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Gorenstein cohomology in abelian categories

By

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Abstract

We investigate relative cohomology functors on subcategories of abelian categories via Auslander-Buchweitz approximations and the resulting strict resolutions. We verify that certain comparison maps between these functors are isomorphisms and introduce a notion of perfection for this context. Our main theorem is a balance result for relative cohomology that simultaneously recovers theorems of Holm and the current authors as special cases.

Introduction

Let $\mathcal{A}$ be an abelian category equipped with subcategories $\mathcal{W}$ and $\mathcal{X}$ such that $\mathcal{X}$ is closed under extensions and $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$. (See Section 1 for definitions and Section 2 for motivating examples from commutative algebra.) Given an object M in $\mathcal{A}$ with finite $\mathcal{X}$-projective dimension, Auslander and Buchweitz's theory of approximations [3] provides a "strict $\mathcal{W}\mathcal{X}$-resolution" of M. Such a resolution enjoys good enough lifting properties to make it unique up to homotopy equivalence and, as such, yields a well-defined relative cohomology functor $\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,-)$ for each integer n. The functors $\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(-,N)$ are defined dually.

These functors have been investigated by numerous authors, beginning with the fundamental work of Butler and Horrocks [6] and Eilenberg and Moore [8]. Our approach to the subject is based on a fusion of the techniques of Avramov and Martsinkovsky [5], Enochs and Jenda [10], and Holm [16].

The contents of this paper are summarized as follows. In Section 3 we present a brief study of the pertinent properties of strict resolutions. Sections 4 focuses on conditions guaranteeing that natural comparison maps are isomorphisms. In Section 5 we introduce a notion of relative perfection and establish a duality between certain classes of relatively perfect objects.

The main theorem of this paper is the following balance result, contained in Theorem 6.7. It showcases the benefit of our approach to studying these functors, as it simultaneously encompasses a result of Holm [16, (3.6)] and our

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own result [21, (5.7)]; see Corollary 6.11 and Remark 6.18.

Main Theorem. *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{W}$ and $\mathcal{V}$ be subcategories of $\mathcal{A}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions, $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, $\mathcal{V}$ is a projective generator for $\mathcal{Y}$, $\mathcal{W} \perp \mathcal{Y}$ and $\mathcal{X} \perp \mathcal{V}$. Assume further $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(T, \mathcal{V}) = 0 = \operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, U)$ for all objects T and U with $\mathcal{W}\text{-}\operatorname{pd}(T) < \infty$ and $\mathcal{V}\text{-}\operatorname{id}(U) < \infty$. If M and N are objects of $\mathcal{A}$ such that $\mathcal{X}\text{-}\operatorname{pd}(M) < \infty$ and $\mathcal{Y}\text{-}\operatorname{id}(N) < \infty$, then there are isomorphisms $\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M, N) \cong \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M, N)$ for all $n \in \mathbb{Z}$.*

1. Categories and resolutions

We begin with notation and terminology for use throughout this paper.

Definition 1.1. Throughout this work $\mathcal{A}$ is an abelian category. We use the term "subcategory" to mean a "full, additive, and essential (closed under isomorphisms) subcategory." Write $\mathcal{P} = \mathcal{P}(\mathcal{A})$ and $\mathcal{I} = \mathcal{I}(\mathcal{A})$ for the subcategories of projective and injective objects in $\mathcal{A}$, respectively.

We fix subcategories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{W}$, and $\mathcal{V}$ of $\mathcal{A}$ such that $\mathcal{W}$ is a subcategory of $\mathcal{X}$ and $\mathcal{V}$ is a subcategory of $\mathcal{Y}$. For an object $M \in \mathcal{A}$, write $M \perp \mathcal{Y}$ (resp., $\mathcal{X} \perp M$) if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(M, Y) = 0$ for each object $Y \in \mathcal{Y}$ (resp., if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(X, M) = 0$ for each object $X \in \mathcal{X}$). Write $\mathcal{X} \perp \mathcal{Y}$ if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(X, \mathcal{Y}) = 0$ for each object $X \in \mathcal{X}$. We say that $\mathcal{W}$ is a *cogenerator* for $\mathcal{X}$ if, for each object $X \in \mathcal{X}$, there exists an exact sequence

$$0 \to X \to W \to X' \to 0$$

with $W \in \mathcal{W}$ and $X' \in \mathcal{X}$. The subcategory $\mathcal{W}$ is an *injective cogenerator* for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{W}$. The terms *generator* and *projective generator* are defined dually.

Definition 1.2. An $\mathcal{A}$-*complex* is a sequence of homomorphisms in $\mathcal{A}$

$$M = \cdots \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots$$

such that $\partial^M_n \partial^M_{n+1} = 0$ for each integer n; the nth *homology object* of M is $\operatorname{H}_n(M) = \operatorname{Ker}(\partial^M_n)/\operatorname{Im}(\partial^M_{n+1})$. We frequently identify objects in $\mathcal{A}$ with complexes concentrated in degree 0. For each integer i, the ith *suspension* (or *shift*) of a complex M, denoted $\Sigma^i M$, is the complex with $(\Sigma^i M)_n = M_{n-i}$ and $\partial^{\Sigma^i M}_n = (-1)^i \partial^M_{n-i}$. The notation ΣX is short for $\Sigma^1 X$.

A complex M is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-*exact* if the complex $\operatorname{Hom}_{\mathcal{A}}(X, M)$ is exact for each object X in $\mathcal{X}$. The term $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{X})$-*exact* is defined dually.

Definition 1.3. Let M, N be $\mathcal{A}$-complexes. The complex $\operatorname{Hom}_{\mathcal{A}}(M, N)$ is the complex of abelian groups with $\operatorname{Hom}_{\mathcal{A}}(M, N)_n = \prod_p \operatorname{Hom}_{\mathcal{A}}(M_p, N_{p+n})$ with $\partial^{\operatorname{Hom}_{\mathcal{A}}(M,N)}_n$ given by $\alpha = \{\alpha_p\} \mapsto \{\partial^N_{p+n}\alpha_p - (-1)^n \alpha_{n-1}\partial^M_p\}$. A *morphism* $M \to N$ is an element of $\operatorname{Ker}(\partial^{\operatorname{Hom}_{\mathcal{A}}(M,N)}_0)$, and a morphism is *null-homotopic* if it is in $\operatorname{Im}(\partial^{\operatorname{Hom}_{\mathcal{A}}(M,N)}_1)$. Two morphisms $\alpha, \alpha' \colon M \to N$ are *homotopic* if $\alpha - \alpha'$ is null-homotopic. The morphism α is a *homotopy equivalence* if there is a morphism $\beta\colon N \to M$ such that $\beta\alpha$ is homotopic to id_M and $\alpha\beta$ is homotopic to id_N.

A morphism $\alpha\colon M \to N$ induces homomorphisms $\mathrm{H}_n(\alpha)\colon \mathrm{H}_n(M) \to \mathrm{H}_n(N)$, and α is a *quasiisomorphism* if each $\mathrm{H}_n(\alpha)$ is bijective. The *mapping cone* of α is the complex $\mathrm{Cone}(\alpha)$ defined as $\mathrm{Cone}(\alpha)_n = N_n \oplus M_{n-1}$ and $\partial_n^{\mathrm{Cone}(\alpha)} = \left(\begin{smallmatrix} \partial_n^N & \alpha_{n-1} \\ 0 & -\partial_{n-1}^M \end{smallmatrix}\right)$. The morphism α is a quasiisomorphism if and only if $\mathrm{Cone}(\alpha)$ is exact.

Definition 1.4. A complex X is *bounded* if $X_n = 0$ for $|n| \gg 0$. When $X_{-n} = 0 = \mathrm{H}_n(X)$ for all $n > 0$, the natural morphism $X \to \mathrm{H}_0(X) \cong M$ is a quasiisomorphism. In this event, the morphism $X \to M$ is an *$\mathcal{X}$-resolution* of M if each X_n is in $\mathcal{X}$, and the exact sequence

$$X^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to M \to 0$$

is the *augmented $\mathcal{X}$-resolution* of M associated to X. We write "projective resolution" in lieu of "$\mathcal{P}$-resolution". The *$\mathcal{X}$-projective dimension* of M is

$$\mathcal{X}\text{-}\mathrm{pd}(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

The objects of $\mathcal{X}$-projective dimension 0 are exactly the objects of $\mathcal{X}$. We let $\operatorname{res}\widehat{\mathcal{X}}$ denote the subcategory of objects M with $\mathcal{X}\text{-}\mathrm{pd}(M) < \infty$. One checks easily that $\operatorname{res}\widehat{\mathcal{X}}$ is additive and contains $\mathcal{X}$.

The terms *$\mathcal{Y}$-coresolution* and *$\mathcal{Y}$-injective dimension* are defined dually. The *augmented $\mathcal{Y}$-coresolution* associated to a $\mathcal{Y}$-coresolution Y is denoted ${}^+Y$, and the $\mathcal{Y}$-injective dimension of M is denoted $\mathcal{Y}\text{-}\mathrm{id}(M)$. The subcategory of R-modules N with $\mathcal{Y}\text{-}\mathrm{id}(N) < \infty$ is $\operatorname{cores}\widehat{\mathcal{Y}}$; it is additive and contains $\mathcal{Y}$.

Definition 1.5. An $\mathcal{X}$-resolution X is *proper* if the augmented resolution X^+ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact. The subcategory of objects admitting a proper $\mathcal{X}$-resolution is denoted $\operatorname{res}\widetilde{\mathcal{X}}$. One checks readily that $\operatorname{res}\widetilde{\mathcal{X}}$ is additive and contains $\mathcal{X}$. Projective resolutions are $\mathcal{P}$-proper, and so $\mathcal{A}$ has enough projectives if and only if $\operatorname{res}\widetilde{\mathcal{P}} = \mathcal{A}$.

Proper coresolutions are defined dually, and we let $\operatorname{cores}\widetilde{\mathcal{Y}}$ denote the subcategory of objects of $\mathcal{A}$ admitting a proper $\mathcal{Y}$-coresolution. Again, $\operatorname{cores}\widetilde{\mathcal{Y}}$ is additive and contains $\mathcal{Y}$ as a subcategory. Injective coresolutions are always $\mathcal{I}$-proper, and so $\mathcal{A}$ has enough injectives if and only if $\operatorname{cores}\widetilde{\mathcal{I}} = \mathcal{A}$.

The next lemmata are standard or have standard proofs: for 1.6 see [3, pf. of (2.3)]; for 1.7 see [3, pf. of (2.1)]; for 1.8 argue as in [5, (4.3)] or [10, pf. of (8.1.3)]; and for the "Horseshoe Lemma" 1.9 see [5, (4.5)] or [10, pf. of (8.2.1)].

Lemma 1.6. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathcal{A}$.*

(a) *If $M_3 \perp \mathcal{X}$, then $M_1 \perp \mathcal{X}$ if and only if $M_2 \perp \mathcal{X}$. If $M_1 \perp \mathcal{X}$ and $M_2 \perp \mathcal{X}$, then $M_3 \perp \mathcal{X}$ if and only if the given sequence is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$ exact.*

(b) *If $\mathcal{X} \perp M_1$, then $\mathcal{X} \perp M_2$ if and only if $\mathcal{X} \perp M_3$. If $\mathcal{X} \perp M_2$ and $\mathcal{X} \perp M_3$, then $\mathcal{X} \perp M_1$ if and only if the given sequence is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X},-)$ exact.*

Lemma 1.7. *If $\mathcal{X} \perp \mathcal{Y}$, then $\mathcal{X} \perp \operatorname{res}\widehat{\mathcal{Y}}$ and $\operatorname{cores}\widehat{\mathcal{X}} \perp \mathcal{Y}$.*

Lemma 1.8. *Let M, M', N, N' be objects in $\mathcal{A}$.*

(a) *Assume that M admits a proper $\mathcal{W}$-resolution $\gamma\colon W \to M$ and M' admits a proper $\mathcal{X}$-resolution $\gamma'\colon X' \to M'$. For each homomorphism $f\colon M \to M'$ there exists a morphism $\overline{f}\colon W \to X'$ unique up to homotopy such that $\gamma'\overline{f} = f\gamma$. If f is an isomorphism, then $\overline{f}$ is a quasiisomorphism. If f is an isomorphism and $\mathcal{X} = \mathcal{W}$, then $\overline{f}$ is a homotopy equivalence.*

(b) *Assume that M admits a projective resolution $\gamma\colon P \to M$ and M' admits a proper $\mathcal{X}$-resolution $\gamma'\colon X' \to M'$. For each homomorphism $f\colon M \to M'$ there exists a morphism $\widetilde{f}\colon P \to X'$ unique up to homotopy such that $\gamma'\widetilde{f} = f\gamma$. If f is an isomorphism, then $\widetilde{f}$ is a quasiisomorphism.*

(c) *Assume that N admits a proper $\mathcal{Y}$-coresolution $\delta\colon N \to Y$ and N' admits a proper $\mathcal{V}$-coresolution $\delta'\colon N' \to V'$. For each homomorphism $g\colon N \to N'$ there exists a morphism $\overline{g}\colon Y \to V'$ unique up to homotopy such that $\overline{g}\delta = \delta' g$. If g is an isomorphism, then $\overline{g}$ is a quasiisomorphism. If g is an isomorphism and $\mathcal{V} = \mathcal{Y}$, then $\overline{g}$ is a homotopy equivalence.*

(d) *Assume that N admits a proper $\mathcal{Y}$-coresolution $\delta\colon N \to Y$ and N' admits an injective resolution $\delta'\colon N' \to I'$. For each homomorphism $g\colon N \to N'$ there exists a morphism $\widetilde{g}\colon Y \to I'$ unique up to homotopy such that $\widetilde{g}\delta = \delta' g$. If g is an isomorphism, then $\widetilde{g}$ is a quasiisomorphism.*

Lemma 1.9. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\mathcal{A}$.*

(a) *Assume that M' and M'' admit proper $\mathcal{X}$-resolutions $\gamma'\colon X' \to M'$ and $\gamma''\colon X'' \to M''$ and that the given sequence is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact. Then M admits a proper $\mathcal{X}$-resolution $\gamma\colon X \to M$ such that there exists a commutative diagram whose top row is degreewise split exact.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}
$$

(b) *Assume that M' and M'' admit proper $\mathcal{Y}$-coresolutions $\delta'\colon M' \to Y'$ and $\delta''\colon M'' \to Y''$ and that the given sequence is $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{Y})$-exact. Then M admits a proper $\mathcal{Y}$-coresolution $\delta\colon M \to Y$ such that there exists a commutative diagram whose bottom row is degreewise split exact.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta''} & & \\
0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0
\end{array}
$$

The last result of this section is for Corollary 6.9; it follows from [22, (2.3)].

