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On Samelson products in $\mathrm{Sp}(n)$

By

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1. Introduction

We work in the category of based spaces and based maps. For based spaces X, Y, $[X, Y]$ denotes the based homotopy set of maps from X to Y. When Y is a group-like space, we will always assume $[X, Y]$ is a group by the natural group structure. We denote Samelson products by $\langle -, - \rangle$. Let Q_n be the symplectic quasi-projective space of rank n, that is, the image of the map $\psi_n \colon S^{4n-1} \times S^3 \to \mathrm{Sp}(n)$ defined by the condition

$$\psi_n(x, \lambda)(v) = \begin{cases} v, & v \text{ is perpendicular to } x \\ x\lambda, & v = x \end{cases}$$

for $v \in \mathbf{H}^n$, where we identify S^{4k-1} with the unit sphere in $\mathbf{H}^k$ and $\mathbf{H}^n$ has the right scalar multiplication. Then Q_n has the cell structure $S^3 \cup e^7 \cup e^{11} \cup \cdots \cup e^{4n-1}$. We denote the inclusion $Q_n \to \mathrm{Sp}(n)$ by ϵ_n. Hamanaka, Kaji and Kono [4] determined the order of $\langle \epsilon_2, \epsilon_2 \rangle$ at the prime three. Hamanaka [3] also studies the Samelson products in the unitary groups localized at a prime p. In this note, we first determine the order of $\langle \epsilon_2, \epsilon_2 \rangle$, not at the prime three, by modifying the calculation in [4].

Theorem 1.1. *The order of the Samelson product* $\langle \epsilon_2, \epsilon_2 \rangle$ *is* 280.

This implies the following result:

Corollary 1.1 ([5]). $\mathrm{Sp}(2)_{(3)}$ *is homotopy commutative, where* $-_{(p)}$ *denotes the localization at a prime* p *in the sense of* [2].

Generalizing this calculation, we discuss the order of $\langle \epsilon_n, \epsilon_n \rangle$ at an odd prime in a certain range of n which is given by p. Using this, we determine the order of $\langle \epsilon_n, \epsilon_n \rangle$ for $n = 3, 4, 5$ at all primes but 2.

Theorem 1.2. *At odd primes, the order of the Samelson products* $\langle \epsilon_3, \epsilon_3 \rangle$, $\langle \epsilon_4, \epsilon_4 \rangle$ *and* $\langle \epsilon_5, \epsilon_5 \rangle$ *are* $31185 = 3^4 \cdot 5 \cdot 7 \cdot 11$, $6081075 = 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ *and* $68746552875 = 3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ *respectively.*

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Remark 1. By Theorem 1.2, it is true for $n \leq 5$ and odd prime p that $\mathrm{Sp}(n)_{(p)}$ is homotopy commutative if and only if the order of $\langle \epsilon_n, \epsilon_n \rangle$ is coprime to p. Then it is plausible to think that it is true in general; We conjecture that $\mathrm{Sp}(2)_{(p)}$ is homotopy commutative whenever the order of $\langle \epsilon_n, \epsilon_n \rangle$ is coprime to p.

The above Remark is suggested by the referee. We would like to acknowledge his valuable comment.

2. Computing method

The coefficient of cohomology will be always the ring of integers $\mathbf{Z}$. Recall that the cohomology of $\mathrm{Sp}(n)$ is

$$H^*(\mathrm{Sp}(n)) = \Lambda(x_3, x_7, \ldots, x_{4n-1}),$$

where x_{4i-1} is the suspension of the universal i-th symplectic Pontrjagin class. Then the cohomology of the infinite Stiefel manifold $X_n = \mathrm{Sp}(\infty)/\mathrm{Sp}(n)$ is

$$H^*(X_n) = \Lambda(y_{4n+3}, y_{4n+7}, \ldots)$$

such that $\pi^*(y_{4i-1}) = x_{4i-1}$ by the projection $\pi\colon \mathrm{Sp}(\infty) \to X_n$. Thus the cohomology of ΩX_n is given by

$$H^*(\Omega X_n) = \langle a_{4n+2}, a_{4n+6}, \ldots, a_{8n+2} \rangle \text{ for } * \leq 8n+3,$$

where a_{4i-2} is the suspension of y_{4i-1} and $\langle e_1, e_2, \ldots \rangle$ denotes the free abelian group with a basis $e_1, e_2, \ldots$.

