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Thin Schubert cells of codimension two

By

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Abstract

A condition on a matroid of rank $n-2$ for the corresponding thin Schubert cell being nonempty is determined. A necessary and sufficient condition for k and n so that the closure of a thin Schubert cell in $G(k,n)$ is always a union of thin Schubert cells is given.

1. Introduction

Thin Schubert cells are introduced in [4], which provides a finer decomposition of a Grassmann variety than Schubert cells. Each thin Schubert cell has a corresponding matroid. That correspondence is not surjective, and it is an open question to determine which matroid has a corresponding thin Schubert cell [6]. In this paper, we give an answer for the case of codimension two (Theorem 3.1). This is the first nontrivial case, and also a special case in the sense that the closure of a thin Schubert cell decomposes into thin Schubert cells (Theorem 4.1).

We fix some notation. We denote the set of integers $\{1,\ldots,n\}$ by $[n]$. We fix a basis $e_1,\ldots,e_n$ of $\mathbb{C}^n$. For each subset I of $[n]$, E_I is the subspace of $\mathbb{C}^n$ spanned by e_i's $(i \in I)$. If $\#I = k$, then E_I is an element of $G(k,n)$, the Grassmann variety of subspaces of dimension k in $\mathbb{C}^n$.

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2. General Theory

This section is based on the works of Gel'fand, MacPherson, Goresky, and Serganova ([1], [3]). The aim of this section is to introduce a useful concept on thin Schubert cells.

We fix an integer n.

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Definition 2.1. Let $\underline{d} = (d_I)_{I \subset [n]}$ be a set of nonnegative integers. We call $\underline{d}$ a *matroid of rank* k if it satisfies the following conditions ([7], [8]):

- $d_{\emptyset} = 0$,
- $d_{[n]} = k$,
- $d_I + d_J \leq d_{I \cup J} + d_{I \cap J}, \quad \forall I, J \subset [n]$.

These conditions are called *the matroid conditions.*

Definition 2.2. We call that two elements $L_1, L_2 \in G(k, n)$ lie in the same thin Schubert cell if they satisfy the following condition:

$$\dim(L_1 \cap E_I) = \dim(L_2 \cap E_I), \quad \forall I \subset [n].$$

This condition is an equivalence relation. We call each equivalence class, regarding as a subset of $G(k, n)$, *a thin Schubert cell.*

For each thin Schubert cell $\mathcal{L}$ with L an element, the set of integers $\underline{d}(\mathcal{L}) = \{d(L)_I \,|\, I \subset [n]\}$ is a matroid, where $d(L)_I := \dim(L \cap E_I)$. Conversely for each matroid $\underline{d}$, we have a subset $\mathcal{L}(\underline{d})$ of $G(k, n)$ as $\{L \in G(k, n) \,|\, d(L)_I = d_I, \ \forall I\}$, which is either a thin Schubert cell or the empty set.

Problem. Find a necessary and sufficient condition of $\underline{d}$ for $\mathcal{L}(\underline{d})$ being nonempty.

In Section 3, we give the answer to this problem for $k = n - 2$.

Definition 2.3. Let $\underline{d}$ be a matroid. The *basis* of $\underline{d}$ is a subset B of $[n]$ consisting of $n - k$ elements such that $d_B = 0$.

We write the set of all the bases as $\mathcal{B}(\underline{d})$ or just $\mathcal{B}$.

Proposition 2.1. *The following map is injective.*

$$\{\underline{d} \,|\, \mathcal{L}(\underline{d}) \neq \emptyset\} \ni \underline{d} \longmapsto \mathcal{B}(\underline{d}) \in \binom{[n]}{n-k}$$

Proof. This proposition is shown in [5] without proof. Note that $d(L)_I \geq k + \#I - n$ holds for every I.