Lemma 1.10. *For each integer $n \geqslant 0$, let $\mathcal{X}_n$ and $\mathcal{Y}_n$ be subcategories of $\mathcal{A}$ such that $\mathcal{X}_n$ and $\mathcal{Y}_n$ are closed under extensions when $n \geqslant 2$.*

(a) *If $\mathcal{X}_n$ is a cogenerator for $\mathcal{X}_{n+1}$ for each $n \geqslant 0$ and $\mathcal{X}_n \perp \mathcal{X}_0$ for each $n \geqslant 1$, then $\mathcal{X}_n$ is an injective cogenerator for $\mathcal{X}_{n+j}$ for each $n, j \geqslant 0$.*

(b) *If $\mathcal{Y}_n$ is a generator for $\mathcal{Y}_{n+1}$ for each $n \geqslant 0$ and $\mathcal{Y}_0 \perp \mathcal{Y}_n$ for each $n \geqslant 1$, then $\mathcal{Y}_n$ is a projective generator for $\mathcal{Y}_{n+j}$ for each $n, j \geqslant 0$.*

2. Categories of interest

Much of the motivation for this work comes from module categories. In reading this paper, the reader may find it helpful to keep in mind the examples of this section, wherein R is a commutative ring. We return to these examples explicitly in Sections 5 and 6.

Definition 2.1. Let $\mathcal{M}(R)$ denote the category of R-modules. For simplicity, we write $\mathcal{P}(R) = \mathcal{P}(\mathcal{M}(R))$ and $\mathcal{I}(R) = \mathcal{I}(\mathcal{M}(R))$. Also set $\mathcal{A}b = \mathcal{M}(\mathbb{Z})$, the category of abelian groups. If $\mathcal{X}(R)$ is a subcategory of $\mathcal{M}(R)$, then $\mathcal{X}^f(R)$ is the subcategory of finitely generated modules in $\mathcal{X}(R)$.

The study of semidualizing modules was initiated independently (with different names) by Foxby [11], Golod [15], and Vasconcelos [24].

Definition 2.2. An R-module C is *semidualizing* if it satisfies:

(1) C admits a (possibly unbounded) resolution by finite rank free R-modules,

(2) the natural homothety map $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism, and

(3) $\operatorname{Ext}_R^{\geqslant 1}(C, C) = 0$.

A finitely generated projective R-module of rank 1 is semidualizing. If R is Cohen-Macaulay, then D is *dualizing* if it is semidualizing and $\operatorname{id}_R(D)$ is finite.

Based on work of Enochs and Jenda [9], the following notions were introduced and studied in this generality by Holm and Jørgensen [18] and White [25].

Definition 2.3. Let C be a semidualizing R-module. An R-module is *C-projective* (resp., *C-injective*) if it is isomorphic to $P \otimes_R C$ for some projective R-module P (resp., $\operatorname{Hom}_R(C, I)$ for some injective R-module I). The categories of C-projective and C-injective R-modules are denoted $\mathcal{P}_C(R)$ and $\mathcal{I}_C(R)$, respectively.

A *complete $\mathcal{P}\mathcal{P}_C$-resolution* is a complex X of R-modules such that:

(1) X is exact and $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact, and

(2) X_n is projective when $n \geqslant 0$ and X_n is C-projective when $n < 0$.

An R-module G is *G_C-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $G \cong \operatorname{Coker}(\partial_1^X)$, in which case X is a *complete $\mathcal{PP}_C$-resolution of G*. We let $\mathcal{GP}_C(R)$ denote the subcategory of G_C-projective R-modules.

The terms *complete $\mathcal{I}_C\mathcal{I}$-coresolution* and *G_C-injective* are defined dually, and $\mathcal{GI}_C(R)$ is the subcategory of G_C-injective R-modules.

Fact 2.4. Let C be a semidualizing R-module. There is a containment $\mathcal{P}(R) \cup \mathcal{P}_C(R) \subseteq \mathcal{GP}_C(R)$, and $\mathcal{P}_C(R)$ is an injective cogenerator for $\mathcal{GP}_C(R)$ by [25, (2.2),(2.6),(2.9)]. Dually, one has $\mathcal{I}(R) \cup \mathcal{I}_C(R) \subseteq \mathcal{GI}_C(R)$, and $\mathcal{I}_C(R)$ is a projective generator for $\mathcal{GI}_C(R)$.

The next definition was first introduced by Auslander and Bridger [1], [2] in the case $C = R$, and in this generality by Golod [15] and Vasconcelos [24].

Definition 2.5. Assume that R is noetherian, and let C be a semidualizing R-module. A finitely generated R-module H is *totally C-reflexive* if

(1) $\operatorname{Ext}_R^{\geqslant 1}(H, C) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(H, C), C)$, and

(2) the natural biduality map $H \to \operatorname{Hom}_R(\operatorname{Hom}_R(H, C), C)$ is an isomorphism.

Let $\mathcal{G}_C(R)$ denote the subcategory of totally C-reflexive R-modules.

Fact 2.6. Assume that R is noetherian and let C be a semidualizing R-module. Then $\mathcal{G}_C(R) = \mathcal{GP}_C^f(R)$ by [25, (4.4)], and so $\mathcal{P}^f(R) \cup \mathcal{P}_C^f(R) \subseteq \mathcal{G}_C(R)$. Also, $\mathcal{P}_C^f(R)$ is an injective cogenerator for $\mathcal{G}_C(R)$ by [25, (2.9),(4.3),(4.4)].

Over a noetherian ring, the next categories were introduced by Avramov and Foxby [4] when C is dualizing, and by Christensen [7] for arbitrary C.[*1] In the non-noetherian setting, these definitions are from [19], [25].

Definition 2.7. Let C be a semidualizing R-module. The *Auslander class* of C is the subcategory $\mathcal{A}_C(R)$ of R-modules M such that

(1) $\operatorname{Tor}^R_{\geqslant 1}(C, M) = 0 = \operatorname{Ext}_R^{\geqslant 1}(C, C \otimes_R M)$, and

(2) The natural map $M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* of C is the subcategory $\mathcal{B}_C(R)$ of R-modules N such that

(1) $\operatorname{Ext}_R^{\geqslant 1}(C, N) = 0 = \operatorname{Tor}^R_{\geqslant 1}(C, \operatorname{Hom}_R(C, N))$, and

(2) The natural evaluation map $C \otimes_R \operatorname{Hom}_R(C, N) \to N$ is an isomorphism.

Fact 2.8. Let C be a semidualizing R-module. If any two R-modules in a short exact sequence are in $\mathcal{A}_C(R)$, respectively $\mathcal{B}_C(R)$, then so is the third; see [19, Cor. 6.3]. There are containments

$$\operatorname{res}\widehat{\mathcal{P}(R)} \cup \operatorname{cores}\widehat{\mathcal{I}_C(R)} \subseteq \mathcal{A}_C(R) \subseteq \operatorname{cores}\widetilde{\mathcal{I}_C} \qquad \text{and}$$
$$\operatorname{res}\widehat{\mathcal{P}_C(R)} \cup \operatorname{cores}\widehat{\mathcal{I}(R)} \subseteq \mathcal{B}_C(R) \subseteq \operatorname{res}\widetilde{\mathcal{P}_C(R)}$$

by [19, Cors. 6.1, 6.2] and [23, (2.4)].

[*1] Note that these works (and others) use the notation $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ for certain categories of complexes, while our categories consist precisely of the modules in these categories by [7, (4.10)].

3. Strict and proper resolutions

This section focuses on the existence of certain proper resolutions which, following [5], we call “strict”. Our treatment focuses on the use of “approximations” (special cases of the “special precovers” of [10]) and blends the approaches of [3], [5], and [10].

Definition 3.1. Fix an object M in $\mathcal{A}$. A *bounded strict* $\mathcal{W}\mathcal{X}$*-resolution of* M is a bounded $\mathcal{X}$-resolution $X \xrightarrow{\simeq} M$ such that X_n is an object in $\mathcal{W}$ for each $n \geqslant 1$. An exact sequence in $\mathcal{A}$

$$0 \to K \to X_0 \to M \to 0$$

such that $K \in \operatorname{res}\widehat{\mathcal{W}}$ and $X_0 \in \mathcal{X}$ is called an $\mathcal{W}\mathcal{X}$*-approximation of* M. The term $\mathcal{W}\mathcal{X}$*-hull of* M is used for an exact sequence in $\mathcal{A}$

$$0 \to M \to K' \to X' \to 0$$

such that $K' \in \operatorname{res}\widehat{\mathcal{W}}$ and $X' \in \mathcal{X}$. The terms *bounded strict* $\mathcal{Y}\mathcal{V}$*-coresolution*, $\mathcal{Y}\mathcal{V}$*-coapproximation* and $\mathcal{Y}\mathcal{V}$*-cohull* are defined dually.

The first result of this section outlines the properness properties of certain (co)resolutions and (co)approximations.

Lemma 3.2. *Assume* $\mathcal{X} \perp \mathcal{W}$ *and* $\mathcal{V} \perp \mathcal{Y}$.

(a) *Bounded* $\mathcal{W}$*-resolutions are* $\mathcal{X}$*-proper and hence* $\mathcal{W}$*-proper.*

(b) *If* $\mathcal{W}$ *is an injective cogenerator for* $\mathcal{X}$*, then bounded strict* $\mathcal{W}\mathcal{X}$*-resolutions are* $\mathcal{X}$*-proper and* $\mathcal{W}\mathcal{X}$*-approximations are* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$*-exact.*

(c) *Bounded* $\mathcal{V}$*-coresolutions are* $\mathcal{Y}$*-proper and hence* $\mathcal{V}$*-proper.*

(d) *If* $\mathcal{V}$ *is a projective generator for* $\mathcal{Y}$*, then bounded strict* $\mathcal{Y}\mathcal{V}$*-coresolutions are* $\mathcal{Y}$*-proper and* $\mathcal{Y}\mathcal{V}$*-coapproximations are* $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{Y})$*-exact.*

Proof. We prove parts (a) and (b); the others are proved dually.

(a) Let M be an object in $\mathcal{A}$ admitting a bounded $\mathcal{W}$-resolution $W \to M$. We need to show that $\operatorname{Hom}_{\mathcal{A}}(X, W^+)$ is exact for each object X in $\mathcal{X}$. Set $M_n = \operatorname{Coker}(\partial^W_{n+2})$ and, when $n \geqslant 0$, consider the associated exact sequence

$$0 \to M_n \to W_n \to M_{n-1} \to 0.$$

The object M_n is in $\operatorname{res}\widehat{\mathcal{W}}$ for each n. Lemma 1.7 implies $\mathcal{X} \perp \operatorname{res}\widehat{\mathcal{W}}$, and so the displayed sequence is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact by Lemma 1.6(b). It follows that W^+ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact as well, that is, the resolution is $\mathcal{X}$-proper.

(b) Let $X \to M$ be a bounded strict $\mathcal{W}\mathcal{X}$-resolution such that $X_i = 0$ for each $i > n$, and set $K = \operatorname{Im}(\partial^X_1)$. The next exact sequence is a bounded $\mathcal{W}$-resolution

$$0 \to X_n \to \cdots \to X_1 \to K \to 0 \tag{3.1}$$

and so part (a) implies that it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact. The following sequence

$$0 \to K \to X_0 \to M \to 0 \tag{3.2}$$

is a $\mathcal{W}\mathcal{X}$-approximation. We show that $\mathcal{W}\mathcal{X}$-approximations are $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact; we then conclude that X is $\mathcal{X}$-proper by splicing (3.1) and (3.2).