Consider the fiber sequence

$$\Omega\mathrm{Sp}(\infty) \xrightarrow{\Omega\pi} \Omega X_n \xrightarrow{\delta} \mathrm{Sp}(n) \xrightarrow{i} \mathrm{Sp}(\infty).$$

Then it induces an exact sequence of groups

$$\widetilde{KSp}^{-2}(X) \xrightarrow{(\Omega\pi)_*} [X, \Omega X_n] \xrightarrow{\delta_*} [X, \mathrm{Sp}(n)] \xrightarrow{i_*} \widetilde{KSp}^{-1}(X) \tag{2.1}$$

where we identify $[\Sigma^k X, B\mathrm{Sp}(\infty)]$ with $\widetilde{KSp}^{-k}(X)$. Let $\gamma\colon \mathrm{Sp}(n) \wedge \mathrm{Sp}(n) \to \mathrm{Sp}(n)$ be the commutator map of $\mathrm{Sp}(n)$. Since $\mathrm{Sp}(\infty)$ is homotopy commutative, we have

$$i_*(\gamma) = 0$$

and then there exists $\tilde{\gamma}\colon \mathrm{Sp}(n) \wedge \mathrm{Sp}(n) \to \Omega X_n$ such that

$$\delta \circ \tilde{\gamma} \simeq \gamma.$$

Thus we obtain

$$\langle \epsilon_n, \epsilon_n \rangle = \delta_*(\tilde{\gamma} \circ (\epsilon_n \wedge \epsilon_n)). \tag{2.2}$$

Denote the equivalence class of $\tilde{\gamma}\circ(\epsilon_n\wedge\epsilon_n)$ in the cokernel of $(\Omega\pi)_*$ by $[\tilde{\gamma}\circ(\epsilon_n\wedge\epsilon_n)]$ and the order of an element g of a group by $\#g$. Then by (2.1) and (2.2), we have

$$\#\langle\epsilon_n,\epsilon_n\rangle=\#[\tilde{\gamma}\circ(\epsilon_n\wedge\epsilon_n)]$$

and thus we will calculate $\#[\tilde{\gamma}\circ(\epsilon_n\wedge\epsilon_n)]$. We will do this by constructing an injection from $[Q_n\wedge Q_n,\Omega X_n]$ to a free module over a ring under consideration using cohomology classes $a_{4n+2},a_{4n+6},\dots,a_{8n-2}\in H^*(\Omega X_n)$. The images of $\tilde{\gamma}\circ(\epsilon_n\wedge\epsilon_n)$ and $\mathrm{Im}(\Omega\pi)_*$ through this map is determined by the following two lemmas and then it only remains a tedious calculation of elementary divisors to determine $\#[\tilde{\gamma}\circ(\epsilon_n\wedge\epsilon_n)]$.

Lemma 2.1 ([6]). *We can choose a lift* $\tilde{\gamma}$ *such that* $\tilde{\gamma}^*(a_{4n+4k-2})=\sum_{i+j=n+k}y_{4i-1}\otimes y_{4j-1}$ *for* $1\le k\le n$.

Let $\mathrm{ad}\colon[\Sigma X,Y]\cong[X,\Omega Y]$ denote the adjoint congruence and let $\Sigma\colon H^*(X)\to H^{*+1}(\Sigma X)$ be the suspension isomorphism. We write by $\mathbf{c}'$ the complexification of quaternions and by $\bar{\mathbf{c}}'$ the map $X_n\to\mathrm{U}(\infty)/\mathrm{U}(2n)$ induced from $\mathbf{c}'$.

Lemma 2.2 ([4, Lemma 3]). *For a map* $\alpha\colon\Sigma^2X\to B\mathrm{Sp}(\infty)$*, we have*

$$(\Omega\pi\circ\mathrm{ad}^2\,\alpha)^*(a_{4n+4k-2})=(-1)^{n+k}(2n+2k-1)!\Sigma^{-2}ch_{2n+2k}(\mathbf{c}'(\alpha)),$$

where ch_i *is the* $2i$*-dimensional part of the Chern character.*

Proof. Note that $(\Omega\pi)^*(a_{4n+4k-2})=(\Omega\pi)^*(\sigma(y_{4n+4k-1}))=\sigma(\pi^*(y_{4n+4k-1}))=\sigma(x_{4n+4k-1})=\sigma^2(q_{n+k})$, where σ and q_i are the cohomology suspension and the universal i-th symplectic Pontrjagin class respectively. Note also that $(\mathbf{c}')^*(c_{2i})=(-1)^iq_i$, where c_j is the universal j-th Chern class. Then it follows that

