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Central limit theorem for linear stochastic evolutions

By

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Abstract

We consider a Markov chain with values in $[0,\infty)^{\mathbb{Z}^d}$. The Markov chain includes some interesting examples such as the oriented site percolation, the directed polymers in random environment, and a time discretization of the binary contact process. We prove a central limit theorem for "the spatial distribution of population" when $d \geq 3$ and a certain square-integrability condition for the total population is satisfied. This extends a result known for the directed polymers in random environment to a large class of models.

1. Introduction

We write $\mathbb{N} = \{0,1,2,\cdots\}$, $\mathbb{N}^* = \{1,2,\cdots\}$ and $\mathbb{Z} = \{\pm x : x \in \mathbb{N}\}$. For $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, $|x|$ stands for the l^1-norm: $|x| = \sum_{i=1}^d |x_i|$. For $\xi = (\xi_x)_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, $|\xi| = \sum_{x\in\mathbb{Z}^d} |\xi_x|$. Let $(\Omega,\mathcal{F},P)$ be a probability space. We write $E[X] = \int X\ dP$ and $E[X : A] = \int_A X\ dP$ for a random variable X and an event A. We denote the constants by C, C_i.

We consider a time discretization of linear systems with values in $[0,\infty)^{\mathbb{Z}^d}$ which is introduced in [10]. A time discretization is defined by iteration of linear transformations for a row vector as follows,

$$N_t = N_0 A_1 \cdots A_t,\ t \in \mathbb{N}^*,$$

where $N_0 \in [0,\infty)^{\mathbb{Z}^d}$ is the initial vector and $A_t = (A_{t,x,y})_{x,y\in\mathbb{Z}^d},\ t = 1,2,\dots$ are i.i.d. random matrices (cf. (1.1)–(1.7) below for more details). This Markov chain contains some examples such as the oriented site percolation, the directed polymers in random environment, and a time discretization of the binary contact path process. We can regard $N_{t,y}$ as the "number of particles" at time-space (t,y). In this paper, we discuss the central limit theorem for "the spatial distribution of particles" in the case where the growth rate of "total numbers of particles" is the same as the mean growth rate with positive probability.

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The central limit theorem for stochastic growth models related to ours are discussed in several papers. To the best of our knowledge, the first results in this direction was obtained by E. Bolthausen for the directed polymers in random environment [2, 4] (see e.g., [3, 12] for more recent progress). The case of the branching random walks in random environment is discussed in [17]. Also, the result for continuous-time binary contact path process can be found in [11].

1.1. Linear stochastic evolution

First, we give the framework introduced by N. Yoshida [15], and in Subsection 1.2, we will give some examples contained in this framework. We call this framework "linear stochastic evolutions", and in the following, we will abbreviate it for LSE.

Let $\{A_t = (A_{t,x,y})_{x,y\in\mathbb{Z}^d} :\ t \geq 1\}$ be a sequence of i.i.d. random matrices with non-negative entries. We denote by $(\Omega, \mathcal{F}, P)$ the probability space on which these random matrices are defined. Here are the set of assumptions we assume throughout this article:

(1.1) For $t \in \mathbb{N}^*$, column vectors $(A_{t,x,y})_{x\in\mathbb{Z}^d}$, $y \in \mathbb{Z}^d$ are independent,

(1.2) $E[A_{1,x,y}^2] < \infty$ for all $x, y \in \mathbb{Z}^d$,

(1.3) $A_{t,x,y} = 0$ a.s. if $|x - y| > r_A$ for some non-random $r_A \in \mathbb{N}$,

(1.4) $A_{1,x,y}$ is not constant a.s. for some $x, y \in \mathbb{Z}^d$,

(1.5) $(A_t \circ \theta_z)_{t\in\mathbb{N}^*} \stackrel{\text{law}}{=} (A_t)_{t\in\mathbb{N}^*}$ for all $z \in \mathbb{Z}^d$ (**shift invariance**),

where $A_t \circ \theta_z = (A_{t,x+z,y+z})_{x,y\in\mathbb{Z}^d}$ for $z \in \mathbb{Z}^d$. We define a $[0,\infty)^{\mathbb{Z}^d}$-valued Markov chain $N_t = (N_{t,y})_{y\in\mathbb{Z}^d}$, $t = 1, 2, \ldots$ by:

$$N_{t+1,y} = \sum_{x\in\mathbb{Z}^d} N_{t,x} A_{t,x,y}\ ,\ t \in \mathbb{N}^*. \tag{1.6}$$

It is easy to check that $(N_t)_{t\in\mathbb{N}}$ is a Markov chain, since $(A_t)_{t\in\mathbb{N}}$ are i.i.d. random variables. Moreover, we suppose that the initial state N_0 is non-negative, non-random, and finite in the following sense,

$$\text{the set } \{x \in \mathbb{Z}^d : N_{0,x} > 0\} \text{ are finite and non-empty.} \tag{1.7}$$

If we regard $N_t \in [0,\infty)^{\mathbb{Z}^d}$ as a row vector, (1.6) can be interpreted as

$$N_t = N_0 A_1 A_2 \cdots A_t, \quad t \in \mathbb{N}^*,$$

i.e., we can consider A_t as a linear transform.