Let us fix an element $L \in \mathcal{L}$. Then the following lemmas show that $d(L)_I$ is uniquely determined from $\mathcal{B}$ and we come to the conclusion. □

Lemma 2.1. *Suppose* $\#I$ *is strictly less than* $n - k$*. Then*

$$d(L)_I \neq 0 \Leftrightarrow d(L)_J \neq 0 \text{ for every } J \supset I.$$

Proof. We will show that $d(L)_I$ is nonzero if all the $d(L)_{I \sqcup \{j\}}$'s are nonzero, where j is an element of $[n] \setminus I$. Suppose $d(L)_I$ is zero. Since $d(L)_{I \sqcup \{j\}}$ is nonzero for each j, there exists a nonzero vector $\mathbf{v}_j$ for each j which is contained in $L \cap (E_{I \sqcup \{j\}} \setminus E_I)$. So $\mathcal{L}$ contains a linear subspace spanned by all the $\mathbf{v}_j$'s, which is $(n - \#I)$ dimensional. Since $n - \#I$ is greater than $k + 1$, we come to the contradiction. □

Remark 1.

$$d(L)_I \neq 0 \text{ if } \#I > n-k+1.$$

Remark 2. Suppose $\mathcal{B}$ is given. Then Lemma 2.1 shows that we can uniquely determine the collection of I which satisfies $d(L)_I = 0$. We shall write this set as $\mathcal{Z}$:

$$\mathcal{Z} = \{I \subset [n] \,|\, \exists J \in \mathcal{B}, J \supset I\} \supset \mathcal{B}.$$

Lemma 2.2.

$$d(L)_I = \min\{\#M \,|\, I \setminus M \in \mathcal{Z}\}.$$

Proof. Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in M(n)$ be the defining matrix of L:

$$L = \left\{ (x_1, \ldots, x_n) \;\middle|\; A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \right\}.$$

For $I = \{i_1, \ldots, i_p\} \in [n]$, put $A_I := (\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_p}) \in M(p, n)$. Then $d(L)_I = \#I - \mathrm{rk} A_I$ and the statement follows. □

Remark 3. Lemma 2.1 shows that $\mathcal{B}(\underline{d})$ is nonempty if we assume that $\mathcal{L}(\underline{d})$ is nonempty.

Another important tool for thin Schubert cells is Gel'fand-MacPherson correspondence, or its extended version. They are, roughly speaking, a correspondence between (an orbit of) an element of a Grassmann variety and a point configuration. The original Gel'fand-MacPherson correspondence, given at [3, 2.2.2], is valid only on general cases and was later extended in [1, 1.5]. Let us introduce the latter.

Note that $GL(n-k)$ canonically acts on $\mathbb{C}^{n-k}$. Thus $PGL(n-k)$ canonically acts on $\hat{\mathbb{P}}^{n-k-1} := \mathbb{P}^{n-k-1} \sqcup \{0\}$ because $\hat{\mathbb{P}}^{n-k-1} = \mathbb{C}^{n-k}/\mathbb{C}^*$. Here, $\{0\}$ corresponds to the origin of $\mathbb{C}^{n-k}$. This action induces the following action:

$$PGL(n-k) \times (\hat{\mathbb{P}}^{n-k-1})^n \ni (g, (x_1, \ldots, x_n)) \mapsto (g \cdot x_1, \ldots, g \cdot x_n) \in (\hat{\mathbb{P}}^{n-k-1})^n.$$

Thus we have the map $p_2 : (\hat{\mathbb{P}}^{n-k-1})^n \to (\hat{\mathbb{P}}^{n-k-1})^n / PGL(n-k)$.

For a given $L \in G(k, n)$ and an isomorphism $\phi : \mathbb{C}^n/L \to \mathbb{C}^{n-k}$, we have a map $v_L^\phi := \phi \circ p_1 : \mathbb{C}^n \to \mathbb{C}^{n-k}$, where $p_1 : \mathbb{C}^n \to \mathbb{C}^n/L$ is a canonical projection.