$$\begin{aligned}(\Omega\pi)^*(a_{4n+4k-2})=\sigma^2(q_{n+k})&=(-1)^{n+k}(\Omega^2\mathbf{c}')^*(\sigma^2(c_{2n+2k}))\\&=(-1)^{n+k}(\Omega^2\mathbf{c}')^*\circ\beta^*(s_{2n+2k-1}),\end{aligned}$$

where $\beta\colon\Omega^2B\mathrm{U}(\infty)\xrightarrow{\cong}\mathbf{Z}\times B\mathrm{U}(\infty)$ is Bott periodicity and s_i is defined by the Newton formula $s_i=-\sum_{j=1}^{i-1}(-1)^js_{i-j}c_j-(-1)^iic_i$ with $s_1=c_1$. Now, for any $\xi\colon X\to B\mathrm{U}(\infty)$, we have $\xi^*(s_i)=i!ch_i(\xi)$. Since the Chern character commutes with Bott periodicity, the proof is completed. □

3. The Samelson product $\langle\epsilon_2,\epsilon_2\rangle$

In this section we calculate the order of the Samelson product $\langle\epsilon_2,\epsilon_2\rangle$ in the group $[Q_2\wedge Q_2,\mathrm{Sp}(2)]$. We first show:

Lemma 3.1. *The group* $[Q_2\wedge Q_2,\Omega X_2]$ *is a free abelian group.*

Proof. The cell structure of X_2 is given by $X_2 = S^{11} \cup e^{15} \cup e^{19} \cup \cdots$. We consider the homotopy fiber F of the inclusion of the bottom cell $S^{11} \to X_2$. Then, by the standard spectral sequence argument and the Hurewicz theorem, we have $\pi_{14}(F) \cong \mathbf{Z}$. Consider the homotopy exact sequence of the fiber sequence $F \to S^{11} \to X_2$:

$$\cdots \to \pi_{15}(S^{11}) \to \pi_{15}(X_2) \to \pi_{14}(F) \to \pi_{14}(S^{11}) \to \cdots.$$

Since $\pi_{15}(S^{11}) = 0$ and $\pi_{14}(S^{11})$ is a torsion group, we obtain $\pi_{15}(X_2) \cong \mathbf{Z}$.

Let A be the three cell complex obtained from $Q_2 \wedge Q_2$ by pinching the bottom cell. We have $A = (S^{10} \vee S^{10}) \cup e^{14}$ and then it is a suspension. Thus $[A, \Omega X_2]$ is an abelian group. Since X_2 is 10-connected, the pinch map of the bottom cell $Q_2 \wedge Q_2 \to A$ induces an isomorphism $[A, \Omega X_2] \cong [Q_2 \wedge Q_2, \Omega X_2]$. Consider the exact sequence induced from the cofiber sequence $S^{10} \vee S^{10} \to A \to S^{14}$:

$$\begin{aligned} \cdots \to \pi_{11}(\Omega X_2) \oplus \pi_{11}(\Omega X_2) \to \pi_{14}(\Omega X_2) \\ \to [A, \Omega X_2] \to \pi_{10}(\Omega X_2) \oplus \pi_{10}(\Omega X_2) \to \cdots. \end{aligned}$$

Since $\pi_i(\Omega X_2)$ are free abelian groups as above for $i = 10, 14$ and $\pi_{11}(\Omega X_2)$ is a torsion group, $[A, \Omega X_2]$ is a free abelian group. Thus the proof is completed. □

It is obvious that the cohomology of Q_2 is

$$H^*(Q_2) = \langle u_3, u_7 \rangle, \ \epsilon_2^*(x_i) = u_i \text{ for } i = 3, 7.$$

Let us define a map $\lambda = (\lambda_1, \lambda_2, \lambda_3) \colon [Q_2 \wedge Q_2, \Omega X_2] \to \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ by

$$\begin{aligned} \alpha^*(a_{10}) &= \lambda_1(\alpha) u_3 \otimes u_7 + \lambda_2(\alpha) u_7 \otimes u_3, \\ \alpha^*(a_{14}) &= \lambda_3(\alpha) u_7 \otimes u_7 \end{aligned}$$

for $\alpha \in [Q_2 \wedge Q_2, \Omega X_2]$. Let y denote the map

$$\left(\prod_{i=2}^{\infty} y_{4i+3} \right) \circ \Delta \colon X_2 \to \prod_{i=2}^{\infty} K(\mathbf{Z}, 4i+3)$$

where Δ is the diagonal map. Since λ is given by the composition

$$[Q_2 \wedge Q_2, \Omega X_2] \xrightarrow{(\Omega y)_*} \prod_{i=2}^{\infty} H^{4i+2}(Q_2 \wedge Q_2) \cong \mathbf{Z}^3,$$