We write

$$a_y = E[A_{1,0,y}],\ |a| = \sum_{y\in\mathbb{Z}^d} a_y. \tag{1.8}$$

In [15], the dual process of $(M_t)_{t\in\mathbb{N}}$ is defined. Let $B_t = (B_{t,x,y})_{x,y\in\mathbb{Z}^d}$ be the transposed matrices of A_t. The dual process is the Markov chain with values in $[0,\infty)^{\mathbb{Z}^d}$ defined by

(1.9) $$\sum_{x\in\mathbb{Z}^d} A_{t,y,x}M_{t-1,x} = \sum_{x\in\mathbb{Z}^d} M_{t-1,x}B_{t,x,y} = M_{t,y}, \quad t\in\mathbb{N},\ y\in\mathbb{Z}^d,$$

where the initial state $M_0 \in [0,\infty)^{\mathbb{Z}^d}$ is non-random and finite. Most properties of LSE also hold for the dual process. However there are differences between LSE and the dual process. One of them is described at the end of Section 2.3.

1.2. Examples

• Oriented site percolation (OSP)

The oriented site percolation is one of the simplest examples of LSE. Let $\{\eta_{t,y} : (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$ be $\{0,1\}$-valued i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in (0,1)$. The site (t,y) with $\eta_{t,y} = 1$ and $\eta_{t,y} = 0$ are referred as *open* and *closed*. A sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N}\times\mathbb{Z}^d$ is called an *open path* from $(0,0)$ to (t,y), if $x_0 = 0$, $x_t = y$, $|x_s - x_{s-1}| = 1$, and $\eta_{s,x_s} = 1$ for all $s = 1,2,\ldots,t$. Let $N_{t,y}$ be the number of open paths from $(0,0)$ to (t,y), and let $|N_t| = \sum_{y\in\mathbb{Z}^d} N_{t,y}$ be the total number of open paths from $(0,0)$ to *level* t. Then, $N_{t,y}$ is LSE with

$$A_{t,x,y} = \eta_{t,y}\mathbf{1}_{|x-y|=1},$$

and $N_{0,x} = \delta_x$, where δ_x is the Dirac function, that is,

$$\delta_{x,y} = \delta_{y-x} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The covariance of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from

(1.10)
$$a_y = p\mathbf{1}_{|y|=1}, \quad E[A_{t,x,y}A_{t,\tilde{x},\tilde{y}}] = \begin{cases} p & \text{if } y=\tilde{y},\ |x-y| = |\tilde{x}-\tilde{y}| = 1, \\ a_{y-x}a_{\tilde{y}-\tilde{x}} & \text{if otherwise.} \end{cases}$$

In particular, we have $|a| = 2dp$. We refer the general $N_{t,y}$ as the "number of particles" at time-space (t,y), and $|N_t|$ as the "total number of particles " as in this example.

The next example is another important one.

• Directed polymers in random environment (DPRE)

Let $\{\eta_{t,y} : (t,y) \in \mathbb{N}\times\mathbb{Z}^d\}$ be $\mathbb{R}$-valued i.i.d. random variables such that

$$e^{\lambda(\beta)} \stackrel{\text{def}}{=} E[\exp(\beta\eta_{t,y})] < \infty \quad \text{for all } \beta \in (0,\infty).$$

We define $N_{t,y}$ by

$$N_{t,y} = E_S^0\Big[\exp\big(\sum_{u=1}^t \beta\eta(u,S_u)\big) : S_t = y\Big], \quad (t,y) \in \mathbb{N}^*\times\mathbb{Z}^d,$$

where (S_t, P_S^x) is a simple random walk on $\mathbb{Z}^d$ which is independent of $\{\eta_{t,y}\}$. Then, we call $|N_t|$ the partition function of the *directed polymers in random environment*. There are many papers on this model [2, 3, 4, 12], and the reader can find more information. Starting from $N_0 = (\delta_x)_{x \in \mathbb{Z}^d}$, the above expectation can be obtained inductively by (1.6) with

$$A_{t,x,y} = \frac{\mathbf{1}_{|x-y|=1}}{2d} \exp(\beta \eta_{t,y}).$$

The covariance of $(A_{t,x,y})_{x,y \in \mathbb{Z}^d}$ can be seen from

$$(1.11)\quad a_y = \frac{e^{\lambda(\beta)} \mathbf{1}_{|y|=1}}{2d}, \quad E[A_{t,x,y} A_{t,\tilde{x},\tilde{y}}] = \begin{cases} a_{y-x} a_{\tilde{y}-\tilde{x}} & \text{if } y \neq \tilde{y}, \\ e^{\lambda(2\beta)-2\lambda(\beta)} a_{y-x} a_{\tilde{y}-\tilde{x}} & \text{if } y = \tilde{y}. \end{cases}$$

In particular, we have $|a| = e^{\lambda(\beta)}$.

- **Binary contact path process (BCPP)**

The binary contact path process is a continuous-time Markov process with values in $\mathbb{N}^{\mathbb{Z}^d}$, originally introduced by D. Griffeath [7]. In this paper, we consider a discrete-time version. Let

$$\{\eta_{x,y} = 0, 1 : (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}, \quad \{\zeta_{t,y} = 0, 1 : (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\},$$
$$\{e_{t,y} : (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$$

be the families of i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in (0,1]$, $P(\zeta_{t,y} = 1) = q \in [0,1]$, and $P(e_{t,y} = e) = \frac{1}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We suppose that these three families are independent of each other. Starting from $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$, we define a Markov chain $(N_t)_{t \in \mathbb{N}}$ with values in $\mathbb{N}^{\mathbb{Z}^d}$ by

$$N_{t+1,y} = \eta_{t+1,y} N_{t,y-e_{t+1,y}} + \zeta_{t+1,y} N_{t,y}, \qquad t \in \mathbb{N}.$$