Put $q : (\mathbb{C}^{n-k})^n \to (\hat{\mathbb{P}}^{n-k-1})^n$ to be the componentwise canonical projection. Then $p_2 \circ q(v_L^\phi(e_1), \ldots v_L^\phi(e_n))$ is an element of $(\hat{\mathbb{P}}^{n-k-1})^n / PGL(n-k)$ and is independent of the choice of ϕ. Let V be a map

$$V : G(k, n) \ni L \mapsto p_2 \circ q(v_L^\phi(e_1), \ldots v_L^\phi(e_n)) \in (\hat{\mathbb{P}}^{n-k-1})^n / PGL(n-k).$$

Note that torus $T := (\mathbb{C}^*)^n$ canonically acts on $\mathbb{C}^n$. Thus there is an action of T on $G(k,n)$. If L and L' are in the same orbit of $G(k,n)$, then $V(L) = V(L')$ holds. So we have a map

$$\bar{V} : G(k,n)/T \to (\hat{\mathbb{P}}^{n-k-1})^n/PGL(n-k).$$

We call the image of $\bar{V}$ a configuration of n points. The following theorem shows that the image consists of a configuration that 'spans $\mathbb{P}^{n-k-1}$'.

Theorem 2.1. *The map $\bar{V}$ is injective. The point $p_2 \circ q(a_1, \dots, a_n)$ is in the image of $\bar{V}$ if and only if there is no nontrivial subspaces of $\mathbb{C}^{n-k}$ containing $\{a_1, \dots, a_n\}$.*

Proof. See [1, 1.5]. □

Remark 4. The subspace $\langle v_L^\phi(e_i) \mid i \in I\rangle \subset \mathbb{C}^{n-k}$ is $(\#I - d(L)_I)$-dimensional regardless to ϕ.

3. The case $G(n-2,n)$

We define a set of integers $\underline{d}^2 \in \mathbb{Z}^{{}_nC_2}$ to be $\underline{d}^2 := (d_I)_{\#I=2}$.

Proposition 3.1. *Let $\underline{d}^2$ be given. Then there exist a thin Schubert cell $\mathcal{L} = \mathcal{L}(\underline{d'})$ in $G(k,n)$ for some k satisfying $d'_I = d_I$ for every I $(\#I = 2)$ if and only if there exists a nonnegative integer m and a subdivision satisfying the followings:*

$$(*) \qquad \begin{cases} A \sqcup B_1 \sqcup \cdots \sqcup B_m \sqcup C = [n] \quad (m \geq 0), \\ \#B_i \geq 2 \text{ for every } i, \\ d_{\{a_1,a_2\}} = 2 \text{ if } a_1, a_2 \in A, \\ d_{\{b_1,b_2\}} = 1 \text{ if } b_1, b_2 \in B_i \text{ for some } i, \\ d_{\{x,y\}} = 0 \quad \text{otherwise.} \end{cases}$$

Remark 5. Both A or C can be empty.

Remark 6. The corresponding thin Schubert cell is not unique in general.

Remark 7. The following table shows the correspondence between the three ways of expressing a thin Schubert cell – a matroid $\underline{d}$, a configuration, and a subdivision.

Proof. Take L in $G(k,n)$. Take a maximal subset A of $[n]$ such that each element a satisfies $e_a \in L$. Next suppose two vectors $e_i + \alpha e_j$, $e_i + \beta e_k$ are contained in L for some $\alpha, \beta \in \mathbb{C}^*$. Then obviously $\alpha e_j - \beta e_k$ is also contained in L. Thus if $d(L)_{\{i,j\}} = d(L)_{\{i,k\}} = 1$ for some $i, j, k \in [n] \setminus A$, $d_{\{j,k\}}(L)$ is also 1. Hence choose B_1 to be a maximal subset of $[n] \setminus A$ such that $d_{\{b_1,b_2\}} = 1$ for

Table 1. The correspondence between $\underline{d}$, a configuration, and a subdivision