Consider a $\mathcal{W}\mathcal{X}$-approximation as in (3.2). Using Lemma 1.7, the assumption $\mathcal{X} \perp \mathcal{W}$ implies $\mathcal{X} \perp K$. Thus, for each $X' \in \mathcal{X}$ the long exact sequence in $\mathrm{Ext}_{\mathcal{A}}(X',-)$ associated to (3.2) implies that (3.2) is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact. □

The next two lemmata provide useful conditions guaranteeing the existence of proper (co)resolutions. Lemma 3.4 is for use in Proposition 4.10.

Lemma 3.3. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions, $\mathcal{W}$ is a cogenerator for $\mathcal{X}$, and $\mathcal{V}$ is a generator for $\mathcal{Y}$. Let M and N be objects in $\mathcal{A}$.*

(a) *If $\mathcal{X}\text{-}\mathrm{pd}(M) < \infty$, then M has a $\mathcal{W}\mathcal{X}$-approximation, a $\mathcal{W}\mathcal{X}$-hull, and a bounded strict $\mathcal{W}\mathcal{X}$-resolution $X \xrightarrow{\simeq} M$ such that $X_i = 0$ for $i > \mathcal{X}\text{-}\mathrm{pd}(M)$.*

(b) *If $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, then $\mathrm{res}\,\widehat{\mathcal{X}}$ is a subcategory of $\mathrm{res}\,\widetilde{\mathcal{X}}$.*

(c) *If $\mathcal{Y}\text{-}\mathrm{id}(N) < \infty$, then N has a $\mathcal{Y}\mathcal{V}$-coapproximation, a $\mathcal{Y}\mathcal{V}$-cohull, and a bounded strict $\mathcal{Y}\mathcal{V}$-coresolution $N \xrightarrow{\simeq} Y$ such that $Y_{-i} = 0$ for $i > \mathcal{Y}\text{-}\mathrm{id}(N)$.*

(d) *If $\mathcal{V}$ is a projective generator for $\mathcal{Y}$, then $\mathrm{cores}\,\widehat{\mathcal{Y}}$ is a subcategory of $\mathrm{cores}\,\widetilde{\mathcal{Y}}$.*

Proof. Parts (a) and (c) follow as in [3, (1.1)]. Parts (b) and (d) follow from (a) and (c) using Lemma 3.2(b) and (d). □

Lemma 3.4. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions, $\mathcal{W}$ is a cogenerator for $\mathcal{X}$, and $\mathcal{V}$ is a generator for $\mathcal{Y}$.*

(a) *If $\mathcal{X}$ is a subcategory of $\mathrm{res}\,\widetilde{\mathcal{W}}$, then $\mathrm{res}\,\widehat{\mathcal{X}}$ is a subcategory of $\mathrm{res}\,\widetilde{\mathcal{W}}$.*

(b) *If $\mathcal{Y}$ is a subcategory of $\mathrm{cores}\,\widetilde{\mathcal{V}}$, then $\mathrm{cores}\,\widehat{\mathcal{Y}}$ is a subcategory of $\mathrm{cores}\,\widetilde{\mathcal{V}}$.*

Proof. We prove part (a); the proof of part (b) is dual. Let M be an object in $\mathrm{res}\,\widehat{\mathcal{X}}$. By Lemma 3.3(a), the object M admits a $\mathcal{W}\mathcal{X}$-approximation

$$0 \to K \to X \to M \to 0. \tag{3.3}$$

Since $\mathcal{X}$ is a subcategory of $\mathrm{res}\,\widetilde{\mathcal{W}}$, the object X admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\simeq} X$. Set $X' = \mathrm{Im}(\partial_1^W)$. Notice that the object X' is in $\mathrm{res}\,\widetilde{\mathcal{W}}$ and the following natural exact sequence is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact

$$0 \to X' \to W_0 \xrightarrow{\tau} X \to 0. \tag{3.4}$$

In the following pullback diagram, each row and column is exact, the bottom row is (3.3), and the middle column is (3.4).

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X' & \xrightarrow{\cong} & X' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U & \longrightarrow & W_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow & \ulcorner & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{3.5}
$$

We show that U is in $\operatorname{res}\widetilde{\mathcal{W}}$ and that the middle row of (3.5) is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact. It is then straightforward to see that a proper $\mathcal{W}$-resolution of M can be obtained by splicing a a proper $\mathcal{W}$-resolution of U with the middle row of (3.5).

Let W' be an object in $\mathcal{W}$. The assumption $\mathcal{X} \perp \mathcal{W}$ implies $\mathcal{W} \perp \mathcal{W}$ and so $\operatorname{Ext}^1_{\mathcal{A}}(W', W_0) = 0$. The long exact sequence in $\operatorname{Ext}_{\mathcal{A}}(W',-)$ associated to the middle column of (3.5) includes the next exact sequence

$$
\operatorname{Hom}_{\mathcal{A}}(W', W_0) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(W',\tau)} \operatorname{Hom}_{\mathcal{A}}(W', X) \to \operatorname{Ext}^1_{\mathcal{A}}(W', X') \to 0.
$$

The middle column of (3.5) is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact, so the map $\operatorname{Hom}_{\mathcal{A}}(W',\tau)$ is surjective, and it follows that $\operatorname{Ext}^1_{\mathcal{A}}(W', X') = 0$. Lemma 1.6(b) implies that the leftmost column of (3.5) is $\operatorname{Hom}_{\mathcal{A}}(W',-)$-exact. Since W' is an arbitrary object of $\mathcal{W}$, this column is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact. The object K is in $\operatorname{res}\widetilde{\mathcal{W}}$ by Lemma 3.2(a). Since X' is also an object in $\operatorname{res}\widetilde{\mathcal{W}}$, we may apply Lemma 1.9(a) to the leftmost column of (3.5) to conclude that U is in $\operatorname{res}\widetilde{\mathcal{W}}$.

To conclude, we show that the middle row of (3.5) is $\operatorname{Hom}_{\mathcal{A}}(W',-)$-exact, that is, that $\operatorname{Hom}_{\mathcal{A}}(W',\pi)$ is surjective. Applying $\operatorname{Hom}_{\mathcal{A}}(W',-)$ to the middle and lower rows of (3.5) yields the next commutative diagram with exact rows.

$$
\begin{array}{ccccccc}
\operatorname{Hom}_{\mathcal{A}}(W', U) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(W', W_0) & \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(W',\pi)} & \operatorname{Hom}_{\mathcal{A}}(W', M) & & \\
\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{A}}(W',\tau)} & & \downarrow{\scriptstyle \cong} & & \\
\operatorname{Hom}_{\mathcal{A}}(W', K) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(W', X) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(W', M) & \longrightarrow & 0
\end{array}
$$

Recalling that $\operatorname{Hom}_{\mathcal{A}}(W',\tau)$ is surjective, chase this last diagram to conclude that $\operatorname{Hom}_{\mathcal{A}}(W',\pi)$ is also surjective. □

4. Relative cohomology

This section contains the foundations of our relative cohomology theories based on the context of Section 3.

Definition 4.1. Let M, M', N, N' be objects in $\mathcal{A}$ with homomorphisms $f\colon M \to M'$ and $g\colon N \to N'$. Assume that M admits a proper $\mathcal{X}$-resolution $\gamma\colon X \to M$, and define the *nth relative $\mathcal{XA}$ cohomology group* as

$$\operatorname{Ext}^n_{\mathcal{XA}}(M, N) = \operatorname{H}_{-n}(\operatorname{Hom}_{\mathcal{A}}(X, N))$$

for each integer n. If M' also admits a proper $\mathcal{X}$-resolution $\gamma'\colon X' \to M'$, let $\overline{f}\colon X \to X'$ be a morphism such that $\gamma'\overline{f} = f\gamma$, as in Lemma 1.8(a), and define

$$\operatorname{Ext}^n_{\mathcal{XA}}(f, N) = \operatorname{H}_{-n}(\operatorname{Hom}_{\mathcal{A}}(\overline{f}, N))\colon \operatorname{Ext}^n_{\mathcal{XA}}(M', N) \to \operatorname{Ext}^n_{\mathcal{XA}}(M, N)$$
$$\operatorname{Ext}^n_{\mathcal{XA}}(M, g) = \operatorname{H}_{-n}(\operatorname{Hom}_{\mathcal{A}}(X, g))\colon \operatorname{Ext}^n_{\mathcal{XA}}(M, N) \to \operatorname{Ext}^n_{\mathcal{XA}}(M, N').$$

We write $\operatorname{Ext}^{\geqslant 1}_{\mathcal{XA}}(M, \mathcal{Y}) = 0$ if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{XA}}(M, Y) = 0$ for each object $Y \in \mathcal{Y}$. When $\mathcal{X} \subseteq \operatorname{res}\widetilde{\mathcal{W}}$, we write $\operatorname{Ext}^{\geqslant 1}_{\mathcal{WA}}(\mathcal{X}, \mathcal{Y}) = 0$ if $\operatorname{Ext}^{\geqslant 1}_{\mathcal{WA}}(X, \mathcal{Y}) = 0$ for each $X \in \mathcal{X}$.

The *nth relative $\mathcal{AY}$-cohomology* $\operatorname{Ext}^n_{\mathcal{AY}}(-,-)$ is defined dually.

Remark 4.2. Definition 4.1 describes well-defined bifunctors

$$\operatorname{Ext}^n_{\mathcal{XA}}(-,-)\colon \operatorname{res}\widetilde{\mathcal{X}} \times \mathcal{A} \to \mathcal{A}b \qquad \operatorname{Ext}^n_{\mathcal{AY}}(-,-)\colon \mathcal{A} \times \operatorname{cores}\widetilde{\mathcal{Y}} \to \mathcal{A}b$$

by Lemma 1.8, and one checks the following natural equivalences readily.

$$\operatorname{Ext}^{\geqslant 1}_{\mathcal{XA}}(\mathcal{X}, -) = 0 = \operatorname{Ext}^{\geqslant 1}_{\mathcal{AY}}(-, \mathcal{Y})$$
$$\operatorname{Ext}^0_{\mathcal{XA}}(-,-) \cong \operatorname{Hom}_{\mathcal{A}}(-,-)|_{\operatorname{res}\widetilde{\mathcal{X}} \times \mathcal{A}}$$
$$\operatorname{Ext}^n_{\mathcal{PA}}(-,-) \cong \operatorname{Ext}^n_{\mathcal{A}}(-,-)|_{\operatorname{res}\widetilde{\mathcal{P}} \times \mathcal{A}}$$
$$\operatorname{Ext}^0_{\mathcal{AY}}(-,-) \cong \operatorname{Hom}_{\mathcal{A}}(-,-)|_{\mathcal{A} \times \operatorname{cores}\widetilde{\mathcal{Y}}}$$
$$\operatorname{Ext}^n_{\mathcal{AI}}(-,-) \cong \operatorname{Ext}^n_{\mathcal{A}}(-,-)|_{\mathcal{A} \times \operatorname{cores}\widetilde{\mathcal{I}}}$$

Lemma 1.9 yields the next long exact sequences as in [10, (8.2.3),(8.2.5)].

Lemma 4.3. *Let M and N be objects in $\mathcal{A}$, and consider an exact sequence in $\mathcal{A}$*

$$\mathbf{L} = \quad 0 \to L' \xrightarrow{f'} L \xrightarrow{f} L'' \to 0.$$

(a) *Assume that the sequence $\mathbf{L}$ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact. If the object M is in $\operatorname{res}\widetilde{\mathcal{X}}$, then $\mathbf{L}$ induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{\mathcal{XA}}(M, L') \xrightarrow{\operatorname{Ext}^n_{\mathcal{XA}}(M,f')} \operatorname{Ext}^n_{\mathcal{XA}}(M, L) \xrightarrow{\operatorname{Ext}^n_{\mathcal{XA}}(M,f)}$$
$$\operatorname{Ext}^n_{\mathcal{XA}}(M, L'') \xrightarrow{\eth^n_{\mathcal{XA}}(M,\mathbf{L})} \operatorname{Ext}^{n+1}_{\mathcal{XA}}(M, L') \xrightarrow{\operatorname{Ext}^{n+1}_{\mathcal{XA}}(M,f')} \cdots$$

(b) *Assume that the sequence* $\mathbf{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact. If the objects* L', L, L'' *are in* $\operatorname{res}\widetilde{\mathcal{X}}$*, then* $\mathbf{L}$ *induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(L'',N) \xrightarrow{\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(f,N)} \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(L,N) \xrightarrow{\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(f',N)}$$
$$\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(L',N) \xrightarrow{\eth^n_{\mathcal{X}\mathcal{A}}(\mathbf{L},N)} \operatorname{Ext}^{n+1}_{\mathcal{X}\mathcal{A}}(L'',N) \xrightarrow{\operatorname{Ext}^{n+1}_{\mathcal{X}\mathcal{A}}(f,N)} \cdots$$

(c) *Assume that the sequence* $\mathbf{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{Y})$*-exact. If the object* N *is in* $\operatorname{cores}\widetilde{\mathcal{Y}}$*, then* $\mathbf{L}$ *induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(L'',N) \xrightarrow{\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(f,N)} \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(L,N) \xrightarrow{\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(f',N)}$$
$$\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(L',N) \xrightarrow{\eth^n_{\mathcal{A}\mathcal{Y}}(\mathbf{L},N)} \operatorname{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(L'',N) \xrightarrow{\operatorname{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(f,N)} \cdots$$

(d) *Assume that the sequence* $\mathbf{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{Y})$*-exact. If the objects* L', L, L'' *are in* $\operatorname{cores}\widetilde{\mathcal{Y}}$*, then* $\mathbf{L}$ *induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,L') \xrightarrow{\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,f')} \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,L) \xrightarrow{\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,f)}$$
$$\operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,L'') \xrightarrow{\eth^n_{\mathcal{A}\mathcal{Y}}(M,\mathbf{L})} \operatorname{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(M,A') \xrightarrow{\operatorname{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(M,f')} \cdots$$

To prove the next "dimension-shifting" lemma, use Lemma 4.3 with the vanishing from Remark 4.2; compare to [10, (8.2.4),(8.2.6)].

