J. Math. Kyoto Univ. (JMKYAZ)
48-4 (2008), 847–855

On Lie algebras of K-invariant functions

By

Ibrahima TOURE and Kinvi KANGNI

Abstract

Let G be a locally compact group and let K be a compact subgroup of $Aut(G)$, the group of automorphisms of G. (G, K) is a Gelfand pair if the algebra $L^1_K(G)$ of K-invariant integrable functions on G is commutative under convolution. In this paper, we give some charactezations of this algebra in the nilpotent case, which generalize some results obtained by C. Benson, J. Jenkins, G. Ratcliff in [1] and obtain a new criterion for Gelfand pairs.

1. Introduction

Let G be a locally compact group and let K be a compact subgroup of automorphisms of G. We designate by $L^1_K(G)$ the algebra (under convolution) of $K-$invariant integrable functions with complex-valued on G. With the Lie algebra structure induced by the convolution product, $L^1_K(G)$ is a Lie algebra. The purpose of this paper is to study and give some characterizations of this algebra when it's nilpotent. When $L^1_K(G)$ is abelian, the pair (G, K) is a Gelfand pair. The notion of Gelfand pair has been sufficiently studied by a number of authors (refer to [1], [2], [3]). So our main result, in this paper, provides a new criterion for Gelfand pairs. The first result gives a necessary condition for $L^1_K(G)$ to be Lie nilpotent. We recall that this result is true in the abelian case.

Theorem 1. *If $L^1_K(G)$ is Lie nilpotent then G is unimodular.*

Now let G be a connected Lie group. We designate by $\mathcal{E}'_K(G)$, the algebra (under convolution) of $K-$invariant compactly supported distributions. With the Lie algebra structure induced by the convolution product, $\mathcal{E}'_K(G)$ is a Lie algebra. We have a following result:

Theorem 2. *Let G be a connected Lie group and let K be a compact subgroup of $Aut(G)$. Then $L^1_K(G)$ is Lie nilpotent with step $\leq p$ if and only if $\mathcal{E}'_K(G)$ is Lie nilpotent with step $\leq p$.*

2000 *Mathematics Subject Classification*(*s*). Primary 430A20, 17B30.
Received April 21, 2008
Revised June 24, 2008

Theorem 1.12 from reference [1] gives a necessary and sufficient condition for (G, K) to be a Gelfand pair. We obtain a similar result as a necessary condition for $L^1_K(G)$ to be Lie nilpotent. Working from these results we show that Lie nilpotent implies abelian and finally obtain a new criterion for Gelfand pairs.

Theorem 3. *If $L^1_K(G)$ is Lie nilpotent then it is abelian. Thus (G, K) is a Gelfand pair if and only if $L^1_K(G)$ is Lie nilpotent.*

Taking $K = \{e\}$ this shows, in particular, that $L^1(G)$ is Lie nilpotent if and only if G is abelian. So we obtain an analytic version of a known algebraic theorem, namely: A group ring $F[G]$ over a field F of characteristic zero is Lie nilpotent if and only if G is abelian. See [6].

2. Notations and setup

We use the notations and setup of this section in the rest of the paper without mentioning.
G is a locally compact group with identity element e. $L^1(G)$ denotes the algebra of all integrable complex-valued functions on G, $C(G)$ (respectively $C^\infty(G)$) the algebra of all continuous, complex-valued functions (respectively infinitely differentiable) on G and $C_c(G)$ (respectively $C_c^\infty(G)$) the subalgebra of $C(G)$ (respectively $C^\infty(G)$) of compactly supported functions. We fix a left-invariant Haar measure dy on G and let $*$ denotes the corresponding convolution of functions.