We interpret this process as a model of the spread of infection, by $N_{t,y}$ infected individuals at time t at the site y. The $\zeta_{t+1,y} N_{t,y}$ term above means that infected individuals $N_{t,y}$ remain infected at time $t+1$ with probability q, and they recover with probability $1-q$. On the other hand, the $\eta_{t+1,y} N_{t,y-e_{t+1,y}}$ term means that, a neighboring site $y - e_{t+1,y}$ is picked at random (say, the wind blows from that direction), and $N_{t,y-e_{t+1,y}}$ individuals at site y are infected anew at time $t+1$ with probability p. This Markov chain is obtained by (1.6) with

$$A_{t,x,y} = \eta_{t,y} \mathbf{1}_{\{e_{t,y} = y-x\}} + \zeta_{t,y} \delta_{y-x}.$$

The covariance of $(A_{t,x,y})_{x,y \in \mathbb{Z}^d}$ can be seen from

$$(1.12)\qquad a_y = \frac{\mathbf{1}_{|y|=1}}{2d} + q\delta_y,$$

$$(1.13)\qquad E[A_{t,x,y} A_{t,\tilde{x},\tilde{y}}] = \begin{cases} a_{y-x} a_{\tilde{y}-\tilde{x}} & \text{if } y \neq \tilde{y}, \\ a_{y-x} & \text{if } x = \tilde{x} \text{ and } y = \tilde{y}, \\ q\delta_{y-x} a_{y-\tilde{x}} + q\delta_{y-\tilde{x}} a_{y-x} & \text{if } x \neq \tilde{x} \text{ and } y = \tilde{y}. \end{cases}$$

In particular, we have $|a| = p + q$.

We give two examples of the dual process.

• **Random walk in time-space random environment**

We denote by $e_i \in \mathbb{Z}^d$, $i = 1, \dots, d$, the basis vectors of $\mathbb{Z}^d$ with $|e_i| = 1$. Let $\{p(t,x,e_i) : (t,x) \in \mathbb{N} \times \mathbb{Z}^d, i = 1, \dots, d\}$ be the $[0,1]$-valued i.i.d. random variables with the uniform distribution on $[0,1)$. We set

$$B_{t,x,y} = \sum_{1 \le i \le d} \Big(p(t-1,x,e_i) \frac{\mathbf{1}_{\{y=x+e_i\}}}{2d} + \big(1 - p(t-1,x,e_i)\big) \frac{\mathbf{1}_{\{y=x-e_i\}}}{2d} \Big).$$

Let us suppose $M_0 = (\delta_x)_{x \in \mathbb{Z}^d}$. Then, it is clear that $\sum_y B_{t,x,y} \equiv 1$ and $|M_t| \equiv 1$ P-a.s. for all $(t,x) \in \mathbb{N}^* \times \mathbb{Z}^d$. From (1.9), we can regard $M_{t,y}$ as the probability of the "t-step" transition from 0 to x. More precisely, the random walk moves according to the transition probabilities $P(S_{t+1} = y | S_t = x) = B_{t+1,x,y}$ for each $t \in \mathbb{N}$, $x, y \in \mathbb{Z}^d$.

• **Directed random walk in oriented bond percolation (DRWOBP)**

Let $\{\eta_{t,x,y} : t \in \mathbb{N}^*,\ x, y \in \mathbb{Z}^d, |x-y| = 1\}$ be the $\{0,1\}$-valued i.i.d. random variables with $P(\eta_{t,x,y} = 1) = p \in [0,1]$. The directed bond $|(t,x),(t+1,y)\rangle$ with $\eta_{t+1,x,y} = 1$ and $\eta_{t+1,x,y} = 0$ are referred as *open* and *closed.* A sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N} \times \mathbb{Z}^d$ is called an *open path* from $(0,0)$ to (t,y), if $x_0 = 0$, $x_t = y$, $|x_s - x_{s-1}| = 1$, and $\eta_{s,x_{s-1},x_s} = 1$ for all $s = 1, 2, \dots, t$. We define $A_{t,y,x} = B_{t,x,y}$ by

$$B_{t,x,y} = \frac{\eta_{t,x,y}}{\deg(t-1,x)} \mathbf{1}_{\{\deg(t-1,x) > 0\}}, \tag{1.14}$$

where $\deg(t,x)$ is the number of open bonds at site (t,x) i.e.

$$\deg(t-1,x) = \sharp\{y \in \mathbb{Z}^d : \eta_{t,x,y} = 1\}.$$

We set $M_{0,y} = (\delta_y)_{yin\mathbb{Z}^d}$. We define the dual process $M_{t,y}$, $t \in \mathbb{N}$, $y \in \mathbb{Z}^d$ by (1.9). The process (M_t) is interpreted as the transition probability of *the directed random walk in oriented bond percolation* starting from the origin. It is described as follows. The random walk starts from $(0,0) \in \mathbb{N} \times \mathbb{Z}^d$. Suppose that the random walk is at time-space (t,x) and let

$$V(t+1,x) = \{y \in \mathbb{Z}^d : \eta_{t+1,x,y} = 1\}.$$

If $V(t+1,x) = \emptyset$, then the random walk is stopped at time t. If, on the other hand, $V(t+1,x) \neq \emptyset$, then the next step of the random walk is determined by choosing the point $y \in V(t+1,x)$ with probability $1/\deg(t,x)$. Also, this random walk is interpreted as the random walk in i.i.d. bond percolation sequences in the following sense. For each time $t \in \mathbb{N}$, we give the p-bond percolation in $\mathbb{Z}^d$, independently. If the random walk starting from 0 is located at site x at time t, then it moves as a simple random walk in bond percolation, but it is killed if the site x has no bond. This random walk coincides with the random walk described above. This follows from the fact that the bond is related to at most one step, since the simple random walk is periodic. We remark that we *cannot* describe two random walks starting from points sitting next by this systems.