$\underline{d}$	a point configuration $a_1, \dots . a_n \in \hat{\mathbb{P}}^{n-k-1}$	a subdivision of $[n]$ $A, B_1, \dots, B_m, C$
$d_{\{i\}} = 1$	$a_i = 0$	$i \in A$
$d_{\{i,j\}} = 1$, $d_{\{i\}} = d_{\{j\}} = 0$	• $a_i = a_j$	$i, j \in B_s$ for some s
$d_{\{i,j\}} = 0$	a_i • a_j • (two different points)	$i, j \in [n] \setminus A$, $\nexists B_s \ni i, j$

every $b_1, b_2 \in B_1$. Repeat this process with $[n]$ replaced by $[n] \setminus B_1$ and so on. Eventually every two elements x, y of $[n] \setminus (A \sqcup B_1 \sqcup \cdots \sqcup B_m)$ satisfies $d_{\{x,y\}} = 0$. Let C be the complement $[n] \setminus (A \sqcup B_1 \sqcup \cdots \sqcup B_m)$. These $A, B_1 \dots, B_m, C$ satisfies the properties $(*)$.

To show the opposite side, we shall construct L from the given $\underline{d}^2$. For $B_s = \{b_1^s, \dots, b_{m_s}^s\}$, define L_s and L as

$$L_s = \left\langle e_{b_1^s} + e_{b_2^s}, e_{b_1^s} + e_{b_3^s}, \dots, e_{b_1^s} + e_{b_{m_s}^s} \right\rangle,$$

$$L = \langle e_a \mid a \in A \rangle + \sum_s L_s \quad \text{(Minkowski sum)}.$$

Then L is an element of $G(k, n)$, where $k = \#A + \sum(\#B_i - 1)$. □

The following lemma is obvious from Proposition 2.1 and Table 1.

Lemma 3.1. *Suppose $n - k = 2$. Then the orbits corresponding to two configurations $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are in the same thin Schubert cell if and only if the following holds*:

- $a_i = 0 \Leftrightarrow b_i = 0$,
- $a_i = a_j \Leftrightarrow b_i = b_j$.

Theorem 3.1. *Let $n - k = 2$ and suppose $\underline{d}$ is given. Then there exists a thin Schubert cell corresponding to $\underline{d}$ if and only if it satisfies the following three conditions*:

- $d_{\{i,j\}} = 0$ *for some* i, j,
- $\underline{d}^2 \subset \underline{d}$ *satisfies condition* $(*)$ *in Proposition* 3.1,
- $d_I = \min\{\#J \mid I \setminus J \subset \{i, j\}$ *for some* i, j *satisfying* $d_{\{i,j\}} = 0\ \}$

Proof. We have already seen from Remark 3, Proposition 3.1, and Lemma 2.2 that the three conditions are necessary. To show the opposite side, let us consider the following configuration of n points $p_1, \dots, p_n \in \hat{\mathbb{P}}^1$:

- $p_i = 0$ if and only if $i \in A$,
- $p_i = p_j \neq 0$ if $i, j \in B_s$ for some s.

From the above lemma, there exists an orbit corresponding to the configuration. □

Remark 8. Since $k = n-2 = \#A + \sum_{i=1}^{m}(\#B_i - 1)$ holds, the subdivision corresponding to $\underline{d}^2$ must satisfy one of the followings:

- $\#A = n-2, m = 0, \#C = 2$,
- $\#A \leq n-3, m = 1, \#C = 1$,
- $\#A \leq n-4, m = 2, C = \emptyset$.

Remark 9. In general, the conditions on the configuration to be in the same thin Schubert cell get more complicated as $\min\{k, n-k\}$ gets larger.

4. The closure of a thin Schubert cell

Throughout the rest of the paper, we assume that every matroid $\underline{d}$ satisfies $\mathcal{L}(\underline{d}) \neq \emptyset$.

For a thin Schubert cell $\mathcal{L} = \mathcal{L}(\underline{d})$, let $\hat{\mathcal{L}} := \{L \in G(k,n) \mid d(L)_I \geq d_I,\ \forall I\}$.