K is a compact subgroup of $Aut(G)$, the group of automorphism of G. Writing the action by $k \in K$ on $x \in G$ as $k.x$ and letting dk denote the normalized Haar measure on K, we put for all function $f \in L^1(G)$,

$$f^K(x) = \int_K f(k.x)dk \quad \text{for all } x \in G.$$

$L^1_K(G)$ denotes the $K-$invariant functions on G, i.e those $f \in L^1(G)$ such that $f^K = f$. We define on $L^1_K(G)$ the following bracket: $[f, g] = f * g - g * f$ for all functions $f, g \in L^1_K(G)$. Let ad denote the adjoint representation of $L^1_K(G)$. Recall the descending central series for $L^1_K(G)$

$$L^1_K(G) = C^{(o)}(L^1_K(G)) \supset C^{(1)}(L^1_K(G)) \supset \ldots \supset C^{(m)}(L^1_K(G)) \supset \ldots$$

where $C^{(m)}(L^1_K(G))$=$[L^1_K(G), C^{(m-1)}(L^1_K(G))]$ for an integer m such that $m \geq 1$. If p is a non-negative integer we say that $L^1_K(G)$ is Lie nilpotent of step p if $C^{(p)}(L^1_K(G)) = \{0\}$ and $C^{(p-1)}(L^1_K(G)) \neq \{0\}$ or equivalently if $ad^p(f) = 0$ for all $f \in L^1_K(G)$.

We denote by $\triangle$ the modular function on G. In the Theorems 2 and 3, G is a connected Lie group. Then we designate by $\mathcal{E}'(G)$ the space of compactly supported distributions. The convolution of distributions $D_1, D_2 \in \mathcal{E}'(G)$ is defined by $\langle D_1 * D_2, f\rangle = \langle D_1, \varphi\rangle$ where $\varphi(x) = \langle D_2, {}_{x^{-1}}f\rangle \forall x \in G$ and ${}_{x^{-1}}f(y) = f(xy) \forall y \in G$.

K is always a compact subgroup of $Aut(G)$. Let $\mathcal{E}'_K(G)$ denote the algebra of $K-$invariant compactly supported distributions that is the distributions $D \in \mathcal{E}'(G)$ such that $D^K = D$ where D^K is defined by $\langle D^K, f\rangle = \langle D,\ f^K\rangle$ for all $f \in C_c^\infty(G)$. We define on $\mathcal{E}'_K(G)$ the following bracket $[D_1, D_2] = D_1 * D_2 - D_2 * D_1$.

If δ_x is the delta function at $x \in G$, then $\delta_x^K \in \mathcal{E}'_K(G)$ has compact support $K.x$. One has for all functions $f \in C_c(G)$,

$$\langle \delta_x^K, f\rangle = \langle \delta_x, f^K\rangle = f^K(x) = \int_K f(k.x)dk.$$

We show that for all $x_1, x_2, \ldots, x_p \in G$,

(2.1)
$$\langle \delta_{x_1}^K * \delta_{x_2}^K * \ldots * \delta_{x_p}^K,\ f\rangle = \int_{K^P} f\left(\left(k_1.x_1\right)\left(k_2.x_2\right)\ldots\left(k_p.x_p\right)\right)\, dk_1\, dk_2 \ldots\, dk_p$$

for all $f \in C_c^\infty(G)$. Let S_p denote the set of all permutations of order p. For $i \in \{1, 2, \ldots, p\}$, let put

$$S_p^i = \{\sigma \in S_p : \sigma(i) = p;\\ \sigma(1) < \sigma(2) < \ldots < \sigma(i) \text{ and } \sigma(i+1) > \sigma(i+2) > \ldots > \sigma(p)\}.$$

We note that :

$$S_p^1 = \{\sigma\} \text{ where } \sigma(1) = p,\ \sigma(2) = p-1, \ldots, \sigma(p) = 1,\\ S_p^p = \{id\}.$$

In general $S_p^i = S_p^{i,1} \cup S_p^{i,1^\sim}$ where

$$S_p^{i,1} = \left\{\sigma \in S_p^i;\ \sigma(1) = 1\right\}$$

and

$$S_p^{i,1^\sim} = \left\{\sigma \in S_p^i;\ \sigma(p) = 1\right\}.$$

3. Proofs of results

In this section we will give the proofs of ours results.