Lemma 4.4. *Let* M *and* N *be objects in* $\mathcal{A}$*, and consider an exact sequence in* $\mathcal{A}$

$$\mathbf{L} = \quad 0 \to L' \xrightarrow{f'} L \xrightarrow{f} L'' \to 0.$$

(a) *Assume that the sequence* $\mathbf{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact and that* M *is in* $\operatorname{res}\widetilde{\mathcal{X}}$*. If* $\operatorname{Ext}^{\geqslant 1}_{\mathcal{X}\mathcal{A}}(M,L) = 0$*, e.g., if* M *is in* $\mathcal{X}$*, then the following map is an isomorphism for each* $n \geqslant 1$

$$\eth^n_{\mathcal{X}\mathcal{A}}(M,\mathbf{L})\colon\ \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,L'') \xrightarrow{\cong} \operatorname{Ext}^{n+1}_{\mathcal{X}\mathcal{A}}(M,L').$$

(b) *Assume that the sequence* $\mathbf{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact and that* L, L', L'' *are in* $\operatorname{res}\widetilde{\mathcal{X}}$*. If* $\operatorname{Ext}^{\geqslant 1}_{\mathcal{X}\mathcal{A}}(L,N) = 0$*, e.g., if* L *is in* $\mathcal{X}$*, then the following map is an isomorphism for each* $n \geqslant 1$

$$\eth^n_{\mathcal{X}\mathcal{A}}(\mathbf{L},N)\colon\ \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(L',N) \xrightarrow{\cong} \operatorname{Ext}^{n+1}_{\mathcal{X}\mathcal{A}}(L'',N).$$

(c) *Assume that the sequence* $\mathbf{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{Y})$*-exact and that* N *is in* $\operatorname{cores}\widetilde{\mathcal{Y}}$*. If* $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{Y}}(L,N) = 0$*, e.g., if* N *is in* $\mathcal{Y}$*, then the following map is an isomorphism for each* $n \geqslant 1$

$$\eth^n_{\mathcal{A}\mathcal{Y}}(\mathbf{L},N)\colon\ \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(L',N) \xrightarrow{\cong} \operatorname{Ext}^{n+1}_{\mathcal{A}\mathcal{Y}}(L'',N).$$

(d) *Assume that the sequence* $\mathbf{L}$ *is* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{Y})$*-exact and that* L, L', L'' *are in* $\operatorname{cores}\widetilde{\mathcal{Y}}$. *If* $\operatorname{Ext}^{\geqslant 1}_{\mathcal{AY}}(M,L)=0$, *e.g., if* L *is in* $\mathcal{Y}$, *then the following map is an isomorphism for each* $n \geqslant 1$

$$\eth^n_{\mathcal{AY}}(M,\mathbf{L}) \quad : \operatorname{Ext}^n_{\mathcal{AY}}(M,L'') \xrightarrow{\cong} \operatorname{Ext}^{n+1}_{\mathcal{AY}}(M,L').$$

The next result is motivated by [5, (4.2.2.a)].

Proposition 4.5. *Let* M *and* N *be objects in* $\operatorname{res}\widetilde{\mathcal{X}}$ *and* $\operatorname{cores}\widetilde{\mathcal{Y}}$, *respectively, and let* n *be a nonnegative integer.*

(a) *Assume that* $\mathcal{X}$ *is closed under direct summands and* $\operatorname{Ext}^{n+1}_{\mathcal{XA}}(M,-)=0$. *If* $X \to M$ *is a proper* $\mathcal{X}$*-resolution, then* $\operatorname{Im}(\partial^X_n) \in \mathcal{X}$ *and* $\mathcal{X}\text{-}\operatorname{pd}(M) \leqslant n$.

(b) *Assume that one of the following conditions holds*:

(1) $\mathcal{X} \perp \mathcal{X}$, *or*

(2) $\mathcal{X}$ *is closed under extensions and* $\mathcal{W}$ *is an injective cogenerator for* $\mathcal{X}$.

Then $\operatorname{Ext}^n_{\mathcal{XA}}(M,-)=0$ *whenever* $n > \mathcal{X}\text{-}\operatorname{pd}(M)$.

(c) *Assume that* $\mathcal{Y}$ *is closed under direct summands and* $\operatorname{Ext}^{n+1}_{\mathcal{AY}}(-,N)=0$. *If* $N \to Y$ *is a proper* $\mathcal{Y}$*-coresolution, then* $\operatorname{Im}(\partial^Y_{-n}) \in \mathcal{Y}$ *and* $\mathcal{Y}\text{-}\operatorname{id}(N) \leqslant n$.

(d) *Assume that one of the following conditions holds*:

(1) $\mathcal{Y} \perp \mathcal{Y}$, *or*

(2) $\mathcal{Y}$ *is closed under extensions and* $\mathcal{V}$ *is a projective cogenerator for* $\mathcal{Y}$.

Then $\operatorname{Ext}^n_{\mathcal{AY}}(-,N)=0$ *whenever* $n > \mathcal{Y}\text{-}\operatorname{id}(N)$.

Proof. We prove parts (a) and (b); the proofs of (c) and (d) are dual.

(a) Let $X \to M$ be a proper $\mathcal{X}$-resolution, and set $M_j = \operatorname{Coker}(\partial^X_{j+2})$ for each j. Note that $M_j \in \operatorname{res}\widetilde{\mathcal{X}}$ and $M \cong M_{-1}$, and consider the exact sequences

$$(*_j) \qquad 0 \to M_j \to X_j \xrightarrow{\epsilon_j} M_{j-1} \to 0$$

when $j \geqslant 0$, which are $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact.

Assume first $\operatorname{Ext}^1_{\mathcal{XA}}(M,-)=0$. An application of Lemma 4.3(a) to the sequence $(*_0)$ yields the following exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(M,M_0) \to \operatorname{Hom}_{\mathcal{A}}(M,X_0) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(M,\epsilon_0)} \operatorname{Hom}_{\mathcal{A}}(M,M) \to 0.$$

Hence, there exists $\phi \in \operatorname{Hom}_{\mathcal{A}}(M,X_0)$ such that $\epsilon_0\phi = \operatorname{id}_M$. Thus M is a direct summand of X_0, and $M \in \mathcal{X}$ because $\mathcal{X}$ is closed under direct summands.

Now assume $\operatorname{Ext}^{n+1}_{\mathcal{XA}}(M,-)=0$. Apply Lemma 4.4(b) to each sequence $(*_j)$ inductively to conclude $\operatorname{Ext}^1_{\mathcal{XA}}(M_{n-1},-)=0$. The previous paragraph implies $\operatorname{Im}(\partial^X_n) = M_{n-1} \in \mathcal{X}$. The conclusion $\mathcal{X}\text{-}\operatorname{pd}(M) \leqslant n$ is now immediate.

(b) Assume without loss of generality that $p = \mathcal{X}\text{-}\operatorname{pd}(M)$ is finite. It suffices to show that M admits a proper $\mathcal{X}$-resolution $X \to M$ such that $X_n = 0$ when $n > p$. If condition (1) holds, then Lemma 3.2(a) implies that every $\mathcal{X}$-resolution $X \to M$ such that $X_n = 0$ for each $n > p$ is proper. If condition (2) holds, then Lemmas 3.2(b) and 3.3(a) yield the desired conclusion. $\square$

The rest of this section is devoted to the study of comparison maps.

Definition 4.6. Let M, N be objects in $\mathcal{A}$.

(a) When M admits a proper $\mathcal{W}$-resolution $\gamma\colon W \to M$ and a proper $\mathcal{X}$-resolution $\gamma'\colon X \to M$, let $\overline{\mathrm{id}_M}\colon W \to X$ be a quasiisomorphism such that $\gamma = \gamma'\overline{\mathrm{id}_M}$, as in Lemma 1.8(a), and set

$$\vartheta^n_{\mathcal{X}\mathcal{W}\mathcal{A}}(M,N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(\overline{\mathrm{id}_M}, N))\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,N) \to \operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M,N).$$

(b) When M admits a projective resolution $\gamma\colon P \to M$ and a proper $\mathcal{X}$-resolution $\gamma'\colon X \to M$, let $\widetilde{\mathrm{id}_M}\colon P \to X$ be a quasiisomorphism such that $\gamma = \gamma'\widetilde{\mathrm{id}_M}$, as in Lemma 1.8(b), and set

$$\varkappa^n_{\mathcal{X}\mathcal{A}}(M,N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(\widetilde{\mathrm{id}_M}, N))\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,N) \to \operatorname{Ext}^n_{\mathcal{A}}(M,N).$$

(c) When N admits a proper $\mathcal{Y}$-coresolution $\delta\colon N \to Y$ and a proper $\mathcal{V}$-coresolution $\delta'\colon N \to V$, let $\overline{\mathrm{id}_N}\colon Y \to V$ be a quasiisomorphism such that $\delta' = \overline{\mathrm{id}_N}\delta$, as in Lemma 1.8(c), and set

$$\vartheta^n_{\mathcal{A}\mathcal{Y}\mathcal{V}}(M,N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(M, \overline{\mathrm{id}_N}))\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,N) \to \operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}(M,N).$$

(d) When N admits a proper $\mathcal{Y}$-coresolution $\delta\colon N \to Y$ and an injective resolution $\delta'\colon N \to I$, let $\widetilde{\mathrm{id}_N}\colon Y \to I$ be a quasiisomorphism such that $\delta' = \widetilde{\mathrm{id}_N}\delta$, as in Lemma 1.8(d), and set

$$\varkappa^n_{\mathcal{A}\mathcal{Y}}(M,N) = \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(M, \widetilde{\mathrm{id}_N}))\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,N) \to \operatorname{Ext}^n_{\mathcal{A}}(M,N).$$

Remark 4.7. Lemma 1.8 shows that Definition 4.6 describes well-defined natural transformations that are independent of resolutions and liftings.

$$\vartheta^n_{\mathcal{X}\mathcal{W}\mathcal{A}}(-,-)\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(-,-)|_{(\operatorname{res}\widetilde{\mathcal{W}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{A}} \to \operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(-,-)|_{(\operatorname{res}\widetilde{\mathcal{W}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{A}}$$
$$\varkappa^n_{\mathcal{X}\mathcal{A}}(-,-)\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(-,-)|_{(\operatorname{res}\widetilde{\mathcal{P}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{A}} \to \operatorname{Ext}^n_{\mathcal{A}}(-,-)|_{(\operatorname{res}\widetilde{\mathcal{P}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{A}}$$
$$\vartheta^n_{\mathcal{A}\mathcal{Y}\mathcal{V}}(-,-)\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(-,-)|_{\mathcal{A}\times(\operatorname{cores}\widetilde{\mathcal{V}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})} \to \operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}(-,-)|_{\mathcal{A}\times(\operatorname{cores}\widetilde{\mathcal{V}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})}$$
$$\varkappa^n_{\mathcal{A}\mathcal{Y}}(-,-)\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(-,-)|_{\mathcal{A}\times(\operatorname{cores}\widetilde{\mathcal{I}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})} \to \operatorname{Ext}^n_{\mathcal{A}}(-,-)|_{\mathcal{A}\times(\operatorname{cores}\widetilde{\mathcal{I}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})}$$

The next result compares to [5, (4.2.3)].

Proposition 4.8. *Assume that $\mathcal{X} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{Y}$, and consider objects $M \in \operatorname{res}\widehat{\mathcal{W}}$ and $N \in \operatorname{cores}\widehat{\mathcal{V}}$.*

(a) *The following natural transormations are isomorphisms for each n*

$$\vartheta^n_{\mathcal{X}\mathcal{W}\mathcal{A}}(M,-)\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,-) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M,-).$$

(b) *The following natural transormations are isomorphisms for each n*

$$\vartheta^n_{\mathcal{A}\mathcal{Y}\mathcal{V}}(-,N)\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(-,N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}(-,N).$$

Proof. We prove part (a); the proof of (b) is dual.

Let $W \to M$ be a bounded $\mathcal{W}$-resolution. Lemma 3.2(a) implies that W is $\mathcal{X}$-proper and $\mathcal{W}$-proper, so $\operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M,-)$ and $\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,-)$ are defined. Further, in the notation of Definition 4.6(a), we can take $\overline{\mathrm{id}_M} = \mathrm{id}_W$, and so there are equalities

$$\vartheta^n_{\mathcal{X}\mathcal{W}\mathcal{A}}(M,-) = \mathrm{H}_{-n}(\operatorname{Hom}(\overline{\mathrm{id}_M},-)) = \mathrm{H}_{-n}(\operatorname{Hom}(\mathrm{id}_W,-)) = \mathrm{id}_{\mathrm{H}_{-n}(\operatorname{Hom}(W,-))}$$

which establish the desired result. □

The next lemma is a tool for the proofs of Propositions 4.10 and 4.11. Note that we do not assume that the complexes satisfy any properness conditions.