Proposition 4.1. *Let $\mathcal{L}$ be a thin Schubert cell. Then $\hat{\mathcal{L}}$ contains the closure $\bar{\mathcal{L}}$ of $\mathcal{L}$.*

Proof. If $L \in \bar{\mathcal{L}}$ satisfies $d(L)_I = 0$, then $d_I = 0$ also holds and the proposition follows from Lemma 2.2. □

Remark 10. There is an example which shows that $\hat{\mathcal{L}}$ and $\bar{\mathcal{L}}$ do not coincide on $G(4,7)$. See [1].

Lemma 4.1. *If $\min(k, n-k) \leq 2$, then $\hat{\mathcal{L}} = \bar{\mathcal{L}}$.*

Proof. Since $G(k,n) \overset{*}{\cong} G(n-k,n)$ holds, it is enough to show the lemma on $G(1,n)$ and $G(n-2,n)$.

Case 1. $G(1,n)$.

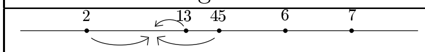
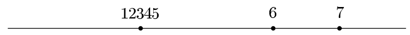
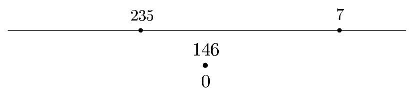
For a thin Schubert cell $\mathcal{L}$, we obviously have

$$\hat{\mathcal{L}} = \bar{\mathcal{L}} = \left\{ \mathbb{C} \cdot \sum_{i \in [n]} a_i e_i \in G(1,n) \mid a_i = 0 \text{ if } [n] \setminus \{i\} \notin \mathcal{B}(\underline{d}(\mathcal{L})) \right\}.$$

Case 2. $G(n-2,n)$.

Take $\hat{L}$ to be an element of $\hat{\mathcal{L}}$. Put $\mathcal{L}' = \mathcal{L}(\underline{d'})$ to be the thin Schubert cell containing $\hat{L}$ as an element. Correspondingly, we have a configuration $\hat{a}_1, \ldots, \hat{a}_n$ and a subdivision $\hat{A}, \hat{B}_1, \ldots, \hat{B}_{\hat{m}}, \hat{C}$ of $\hat{L}$ (see table 1). Let $A, B_1, \ldots, B_m, C$ be the subdivision corresponding to $\mathcal{L}$. From Theorem 3.1 we know that there exists integers z_1, z_2 satisfying $d'_{\{z_1,z_2\}} = 0 = d_{\{z_1,z_2\}}$. We will prove $\hat{L} \in \bar{\mathcal{L}}$ in two steps as in the following table.

Table 2. $\hat{L} \in \bar{\mathcal{L}}$

$\mathcal{L} : A = \emptyset, B_1 = \{1,3\}, B_2 = \{4,5\}, C = \{2,6,7\}$ $\hat{L} = \langle e_1, e_4, e_6, e_2 + e_3, e_3 + e_5 \rangle$	
configuration	subdivision
 ↓ Step 2	$A = \emptyset, B_1 = \{1,3\}, B_2 = \{4,5\},$ $C = \{2,6,7\}$
 ↓ Step 1	$A = \emptyset, B_1 = \{1,2,3,4,5\}, C = \{6,7\}$
	$\hat{A} = \{1,4,6\}, \hat{B}_1 = \{2,3,5\}, \hat{C} = \{7\}$

Step 1.

Consider the following configuration $a'_1, \ldots, a'_n$ defined recursively by

$$a'_i = \begin{cases} \hat{a}_j & \text{if } i \in \hat{A} \setminus A \text{ and there exists } \ j \notin \hat{A} \\ & \qquad\qquad \text{satisfying } i, j \in B_s \text{ for some } s, \\ a'_k & \text{if } i \in \hat{A} \setminus A \text{ and } i, k \in B_s \text{ for some } s \text{ and some } k < i, \\ \text{a general point} & \text{if } i \in \hat{A} \setminus A \text{and does not satisfy} \\ & \qquad\qquad \text{either of the above conditions,} \\ \hat{a}_i & \text{otherwise.} \end{cases}$$

Then there exists a corresponding orbit $\mathcal{O}'$ since a'_{z_1} and a'_{z_2} span $\mathbb{P}^1$. From the construction, $(\hat{\mathcal{L}} \supset)\bar{\mathcal{O}}' \ni \hat{L}$ holds because every nonempty closed subset of $\hat{\mathbb{P}}^1/PGL(2)$ contains 0. Thus we may assume $\hat{A}$ to be A.