λ is a homomorphism of groups. Since the rationalization of y is a rational homotopy equivalence, it follows that λ is an isomorphism after tensoring $\mathbf{Q}$. Thus, by Lemma 3.1, we obtain:

Proposition 3.1. *The map λ is a monomorphism of abelian groups.*

We will calculate the order of $\tilde{\gamma} \circ (\epsilon_2 \wedge \epsilon_2)$ in the cokernel of $(\Omega\pi)_*\colon \widetilde{KSp}^{-2}(Q_2 \wedge Q_2) \to [Q_2 \wedge Q_2, \Omega X_2]$ by making use of this map λ. To do so, we next calculate $\widetilde{KSp}^{-2}(Q_2 \wedge Q_2)$. Let $\iota\colon \Sigma Q_2 \to B\mathrm{Sp}(\infty)$ be the adjoint of the composition of the inclusions $Q_2 \to \mathrm{Sp}(2) \to \mathrm{Sp}(\infty)$ and let $\alpha\colon \Sigma Q_2 \to B\mathrm{Sp}(\infty)$ be the pinch map of the bottom cell $q\colon \Sigma Q_2 \to S^8$ followed by a generator of $\pi_8(B\mathrm{Sp}(\infty)) \cong \mathbf{Z}$. Then a standard argument shows that $\widetilde{KSp}(\Sigma Q_2)$ is a free abelian group with a basis ι, α. Define maps $\theta_i\colon \Sigma Q_2 \wedge \Sigma Q_2 \to B\mathrm{Sp}(\infty)$ for $i = 1, 2, 3, 4$ as follows. Let $\mathbf{q}$ denote the quaternionization and let β and β' be generators of $\widetilde{KO}(S^8)$ and $\widetilde{KSp}(S^{16})$ respectively. Then we put

$$\theta_1 = \mathbf{q}(\mathbf{c}'(\iota) \wedge \mathbf{c}'(\iota)),\ \theta_2 = (q \wedge 1)^*(\beta \wedge \iota),\ \theta_3 = (1 \wedge q)^*(\iota \wedge \beta),\ \theta_4 = (q \wedge q)^*(\beta').$$

Lemma 3.2. *$\widetilde{KSp}(\Sigma Q_2 \wedge \Sigma Q_2)$ is a free abelian group with a basis $\theta_1, \ldots, \theta_4$.*

Proof. Write $X = \Sigma Q_2 \wedge \Sigma Q_2$. The cell structure of X is $S^8 \cup e^{12} \cup e^{12} \cup e^{16}$. Compute $\widetilde{KSp}(X)$ by the exact sequences induced from cofiber sequences

$$S^8 \to X \to X/S^8,\ S^{12} \vee S^{12} \to X/S^8 \to S^{16}.$$

Then we see that $\widetilde{KSp}(X)$ is a free abelian group with a basis $\xi_1, \ldots, \xi_4$ which are characterized by:

1. $\xi_1|_{S^8}$ is a generator of $\widetilde{KSp}(S^8) \cong \mathbf{Z}$.
2. There exist $\eta_1, \eta_2 \in \widetilde{KSp}(X/S^8)$ such that $\eta_1|_{S^{12} \vee S^{12}}$ and $\eta_2|_{S^{12} \vee S^{12}}$ form a basis of $\widetilde{KSp}(S^{12} \vee S^{12}) \cong \mathbf{Z} \oplus \mathbf{Z}$ and η_1, η_2 pull back to ξ_2, ξ_3 respectively by the pinch map $X \to X/S^8$.
3. A generator of $\widetilde{KSp}(S^{16}) \cong \mathbf{Z}$ pulls back to ξ_4 by the pinch map $X \to S^{16}$.