Lemma 4.9. *Let M, N be objects in $\mathcal{A}$, and assume $\mathcal{X} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{Y}$.*

(a) *Let $\alpha\colon X \to X'$ be a quasiisomorphism between bounded below complexes in $\mathcal{X}$. If $\mathcal{W}\text{-}\operatorname{pd}(N) < \infty$, then the morphism $\operatorname{Hom}_{\mathcal{A}}(\alpha,N)\colon \operatorname{Hom}_{\mathcal{A}}(X',N) \to \operatorname{Hom}_{\mathcal{A}}(X,N)$ is a quasiisomorphism.*

(b) *Let $\beta\colon Y \to Y'$ be a quasiisomorphism between bounded above complexes in $\mathcal{Y}$. If $\mathcal{V}\text{-}\operatorname{id}(M) < \infty$, then the morphism $\operatorname{Hom}_{\mathcal{A}}(M,\beta)\colon \operatorname{Hom}_{\mathcal{A}}(M,Y) \to \operatorname{Hom}_{\mathcal{A}}(M,Y')$ is a quasiisomorphism.*

Proof. We prove part (a); the proof of part (b) is dual. It suffices to show that $\operatorname{Cone}(\operatorname{Hom}_{\mathcal{A}}(\alpha,N))$ is exact. From the next isomorphism

$$\operatorname{Cone}(\operatorname{Hom}_{\mathcal{A}}(\alpha,N)) \cong \mathbf{\Sigma}\operatorname{Hom}_{\mathcal{A}}(\operatorname{Cone}(\alpha),N)$$

we need to show that $\operatorname{Hom}_{\mathcal{A}}(\operatorname{Cone}(\alpha),N)$ is exact. Note that $\operatorname{Cone}(\alpha)$ is an exact, bounded below complex in $\mathcal{X}$. Set $M_j = \operatorname{Ker}(\partial_j^{\operatorname{Cone}(\alpha)})$ for each integer j, and note $M_{j-1} \in \mathcal{X}$ for $j \ll 0$. Consider the exact sequences

$$(*_j) \qquad 0 \to M_j \to \operatorname{Cone}(\alpha)_j \to M_{j-1} \to 0.$$

The condition $\mathcal{X} \perp \mathcal{W}$ implies $\mathcal{X} \perp N$ by Lemma 1.7. Hence, induction on j using Lemma 1.6(a) implies $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(M_j,N) = 0$ for each j and so each sequence $(*_j)$ is $\operatorname{Hom}_{\mathcal{A}}(-,N)$-exact. It follows that $\operatorname{Hom}_{\mathcal{A}}(\operatorname{Cone}(\alpha),N)$ is exact. □

The next two results compare to [5, (4.2.4)]. Note that Lemmas 3.3 and 3.4 provide conditions implying the containments $\operatorname{res}\widehat{\mathcal{X}} \subseteq \operatorname{res}\widetilde{\mathcal{X}} \cap \operatorname{res}\widetilde{\mathcal{W}}$ and $\operatorname{cores}\widehat{\mathcal{Y}} \subseteq \operatorname{cores}\widetilde{\mathcal{Y}} \cap \operatorname{cores}\widetilde{\mathcal{V}}$.

Proposition 4.10. *Let M, N be objects in $\mathcal{A}$, and let $\mathcal{X} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{Y}$.*

(a) *If M is in $\operatorname{res}\widetilde{\mathcal{X}} \cap \operatorname{res}\widetilde{\mathcal{W}}$ and N is in $\operatorname{res}\widehat{\mathcal{W}}$, then the following natural map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{X}\mathcal{W}\mathcal{A}}(M,N)\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M,N).$$

(b) *If M is in $\operatorname{cores}\widehat{\mathcal{V}}$ and N is in $\operatorname{cores}\widetilde{\mathcal{Y}} \cap \operatorname{cores}\widetilde{\mathcal{V}}$, then the following natural map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{A}\mathcal{Y}\mathcal{V}}(M,N)\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}(M,N).$$

Proof. We prove part (a); the proof of part (b) is dual. The object M has a proper $\mathcal{W}$-resolution $\gamma\colon W \to M$ and a proper $\mathcal{X}$-resolution $\gamma'\colon X \to M$. Lemma 1.8(a) yields a quasiisomorphism $\overline{\mathrm{id}_M}\colon W \to X$ such that $\gamma = \gamma'\overline{\mathrm{id}_M}$, and Lemma 4.9(a) implies that the morphism $\operatorname{Hom}_{\mathcal{A}}(\overline{\mathrm{id}_M}, N)$ is a quasiisomorphism. The result now follows from the definition of $\vartheta^n_{\mathcal{X}\mathcal{W}\mathcal{A}}(M,N)$. □

Proposition 4.11. *Let M, N be objects in $\mathcal{A}$, and let $\mathcal{X} \perp \mathcal{W}$ and $\mathcal{V} \perp \mathcal{Y}$.*

(a) *If M is in $\operatorname{res}\widetilde{\mathcal{X}} \cap \operatorname{res}\widetilde{\mathcal{P}}$ and N is in $\operatorname{res}\widehat{\mathcal{W}}$, then the following natural map is an isomorphism for each n*

$$\varkappa^n_{\mathcal{X}\mathcal{A}}(M,N)\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{A}}(M,N).$$

(b) *If M is in $\operatorname{cores}\widehat{\mathcal{V}}$ and N is in $\operatorname{cores}\widetilde{\mathcal{Y}} \cap \operatorname{cores}\widetilde{\mathcal{I}}$, then the following natural map is an isomorphism for each n*

$$\varkappa^n_{\mathcal{A}\mathcal{Y}}(M,N)\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{A}}(M,N).$$

Proof. Argue as in Proposition 4.10. When invoking Lemma 4.9(a), use the category $\mathcal{X} \oplus \mathcal{P}$ whose objects are precisely those of the of the form $X \oplus P$ for some $X \in \mathcal{X}$ and $P \in \mathcal{P}$. □

The next two lemmata are tools for Proposition 4.14 and Theorem 6.7.

Lemma 4.12. *Let $\mathcal{W}$ be a cogenerator for $\mathcal{X}$ and $\mathcal{V}$ a generator for $\mathcal{Y}$.*

(a) *If $\mathcal{W} \perp (\mathcal{W} \cup \mathcal{Y})$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0$, then $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{Y}) = 0$.*

(b) *If $(\mathcal{X} \cup \mathcal{V}) \perp \mathcal{V}$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \operatorname{cores}\widehat{\mathcal{V}}) = 0$, then $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{X}, \operatorname{cores}\widehat{\mathcal{V}}) = 0$.*

Proof. We prove part (a); part (b) is proved dually. Fix objects M in $\operatorname{res}\widehat{\mathcal{W}}$ and Y in $\mathcal{Y}$, and set $Y_0 = Y$. Because $\mathcal{V}$ is a generator for $\mathcal{Y}$ there exist exact sequences

$$0 \to Y_{n+1} \to V_n \to Y_n \to 0$$

with V_n in $\mathcal{V}$ and Y_{n+1} in $\mathcal{Y}$. The assumption $\mathcal{W} \perp \mathcal{Y}$ implies that each of these sequences is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact by Lemma 1.6(b). Fix an integer $j \geqslant 1$ and set $p = \mathcal{W}\text{-}\operatorname{pd}(M)$. The vanishing hypothesis implies $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(M, V_n) = 0$ for each n, and so Lemma 4.4(a) yields the isomorphism in the following sequence

$$\operatorname{Ext}^{j}_{\mathcal{W}\mathcal{A}}(M, Y) = \operatorname{Ext}^{j}_{\mathcal{W}\mathcal{A}}(M, Y_0) \cong \operatorname{Ext}^{j+p}_{\mathcal{W}\mathcal{A}}(M, Y_p) = 0$$

where the last equality is from Proposition 4.5(b) because $\mathcal{W} \perp \mathcal{W}$. □

Lemma 4.13. *Assume that $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{V}$ is a generator for $\mathcal{Y}$. Let M, N be objects in $\mathcal{A}$ with $\mathcal{W}\text{-}\operatorname{pd}(M) < \infty$ and $\mathcal{V}\text{-}\operatorname{id}(N) < \infty$.*

(a) *Assume $(\mathcal{X} \cup \mathcal{V}) \perp \mathcal{V}$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \operatorname{cores}\widehat{\mathcal{V}}) = 0$. If $\alpha\colon X \xrightarrow{\simeq} X'$ is a quasiisomorphism between bounded below complexes in $\mathcal{X}$, then the morphism $\operatorname{Hom}_{\mathcal{A}}(\alpha, N)\colon \operatorname{Hom}_{\mathcal{A}}(X', N) \to \operatorname{Hom}_{\mathcal{A}}(X, N)$ is a quasiisomorphism.*

(b) *Assume $\mathcal{W} \perp (\mathcal{W} \cup \mathcal{Y})$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0$. If $\beta\colon Y \xrightarrow{\simeq} Y'$ is a quasiisomorphism between bounded above complexes in $\mathcal{Y}$, then the morphism $\operatorname{Hom}_{\mathcal{A}}(M, \beta)\colon \operatorname{Hom}_{\mathcal{A}}(M, Y) \to \operatorname{Hom}_{\mathcal{A}}(M, Y')$ is a quasiisomorphism.*

Proof. We prove part (a); the proof of part (b) is dual.

Set $M_j = \operatorname{Ker}(\partial^{\operatorname{Cone}(\alpha)}_j)$ for each j, and note $M_j \in \mathcal{X}$ for $j \ll 0$. As in the proof of Lemma 4.9, it suffices to show that each of the following sequences

$$(*_j) \qquad 0 \to M_j \to \operatorname{Cone}(\alpha)_j \to M_{j-1} \to 0.$$

is $\operatorname{Hom}_{\mathcal{A}}(-, N)$-exact. The condition $\mathcal{X} \perp \mathcal{V}$ implies $M_j \perp \mathcal{V}$ for $j \ll 0$ and $\operatorname{Cone}(\alpha)_j \perp \mathcal{V}$ for all $j \in \mathbb{Z}$. Applying Lemma 1.6(a) to the sequences $(*_j)$ inductively implies $M_j \perp \mathcal{V}$ for all $j \in \mathbb{Z}$ and so each $(*_j)$ is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{V})$-exact. Lemma 4.12(b) implies $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(M_j, N) = 0$ for $j \ll 0$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\operatorname{Cone}(\alpha)_j, N) = 0$ for all $j \in \mathbb{Z}$. Applying Lemma 4.4(c) to $(*_j)$ yields $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(M_j, N) = 0$ for all $n \in \mathbb{Z}$. Thus, each sequence $(*_j)$ is $\operatorname{Hom}_{\mathcal{A}}(-, N)$-exact, as desired. □

The next result is proved like Proposition 4.10, using Lemma 4.13 in place of Lemma 4.9.

Proposition 4.14. *Assume that $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{V}$ is a generator for $\mathcal{Y}$. Let M and N be objects in $\mathcal{A}$.*

(a) *Assume $(\mathcal{X} \cup \mathcal{V}) \perp \mathcal{V}$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \operatorname{cores}\widehat{\mathcal{V}}) = 0$. If M is in $\operatorname{res}\widetilde{\mathcal{X}} \cap \operatorname{res}\widetilde{\mathcal{W}}$ and N is in $\operatorname{cores}\widehat{\mathcal{V}}$, then the next map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{X}\mathcal{W}\mathcal{A}}(M, N)\colon \operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M, N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{W}\mathcal{A}}(M, N).$$

(b) *Assume $\mathcal{W} \perp (\mathcal{W} \cup \mathcal{Y})$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0$. If M is in $\operatorname{res}\widehat{\mathcal{W}}$ and N is in $\operatorname{cores}\widetilde{\mathcal{Y}} \cap \operatorname{cores}\widetilde{\mathcal{V}}$, then the next map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{A}\mathcal{Y}\mathcal{V}}(M, N)\colon \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M, N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{A}\mathcal{V}}(M, N).$$

5. Relative perfection

This section is concerned with a relative notion of perfection akin to the Gorenstein perfection of [5], the quasi-perfection of [12] and the generalized perfection of [15]. We begin with the relevant definitions.

Definition 5.1. Let $\mathcal{A}^o$ be another abelian category with subcategory $\mathcal{X}^o$ and let T and T^o be objects in $\mathcal{X}$ and $\mathcal{X}^o$, respectively. The pair (T, T^o) is a *relative cotilting pair* for the quadruple $(\mathcal{A}, \mathcal{X}, \mathcal{A}^o, \mathcal{X}^o)$ when the next conditions are satisfied:

(1) The functor $\operatorname{Hom}_{\mathcal{A}}(-, T)$ maps $\mathcal{A}$ to $\mathcal{A}^o$ and $\mathcal{X}$ to $\mathcal{X}^o$.