Step 2.

Since $d'_{\{i,j\}} \geq d_{\{i,j\}}$ holds for every i, j, each B'_s can be written as a sum of B_i's and $B'_s \cap C$, i.e.

$$B'_s = (\sqcup_{i \in I(s)} B_i) \sqcup (B'_s \cap C)$$

for some subset $I(s)$ of $[m]$. Without loss of generality, we may assume that the configuration satisfies $\hat{a}_1, \ldots, \hat{a}_n \in \mathbb{C} \sqcup \{0\} \subset \hat{\mathbb{P}}^1$. Take ϵ small enough so that if we write by $B(\hat{a}_i)$ the ϵ-ball centered at $\hat{a}_i$,

$$B(\hat{a}_i) \cap B(\hat{a}_j) \neq \emptyset \Leftrightarrow \hat{a}_i = \hat{a}_j.$$

Then we define a family of configurations $\{a_i(t)\}_{t\in[0,1]}$ as follows:

$$a_i(t) = \begin{cases} \hat{a}_i & \text{if } i \in A \text{ or } i \in \hat{C} \\ \hat{a}_i + (1-t)p\epsilon/n & \text{if } i \in \hat{B}_p \cap B_q \text{ for some } p, q, \\ \hat{a}_i - (1-t)i\epsilon/n & \text{if } i \in \hat{B}_p \cap C \text{ for some } p. \end{cases}$$

Then for $t \in [0, 1)$ we have

$$a_i(t) = a_j(t) \text{ if } i, j \in A \text{ or } i, j \in B_s \text{ for some } s,$$

whereas for $t = 1$,

$$(a_1(t), \ldots, a_n(t)) = (\hat{a}_1, \ldots, \hat{a}_n)$$

and thus we have shown that $\hat{L} \in \bar{\mathcal{L}}$. □

For a thin Schubert cell $\mathcal{L}$, there exists a subdivision $A, \ldots, B_1, \ldots, B_m, C$ of $[n]$ from Proposition 3.1. Let $\nu(\mathcal{L})$ denote $\#C$. This integer $\nu(\mathcal{L})$ equals the number of independent points in $\mathbb{P}^{n-k-1}$ of the configuration corresponding to $\mathcal{L}$.

In the following three lemmas, we assume $k = 3, n = 6$.

Lemma 4.2. *If $\nu(\mathcal{L}) \leq 5$, then $\hat{\mathcal{L}} = \bar{\mathcal{L}}$.*

There are nine types of configurations of six independent unlabeled points in $\mathbb{P}^2$ as shown in Table 3, where points being on a same line means that they are collinear.

Lemma 4.3. *Suppose that thin Schubert cells $\mathcal{L}'$ and $\mathcal{L}$ and a subspace $L \in \mathcal{L}'$ satisfy $L \in \hat{\mathcal{L}},\ \nu(\mathcal{L}') \leq 4,\ \nu(\mathcal{L}) = 6$. Then with one exception, there exists a thin Schubert cell $\mathcal{L}_1$ such that $L \in \bar{\mathcal{L}_1} \subset \hat{\mathcal{L}},\ \nu(\mathcal{L}_1) = 5$.*