The covariance of $(B_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$b_y \stackrel{\text{def}}{=} E[B_{1,0,y}] = \frac{1}{2d}\big(1-(1-p)^{2d}\big)\mathbf{1}_{|y|=1}, \tag{1.15}$$

and

(1.16)

$$E[B_{t,x,y}B_{t,\tilde{x},\tilde{y}}] = \begin{cases} \sum_{k=1}^{2d}\frac{1}{k^2}p^k(1-p)^{2d-k}\binom{2d-1}{k-1} & \text{if } x=\tilde{x}, y=\tilde{y}, |x-y|=1, \\ \sum_{k=1}^{2d}\frac{1}{k^2}p^k(1-p)^{2d-k}\binom{2d-2}{k-2} & \text{if } x=\tilde{x}, y\neq\tilde{y}, |x-y|=1, |\tilde{x}-\tilde{y}|=1, \\ 1 & \text{if } x\neq\tilde{x},\ |x-y|=1,\ |\tilde{x}-\tilde{y}|=1, \\ 0 & \text{otherwise.} \end{cases}$$

These are the examples of LSE and the dual process. In the following Subsection 1.3, we discuss some properties for LSE, which also hold for the dual process.

1.3. Some basic properties

In this subsection, we discuss the basic properties of LSE.

First of all, we give another representation of $N_{t,y}$ by using the random walk $((S_t)_{t\in\mathbb{N}}, P_S^x)$ on $\mathbb{Z}^d$ defined by

$$P_S^x(S_0=x)=1 \textit{ and } P_S^x(S_1=y)=\overline{a}_{y-x} \stackrel{\text{def}}{=} a_{y-x}/|a|, \tag{1.17}$$

and we identify the mean growth rate of $|N_t|$ with $|a|^t$. We introduce notation and definition:

$$\zeta_0 = 1 \quad \text{for } t=0 \tag{1.18}$$

$$\zeta_t = \prod_{1\le u\le t}\frac{A_{u,S_{u-1},S_u}}{a_{S_u-S_{u-1}}} \quad \text{for } t\ge 1. \tag{1.19}$$

We denote by $\mathcal{F}_t$ and $\mathcal{G}_t$, $t\in\mathbb{N}$, the σ-field generated by $A_1,\dots,A_t$ and $S_1,\dots,S_t$, respectively. For $t=0$, we define $\mathcal{F}_0=\{\Omega,\emptyset\}$ and $\mathcal{G}_0=\{\tilde{\Omega},\emptyset\}$, where $(\tilde{\Omega},\mathcal{G})$ is the probability space on which $(S_t,P_S^x)_{x\in\mathbb{Z}^d}$ are defined. Let $\mathcal{H}_t$ be the product σ-field of $\mathcal{F}_t$ and $\mathcal{G}_t$ and we define $P_{A,S}^x$ and $E_{A,S}^x$ as the product measure of P and P_S^x and its expectation, respectively.

Remark 1. It is easy to see that $a_{S_t-S_{t-1}}\neq 0$ a.s. and

$$A_{t+1,z,y} = |a|E_S^z\Big[\frac{A_{t+1,S_0,S_1}}{a_{S_1-S_0}} : S_1=y\Big] \text{ a.s. for all } t\ge 1 \text{ and } z,y\in\mathbb{Z}^d \tag{1.20}$$

The following lemma gives the representation of $N_{t,y}$ by using ζ_t.

Lemma 1.1. *ζ_t is a martingale with respect to $\mathcal{H}_t$. Moreover, we have that*

$$N_{t,y} = |a|^t \sum_{x\in\mathbb{Z}^d} N_{0,x} E_S^x[\zeta_t : S_t = y], \qquad P\text{-}a.s. \tag{1.21}$$

Proof. The martingale property of ζ_t follows from the fact that it is a product of adapted, mean-one random variables. The latter equality (1.21) can be obtained by induction. It is easy to see that (1.21) holds for $t = 0$. If (1.21) holds for $t \geq 0$, then

$$\begin{aligned} N_{t+1,y} &= \sum_{z\in\mathbb{Z}^d} N_{t,z} A_{t+1,z,y} \\ &= \sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} |a|^t N_{0,x} E_S^x[\zeta_t : S_t = z] A_{t+1,z,y}. \end{aligned}$$

Considering the equality (1.20), we have that

$$N_{t+1,y} = |a|^{t+1} \sum_{x\in\mathbb{Z}^d} N_{0,x} E_S^x[\zeta_{t+1} : S_{t+1} = y].$$

□

The following lemma is obtained as a corollary.

Lemma 1.2. *$(|\overline{N}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ is a martingale, where $\overline{N}_t = (\overline{N}_{t,x})_{x\in\mathbb{Z}^d}$ is defined by*

$$\overline{N}_{t,x} = |a|^{-t} N_{t,x}.$$

Proof. This lemma is an immediate consequence of Lemma 1.1. □

From Lemma 1.2, we see that the mean growth rate of $|N_t|$ coincides with $|a|^t$. In the following lemma, we compare $|N_t|$ and its mean growth rate $|a|^t$.

Lemma 1.3. *Referring to Lemma 1.2, the limit*

$$|\overline{N}_\infty| = \lim_{t\to\infty} |\overline{N}_t| \tag{1.22}$$

exists a.s. and

$$E[|\overline{N}_\infty|] = |N_0| \text{ or } 0. \tag{1.23}$$

Moreover, $E[|N_\infty|] = |N_0|$ if and only if the limit (1.22) is convergent in $\mathbb{L}^1(P)$.