One can easily check that $\theta_1, \ldots, \theta_4$ satisfy above properties and then the proof is completed. □

By Proposition 3.1, our remaining task is to compute $\lambda((\Omega\pi)_*(\theta_i))$ for $i = 1, 2, 3, 4$ and $\lambda(\tilde{\gamma} \circ (\epsilon_n \wedge \epsilon_n))$. It is easy to see that

$$ch(\mathbf{c}'(\iota)) = \Sigma u_3 - \frac{1}{6}\Sigma u_7.$$

Then it follows that

$$\begin{aligned} ch(\mathbf{c}'(\theta_1)) &= ch(\mathbf{c}'\mathbf{q}(\mathbf{c}'(\iota) \wedge \mathbf{c}'(\iota)))) \\ &= ch(\mathbf{c}'(\iota) \wedge \mathbf{c}'(\iota) + \overline{\mathbf{c}'(\iota) \wedge \mathbf{c}'(\iota)}) \\ &= 2\Sigma u_3 \otimes \Sigma u_3 - \frac{1}{3}\Sigma u_3 \otimes \Sigma u_7 - \frac{1}{3}\Sigma u_7 \otimes \Sigma u_3 + \frac{1}{18}\Sigma u_7 \otimes \Sigma u_7. \end{aligned}$$

By [1], we also have

$$\begin{aligned} ch(\mathbf{c}'(\theta_2)) &= \Sigma u_7 \otimes \Sigma u_3 - \frac{1}{6}\Sigma u_7 \otimes \Sigma u_7, \\ ch(\mathbf{c}'(\theta_3)) &= \Sigma u_3 \otimes \Sigma u_7 - \frac{1}{6}\Sigma u_7 \otimes \Sigma u_7, \\ ch(\mathbf{c}'(\theta_4)) &= 2\Sigma u_7 \otimes \Sigma u_7. \end{aligned}$$

Thus by Lemma 2.2, $\lambda((\Omega\pi)_*(\theta_i))$ is

$$\left(\frac{5!}{3}, \frac{5!}{3}, \frac{7!}{18}\right), \left(-5!, 0, -\frac{7!}{6}\right), \left(0, -5!, -\frac{7!}{6}\right), (0, 0, 2\cdot 7!)$$

respectively for $i = 1, 2, 3, 4$. On the other hand, it follows from Lemma 2.1 that

$$\lambda(\tilde{\gamma} \circ (\epsilon_2 \wedge \epsilon_2)) = (1, 1, 1)$$

and thus, by an easy calculation, we establish Theorem 1.1. □

Proof of Corollary 1.1. By the above result, $\langle \epsilon_{2(3)}, \epsilon_{2(3)} \rangle = 0$. Taking the adjoint, the Whitehead product $[\iota_{2(3)}, \iota_{2(3)}]$ vanishes for $\iota_2 = \mathrm{ad}^{-1}\,\epsilon_2 \colon \Sigma Q_2 \to B\mathrm{Sp}(2)$. Note that $\Sigma Q_{2(3)}$ is a retract of $\Sigma \mathrm{Sp}(2)_{(3)}$ and the retraction r can be chosen to satisfy $\iota_{2(3)} \circ r = \mathrm{ad}^{-1}\,1$. Thus $[\mathrm{ad}^{-1}\,1, \mathrm{ad}^{-1}\,1]$ also vanishes and the proof is completed. □

4. The Samelson product $\langle \epsilon_n, \epsilon_n \rangle$ at odd primes

In this section, we consider the Samelson product $\langle \epsilon_n, \epsilon_n \rangle$ at an odd prime in some range of n. For $n = 3, 4, 5$, the order of $\langle \epsilon_n, \epsilon_n \rangle$ will be given at any odd primes.

As in the previous section, we need the following lemma to construct an injection $[Q_n \wedge Q_n, \Omega X_n]_{(p)} \to \mathbf{Z}_{(p)}^{n(n+1)/2}$.

Lemma 4.1. *For an odd prime p, the group $[Q_n \wedge Q_n, \Omega X_n]_{(p)}$ is a free $\mathbf{Z}_{(p)}$-module for $n < (p-1)p$.*

Proof. We denote the k-skeleton of CW-complex X by $X^{(k)}$. Note that $Q_n \wedge Q_n$ has cells in dimensions $6, 10, 14, \ldots, 8n-2$ and ΩX_n is $(4n+1)$-connected. Thus

$$q^* \colon [(Q_n \wedge Q_n)/(Q_n \wedge Q_n)^{(4n)}, \Omega X_n] \cong [Q_n \wedge Q_n, \Omega X_n]$$

by the pinch map of the $4n$-skeleton $q\colon Q_n \wedge Q_n \to (Q_n \wedge Q_n)/(Q_n \wedge Q_n)^{(4n)}$. We write $(Q_n \wedge Q_n)/(Q_n \wedge Q_n)^{(4n)}$ by Y. Since Y is $(8n-2)$-dimensional and $(4n+1)$-connected, it is a suspension. Then $[Y, \Omega X_n]$ is an abelian group. By the rational cohomology, we have

$$(4.1) \qquad \pi_{4n+4}(X_n)_{(0)} = \pi_{4n+8}(X_n)_{(0)} = \cdots = \pi_{8n}(X_n)_{(0)} = 0.$$