Theorem 3.1. *If $L_K^1(G)$ is Lie nilpotent then G is unimodular.*

Proof. Suppose $L_K^1(G)$ is Lie nilpotent of step p then $C^{(p-1)}(L_K^1(G)) \neq \{0\}$ and $C^{(p)}(L_K^1(G)) = \{0\}$. Thus for all function $f \in L_K^1(G)$ and for all function $g \in C^{(p-1)}(L_K^1(G))$ not identically zero, one has for all $x \in G$, $f * g(x) = g * f(x)$ i.e

$$\int_G f(y)g(y^{-1}x)dy = \int_G \triangle\left(y^{-1}\right) f(y)g(xy^{-1})dy,$$

which implies

$$\int_G f(y)\left(g(y^{-1}x) - \triangle\left(y^{-1}\right)\ g(xy^{-1})\right) dy = 0.$$

Thus for all function $h \in L^1(G)$,

$$\begin{aligned} 0 &= \int_G h^K(y)\ \ \left(g(y^{-1}x) - \triangle\left(y^{-1}\right)\ g(xy^{-1})\right) dy \\ &= \int_G\int_K h(y)\ \ \left(g(y^{-1}(k.x)) - \triangle\left(y^{-1}\right)\ g((k.x)\,y^{-1})\right)\ dk\,dy \end{aligned}$$

which implies that

$$\int_K \left(g(y^{-1}(k.x)) - \triangle\left(y^{-1}\right)\ g((k.x)\,y^{-1})\ \right) dk = 0$$

for all $y \in G$ and for all function $g \in C^{(p-1)}(L^1_K(G))$. In particular, for $x = e$, we obtain $g(y^{-1}) - \triangle\left(y^{-1}\right)\ g(\,y^{-1}) = 0$ for all $y \in G$. We deduce that $\triangle\,(y) = 1$, for all $y \in G$ and so G is unimodular. □

We will first show the following result which is useful for the proof of Theorem 2.

Lemma 3.1. *For all $x_1, x_2, \ldots, x_{p+1} \in G$,*

$$ad\left(\delta^K_{x_1}\right) ad\left(\delta^K_{x_2}\right)\ldots ad\left(\delta^K_{x_p}\right)\delta^K_{x_{p+1}} = \sum_{i=1}^{p+1}\sum_{\sigma\in S^i_{p+1}} (-1)^{p+1-i}\,\delta^K_{x_{\sigma(1)}} * \ldots * \delta^K_{x_{\sigma(p+1)}}.$$

Proof. For $p = 1$, one has

$$ad\left(\delta^K_{x_1}\right)\ \delta^K_{x_2} = \delta^K_{x_1} * \delta^K_{x_2} - \delta^K_{x_2} * \delta^K_{x_1}$$

and

$$\sigma \in S^1_2 \Longrightarrow \sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$
$$\sigma \in S^2_2 \Longrightarrow \sigma = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Thus

$$\sum_{i=1}^{2}\sum_{\sigma\in S^i_2} (-1)^i\,\delta^K_{x_{\sigma(1)}} * \delta^K_{x_{\sigma(2)}} = -\delta^K_{x_2} * \delta^K_{x_1} + \delta^K_{x_1} * \delta^K_{x_2}.$$

Let's assum that the equality is true for some p and let's consider $x_1, x_2, \ldots,$ $x_{p+2} \in G$, we have.

$$\begin{aligned} A &= ad\left(\delta^K_{x_1}\right)\ ad\left(\delta^K_{x_2}\right)\ldots ad\left(\delta^K_{x_{p+1}}\right)\delta^K_{x_{p+2}} \\ &= ad\left(\delta^K_{x_1}\right)\left[ad\left(\delta^K_{x_2}\right)\ldots ad\left(\delta^K_{x_{p+1}}\right)\delta^K_{x_{p+2}}\right]. \end{aligned}$$

and puting $y_1 = x_2$, $y_2 = x_3, \ldots, y_{p+1} = x_{p+2}$, we have

$$\begin{aligned} B &= ad\left(\delta^K_{x_2}\right)\ ad\left(\delta^K_{x_3}\right)\ldots ad\left(\delta^K_{x_{p+1}}\right)\delta^K_{x_{p+2}} \\ &= ad\left(\delta^K_{y_1}\right)\ ad\left(\delta^K_{y_2}\right)\ldots ad\left(\delta^K_{y_p}\right)\delta^K_{y_{p+1}} \\ &= \sum_{i=1}^{p+1}\sum_{\sigma\in S^i_{p+1}}(-1)^{p+1-i}\,\delta^K_{y_{\sigma(1)}} * \ldots * \delta^K_{y_{\sigma(p+1)}}. \end{aligned}$$