Proof. We refer the proof to [15, Lemma 1.3.2]. □

We introduce some more notations. For $(s,z)\in\mathbb{N}\times\mathbb{Z}^d$, we define $N_t^{s,z}=(N_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$ and $\overline{N}_t^{s,z}=(\overline{N}_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$, $t\in\mathbb{N}$ respectively by

$$(1.24)\qquad \begin{aligned} &N_{0,y}^{s,z}=\delta_{z,y},\quad N_{t+1,y}^{s,z}=\sum_{x\in\mathbb{Z}^d}N_{t,x}^{s,z}A_{s+t+1,x,y},\\ &\text{and } \overline{N}_{t,y}^{s,z}=|a|^{-t}N_{t,y}^{s,z}.\end{aligned}$$

Then, we can write

$$(1.25)\qquad |\overline{N}_{s+t}|=\sum_{y\in\mathbb{Z}^d}\overline{N}_{s,y}|\overline{N}_t^{s,y}|.$$

In particular, $(N_y^{0,z})$ is the Markov chain (1.6) with the initial state $N_0^{0,z}=(\delta_{z,y})_{y\in\mathbb{Z}^d}$. Moreover, we have

$$(1.26)\qquad N_{t,y}=\sum_{x\in\mathbb{Z}^d}N_{0,x}N_{t,y}^{0,x}\quad \text{for any initial state } N_0.$$

Now, it follows that

$$E[|\overline{N}_\infty^{0,0}|]=1 \text{ or } 0$$

from Lemma1.3. We will refer to the former case as *the regular growth phase* and the latter as *the slow growth phase*. By (1.25) and the shift invariance, $E[|\overline{N}_\infty|]=|N_0|$ for all N_0 in the regular growth phase and $E[|\overline{N}_\infty|]=0$ for all N_0 in the slow growth phase. The regular growth phase means that the growth rate of $|N_t|$ is the same order as its expectation $|a|^t|N_0|$. And the slow growth phase means that, almost surely, the growth rate is slower than that of its expectation.

We discuss the case in the regular growth phase in this article. We refer the reader to the paper [15] for information on the slow growth phase. In the following section, we will give a sufficient condition of regular growth phase for OSP, DPRE, and BCPP.

2. Regular growth phase

2.1. Preparation

We introduce some notations and prove a lemma in this subsection to discuss the main theorem in this paper. It is convenient to introduce the following notation:

(2.1)

$$w(x,\tilde{x},y,\tilde{y})=\begin{cases}E\left[\frac{A_{1,x,y}A_{1,\tilde{x},\tilde{y}}}{a_{y-x}a_{\tilde{y}-\tilde{x}}}\right]=\left(E\left[\frac{A_{1,x-y,0}A_{1,\tilde{x}-y,0}}{a_{y-x}a_{y-\tilde{x}}}\right]\right)^{\delta_{y,\tilde{y}}} & \text{if } a_{y-x}a_{\tilde{y}-\tilde{x}}\neq 0,\\ 0 & \text{if } a_{y-x}a_{\tilde{y}-\tilde{x}}=0.\end{cases}$$

Remark 2. Suppose that A_t satisfies the equation

(2.2)
$$E[A_{t,x,y}A_{t,\tilde{x},\tilde{y}}] = \begin{cases} \gamma^{\delta_{y,\tilde{y}}} a_{y-x}a_{\tilde{y}-\tilde{x}} & \text{if } a_{y-x}a_{\tilde{y}-\tilde{x}} \neq 0, \\ 0 & \text{if } a_{y-x}a_{\tilde{y}-\tilde{x}} = 0, \end{cases} \quad \text{for all } x,\tilde{x},y,\tilde{y} \in \mathbb{Z}^d,$$

where γ is a positive constant. Then, we can rewrite $w(x,\tilde{x},y,\tilde{y})$ as follows.

$$w(x,\tilde{x},y,\tilde{y}) = \begin{cases} \gamma^{\delta_{y,\tilde{y}}} & \text{if } a_{y-x}a_{\tilde{y}-\tilde{x}} \neq 0, \\ 0 & \text{if } a_{y-x}a_{\tilde{y}-\tilde{x}} = 0. \end{cases} \tag{2.3}$$

It can be seen from (1.10) and (1.11) that the equation (2.2) holds for OSP and DPRE, where $\gamma = 1/p$ and $\exp(\lambda(2\beta) - 2\lambda(\beta))$ respectively.

Let $(S,\tilde{S}) = ((S_t,\tilde{S}_t)_{t\in\mathbb{N}}, P^{x,\tilde{x}}_{S,\tilde{S}})$ denote the independent product of the random walks given in (1.17). Then, we obtain the following Feynmann-Kac formula.

Lemma 2.1.

$$E[N_{t,y}N_{t,\tilde{y}}] = |a|^{2t} \sum_{x_0,\tilde{x}_0\in\mathbb{Z}^d} N_{0,x_0}N_{0,\tilde{x}_0}E^{x_0,\tilde{x_0}}_{S,\tilde{S}}[e_t : (S_t,\tilde{S}_t) = (y,\tilde{y})] \tag{2.4}$$

for all $t \in \mathbb{N}$, *and* $y,\tilde{y} \in \mathbb{Z}^d$, *where*

$$e_t = \prod_{1\le u\le t} w(S_{u-1},\tilde{S}_{u-1},S_u,\tilde{S}_u), \ \textit{for } t \geq 1. \tag{2.5}$$

Consequently,

$$E[|\overline{N}_t|^2] = \sum_{x_0,\tilde{x_0}} N_{0,x_0}N_{0,\tilde{x}_0}E^{x_0,\tilde{x_0}}_{S,\tilde{S}}[e_t],$$

and

$$\begin{aligned} \sup_{t\in\mathbb{N}} E[|\overline{N}_t|^2] < \infty \Leftrightarrow \sup_{t\in\mathbb{N}} E^{0,0}_{S,\tilde{S}}[e_t] < \infty, \\ \Rightarrow E[|\overline{N}_\infty|] = |N_0|. \end{aligned} \tag{2.6}$$

