}, \ldots, t^{j+1}_{a_1}; t^j_{s_j}, \ldots, t^j_1)$ over the sets of variables $\{t^j_{s_{i-1}+1}, \ldots, t^j_{s_i}\}$

for $i=1,\dots,j$. We have the identity of the formal series proved in [KP]:

$$\begin{aligned}
&\prod_{i=1}^{j}\prod_{s_{i-1}<\ell<\ell'\le s_i}\alpha_3^{-1}(t^j_\ell/t^j_{\ell'})\prod_{\substack{\ell<\ell'\\ \ell,\ell'\in\bar A_i}}\alpha_1(t^{j+1}_\ell/t^{j+1}_{\ell'})\\
&\quad\times\overline{\mathrm{Sym}}_{\ \bar A_j}\cdots\overline{\mathrm{Sym}}_{\ \bar A_1}Y(t^{j+1}_{a_{s_j}},\dots,t^{j+1}_{a_1};\bar t^j_{[s_j]})\\
&\quad=\ \overline{\mathrm{Sym}}_{\ \bar t^j_{[s_{j-1},s_j]}}\cdots\overline{\mathrm{Sym}}_{\ \bar t^j_{[s_1,s_2]}}\overline{\mathrm{Sym}}_{\ \bar t^j_{[s_1]}}\left(Y(t^{j+1}_{a_{s_j}},\dots,t^{j+1}_{a_1};\bar t^j_{[s_j]})\right).
\end{aligned}\tag{C.8}$$

This identity allows to present the r.h.s. of (C.7) in the form

$$\begin{aligned}
&\sum_{\bar A_j}\cdots\sum_{\bar A_2}\sum_{\bar A_1}\prod_{\substack{\ell<\ell'\\ \ell\in A^{\perp}_{i-1}\ \ell'\in\bar A_i}}\frac{q^{-1}-qt^{j+1}_\ell/t^{j+1}_{\ell'}}{q-q^{-1}t^{j+1}_\ell/t^{j+1}_{\ell'}}\\
&\times\ \underbrace{F_{j+1}(t^{j+1}_{s_{j+1}})\cdots F_{j+1}(t^{j+1}_1)}_{F_{j+1}(t^{j+1}_{a_1}),\dots,F_{j+1}(t^{j+1}_{a_{s_j}})\text{ are omitted}}\ \prod^{\longleftarrow}_{1\le i\le j}\left(F_{j+2,i}(t^{j+1}_{a_{s_i}})\cdots F_{j+2,i}(t^{j+1}_{a_{s_{i-1}+1}})\right)\\
&\times\ \overline{\mathrm{Sym}}_{\ \bar t^j_{[s_{j-1},s_j]}}\cdots\overline{\mathrm{Sym}}_{\ \bar t^j_{[s_1,s_2]}}\overline{\mathrm{Sym}}_{\ \bar t^j_{[s_1]}}\left(Y(t^{j+1}_{a_{s_j}},\dots,t^{j+1}_{a_1};t^j_{s_j},\dots,t^j_1)\right).
\end{aligned}\tag{C.9}$$

In its turn, the summation over ordered sets $\bar A_i$ in (C.9) can be written as q-symmetrization over the set of variables $\{t^{j+1}_{[s_{j+1}]}\}$, so finally we obtain the statement of Proposition C.1. □

Let $\bar t_{[\bar s]}$ and $\bar t_{[\bar s']}$ be the sets of variables defined by collections of segments $[\bar 0,\bar s]\equiv[\bar s]$ and $[\bar 0,\bar s']\equiv[\bar s]$ respectively. Proposition C.1 implies the following

Proposition C.2.

$$\begin{aligned}
&\mathcal F(\bar t^{j+1}_{[s_{j+1}]})\cdot\overline{\mathrm{Sym}}_{\ \bar t_{[\bar s']}}\left(X(\bar t_{[\bar s']})\cdot P^-\left(\mathcal F^j_{\bar s'}(\bar t^j_{[s_j]})\right)\right)=\\
&\qquad=\frac{1}{(s_{j+1}-s_j)!}\,\overline{\mathrm{Sym}}_{\ \bar t_{[\bar s]}}\left(X(\bar t_{[\bar s]})\cdot\mathcal F^{j+1}_{\bar s}(\bar t^{j+1}_{[s_{j+1}]})\right)
\end{aligned}\tag{C.10}$$

modulo elements of the form $P^-\left(U^{\varepsilon}_j\right)\cdot U_{j+1}$.

Note that the q-symmetrization $\overline{\mathrm{Sym}}_{\ \bar t_{[\bar s']}}$ in (C.10) goes over the set of variables $\bar t_{[\bar s']}$ which does not include the variables $\bar t^{j+1}_{[s_{j+1}]}$. This means that the left hand side of (C.10) can be written in the form

$$\overline{\mathrm{Sym}}_{\ \bar t_{[\bar s']}}\left(X(\bar t_{[\bar s']})\cdot\mathcal F(\bar t^{j+1}_{[s_{j+1}]})\cdot P^-\left(\mathcal F^j_{\bar s'}(\bar t^j_{[s_j]})\right)\right).$$

To prove Proposition C.2 we have to substitute (C.2) into this expression. We need the following

Lemma C.2. *Rational series* $\overline{\mathrm{Sym}}_{\,\bar t^{j-1}_{[s_{j-1}]}}\cdots\overline{\mathrm{Sym}}_{\,\bar t^{2}_{[s_2]}}\overline{\mathrm{Sym}}_{\,\bar t^{1}_{[s_1]}}X(\bar t_{\bar s'})$ *is symmetric in each group of variables* $\{t^j_{s_{i-1}+1},\ldots,t^j_{s_i}\}$, $i=1,\ldots,j$.

Proof. of this Lemma results from the definition of the rational series (4.15) and from the following corollary of (C.8): the q-symmetrization $\overline{\mathrm{Sym}}_{\bar v}Y(\bar u;\bar v)$ is a symmetric series on the set of variables $\bar u$. □

Statement of Proposition C.2 follows now from the identity for the series

$$\overline{\mathrm{Sym}}_{\,\bar t^{j}_{[s_j]}}\Bigg(\overline{\mathrm{Sym}}_{\,\bar t^{j-1}_{[s_{j-1}]}}\cdots\overline{\mathrm{Sym}}_{\,\bar t^{1}_{[s_1]}}X(\bar t_{\bar s'})\times$$
$$\times\overline{\mathrm{Sym}}_{\,\bar t^{j}_{[s_{j-1},s_j]}}\cdots\overline{\mathrm{Sym}}_{\,\bar t^{j}_{[s_1,s_2]}}\overline{\mathrm{Sym}}_{\,\bar t^{j}_{[s_1]}}Y(\bar t^{j+1}_{[s_j]};\bar t^{j}_{[s_j]})\Bigg)=$$
$$=\prod_{i=1}^{j}(s_i-s_{i-1})!\ \ \overline{\mathrm{Sym}}_{\,\bar t_{[\bar s']}}\left(X(\bar t_{\bar s})\right).$$

□

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