Consider the exact sequence

$$\cdots \to \bigoplus^{2n-k+1} \pi_{4k}(X_n) \to [\Sigma^2 Y^{(4k-2)}, X_n] \to [\Sigma^2 Y^{(4k-6)}, X_n] \to \cdots$$

induced from the cofiber sequence $Y^{(4k-6)} \to Y^{(4k-2)} \to \bigvee^{2n-k+1} S^{4k-2}$ for $k = n+1, \ldots, 2n$. Then, by (4.1), we have obtained that $[\Sigma^2 Y^{(4k-2)}, X_n]$ are torsion groups for $k = n+1, \ldots, 2n$. Next we consider the exact sequence

$$\cdots \to [\Sigma^2 Y^{(4k-2)}, X_n] \to \bigoplus^{2n-k} \pi_{4k+3}(X_n) \\ \to [\Sigma Y^{(4k+2)}, X_n] \to [\Sigma Y^{(4k-2)}, X_n] \to \cdots$$

induced by the cofiber sequence $Y^{(4k-2)} \to Y^{(4k+2)} \to \bigvee^{2n-k} S^{4k+2}$ for $k = n+1, \ldots, 2n-1$. If $\pi_{4k+3}(X_n)_{(p)}$ are free $\mathbf{Z}_{(p)}$-modules for $k = n, \ldots, 2n-1$, $[\Sigma Y, X_n]_{(p)}$ is a free $\mathbf{Z}_{(p)}$-module by induction since $[\Sigma^2 Y^{(4k-2)}, X_n]$ are torsion groups as above. Then we shall show that $\pi_{4k+3}(X_n)_{(p)}$ are free $\mathbf{Z}_{(p)}$-modules. To do so, we consider the homotopy fiber F_k of the inclusion of the $(4k-1)$-skeleton $X_n^{(4k-1)} \to X_n$ for $k = n+1, \ldots, 2n-1$. By the standard spectral sequence argument, we have

$$\pi_{4k+2}(F_k) \cong \mathbf{Z}. \tag{4.2}$$

Since $X_n^{(4k-1)}$ is $(4n+2)$-connected, $\pi_{4k+3}(X_n^{(4k-1)})$ is in the stable range. Then the cofiber sequence $X_n^{(4i-5)} \to X_n^{(4i-1)} \to S^{4i-1}$ induces an exact sequence

$$\cdots \to \pi_{4k+3}(X_n^{(4i-5)}) \to \pi_{4k+3}(X_n^{(4i-1)}) \to \pi_{4k+3}(S^{4i-1}) \to \cdots$$

for $i = n+2, \ldots, k$. Recall that the p-primary component of the $4l$-dimensional stable homotopy groups of spheres are trivial for $l < (p-1)p - 1$. Then by induction we have

$$\pi_{4k+3}(X_n^{(4k-1)})_{(p)} = 0. \tag{4.3}$$

Now the fiber sequence $F_k \to X_n^{(4k-1)} \to X_n$ induces the homotopy exact sequence

$$\cdots \to \pi_{4k+3}(F_k) \to \pi_{4k+3}(X_n^{(4k-1)}) \to \pi_{4k+3}(X_n) \to \pi_{4k+2}(F_k) \to \cdots .$$

Thus, by (4.2) and (4.3), $\pi_{4k+3}(X_n)_{(p)}$ is a free $\mathbf{Z}_{(p)}$-module and the proof is completed. □

Let p be an odd prime and $n < (p-1)p$ as above. Then we define $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n(n+1)/2})\colon\ [Q_n \wedge Q_n, \Omega X_n]_{(p)} \to \mathbf{Z}_{(p)}^{n(n+1)/2}$ as

$$\alpha^*(a_{4n+2}) = \lambda_1(\alpha) u_3 \otimes u_{4n-1} + \lambda_2(\alpha) u_7 \otimes u_{4n-5} + \cdots + \lambda_n(\alpha) u_{4n-1} \otimes u_3,$$
$$\alpha^*(a_{4n+6}) = \lambda_{n+1}(\alpha) u_7 \otimes u_{4n-1} + \lambda_2(\alpha) u_{11} \otimes u_{4n-5} + \cdots + \lambda_{2n-1}(\alpha) u_{4n-1} \otimes u_7,$$
$$\vdots$$
$$\alpha^*(a_{8n-2}) = \lambda_{n(n+1)/2}(\alpha) u_{4n-1} \otimes u_{4n-1}.$$