Proof. From Lemma 1.1, we have that

$$\begin{aligned} E[N_{t,y}N_{t,\tilde{y}}] &= |a|^{2t} \sum_{x_0\in\mathbb{Z}^d}\sum_{\tilde{x}_0\in\mathbb{Z}^d} N_{0,x_0}N_{0,\tilde{x}_0}E\Big[E^{x_0}_S[\zeta_t : S_t = y]E^{\tilde{x}_0}_S[\zeta_t : S_t = \tilde{y}]\Big] \\ &= |a|^{2t} \sum_{x_0\in\mathbb{Z}^d}\sum_{\tilde{x}_0\in\mathbb{Z}^d} N_{0,x_0}N_{0,\tilde{x}_0}E^{x_0,\tilde{x}_0}_{S,\tilde{S}}\Big[E[\zeta_t\tilde{\zeta}_t] : (S_t,\tilde{S}_t) = (y,\tilde{y})\Big]. \end{aligned}$$

Now, we can easily check the equation:

$$E[\zeta_t\tilde{\zeta}_t] = e_t, \qquad P^{x,\tilde{x}}_{S,\tilde{S}}\text{-a.s. for all } x,\tilde{x} \in \mathbb{Z}^d. \tag{2.7}$$

□

Remark 3. When (2.2) holds, we have

$$e_t = \gamma^{\sum_{u=1}^{t} \delta_{S_u,\tilde{S}_u}}. \tag{2.8}$$

Let us assume (2.8) and set

$$\pi_x = P^{x,0}_{S,\tilde{S}}(S_t = \tilde{S}_t \text{ for some } t \in \mathbb{N}^*).$$

We then have for all $x \in \mathbb{Z}^d$ that

$$P^{x,0}_{S,\tilde{S}}\Big(\sum_{u=1}^{\infty} \delta_{S_u,\tilde{S}_u} = k\Big) = \begin{cases} 1-\pi_x & \text{for } k=0, \\ \pi_x \pi_0^{k-1}(1-\pi_0) & \text{for } k=1,2,\dots, \end{cases}$$

and hence that

$$\begin{aligned} &\sup_{t\in\mathbb{N}} P^{0,0}_{S,\tilde{S}}[e_t] < \infty \Leftrightarrow \pi_0\gamma < 1 \\ &\qquad \Rightarrow \lim_{t\to\infty} P^{x,0}_{S,\tilde{S}}[e_t] = 1 + \frac{\pi_x(\gamma-1)}{1-\pi_0\gamma}. \end{aligned} \tag{2.9}$$

On the other hand, it can be seen from (2.4) that

$$E[|\overline{N}^{0,x}_\infty||\overline{N}^{0,\tilde{x}}_\infty|] = \lim_{t\to\infty} P^{x,\tilde{x}}_{S,\tilde{S}}[e_t],$$

using the notation (1.24). Also, it follows from (2.6) and (2.9) that

$$\sup_{t\in\mathbb{N}} E[|\overline{N}_t|^2] < \infty \Leftrightarrow d \geq 3 \text{ and } \begin{cases} p > \pi_0 & \text{for OSP}, \\ \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_0) & \text{for DPRE}. \end{cases} \tag{2.10}$$

2.2. Results

In this subsection, we state the main theorem in this article. We show the central limit theorem for the spatial distribution of particles. More precisely, the theorem is given as follows:

Theorem 2.1. *Suppose that $d \geq 3$ and (2.6).*
Then, for all $f \in C_b(\mathbb{R}^d)$,

$$\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\Big(\frac{x-mt}{\sqrt{t}}\Big)\overline{N}_{t,x} = |\overline{N}_\infty| \int_{\mathbb{R}^d} f(x)d\nu(x), \quad P\text{-a.s.}, \tag{2.11}$$

where $C_b(\mathbb{R}^d)$ stands for the set of bounded continuous functions on $\mathbb{R}^d$,

$$m = (m_1,\dots,m_d) = \sum_{x\in\mathbb{Z}^d} x\overline{a}_x, \tag{2.12}$$

and ν is the Gaussian measure with

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu = \sum_{x\in\mathbb{Z}^d} (x_i - m_i)(x_j - m_j)\overline{a}_x, \quad i,j = 1,\dots,d. \tag{2.13}$$

Remark 4. From Lemma 1.1, we can rewrite (2.11) as

$$\lim_{t\to\infty}\sum_{x\in\mathbb{Z}^d} N_{0,x}E_S^x\Big[f\Big(\frac{S_t-mt}{\sqrt{t}}\Big)\zeta_t\Big]=|\overline{N}_\infty|\int_{\mathbb{R}^d} f(x)d\nu(x),\quad P\text{-a.s.} \tag{2.14}$$

If we set $\rho_{t,x}=\frac{N_{t,x}}{|N_t|}\mathbf{1}_{\{N_t>0\}}$, then we consider $\rho_{t,x}$ as the density of the particles. From this observation, we can regard Theorem 2.1 as the central limit theorem for probability measures with the density $\rho_{t,\sqrt{t}x}$ on $\{|\overline{N}_\infty|>0\}$.