(2) The functor $\operatorname{Hom}_{\mathcal{A}^o}(-, T^o)$ maps $\mathcal{A}^o$ to $\mathcal{A}$ and $\mathcal{X}^o$ to $\mathcal{X}$.

(3) There are natural isomorphisms $\operatorname{Hom}_{\mathcal{A}^o}(\operatorname{Hom}_{\mathcal{A}}(-, T), T^o)|_{\mathcal{X}} \cong \operatorname{id}_{\mathcal{X}}$ and $\operatorname{Hom}_{\mathcal{A}}(\operatorname{Hom}_{\mathcal{A}^o}(-, T^o), T)|_{\mathcal{X}^o} \cong \operatorname{id}_{\mathcal{X}^o}$.

The term *relative tilting pair* is defined dually.

Definition 5.2. Let T be an object in $\mathcal{A}$. An object M in $\mathcal{A}$ with $g = \mathcal{X}\text{-}\operatorname{pd}(M) < \infty$ is *$\mathcal{X}T$-perfect of grade g* if $\operatorname{Ext}^n_{\mathcal{A}}(M, T) = 0$ for each $n \neq g$. The term *$T\mathcal{Y}$-coperfect of cograde g* is defined dually.

Our motivating example comes from our categories of interest.

Example 5.3. If R is noetherian and C is a semidualizing R-module, then (C, C) is a relative cotilting pair for $(\mathcal{M}(R), \mathcal{G}_C(R), \mathcal{M}(R), \mathcal{G}_C(R))$. (More generally, if C is a semidualizing RS-bimodule as in [19], then the pair $({}_RC, C_S)$ is a relative cotilting pair for $(\mathcal{M}(R), \mathcal{G}_C(R), \mathcal{M}(S^o), \mathcal{G}_C(S^o))$.) In this case, we write "$\mathcal{G}_C$-perfect" instead of "$\mathcal{G}_C(R)C$-perfect". The class of $\mathcal{G}_C$-perfect R-modules includes the totally C-reflexive R-modules and the perfect R-modules. When $C = R$, this notion recovers the G-perfect modules of [5].

Our main result on relative perfection establishes a duality between categories of relatively perfect objects.

Proposition 5.4. *Let M be an object in $\mathcal{A}$, and let $\mathcal{A}^o$ be an abelian category with subcategories $\mathcal{X}^o$ and $\mathcal{Y}^o$.*

(a) *Let (T, T^o) be a relative cotilting pair for $(\mathcal{A}, \mathcal{X}, \mathcal{A}^o, \mathcal{X}^o)$ such that $\mathcal{X} \perp T$ and $\mathcal{X}^o \perp T^o$. Assume that $\mathcal{A}$ and $\mathcal{A}^o$ have enough projectives. If M is $\mathcal{X}T$-perfect of grade g, then $\operatorname{Ext}^g_{\mathcal{A}}(M, T)$ is an object of $\mathcal{A}^o$ that is $\mathcal{X}^oT^o$-perfect of grade g, and $\operatorname{Ext}^g_{\mathcal{A}^o}(\operatorname{Ext}^g_{\mathcal{A}}(M, T), T^o) \cong M$.*

(b) *Let (U, U^o) be a relative tilting pair for $(\mathcal{A}, \mathcal{Y}, \mathcal{A}^o, \mathcal{Y}^o)$ such that $U \perp \mathcal{Y}$ and $U^o \perp \mathcal{Y}^o$, and assume that $\mathcal{A}$ and $\mathcal{A}^o$ have enough injectives. If M is $U\mathcal{Y}$-coperfect of cograde g, then $\operatorname{Ext}^g_{\mathcal{A}}(U, M)$ is an object of $\mathcal{A}^o$ that is $U^o\mathcal{Y}^o$-coperfect of cograde g, and $\operatorname{Ext}^g_{\mathcal{A}^o}(U^o, \operatorname{Ext}^g_{\mathcal{A}}(U, M)) \cong M$.*

Proof. We prove part (a); the proof of part (b) is dual.

The result is trivial if $M = 0$, so assume $M \neq 0$. Let $X \xrightarrow{\simeq} M$ be an $\mathcal{X}$-resolution such that $X_n = 0$ for each $n > g = \mathcal{X}\text{-}\operatorname{pd}(M)$. By assumption, the complex $\operatorname{Hom}_{\mathcal{A}}(X,T)$ consists of objects and morphisms in $\mathcal{X}^o$.

As in the proof of Proposition 4.11, Lemma 4.9(a) yields an isomorphism

$$\mathrm{H}_{-n}(\operatorname{Hom}_{\mathcal{A}}(X,T)) \cong \operatorname{Ext}^n_{\mathcal{A}}(M,T)$$

for each n. Because M is $\mathcal{X}T$-perfect of grade g, we conclude that the complex $\Sigma^g \operatorname{Hom}_{\mathcal{A}}(X,T)$ is an $\mathcal{X}^o$-resolution of $\operatorname{Ext}^g_{\mathcal{A}}(M,T)$ with $(\Sigma^g \operatorname{Hom}_{\mathcal{A}}(X,T))_n = 0$ for each $n > g$. In particular, the object $\operatorname{Ext}^g_{\mathcal{A}}(M,T) \cong \operatorname{Coker}(\operatorname{Hom}_{\mathcal{A}}(\partial^X_g,T))$ is in $\mathcal{A}^o$ and $g^o = \mathcal{X}^o\text{-}\operatorname{pd}(\operatorname{Ext}^g_{\mathcal{A}}(M,T)) \leqslant g < \infty$.

Similarly, we conclude that there is an isomorphism

$$\mathrm{H}_{g-n}(\operatorname{Hom}_{\mathcal{A}^o}(\operatorname{Hom}_{\mathcal{A}}(X,T),T^o)) \cong \operatorname{Ext}^n_{\mathcal{A}^o}(\operatorname{Ext}^g_{\mathcal{A}}(M,T),T^o)$$

for each n. Our assumptions yield the isomorphism in the next sequence

$$\operatorname{Hom}_{\mathcal{A}^o}(\operatorname{Hom}_{\mathcal{A}}(X,T),T^o) \cong X \simeq M$$

while the quasiisomorphism is by construction. These displays imply

$$\operatorname{Ext}^n_{\mathcal{A}^o}(\operatorname{Ext}^g_{\mathcal{A}}(M,T),T^o) \cong \begin{cases} 0 & \text{if } n \neq g \\ M & \text{if } n = g. \end{cases}$$

It remains to justify the equality $g^0 = g$. We already know $g^o \leqslant g$, so suppose $g^o < g$. Using Lemma 4.9(a) as above, this implies $\operatorname{Ext}^n_{\mathcal{A}^o}(\operatorname{Ext}^g_{\mathcal{A}}(M,T),T^o) = 0$ for each $n \geqslant g$. In particular, we have a contradiction from the sequence $0 = \operatorname{Ext}^g_{\mathcal{A}^o}(\operatorname{Ext}^g_{\mathcal{A}}(M,T),T^o) \cong M$. ☐

We conclude this section with the special case of Proposition 5.4 for our categories of interest. The special case $C = R$ recovers [5, (6.3.1,2)].

Corollary 5.5. *Let R be a commutative noetherian ring and let C, M be finitely generated R-modules with C semidualizing and $\mathrm{G}_C\text{-}\dim_R(M) < \infty$.*

(a) *There is an inequality $\operatorname{grade}_R(M) \leqslant \mathrm{G}_C\text{-}\dim_R(M)$, and M is G_C-perfect of grade g if and only if $\operatorname{grade}_R(M) = \mathrm{G}_C\text{-}\dim_R(M) = g$.*

(b) *If M is G_C-perfect of grade g, then so is the R-module $\operatorname{Ext}^g_R(M,C)$, and there is an isomorphism $M \cong \operatorname{Ext}^g_R(\operatorname{Ext}^g_R(M,C),C)$.*

Proof. Part (a) is established in the next sequence:

$$\begin{aligned} \operatorname{grade}_R(M) &= \operatorname{depth}_{\operatorname{Ann}_R(M)}(R) \\ &= \operatorname{depth}_{\operatorname{Ann}_R(M)}(C) \\ &= \inf\{n \geqslant 0 \mid \operatorname{Ext}^n_R(M,C) \neq 0\} \\ &\leqslant \sup\{n \geqslant 0 \mid \operatorname{Ext}^n_R(M,C) \neq 0\} \\ &= \mathrm{G}_C\text{-}\dim_R(M). \end{aligned}$$

The first equality is by definition. The second equality follows from the fact that a sequence is R-regular if and only if it is C-regular; see [15, p. 68]. The third equality and the inequality are standard. The last equality is in [13, (2.1)].

Part (b) follows immediately from Proposition 5.4(a); see Example 5.3. $\square$

6. Balanced properties for relative cohomology

Definition 6.1. Fix subcategories $\mathcal{X}' \subseteq \operatorname{res}\widetilde{\mathcal{X}}$ and $\mathcal{Y}' \subseteq \operatorname{cores}\widetilde{\mathcal{Y}}$. We say that $\operatorname{Ext}_{\mathcal{X}\mathcal{A}}$ and $\operatorname{Ext}_{\mathcal{A}\mathcal{Y}}$ are *balanced* on $\mathcal{X}' \times \mathcal{Y}'$ if the following condition holds: For each object M in $\mathcal{X}'$ and N in $\mathcal{Y}'$, if $X \to M$ is a proper $\mathcal{X}$-resolution, and $N \to Y$ a proper $\mathcal{Y}$-coresolution, then the induced morphisms of complexes

$$\operatorname{Hom}_{\mathcal{A}}(M, Y) \to \operatorname{Hom}_{\mathcal{A}}(X, Y) \leftarrow \operatorname{Hom}_{\mathcal{A}}(X, N)$$

are quasiisomorphisms.

Remark 6.2. Fix objects $M \in \mathcal{X}'$ and $N \in \mathcal{Y}'$. If $\operatorname{Ext}_{\mathcal{X}\mathcal{A}}$ and $\operatorname{Ext}_{\mathcal{A}\mathcal{Y}}$ are balanced on $\mathcal{X}' \times \mathcal{Y}'$, then $\operatorname{Ext}^n_{\mathcal{X}\mathcal{A}}(M, N) \cong \operatorname{Ext}^n_{\mathcal{A}\mathcal{Y}}(M, N)$ for all and all $n \in \mathbb{Z}$.

The next four lemmata are tools for the proof of our Main Theorem.

Lemma 6.3. *Assume* $\mathcal{W} \perp \mathcal{V}$.

(a) *If* $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0$ *and* $\mathcal{W} \perp \mathcal{W}$, *then* $\operatorname{res}\widehat{\mathcal{W}} \perp \mathcal{V}$.

(b) *If* $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \operatorname{cores}\widehat{\mathcal{V}}) = 0$ *and* $\mathcal{V} \perp \mathcal{V}$, *then* $\mathcal{W} \perp \operatorname{cores}\widehat{\mathcal{V}}$.

Proof. We prove part (a); part (b) is verified similarly. Fix objects M in $\operatorname{res}\widehat{\mathcal{W}}$ and V in $\mathcal{V}$ and set $n = \mathcal{W}\text{-}\operatorname{pd}(M)$. We proceed by induction on n. If $n = 0$, then $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(M, V) = 0$ since $\mathcal{W} \perp \mathcal{V}$. So assume $n \geqslant 1$. There exists an exact sequence

$$0 \to M' \xrightarrow{\epsilon} W \to M \to 0 \tag{6.1}$$

such that W is an object in $\mathcal{W}$ and $\mathcal{W}\text{-}\operatorname{pd}(M') = n - 1$. The induction hypothesis implies $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}}(M', V) = 0$. Fix an integer $i \geqslant 1$. Using the hypothesis $\mathcal{W} \perp \mathcal{V}$, a standard dimension-shifting argument yields $0 = \operatorname{Ext}^i_{\mathcal{A}}(M', V) \cong \operatorname{Ext}^{i+1}_{\mathcal{A}}(M, V)$, so it remains to show $\operatorname{Ext}^1_{\mathcal{A}}(M, V) = 0$.