Thus

$$\begin{aligned} A &= ad\left(\delta^K_{x_1}\right)\ B \\ &= \delta^K_{x_1} * B - B * \delta^K_{x_1} \\ &= \sum_{i=1}^{p+1}\sum_{\sigma\in S^i_{p+1}}(-1)^{p+1-i}\,\delta^K_{x_1} * \delta^K_{y_{\sigma(1)}} * \delta^K_{y_{\sigma(2)}} * \ldots * \delta^K_{y_{\sigma(p+1)}} \\ &\quad - \sum_{i=1}^{p+1}\sum_{\sigma\in S^i_{p+1}}(-1)^{p+1-i}\,\delta^K_{y_{\sigma(1)}} * \ldots * \delta^K_{y_{\sigma(p+1)}} * \delta^K_{x_1}. \end{aligned}$$

Let put

$$\sigma'(1) = 1,\ \sigma'(i) = \sigma(i-1) + 1 \text{ for all } i \in \{2, 3, \ldots, p+2\}$$

and

$$\sigma''(i) = \sigma(i) + 1, \text{for all } i \in \{1, 2, \ldots, p+1\},\ \sigma''(p+2) = 1.$$

One has $\sigma' \in S^{i+1}_{p+2}$ and $\sigma'' \in S^i_{p+2}$.
Hence

$$\begin{aligned} A &= \sum_{i=1}^{p+1}\sum_{\sigma'\in S^{i+1}_{p+2}}(-1)^{p+1-i}\,\delta^K_{x_{\sigma'(1)}} * \delta^K_{y_{\sigma'(2)-1}} * \delta^K_{y_{\sigma'(3)-1}} * \ldots * \delta^K_{y_{\sigma'(p+2)-1}} \\ &\quad - \sum_{i=1}^{p+1}\sum_{\sigma''\in S^i_{p+2}}(-1)^{p+1-i}\,\delta^K_{y_{\sigma''(1)-1}} * \ldots * \delta^K_{y_{\sigma''(p+1)-1}} * \delta^K_{x_{\sigma''(p+2)}}. \\ &= \sum_{i=1}^{p+1}\sum_{\substack{\sigma'\in S^{i+1}_{p+2} \\ \sigma'(1)=1}}(-1)^{p+1-i}\,\delta^K_{x_{\sigma'(1)}} * \ldots * \delta^K_{x_{\sigma'(p+2)}} \\ &\quad - \sum_{i=1}^{p+1}\sum_{\substack{\sigma''\in S^i_{p+2} \\ \sigma''(p+2)=1}}(-1)^{p+1-i}\,\delta^K_{x_{\sigma''(1)}} * \ldots * \delta^K_{x_{\sigma''(p+2)}} \\ &= \sum_{i=2}^{p+2}\sum_{\sigma'\in S^{i,1}_{p+2}}(-1)^{p+2-i}\,\delta^K_{x_{\sigma'(1)}} * \ldots * \delta^K_{x_{\sigma'(p+2)}} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p+1}\sum_{\sigma''\in S^{i,1\sim}_{p+2}}(-1)^{p+2-i}\,\delta^K_{x_{\sigma''(1)}}*\ldots*\delta^K_{x_{\sigma''(p+2)}}\\
&=\sum_{i=1}^{p+2}\sum_{\sigma\in S^{i}_{p+2}}(-1)^{p+2-i}\,\delta^K_{x_{\sigma(1)}}*\ldots*\delta^K_{x_{\sigma(p+2)}}
\end{aligned}
$$

so we have the equality for $p+1$, which proves the lemma. □

Theorem 3.2. *Let G be a connected Lie group and let K be a compact subgroup of $Aut(G)$. Then $L^1_K(G)$ is Lie nilpotent of step $\leq p$ if and only if $\mathcal{E}'_K(G)$ is Lie nilpotent of step $\leq p$.*