We will give the proof of Theorem 2.1 in Subsection 2.3. First, we make some important observations. By Fatou's lemma, we obtain

$$E_{S,\tilde{S}}^{x,\tilde{x}}[\varliminf_{t\to\infty} e_t]\le \varliminf_{t\to\infty} E_{S,\tilde{S}}^{x,\tilde{x}}[e_t]\le \sup_{t\ge 1} E_{S,\tilde{S}}^{x,\tilde{x}}[e_t]<\infty\ .$$

We have that

$$e_t=e_{t+1}\quad\text{on } \{S_{t+1}\neq\tilde{S}_{t+1}\}, \tag{2.15}$$

and hence there exists the limit

$$e_\infty=\lim_{t\to\infty} e_t$$

almost surely because of the transience of random walk on $\mathbb{Z}^d$ for $d\ge 3$. We define k-th meeting time τ_k by

$$\tau_0=0,\ \text{and } \tau_{k+1}=\inf\{t>\tau_k : S_t=\tilde{S}_t\}, \tag{2.16}$$

where $\inf\emptyset=\infty$. Then, from the Markov property and the shift invariance, we can write that

$$E_{S,\tilde{S}}^{0,0}[e_\infty]=\sum_{k=0}^\infty\Big(E_{S,\tilde{S}}^{0,0}[e_{\tau_1}:\tau_1<\infty]\Big)^k P_{S,\tilde{S}}^{0,0}(\tau_{k+1}=\infty)<\infty.$$

Therefore, we conclude that

$$\eta=E_{S,\tilde{S}}^{0,0}[e_{\tau_1}:\tau_1<\infty]<1. \tag{2.17}$$

Lemma 2.2. *Suppose that $d\ge 3$ and (2.6). Then, there is a constant C such that*

$$E_{S,\tilde{S}}^{x,x}[e_t:S_t=\tilde{S}_t]=E_{S,\tilde{S}}^{0,0}[e_t:S_t=\tilde{S}_t]\le Ct^{-d/2}. \tag{2.18}$$

Proof. From the Markov property and the observation (2.15), we can decompose $E_{S,\tilde{S}}^{0,0}[e_t:S_t=\tilde{S}_t]$ into

$$E_{S,\tilde{S}}^{0,0}[e_t:S_t=\tilde{S}_t]=\sum_{k=1}^t\Big(\sum_{t_1+\cdots+t_k=t}\prod_{i=1}^k E_{S,\tilde{S}}^{0,0}[e_{t_i}:\tau_1=t_i]\Big).$$

We define a_t by

$$a_t = E^{0,0}_{S,\tilde{S}}[e_t : \tau_1 = t].$$

We remark that

$$a_t \le c_1 t^{-d/2}, \quad \sum_{t\ge 1} a_t = \eta < 1, \text{ and } \sum_{t\ge 1}\sum_{t_1+\cdots+t_k=t} a_{t_1}\cdots a_{t_k} = \eta^k. \tag{2.19}$$

It is enough to show that there exists some $\alpha < 1$ and $C_1 > 0$ such that

$$\sum_{t_1+\cdots+t_k=t} a_{t_1}\cdots a_{t_k} \le C_1\alpha^k t^{-d/2}, \quad \text{for all } t \ge 1. \tag{2.20}$$

For $k = 1$, this inequality holds. We consider the sequence $\{c_k\}_{k\ge 1}$ satisfying that for some $0 < \epsilon < 1$,

$$c_{k+1} = \frac{c_1}{(1-\epsilon)^{d/2}}\eta^k + \frac{c_k}{\epsilon^{d/2}}\eta, \tag{2.21}$$

where c_1 is given in (2.19). We suppose that for $k \ge 1$ the following inequality holds,

$$\sum_{t_1+\cdots+t_k=t} a_{t_1}\cdots a_{t_k} \le c_k t^{-d/2}, \quad \text{for all } t \ge 1. \tag{2.22}$$

Then, we have the inequality from (2.19) that

$$\begin{aligned}
&\sum_{t_1+\cdots+t_{k+1}=t} a_{t_1}\cdots a_{t_{k+1}}\\
&= \sum_{s=k}^{t-1}\Big(\sum_{t_1+\cdots+t_k=s} a_{t_1}\cdots a_{t_k}\Big)a_{t-s}\\
&\le \sum_{s\le \epsilon t}\Big(\sum_{t_1+\cdots+t_k=s} a_{t_1}\cdots a_{t_k}\Big)c_1(t-s)^{-d/2} + \sum_{\epsilon t\le s\le t} c_k s^{-d/2}a_{t-s}\\
&\le \sum_{s\le \epsilon t}\Big(\sum_{t_1+\cdots+t_k=s} a_{t_1}\cdots a_{t_k}\Big)c_1(t-\epsilon t)^{-d/2} + \sum_{\epsilon t\le s\le t} c_k (\epsilon t)^{-d/2}a_{t-s}\\
&\le \eta^k c_1(t-\epsilon t)^{-d/2} + \eta c_k(\epsilon t)^{-d/2}\\
&= c_{k+1}t^{-d/2},
\end{aligned}$$

and hence (2.22) holds for $k+1$. We choose ϵ such that $\eta < \epsilon^{d/2} < 1$. Then, we have $c_k \le C_2\left(\frac{\eta}{\epsilon^{d/2}}\right)^k$ for all $k \ge 1$ by simple calculation. □

We remark that we can show that $E^{0,0}_{S,\tilde{S}}[e_{t-1} : S_t = \tilde{S}_t] \le Ct^{-d/2}$ by a similar argument.

Lemma 2.2 means delocalization. This can be seen as follows.

For any $t > 0$, we define the probability measure μ_t on the product of the path space by

$$\mu_t(A) = \frac{1}{E^{x,\tilde{x}}_{S,\tilde{S}}[e_t]} E^{x,\tilde{x}}_{S,\tilde{S}}[\mathbf{1}_A e_t], \quad \text{for any } A \in \mathcal{G}_t \otimes \tilde{\mathcal{G}}_t.$$

Then, we can see that $\mu_t(S_t = \tilde{S}_t)$ is the probability that two paths meet at time t. It follows from Lemma 2.2 that the expected amount of the meeting is finite under $d \geq 3$ and (2.6).