By Lemma 1.7 we know $\mathcal{W} \perp \mathcal{W}$ implies $\mathcal{W} \perp \operatorname{res}\widehat{\mathcal{W}}$. Hence, the sequence (6.1) is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact by Lemma 1.6(b). By assumption, we have $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(M, V) = 0$ and so the long exact sequence in $\operatorname{Ext}_{\mathcal{W}\mathcal{A}}(-, V)$ associated to (6.1) has the form

$$0 \to \operatorname{Hom}(M, V) \to \operatorname{Hom}(W, V) \xrightarrow{\operatorname{Hom}(\epsilon, V)} \operatorname{Hom}(M', V) \to 0.$$

Thus, the map $\operatorname{Hom}(\epsilon, V)$ is surjective. The assumption $\mathcal{W} \perp \mathcal{V}$ implies that the long exact sequence in $\operatorname{Ext}(-, V)$ associated to (6.1) starts as

$$0 \to \operatorname{Hom}(M, V) \to \operatorname{Hom}(W, V) \xrightarrow{\operatorname{Hom}(\epsilon, V)} \operatorname{Hom}(M', V) \to \operatorname{Ext}^1(M, V) \to 0.$$

Since $\operatorname{Hom}_{\mathcal{A}}(\epsilon, V)$ is surjective, this implies $\operatorname{Ext}^1_{\mathcal{A}}(M, V) = 0$ as desired. $\square$

Lemma 6.4. *Let $\mathcal{W}$ be a cogenerator for $\mathcal{X}$ and let $\mathcal{V}$ be a generator for $\mathcal{Y}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions.*

(a) *If $\mathcal{W} \perp \mathcal{W}$ and $\mathcal{X} \perp \mathcal{V}$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0$, then $\operatorname{res}\widehat{\mathcal{X}} \perp \mathcal{V}$.*

(b) *If $\mathcal{V} \perp \mathcal{V}$ and $\mathcal{W} \perp \mathcal{Y}$ and $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \operatorname{cores}\widehat{\mathcal{V}}) = 0$, then $\mathcal{W} \perp \operatorname{cores}\widehat{\mathcal{Y}}$.*

Proof. We prove part (a); the proof of part (b) is dual. Fix an object $M \in \operatorname{res}\widehat{\mathcal{X}}$ and, using Lemma 3.3(a), a $\mathcal{W}\mathcal{X}$-hull

$$0 \to M \to K' \to X' \to 0.$$

Because X' is in $\mathcal{X}$, we have $X' \perp \mathcal{V}$. Lemma 6.3(a) implies $K' \perp \mathcal{V}$ and so Lemma 1.6(a) guarantees $M \perp \mathcal{V}$, as desired. □

Lemma 3.3 provides the proper resolutions and coresolutions in the next two lemmata which are the primary tools for proving the Main Theorem.

Lemma 6.5. *Assume the following: $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions, $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, $\mathcal{V}$ is a projective generator for $\mathcal{Y}$, $\mathcal{W} \perp \mathcal{Y}$ and $\mathcal{X} \perp \mathcal{V}$.*

(a) *Assume $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}}, \mathcal{V}) = 0$. If M is an object in $\operatorname{res}\widehat{\mathcal{X}}$ with proper $\mathcal{X}$-resolution $X \to M$, then X^+ is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{Y})$-exact.*

(b) *Assume $\operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W}, \operatorname{cores}\widehat{\mathcal{V}}) = 0$. If N is an object in $\operatorname{cores}\widehat{\mathcal{Y}}$ with proper $\mathcal{Y}$-coresolution $N \to Y$, then ${}^+Y$ is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact.*

Proof. We proof part (a); the proof of (b) is dual. Lemma 3.3(a) yields a strict $\mathcal{W}\mathcal{X}$-resolution $X' \to M$, and Lemma 3.2(b) implies that this resolution is $\mathcal{X}$-proper. Lemma 1.8(a) shows that X and X' are homotopy equivalent, so we may replace X with X' to assume that $X \to M$ is a strict $\mathcal{W}\mathcal{X}$-resolution.

Fix an object $Y \in \mathcal{Y}$. For each n, set $M_n = \operatorname{Coker}(\partial^X_{n+2})$, noting $M_{-1} \cong M$. When $n \geqslant 0$, we have $\mathcal{W}\text{-}\operatorname{pd}(M_n) < \infty$ and we consider the exact sequences

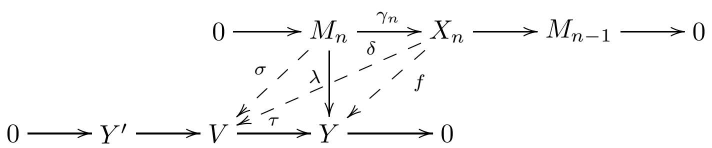
$$0 \to M_n \xrightarrow{\gamma_n} X_n \to M_{n-1} \to 0. \tag{6.2}$$

It suffices to show that each of these sequences is $\operatorname{Hom}_{\mathcal{A}}(-, Y)$-exact, that is, that the map $\operatorname{Hom}_{\mathcal{A}}(\gamma_n, Y)\colon \operatorname{Hom}_{\mathcal{A}}(X_n, Y) \to \operatorname{Hom}_{\mathcal{A}}(M_n, Y)$ is surjective. Since $\mathcal{V}$ is a generator for $\mathcal{Y}$ and Y is in $\mathcal{Y}$, there is an exact sequence

$$0 \to Y' \to V \xrightarrow{\tau} Y \to 0 \tag{6.3}$$

such that Y' is an object in $\mathcal{Y}$ and V is an object in $\mathcal{V}$. The assumption $\mathcal{W} \perp \mathcal{Y}$ implies that this sequence is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact by Lemma 1.6(b).

Fix an element $\lambda \in \operatorname{Hom}_{\mathcal{A}}(M_n, Y)$. The proof will be complete once we find $f \in \operatorname{Hom}_{\mathcal{A}}(X_n, Y)$ such that $\lambda = f\gamma_n$. The following diagram is our guide



wherein the top row is (6.2) and the bottom row is (6.3).

Since the sequence (6.3) is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W},-)$-exact, it yields a long exact sequence in $\mathrm{Ext}_{\mathcal{W}\mathcal{A}}(M_n,-)$ by Lemma 4.3(a). From Lemma 4.12(a) we conclude $\mathrm{Ext}^1_{\mathcal{W}\mathcal{A}}(M_n,Y')=0$, so this long exact sequence begins as follows

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M_n,Y') \to \mathrm{Hom}_{\mathcal{A}}(M_n,V) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(M_n,\tau)} \mathrm{Hom}_{\mathcal{A}}(M_n,Y) \to 0.$$

Hence, there exists $\sigma \in \mathrm{Hom}_{\mathcal{A}}(M_n,V)$ such that $\lambda = \tau\sigma$.

Lemma 6.4(a) implies $\mathrm{Ext}^1_{\mathcal{A}}(M_{n-1},V)=0$, so an application of $\mathrm{Ext}_{\mathcal{A}}(-,V)$ to the sequence (6.2) yields the next exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M_{n-1},V) \to \mathrm{Hom}_{\mathcal{A}}(X_n,V) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\gamma_n,V)} \mathrm{Hom}_{\mathcal{A}}(M_n,V) \to 0.$$

Hence, there exists $\delta \in \mathrm{Hom}_{\mathcal{A}}(X_n,V)$ such that $\sigma = \delta\gamma_n$. It follows that

$$(\tau\delta)\gamma_n = \tau\sigma = \lambda$$

and so $f = \tau\delta \in \mathrm{Hom}_{\mathcal{A}}(X_n,V)$ has the desired property. □

Lemma 6.6. *Assume the following: $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions, $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, $\mathcal{V}$ is a projective generator for $\mathcal{Y}$, $\mathcal{W} \perp \mathcal{Y}$ and $\mathcal{X} \perp \mathcal{V}$.*

(a) *Let M be an object in $\operatorname{res}\widehat{\mathcal{X}}$ with proper $\mathcal{X}$-resolution $\alpha\colon X \to M$. If Y' is a bounded above complex in $\mathcal{Y}$ and $\mathrm{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}},\mathcal{V})=0$, then the induced map $\mathrm{Hom}_{\mathcal{A}}(M,Y') \to \mathrm{Hom}_{\mathcal{A}}(X,Y')$ is a quasiisomorphism.*

(b) *Let N be an object in $\operatorname{cores}\widehat{\mathcal{Y}}$ with proper $\mathcal{Y}$-coresolution $\alpha\colon N \to Y'$. If X' is a bounded below complex in $\mathcal{X}$ and $\mathrm{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W},\operatorname{cores}\widehat{\mathcal{V}})=0$, then the induced map $\mathrm{Hom}_{\mathcal{A}}(X',N) \to \mathrm{Hom}_{\mathcal{A}}(X',Y)$ is a quasiisomorphism.*

Proof. We proof part (a); the proof of (b) is dual. Lemma 6.5(a) shows that the complex $\mathrm{Hom}_{\mathcal{A}}(X^+,Y_n)$ is exact for each n, and a standard argument demonstrates that $\mathrm{Hom}_{\mathcal{A}}(X^+,Y)$ is exact. From the following isomorphisms of complexes

$$\mathrm{Cone}(\mathrm{Hom}_{\mathcal{A}}(\alpha,Y)) \cong \Sigma\,\mathrm{Hom}_{\mathcal{A}}(\mathrm{Cone}(\alpha),Y) \cong \Sigma\,\mathrm{Hom}_{\mathcal{A}}(X^+,Y) \simeq 0$$

one concludes that $\mathrm{Hom}_{\mathcal{A}}(\alpha,Y)$ is a quasiisomorphism. □

The next result contains the Main Theorem from the introduction.

Theorem 6.7. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions, $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, $\mathcal{V}$ is a projective generator for $\mathcal{Y}$, $\mathcal{W} \perp \mathcal{Y}$, $\mathcal{X} \perp \mathcal{V}$ and $\mathrm{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}},\mathcal{V}) = 0 = \mathrm{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W},\operatorname{cores}\widehat{\mathcal{V}})$. Then $\mathrm{Ext}_{\mathcal{X}\mathcal{A}}$ and $\mathrm{Ext}_{\mathcal{A}\mathcal{Y}}$ are balanced on $\operatorname{res}\widehat{\mathcal{X}} \times \operatorname{cores}\widehat{\mathcal{Y}}$. In particular, there are isomorphisms $\mathrm{Ext}^n_{\mathcal{X}\mathcal{A}}(M,N) \cong \mathrm{Ext}^n_{\mathcal{A}\mathcal{Y}}(M,N)$ for all objects M in $\operatorname{res}\widehat{\mathcal{X}}$ and N in $\operatorname{cores}\widehat{\mathcal{Y}}$ and for all $n \in \mathbb{Z}$.*

Proof. Fix objects M in $\operatorname{res}\widehat{\mathcal{X}}$ and $N \in \operatorname{cores}\widehat{\mathcal{Y}}$. Using Lemma 3.3, we have a proper $\mathcal{X}$-resolution $\alpha\colon X \to M$ and a proper $\mathcal{Y}$-coresolution $\beta\colon N \to Y$. Lemma 6.6 implies that the induced morphisms

$$\operatorname{Hom}_{\mathcal{A}}(M,Y) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(\alpha,Y)} \operatorname{Hom}_{\mathcal{A}}(X,Y) \xleftarrow{\operatorname{Hom}_{\mathcal{A}}(X,\beta)} \operatorname{Hom}_{\mathcal{A}}(X,N)$$

are quasiisomorphisms, and hence the desired conclusion. □

Remark 6.8. Under the hypotheses of Theorem 6.7, it follows almost immediately from Proposition 4.8 that $\operatorname{Ext}_{\mathcal{W}\mathcal{A}}$ and $\operatorname{Ext}_{\mathcal{A}\mathcal{V}}$ are balanced on $\operatorname{res}\widehat{\mathcal{W}} \times \operatorname{cores}\widehat{\mathcal{V}}$. This conclusion also follows from the weaker hypothesis $\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}},\mathcal{V}) = 0 = \operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W},\operatorname{cores}\widehat{\mathcal{V}})$ using [10, (8.2.14)].

The next result follows from Lemma 1.10 and Thoerem 6.7.

Corollary 6.9. *For $n = 0,1,2,\ldots$, let $\mathcal{X}_n$ and $\mathcal{Y}_n$ be subcategories of $\mathcal{A}$ such that $\mathcal{X}_n$ and $\mathcal{Y}_n$ are closed under extensions when $n \geqslant 1$. Assume that $\mathcal{X}_n$ is an injective cogenerator for $\mathcal{X}_{n+1}$ and $\mathcal{Y}_n$ is a projective generator for $\mathcal{Y}_{n+1}$ for each $n \geqslant 0$. Assume $\mathcal{X}_n \perp \mathcal{Y}_0$ and $\mathcal{X}_0 \perp \mathcal{Y}_n$ for each $n \geqslant 0$. If $\operatorname{Ext}^{\geqslant 1}_{\mathcal{X}_0\mathcal{A}}(\operatorname{res}\widehat{\mathcal{X}_0},\mathcal{Y}_0) = 0 = \operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{Y}_0}(\mathcal{X}_0,\operatorname{cores}\widehat{\mathcal{Y}_0})$, then $\operatorname{Ext}_{\mathcal{X}_m\mathcal{A}}$ and $\operatorname{Ext}_{\mathcal{A}\mathcal{Y}_n}$ are balanced on $\operatorname{res}\widehat{\mathcal{X}_m} \times \operatorname{cores}\widehat{\mathcal{Y}_n}$ for each $m,n \geqslant 0$.*

We conclude with special cases of Theorem 6.7 for our categories of interest.