Proof. The convolution product is continuous as is the adjoint representation ad of $\mathcal{E}'_K(G)$ (resp. of $L^1_K(G)$). Let $D\in\mathcal{E}'_K(G)$, there exists a net $\{f_n\}\subset C^\infty_c(G)$ such that $\{f_n\}$ converges to D.
For $\varphi\in C^\infty_c(G)$ one has:

$$
\begin{aligned}
\lim_{n\longrightarrow+\infty}\langle f^K_n,\varphi\rangle &= \lim_{n\longrightarrow+\infty}\langle f_n,\varphi^K\rangle\\
&=\langle D,\varphi\rangle.
\end{aligned}
$$

Then $\{f^K_n\}\subset C^\infty_K(G)$ converges to D. Since $C^\infty_K(G)\subset L^1_K(G)$ and $L^1_K(G)$ is Lie nilpotent of step $\leq p$ then $ad^p(D)=ad^p(\lim_{n\longrightarrow+\infty}f^K_n)=\lim_{n\longrightarrow+\infty}(ad^p(f^K_n))=0$.
So $\mathcal{E}'_K(G)$ is Lie nilpotent of step $\leq p$.
Conversely, if $\mathcal{E}'_K(G)$ is Lie nilpotent of step $\leq p$, we have for $h\in C^\infty_c(G)$ and for all $x_1,x_2,\ldots,x_{p+1}\in G$,

$$
\langle ad\left(\delta^K_{x_1}\right)ad\left(\delta^K_{x_2}\right)\ldots ad\left(\delta^K_{x_p}\right)\delta^K_{x_{p+1}},h\rangle=0
$$

and thanks to Lemma 3.1 and the equality (2.1) we have

$$
\begin{aligned}
&\sum_{i=1}^{p+1}\sum_{\sigma\in S^i_{p+1}}(-1)^{p+1-i}\\
&\times\int_{K^{p+1}}h((k_{\sigma(1)}.x_{\sigma(1)})(k_{\sigma(2)}.x_{\sigma(2)})\ldots(k_{\sigma(p+1)}.x_{\sigma(p+1)}))dk_{\sigma(1)}\ldots dk_{\sigma(p+1)}=0
\end{aligned}
$$

which implies that

(3.1)

$$
\begin{aligned}
&\sum_{i=1}^{p}\sum_{\sigma\in S^i_{p+1}}(-1)^{p+1-i}\\
&\quad\times\int_{K^{p+1}}h((k_{\sigma(1)}.x_{\sigma(1)})(k_{\sigma(2)}.x_{\sigma(2)})\ldots(k_{\sigma(p+1)}.x_{\sigma(p+1)}))dk_{\sigma(1)}\ldots dk_{\sigma(p+1)}\\
&\quad=-\int_{K^{p+1}}h((k_1.x_1)(k_2.x_2)\ldots(k_{p+1}.x_{p+1}))dk_1dk_2\ldots dk_{p+1}.
\end{aligned}
$$

For $f_1,\, f_2, \dots, f_{p+1} \in L^1_K(G)$ and $h \in C_c^\infty(G)$, we have thanks to the equality (3.1)