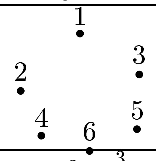
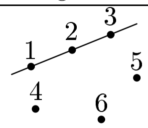
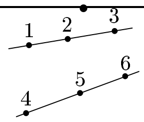
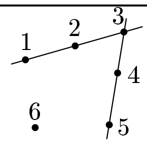
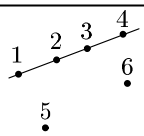
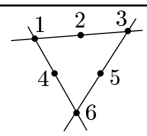
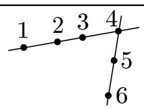
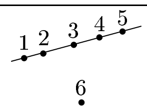
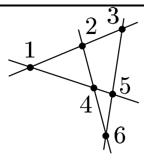
The exceptional case is that the configuration corresponding to $\mathcal{L}$ is of type [4] *in Table* 3. *In this case, if we denote by $\mathcal{L}_1$ the thin Schubert cell corresponding to Figure* 1, *we have $L \in \bar{\mathcal{L}_1} \subset \bar{\mathcal{L}} = \hat{\mathcal{L}}$.*

Lemma 4.4. *Suppose that thin Schubert cells $\mathcal{L}'$ and $\mathcal{L}$ satisfy $\mathcal{L}' \subset \hat{\mathcal{L}},\ 5 \leq \nu(\mathcal{L}') \leq \nu(\mathcal{L}) = 6$. Then $\mathcal{L}' \subset \bar{\mathcal{L}}$ holds.*

Proof. The first lemma comes from the fact that a configuration of r independent points in $\mathbb{P}^2$ is essentially same as that of $G(r-3, r)$. The second and the third lemma will be shown by checking all the pairs $(\mathcal{L}', \mathcal{L})$ one by one using the configurations.

Figure 2 illustrates the way to obtain a given [4] as a limit of [3-1]. Starting from the configuration [4], slide the point on the center along either of the two lines. We have the configuration [3-1]. By reversing the movement, we return to the configuration [4] as a limit of [3-1]. The other cases can be shown similarly. □

Table 3. The configuration of six independent points and its bases $\mathcal{B}$

	configuration	$\binom{[6]}{3} \setminus \mathcal{B}$		configuration	$\binom{[6]}{3} \setminus \mathcal{B}$
[0]		$\emptyset$	[1]		$\{1,2,3\}$
[2-1]		$\{1,2,3\}$, $\{4,5,6\}$	[2-2]		$\{1,2,3\}$, $\{3,4,5\}$
[2-3]		$\binom{[4]}{3}$	[3-1]		$\{1,2,3\}$, $\{1,4,6\}$, $\{3,5,6\}$
[3-2]		$\binom{[4]}{3}$, $\{4,5,6\}$	[3-3]		$\binom{[5]}{3}$
[4]		$\{1,2,3\}$, $\{1,4,5\}$, $\{2,4,6\}$, $\{3,5,6\}$			

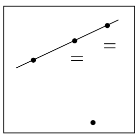


Figure 1. The exceptional configuration ("=" denotes a double point)

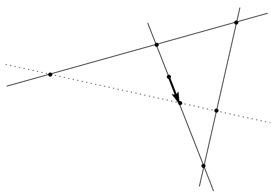


Figure 2. [3-1] converging to [4]

Theorem 4.1. *If* $\min(k, n-k) \leq 2$ *or* $(k, n) = (3, 6)$, *then* $\hat{\mathcal{L}} = \bar{\mathcal{L}}$ *holds for every thin Schubert cell* $\mathcal{L}$. *The opposite is also true.*

Proof. Suppose that $n - k \geq 3$ and $k \geq 4$. Let $\mathcal{L} \subset \langle e_1, \ldots, e_7 \rangle$ be the thin Schubert cell satisfying $\bar{\mathcal{L}} \subsetneq \hat{\mathcal{L}}$ as in the remark of Proposition 4.1. Define

$\mathcal{L}'$ as

$$\mathcal{L}' = \begin{cases} \mathcal{L} \subset \langle e_1, \ldots, e_n \rangle, & \text{if } k = 4, \\ \mathcal{L} + \langle e_8, \ldots, e_{k+4} \rangle \text{ (Minkowski sum)}, & \text{otherwise.} \end{cases}$$

Then $\bar{\mathcal{L}}' \subsetneq \hat{\mathcal{L}}'$. Thus we know that there exists a thin Schubert cell satisfying

$$\hat{\mathcal{L}} \neq \bar{\mathcal{L}} \text{ if } n - k \geq 3 \text{ and } k \geq 4.$$