Definition 6.10. We simplify our notation for certain relative cohomology functors and for some of the connecting maps from Definition 4.6

$$\begin{aligned}
\operatorname{Ext}^n_{\mathcal{P}_C}(-,-) &= \operatorname{Ext}^n_{\mathcal{P}_C(R)R}(-,-) & \operatorname{Ext}^n_{\mathcal{I}_C}(-,-) &= \operatorname{Ext}^n_{R\mathcal{I}_C(R)}(-,-)\\
\operatorname{Ext}^n_{\mathcal{G}\mathcal{P}_C}(-,-) &= \operatorname{Ext}^n_{\mathcal{G}\mathcal{P}_C(R)R}(-,-) & \operatorname{Ext}^n_{\mathcal{G}\mathcal{I}_C}(-,-) &= \operatorname{Ext}^n_{R\mathcal{G}\mathcal{I}_C(R)}(-,-)\\
\operatorname{Ext}^n_{\mathcal{G}\mathcal{P}}(-,-) &= \operatorname{Ext}^n_{\mathcal{G}\mathcal{P}(R)R}(-,-) & \operatorname{Ext}^n_{\mathcal{G}\mathcal{I}}(-,-) &= \operatorname{Ext}^n_{R\mathcal{G}\mathcal{I}(R)}(-,-)\\
\varkappa^n_{\mathcal{P}_C} &= \varkappa^n_{\mathcal{P}_C(R)R} & \varkappa^n_{\mathcal{I}_C} &= \varkappa^n_{R\mathcal{I}_C(R)}.
\end{aligned}$$

We now show how Theorem 6.7 recovers [16, (3.6)].

Corollary 6.11. *If R is a commutative ring, then $\operatorname{Ext}_{\mathcal{G}\mathcal{P}}$ and $\operatorname{Ext}_{\mathcal{G}\mathcal{I}}$ are balanced on $\operatorname{res}\widehat{\mathcal{G}\mathcal{P}(R)} \times \operatorname{cores}\widehat{\mathcal{G}\mathcal{I}(\mathcal{R})}$.*

Proof. Set $\mathcal{X} = \mathcal{G}\mathcal{P}(R)$, $\mathcal{Y} = \mathcal{G}\mathcal{I}(R)$, $\mathcal{W} = \mathcal{P}(R)$ and $\mathcal{V} = \mathcal{I}(R)$. From [17, (2.5),(2.6)] we know that $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions. Fact 2.4 implies that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ and $\mathcal{V}$ is a projective generator for $\mathcal{Y}$. Clearly, we have $\mathcal{W} \perp \mathcal{Y}$ and $\mathcal{X} \perp \mathcal{V}$. The natural isomorphisms

$$\operatorname{Ext}^n_{\mathcal{P}(R)\mathcal{M}(R)}(-,-) \cong \operatorname{Ext}^n_R(-,-) \cong \operatorname{Ext}^n_{\mathcal{M}(R)\mathcal{I}(R)}(-,-)$$

from Remark 4.2 yield

$$\operatorname{Ext}^{\geqslant 1}_{\mathcal{W}\mathcal{A}}(\operatorname{res}\widehat{\mathcal{W}},\mathcal{V}) = 0 = \operatorname{Ext}^{\geqslant 1}_{\mathcal{A}\mathcal{V}}(\mathcal{W},\operatorname{cores}\widehat{\mathcal{V}}).$$

Hence, Theorem 6.7 yields the desired conclusion. □

The next lemmata are for use in Corollary 6.16.

Lemma 6.12. *Let R be a commutative ring and let B and B' be semidualizing R-modules. If* $\operatorname{Tor}^R_{\geqslant 1}(B,B')=0$*, then* $\mathcal{P}_B(R)\perp\mathcal{I}_{B'}(R)$.

Proof. Let P be a projective R-module and I an injective R-module. For each $i\geqslant 1$, the first isomorphism in the following sequence is a standard form of adjunction using the fact that P is projective and I is injective

$$\begin{aligned}\operatorname{Ext}^i_R(P\otimes_R B,\operatorname{Hom}_R(B',I)) &\cong \operatorname{Hom}_R(\operatorname{Tor}^R_i(P\otimes_R B,B'),I)\\ &\cong \operatorname{Hom}_R(P\otimes_R \operatorname{Tor}^R_i(B,B'),I)\\ &=0.\end{aligned}$$

The second isomorphism follows from the fact that P is projective, and the vanishing is by assumption. □

The next example shows how to construct semidualizing R-modules satisfying the hypotheses of Lemma 6.12.

Example 6.13. Let R be a commutative ring and let B and C be semidualizing R-modules. One has $C\in\mathcal{B}_B(R)$ if and only if $B\in\mathcal{G}_C(R)$ by [21, (3.14)]. Assume $C\in\mathcal{B}_B(R)$. From [7, (2.11)], we conclude that the R-module $B^{\dagger_C}=\operatorname{Hom}_R(B,C)$ is semidualizing, and [13, (3.1.b)] yields $B^{\dagger_C}\in\mathcal{A}_B(R)$ and $B\in\mathcal{A}_{B^{\dagger_C}}(R)$. In particular, we conclude $\operatorname{Tor}^R_{\geqslant 1}(B,B^{\dagger_C})=0$.

For example, one always has $C\in\mathcal{B}_R(R)=\mathcal{M}(R)$. If R is Cohen-Macaulay and D is dualizing, then $D\in\mathcal{B}_C(R)$. For discussions of methods for generating other semidualizing modules B and C such that $C\in\mathcal{B}_B(R)$, see [13], [14], [20].

Lemma 6.14. *Let R be a commutative ring and let B and C be semidualizing R-modules such that $C\in\mathcal{B}_B(R)$. With* $B^{\dagger_C}=\operatorname{Hom}_R(B,C)$*, there are containments* $\operatorname{res}\widehat{\mathcal{P}_B(R)}\subseteq\mathcal{B}_B(R)\cap\mathcal{A}_{B^{\dagger_C}}(R)\supseteq\operatorname{cores}\widehat{\mathcal{I}_{B^{\dagger_C}}(R)}$.

Proof. We verify the first containment; the second one is dual. Fact 2.8 implies $\operatorname{res}\widehat{\mathcal{P}_B(R)}\subseteq\mathcal{B}_B(R)$. Example 6.13 shows that $B\in\mathcal{A}_{B^{\dagger_C}}(R)$, and this implies $\mathcal{P}_B(R)\subseteq\mathcal{A}_{B^{\dagger_C}}(R)$. Fact 2.8 then yields $\operatorname{res}\widehat{\mathcal{P}_B(R)}\subseteq\mathcal{A}_{B^{\dagger_C}}(R)$. □

Lemma 6.15. *Let R be a commutative ring and let B and C be semidualizing R-modules such that $C\in\mathcal{B}_B(R)$. If* $B^{\dagger_C}=\operatorname{Hom}_R(B,C)$*, then* $\operatorname{Ext}_{\mathcal{P}_B}$ *and* $\operatorname{Ext}_{\mathcal{I}_{B^{\dagger_C}}}$ *are balanced on* $\operatorname{res}\widehat{\mathcal{P}_B(R)}\times\operatorname{cores}\widehat{\mathcal{I}_{B^{\dagger_C}}(R)}$.

Proof. Let M,N be R-modules with $\mathcal{P}_B\text{-}\operatorname{pd}_R(M)$ and $\mathcal{I}_{B^{\dagger_C}}\text{-}\operatorname{id}_R(N)$ both finite. From Lemma 6.14 we conclude $M,N\in\mathcal{B}_B(R)\cap\mathcal{A}_{B^{\dagger_C}}(R)$ and so [21, (4.1)] implies that the following natural maps are isomorphisms for each $n\in\mathbb{Z}$

$$\operatorname{Ext}^n_{\mathcal{P}_B}(M,N)\xrightarrow[\cong]{\varkappa^n_{\mathcal{P}_C}(M,N)}\operatorname{Ext}^n_R(M,N)\xleftarrow[\cong]{\varkappa^n_{\mathcal{I}_{B^{\dagger_C}}}(M,N)}\operatorname{Ext}^n_{\mathcal{I}_{B^{\dagger_C}}}(M,N).$$

In particular, we have

$$\operatorname{Ext}^n_{\mathcal{P}_B}(\operatorname{res}\widehat{\mathcal{P}_B(R)},\mathcal{I}_{B^{\dagger_C}}(R))=0=\operatorname{Ext}^n_{\mathcal{I}_{B^{\dagger_C}}}(\mathcal{P}_B(R),\operatorname{cores}\widehat{\mathcal{I}_{B^{\dagger_C}}(R)})$$

and the desired conclusion follows from [10, (8.2.14)]. □

Theorem 6.7 and Lemma 6.15 yield the next result.

Corollary 6.16. *Let R be a commutative ring and let B and C be semidualizing R-modules such that $C \in \mathcal{B}_B(R)$. Set $B^{\dagger_C} = \operatorname{Hom}_R(B, C)$ and assume $\mathcal{P}_B(R) \perp \mathcal{GI}_{B^{\dagger_C}}(R)$ and $\mathcal{GP}_B(R) \perp \mathcal{I}_{B^{\dagger_C}}(R)$. Then $\operatorname{Ext}_{\mathcal{GP}_B}$ and $\operatorname{Ext}_{\mathcal{GI}_{B^{\dagger_C}}}$ are balanced on $\operatorname{res}\widehat{\mathcal{GP}_B(R)} \times \operatorname{cores}\widehat{\mathcal{GI}_{B^{\dagger_C}}(R)}$.*

Question 6.17. Let R be a commutative ring and let B and C be semidualizing R-modules such that $C \in \mathcal{B}_B(R)$. With $B^{\dagger_C} = \operatorname{Hom}_R(B, C)$, must one have $\mathcal{P}_B(R) \perp \mathcal{GI}_{B^{\dagger_C}}(R)$ and $\mathcal{GP}_B(R) \perp \mathcal{I}_{B^{\dagger_C}}(R)$?

If the answer to this question is "yes" then the assumptions $\mathcal{P}_B(R) \perp \mathcal{GI}_{B^{\dagger_C}}(R)$ and $\mathcal{GP}_B(R) \perp \mathcal{I}_{B^{\dagger_C}}(R)$ can be removed from Corollary 6.16. Next we discuss one case where this is known, showing that [21, (5.7)] is a special case of Corollary 6.16.

Remark 6.18. Let R be a commutative Cohen-Macaulay ring with a dualizing module D. Let B be a semidualizing R-module. The membership $D \in \mathcal{B}_B(R)$ is in [7, (4.4)]. The conditions $\mathcal{P}_B(R) \perp \mathcal{GI}_{B^{\dagger_D}}(R)$ and $\mathcal{GP}_B(R) \perp \mathcal{I}_{B^{\dagger_D}}(R)$ follow from the containments $\mathcal{GI}_{B^{\dagger_D}}(R) \subseteq \mathcal{B}_B(R)$ and $\mathcal{GP}_B(R) \subseteq \mathcal{A}_{B^{\dagger_D}}(R)$ in [18, (4.6)]. It follows that $\operatorname{Ext}_{\mathcal{GP}_C}$ and $\operatorname{Ext}_{\mathcal{GI}_{C^{\dagger_D}}}$ are balanced on $\operatorname{res}\widehat{\mathcal{GP}_C(R)} \times \operatorname{cores}\widehat{\mathcal{GI}_{C^{\dagger_D}}(R)}$.

The following question is from the folklore of this subject and is related to the composition question for ring homomorphisms of finite G-dimension; see [4, (4.8)]. Remark 6.20 addresses its relevance to Corollary 6.16 and Question 6.17.

Question 6.19. Let R be a commutative ring and let B and C be semidualizing R-modules such that $C \in \mathcal{B}_B(R)$. Must the following hold?

$$\mathcal{GP}_B(R) \subseteq \mathcal{GP}_C(R) \qquad \mathcal{GI}_B(R) \subseteq \mathcal{GI}_C(R)$$
$$\mathcal{A}_C(R) \subseteq \mathcal{A}_B(R) \qquad \mathcal{B}_C(R) \subseteq \mathcal{B}_B(R)$$

Remark 6.20. Let R be a commutative Cohen-Macaulay ring with a dualizing module D. Let B and C be semidualizing R-modules such that $C \in \mathcal{B}_B(R)$. Arguing as in [13, (3.9)], one concludes $B^{\dagger_D} \in \mathcal{B}_{B^{\dagger_C}}(R)$ and $B \in \mathcal{B}_{B^{\dagger_C \dagger_D}}(R)$. Assume that the answer to Question 6.19 is "yes". Then we have

$$\mathcal{GP}_B(R) \subseteq \mathcal{A}_{B^{\dagger_D}}(R) \subseteq \mathcal{A}_{B^{\dagger_C}}(R)$$
$$\mathcal{GI}_{B^{\dagger_C}}(R) \subseteq \mathcal{B}_{B^{\dagger_C \dagger_D}}(R) \subseteq \mathcal{B}_B(R)$$

by [18, (4.6)]. One concludes $\mathcal{P}_B(R) \perp \mathcal{GI}_{B^{\dagger_C}}(R)$ and $\mathcal{GP}_B(R) \perp \mathcal{I}_{B^{\dagger_C}}(R)$ from the easily verified conditions $\mathcal{P}_B(R) \perp \mathcal{B}_B(R)$ and $\mathcal{A}_{B^{\dagger_C}}(R) \perp \mathcal{I}_{B^{\dagger_C}}(R)$.

In particular, if the answer to Question 6.19 is "yes", then the same is true for Question 6.17 and the assumptions $\mathcal{P}_B(R) \perp \mathcal{GI}_{B^{\dagger_C}}(R)$ and $\mathcal{GP}_B(R) \perp \mathcal{I}_{B^{\dagger_C}}(R)$ can be removed from Corollary 6.16.

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