$$\begin{aligned}
&\int_G f_1 * f_2 * \dots * f_{p+1}(x_{p+1}) h(x_{p+1}) dx_{p+1} \\
&= \int_{G^{p+1}} f_1(x_1) f_2(x_2) \dots f_p(x_p) f_{p+1}(x_p^{-1} \dots x_1^{-1} x_{p+1}) h(x_{p+1}) dx_1 \dots dx_{p+1} \\
&= \int_{G^{p+1}} f_1(x_1) f_2(x_2) \dots f_p(x_p) f_{p+1}(x_{p+1}) h(x_1 x_2 \dots x_{p+1}) dx_1 \dots dx_{p+1} \\
&= \int_{G^{p+1}} f_1(x_1) \dots f_{p+1}(x_{p+1}) \\
&\quad \times \int_{K^{p+1}} h((k_1.x_1)(k_2.x_2) \dots (k_{p+1}.x_{p+1})) dk_1 \dots dk_{p+1} dx_1 \dots dx_{p+1} \\
&= -\sum_{i=1}^{p} \sum_{\sigma \in S^i_{p+1}} (-1)^{p+1-i} \\
&\quad \times \int_{G^{p+1}} f_{\sigma(1)}(x_{\sigma(1)}) \dots f_{\sigma(p+1)}(x_{\sigma(p+1)}) H(X) dx_{\sigma(1)} \dots dx_{\sigma(p+1)} \\
&= -\sum_{i=1}^{p} \sum_{\sigma \in S^i_{p+1}} (-1)^{p+1-i} \\
&\quad \times \int_{G^{p+1}} f_{\sigma(1)}(x_{\sigma(1)}) \dots f_{\sigma(p+1)}(x_{\sigma(p+1)}) \\
&\quad \times h(x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(p+1)}) dx_{\sigma(1)} \dots dx_{\sigma(p+1)} \\
&= -\sum_{i=1}^{p} \sum_{\sigma \in S^i_{p+1}} (-1)^{p+1-i} \\
&\quad \times \int_{G^{p+1}} f_{\sigma(1)}(x_{\sigma(1)}) \dots f_{\sigma(p+1)}(x^{-1}_{\sigma(p)} \dots x^{-1}_{\sigma(1)} x_{\sigma(p+1)}) \\
&\quad \times h(x_{\sigma(p+1)}) dx_{\sigma(1)} \dots dx_{\sigma(p+1)} \\
&= \int_G -\sum_{i=1}^{p} \sum_{\sigma \in S^i_{p+1}} (-1)^{p+1-i} f_{\sigma(1)} * f_{\sigma(2)} * \dots * f_{\sigma(p+1)}(x_{\sigma(p+1)}) \\
&\quad \times h(x_{\sigma(p+1)}) dx_{\sigma(p+1)}
\end{aligned}$$

where

$$\begin{aligned}
&H(X) \\
&\quad = \int_{K^{p+1}} h((k_{\sigma(1)}.x_{\sigma(1)})(k_{\sigma(2)}.x_{\sigma(2)}) \dots (k_{\sigma(p+1)}.x_{\sigma(p+1)})) dk_{\sigma(1)} \dots dk_{\sigma(p+1)}.
\end{aligned}$$

Hence we have

$$f_1 * f_2 * \ldots * f_{p+1} = -\sum_{i=1}^{p} \sum_{\sigma \in S_{p+1}^{i}} (-1)^{p+1-i} f_{\sigma(1)} * f_{\sigma(2)} * \ldots * f_{\sigma(p+1)}.$$

If we substitute $\delta_{x_i}^K$ by f_i in the Lemma 3.1, we have

$$ad(f_1)ad(f_2)ad(f_3)\ldots ad(f_p)f_{p+1} = 0.$$

So $L_K^1(G)$ is Lie nilpotent of step less than p. □

Theorem 3.3. *Let G be a connected Lie group and K a compact subgroup of $Aut(G)$. If $L_K^1(G)$ is Lie nilpotent then it is abelian. Thus (G,K) is a Gelfand pair if and only if $L_K^1(G)$ is Lie nilpotent.*

Proof. Let us show that if $L_K^1(G)$ is Lie nilpotent of step p then for all $x_1, x_2, \ldots, x_{p+1} \in G$, there exists an integer $m \in \{1, 2, 3, \ldots, p\}$ and a permutation $\sigma \in S_{p+1}^m$ such that : $x_1 x_2 x_3 \ldots x_{p+1} \in \big(K.x_{\sigma(1)}\big)\big(K.x_{\sigma(2)}\big)\ldots\big(K.x_{\sigma(p+1)}\big)$. In fact, if $L_K^1(G)$ is Lie nilpotent of step p then $\mathcal{E}_K'(G)$ is Lie nilpotent of step $\leq p$ according to Theorem 3.2. Thus for all $x_1, x_2, \ldots, x_{p+1} \in G$ one has:

$$ad\left(\delta_{x_1}^K\right)\, ad\left(\delta_{x_2}^K\right)\ldots ad(\delta_{x_p}^K)\,\delta_{x_{p+1}}^K = 0. \tag{3.2}$$

Let's suppose that there exists some elements $x_1, x_2, \ldots, x_{p+1} \in G$ such that for all $m \in \{1, 2, \ldots, p\}$ and for all permutation $\sigma \in S_{p+1}^m$, we have

$$x_1 x_2 \ldots x_{p+1} \notin \left(K.x_{\sigma(1)}\right)\left(K.x_{\sigma(2)}\right)\ldots\left(K.x_{\sigma(p+1)}\right).$$