Note that $\lambda_{(0)}\colon [Q_n \wedge Q_n, \Omega X_n]_{(0)} \to \mathbf{Q}^{n(n+1)/2}$ is an isomorphism as in Section 3. Thus by Lemma 4.1, we have:

Proposition 4.1. *The map* $\lambda\colon [Q_n \wedge Q_n, \Omega X_n]_{(p)} \to \mathbf{Z}_{(p)}^{n(n+1)/2}$ *is monic.*

Next, we need to calculate $\widetilde{KSp}(\Sigma Q_n \wedge \Sigma Q_n)$. If we localize at an odd prime p, the complexification $\mathbf{c}'_{(p)}\colon \widetilde{KSp}(\Sigma Q_n \wedge \Sigma Q_n)_{(p)} \to \widetilde{K}(\Sigma Q_n \wedge \Sigma Q_n)_{(p)}$ is an isomorphism and thus we consider $\widetilde{K}(\Sigma Q_n \wedge \Sigma Q_n)$ instead of $\widetilde{KSp}(\Sigma Q_n \wedge \Sigma Q_n)$. Let η be the canonical line bundle on $\mathbf{C}P^{2n-1}$ minus the trivial line bundle. Then we have $\widetilde{H}^*(\Sigma \mathbf{C}P^{2n-1}) = \langle \Sigma c_1(\eta), \Sigma c_1(\eta)^2, \ldots, \Sigma c_1(\eta)^{2n-1}\rangle$. By Kozima and Toda [7], there is a map $t_n\colon Q_n \to \Sigma \mathbf{C}P^{2n-1}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Sp}(n) & \xrightarrow{\mathbf{c}'} & \mathrm{U}(2n) \\ {\scriptstyle \epsilon_n}\uparrow & & \uparrow \\ Q_n & \xrightarrow{t_n} & \Sigma \mathbf{C}P^{2n-1} \end{array}$$

where the right vertical arrow is the inclusion.

Proposition 4.2. *The group* $\widetilde{K}(\Sigma Q_n)$ *is a free abelian group with a basis* $t_n^*(\Sigma^2\eta), t_n^*(\Sigma^2\eta^3), \ldots,\ t_n^*(\Sigma^2\eta^{2n-1})$.

Proof. We have

$$H^*(\Sigma Q_n) = \langle \Sigma u_3, \Sigma u_7, \ldots \Sigma u_{4n-1}\rangle,$$

and

$$t_n^*(\Sigma^2 c_1(\eta)^{2i-1}) = (-1)^i \Sigma u_{4i-1}.$$

By the naturality of the Chern character, we have established Proposition 4.3. □

Put $\theta_{ij} = t_n^*(\Sigma\eta^{2i-1}) \wedge t_n^*(\Sigma\eta^{2j-1})$. Then it follows that θ_{ij} $(1 \le i, j \le n)$ form a basis of $\widetilde{K}(\Sigma Q_n \wedge \Sigma Q_n)$. We set

$$\hat{\theta}_1 = \theta_{11}, \hat{\theta}_2 = \theta_{12}, \ldots \hat{\theta}_n = \theta_{1n}, \hat{\theta}_{n+1} = \theta_{21}, \ldots \hat{\theta}_{n^2} = \theta_{nn}.$$

Define an $\frac{n(n+1)}{2} \times n^2$ matrix $A_n = (a_{ij})$ by

$$\lambda \circ \Omega\pi(\hat{\theta}_j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{n(n+1)/2,j} \end{pmatrix}.$$

On the other hand, we have $\lambda \circ \Omega\pi(\tilde{\gamma} \circ \langle \epsilon_n, \wedge \epsilon_n\rangle)$. Therefore we can determine the order of $\langle \epsilon_n, \epsilon_n\rangle$ in principle. As examples, we shall calculate the order for $n = 3, 4, 5$ and prove Theorem 1.2.