Since $G(k, n)$ is dual to $G(n - k, n)$, we also know that there exists a thin Schubert cell satisfying

$$\hat{\mathcal{L}} \neq \bar{\mathcal{L}} \text{ if } n - k \geq 4 \text{ and } k \geq 3.$$

Thus the remaining case is that $(k, n) = (3, 6)$, which is a consequence of the previous three lemmas. □

5. Thin Schubert Cells and Schubert Varieties

Let us fix a complete flag

$$V_* : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n.$$

We define a Schubert variety $\Omega(V_{i_1}, V_{i_2}, \ldots, V_{i_k})$ $(1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n)$ as a subvariety of $G(k, n)$ such that every element L satisfies $\dim(L \cap V_{i_s}) \geq s$ for every s.

From now on, let k be $n-2$ as before and $V_i = E_{[i]}$. Let a and b be integers satisfying

$$(**) \qquad \begin{cases} 0 \leq a \leq b \leq n, \\ b - a \neq 1, \\ a \neq n. \end{cases}$$

Let A be $[a]$, B_1 be $[b] \setminus [a]$ and C be $[n] \setminus [b]$. From Theorem 3.1, there exists a corresponding thin Schubert cell $\mathcal{L}$ of $G(k, n)$. Put $b' = \max\{a + 1, b\}$.

Proposition 5.1. *The thin Schubert cell $\mathcal{L}$ is a dense subset of a Schubert variety $\Omega(V_1, \ldots, \check{V}_{a+1}, \ldots, \check{V}_{b'+1}, \ldots, V_n)$, where $\check{V}_i$ means omitting V_i.*

Proof. Denote simply by Ω the above Schubert variety $\Omega(V_1, \ldots, \check{V}_{a+1}, \ldots, \check{V}_{b'+1}, \ldots, V_n)$. First, we will show that $\mathcal{L}$ is a subset of Ω. For every element i of $[a]$, we have $d(\mathcal{L})_{\{i\}} = 1$ and therefore $d(\mathcal{L})_{[i]} = i$. Let N be an integer less than b'. For each elements p, q of $[N]$, $d'_{\{p,q\}}$ is nonzero and therefore $d_N \geq N - 1$ from the proof of Proposition 2.1. Thus $\mathcal{L}$ is a subset of Ω.

We now show that $\mathcal{L}$ is dense in Ω. Fix an element L of Ω and let $\mathcal{L}'$ be a thin Schubert cell containing L. Then from Lemma 4.1, it is enough to show that $d(\mathcal{L}')_{\{i,j\}} \geq d(\mathcal{L})_{\{i,j\}}$ for every $i, j \ \ (i < j)$.

Note that $d(\mathcal{L})_{\{i,j\}} \leq 2$ holds for every i, j. Thus if $d(\mathcal{L}')_{\{i,j\}} = 2$, there is nothing to prove. Suppose that $d(\mathcal{L}')_{\{i,j\}} = 1$. Then j is strictly greater than a since L contains axes $e_1, \ldots, e_a$. Thus $d(\mathcal{L})_{\{i,j\}}$ cannot be 2. If $d(\mathcal{L}')_{\{i,j\}} = 0$, then both i and j must be greater than a. Suppose that j is less than b. Then from the proof of Proposition 2.1, we have $d(\mathcal{L}')_{[j]} \leq j - 2$ which is a contradiction. Thus j is strictly greater than b and we have $d(\mathcal{L})_{\{i,j\}} = 0$. □

Corollary 5.1. *A thin Schubert cell is a dense subset of a Schubert variety if and only if the corresponding subdivision is* $[n] = A \sqcup B_1 \sqcup C$*, where* $A = [a]$ *and* $B = [b] \setminus [a]$ *for some* a, b *satisfying* $a \leq b$.

Proof. The condition $(**)$ is automatically satisfied if there exists a corresponding thin Schubert cell. □

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