Then there exists a compactly supported non negative function f such that

$$f\left(x_1 x_2 \ldots x_{p+1}\right) = 1$$

and

$$f\left(\left(K.x_{\sigma(1)}\right)\left(K.x_{\sigma(2)}\right)\ldots\left(Kx_{\sigma(p+1)}\right)\right) = 0$$

for all $\sigma \in S_{p+1}^m$. Hence

$$\begin{aligned}
&\langle ad\left(\delta_{x_1}^K\right)\, ad\left(\delta_{x_2}^K\right)\ldots ad\left(\delta_{x_p}^K\right)\, \delta_{x_{p+1}}^K, f\rangle \\
&= \left\langle \sum_{m=1}^{p+1} \sum_{\sigma \in S_{p+1}^m} (-1)^{p+1-m}\, \delta_{x_{\sigma(1)}}^K * \ldots * \delta_{x_{\sigma(p+1)}}^K, f \right\rangle \\
&= \left\langle \sum_{m=1}^{p} \sum_{\sigma \in S_{p+1}^m} (-1)^{p+1-m}\, \delta_{x_{\sigma(1)}}^K * \ldots * \delta_{x_{\sigma(p+1)}}^K, f \right\rangle \\
&\quad + \langle \delta_{x_1}^K * \ldots * \delta_{x_{p+1}}^K,\, f\rangle \\
&= \langle \delta_{x_1}^K * \ldots * \delta_{x_{p+1}}^K,\, f\rangle > 0
\end{aligned}$$

which is impossible according to (3.2).

This is a necessary condition for $L^1_K(G)$ to be Lie nilpotent. So suppose that $L^1_K(G)$ is Lie nilpotent of step $p \geq 1$ and let x, y be any pair of elements in G. Take $x_p = x$, $x_{p+1} = y$ and $x_j = e$ for $1 \leq j \leq p-1$ in the above statement. We have

$$xy = x_1 x_2 \dots x_{p+1} \in \left(K.x_{\sigma(1)}\right)\left(K.x_{\sigma(2)}\right)\dots\left(K.x_{\sigma(p+1)}\right) \tag{3.3}$$

for some permutation $\sigma \in S^m_{p+1}$ with $1 \leq m \leq p$.

But $K.e = \{e\}$ and $\sigma(p+1) < \sigma(p)$ holds for all such permutations σ. So (3.3) reads $xy \in (K.y)(K.x)$. Theorem 1.12 from reference [1] now implies (G, K) is a Gelfand pair. □

Acknowledgements. The authors would like to thank the referee for his helpful suggestion and his great contribution for the result of Theorem 3.

Université de COCODY - ABIDJAN
UFR de Mathématiques et Informatique
22 Bp 582 ABIDJAN 22
CÔTE D'IVOIRE
e-mail: toureibt@yahoo.fr

Université de COCODY - ABIDJAN
UFR de Mathématiques et Informatique
22 Bp 582 ABIDJAN 22
CÔTE D'IVOIRE
e-mail: kangnikinvi@yahoo.fr

References

[1] C. Benson, J. Jenkins and G. Ratcliff, *On Gelfand pairs associated with solvable Lie groups*, Trans. Amer. Maths. Soc. **321** (1990), 85–116.

[2] ———, *The orbit method and Gelfand pairs associated with nilpotent Lie groups*, J. Geom. Anal. **9**-4 (1999), 569–582.

[3] J. Faraut, *Analyse Harmonique sur les paires de Gelfand*, les cours du CIMPA, 1980.

[4] I. M. Gelfand, *Spherical functions on symetric spaces*, Dokl. Akad. Nauk USSR **70** (1960) 5–8.

[5] K. Kangni, *Transformation et distribution sphérique de type* δ, Thèse de Doctorat d'Etat, Abidjan, Mai 2000.

[6] I. B. S. Passi, D. S. Passman and S. K. Sehgal, *Lie solvable group rings*, Canad. J. Math. **25** (1973), 748–757.

[7] G. Warner, *Harmonic Analysis on semi-simple Lie Group*, Tome I, Springer-verlag, Berlin Heidelberg, New-York, 1972.