4.1. The case n=3

For $n = 3$, we first compute Chern characters of $t_3^*(\Sigma^2\eta^{2i-1})$ using the equation $\Sigma u_{4i-1} = (-1)^i t_3^*(\Sigma^2 c_1(\eta)^{2i-1})$. The results are

$$\begin{aligned} ch(t_3^*(\Sigma^2\eta)) &= -\Sigma u_3 + \frac{1}{6}\Sigma u_7 - \frac{1}{120}\Sigma u_{11}, \\ ch(t_3^*(\Sigma^2\eta^3)) &= \Sigma u_7 - \frac{5}{4}\Sigma u_{11}, \\ ch(t_3^*(\Sigma^2\eta^5)) &= -\Sigma u_{11}. \end{aligned}$$

Then, by Lemma 2.2, we have

$$A_3 = \begin{pmatrix} 84 & 12600 & 10080 & 0 & 0 & 0 & 0 & 0 & 0 \\ 280 & 1680 & 0 & 1680 & 10080 & 0 & 0 & 0 & 0 \\ 84 & 0 & 0 & 12600 & 0 & 0 & 10080 & 0 & 0 \\ 1008 & 151200 & 120960 & 6048 & 907200 & 725760 & 0 & 0 & 0 \\ 1008 & 6048 & 0 & 151200 & 907200 & 0 & 120960 & 725760 & 0 \\ 5544 & 831600 & 665280 & 831600 & 124740000 & 99792000 & 665280 & 99792000 & 79833600 \end{pmatrix}.$$

On the other hand, we have $\lambda(\tilde{\gamma} \circ (\epsilon_3 \wedge \epsilon_3)) = {}^t(1,1,1,1,1,1)$ by Lemma 2.1. Then it follows that $\#[\tilde{\gamma} \circ (\epsilon_3 \wedge \epsilon_3)] = 3^4 \cdot 5 \cdot 7 \cdot 11$.

4.2. The case n=4

In this case, the Chern characters of $t_4^*(\Sigma^2\eta^{2i-1})$ are given as

$$\begin{aligned} ch(t_4^*(\Sigma^2\eta)) &= -\Sigma u_3 + \frac{1}{3!}\Sigma u_7 - \frac{1}{5!}\Sigma u_{11} + \frac{1}{7!}\Sigma u_{15}, \\ ch(t_4^*(\Sigma^2\eta^3)) &= \Sigma u_7 - \frac{5}{4}\Sigma u_{11} + \frac{43}{120}\Sigma u_{15}, \\ ch(t_4^*(\Sigma^2\eta^5)) &= -\Sigma u_{11} + \frac{10}{3}\Sigma u_{15}, \\ ch(t_4^*(\Sigma^2\eta^7)) &= \Sigma u_{15}. \end{aligned}$$

Then it is easy to obtain A_4 by Lemma 2.2. We also have $\lambda(\tilde{\gamma} \circ (\epsilon_4 \wedge \epsilon_4)) = {}^t(1,1,1,1,1,1,1,1,1,1)$ by Lemma 2.1. Thus we obtain

$$\#[\tilde{\gamma} \circ (\epsilon_4 \wedge \epsilon_4)] = 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13.$$

4.3. The case n=5

In this case, the Chern characters of $t_5^*(\Sigma^2\eta^{2i-1})$ are given as

$$\begin{aligned} ch(t_5^*(\Sigma^2\eta)) &= -\Sigma u_3 + \frac{1}{3!}\Sigma u_7 - \frac{1}{5!}\Sigma u_{11} + \frac{1}{7!}\Sigma u_{15} - \frac{1}{9!}u_{19}, \\ ch(t_5^*(\Sigma^2\eta^3)) &= \Sigma u_7 - \frac{5}{4}\Sigma u_{11} + \frac{43}{120}\Sigma u_{15} - \frac{605}{1296}\Sigma u_{19}, \\ ch(t_5^*(\Sigma^2\eta^5)) &= -\Sigma u_{11} + \frac{10}{3}\Sigma u_{15} - \frac{331}{144}\Sigma u_{19}, \\ ch(t_5^*(\Sigma^2\eta^7)) &= \Sigma u_{15} - \frac{77}{12}\Sigma u_{19}, \\ ch(t_5^*(\Sigma^2\eta^9)) &= -\Sigma u_{19}. \end{aligned}$$

Then we obtain A_5 as before. We also have $\lambda(\tilde{\gamma} \circ (\epsilon_5 \wedge \epsilon_5)) = {}^t(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$. By a tedious calculation of linear algebra, we obtain the result:

$$\#[\tilde{\gamma} \circ (\epsilon_5 \wedge \epsilon_5)] = 3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$$

□

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