Remark 5. In contrast to the delocalization result we obtained, it is shown in [16] that localization occurs when $|\overline{N}_\infty| = 0$ a.s. thus, when $d = 1, 2$.

In the end, we consider a sufficient condition for the regular growth phase for BCPP. For BCPP, $w(x, \tilde{x}, y, \tilde{y})$ is given as follows:

$$w(x,\tilde{x},y,\tilde{y}) = \begin{cases} 1/q & \text{if } x = \tilde{x} = y = \tilde{y}, \\ 2d/p & \text{if } x = \tilde{x},\ y = \tilde{y},\ |x-y| = 1, \\ 1 & \text{if } y = \tilde{y} = x,\ |\tilde{x}-y| = 1, \\ 1 & \text{if } y = \tilde{y} = \tilde{x},\ |x-y| = 1, \\ 1 & \text{if } y \neq \tilde{y} \text{ and } a_{y-x}a_{\tilde{y}-\tilde{x}}, \\ 0 & \text{if otherwise,} \end{cases} \tag{2.23}$$

where $1/q$ on the first line is replaced by 0 if $q = 0$.

In [15], it is shown that (2.17) is necessary and sufficient condition for the uniform square integrability of $|\overline{N}_t|$. Now, we will estimate $\eta = E^{0,0}_{S,\tilde{S}}[e_\tau : \tau < \infty]$. We can write from (2.23) and the Markov property that

$$\begin{aligned} \eta &= E^{0,0}_{S,\tilde{S}}[e_1 : \tau = 1] + E^{0,0}_{S,\tilde{S}}[e_\tau : 1 < \tau < \infty] \\ &= \frac{1}{p+q} + E^{0,0}_{S,\tilde{S}}[e_\tau : 1 < \tau < \infty] \\ &= \frac{1}{p+q} + E^{0,0}_{S,\tilde{S}}\big[E^{S_1,\tilde{S}_1}_{S,\tilde{S}}[e_\tau : \tau < \infty] : \tau \neq 1\big] \\ &= \frac{1}{p+q} + E^{0,0}_{S,\tilde{S}}\Big[P^{S_1,\tilde{S}_1}_{S,\tilde{S}}\big(|S_{\tau-1} - \tilde{S}_{\tau-1}| = 1, \tau < \infty\big) : \tau \neq 1\Big] \\ &\leq \frac{1}{p+q} + \sup_{\substack{|x|,|\tilde{x}| \leq 1, \\ |x-\tilde{x}| \geq 1}} \Big\{P^{x,\tilde{x}}_{S,\tilde{S}}\big(\tau < \infty\big)\Big\}\Big(1 - \frac{1}{2d}\Big(\frac{p}{p+q}\Big)^2 - \Big(\frac{q}{p+q}\Big)^2\Big). \end{aligned} \tag{2.24}$$

We have that if $x \neq 0$, then

$$\pi(x) \leq C_3(1/d) \quad \text{as } d \nearrow \infty.$$

We show this in Lemma 3.2. From this and (2.24), a sufficient condition is that

$$p + q > 1 \text{ and } d = d_{p,q} \geq 3 \text{ is large enough.} \tag{2.25}$$

Also, by numerical calculation, we find that the following condition is sufficient:

$$d \geq 3, \text{ and } (p,q) \in (1-\epsilon, 1] \times (1-\epsilon, 1], \tag{2.26}$$

where $\epsilon > 0$ is small enough (see, Appendix 3.1).

2.3. Proof of Theorem 2.1

We now show Theorem 2.1 by using the argument in [2]. First, we introduce some notations. Let $\{\xi_t\}_{t\geq 1}$ be i.i.d. random variables with values in $\mathbb{R}^d$. We denote by X_t the random walk with each step given by ξ_t. Moreover, we assume that $E[\exp(\theta \cdot \xi_1)] < \infty$ for θ in a neighborhood of 0 in $\mathbb{R}^d$. We define $\rho(\theta)$ by

$$\rho(\theta) = \ln E[\exp(\theta \cdot \xi_1)]. \tag{2.27}$$

Then, it is obvious that

$$\exp\Big(\theta \cdot X_t - t\rho(\theta)\Big)$$

is a martingale with respect to the filtration of the random walk.

We will use standard notation $x^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$ and $\left(\frac{\partial}{\partial x}\right)^{\mathbf{n}} = \left(\frac{\partial}{\partial x_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{n_d}$ for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, the polynomial $W_{\mathbf{n}}(t,x)$ is defined by

$$W_{\mathbf{n}}(t,x) = \left(\frac{\partial}{\partial\theta}\right)^{\mathbf{n}} \exp\Big(\theta \cdot x - t\rho(\theta)\Big)\Big|_{\theta=0},$$

where $|\mathbf{n}| = n_1 + \cdots + n_d$. We write

$$W_{\mathbf{n}}(t,x) = \sum A_{\mathbf{n}}(\mathbf{i}, j) x^{\mathbf{i}} t^j. \tag{2.28}$$

The coefficients $A_{\mathbf{n}}(\mathbf{i}, j)$ depend on the derivatives of ρ at 0. The following lemma gives some useful properties of $W_{\mathbf{n}}(t,x)$.

Lemma 2.3. *For a general random walk with* $\exp(\rho(\theta)) < \infty$ *for* θ *in a neighborhood of* 0 *and* $E[\xi_1] = 0$, *we have*

(a) *if* $|\mathbf{i}| + 2j > |\mathbf{n}|$, *then* $A_{\mathbf{n}}(\mathbf{i}, j) = 0$.

(b) *The coefficient with* $|\mathbf{i}| + 2j = |\mathbf{n}|$ *depends only on the second derivatives of* ρ *at 0, that is, on the covariance of* ξ_1.

(c) *If* $|\mathbf{i}| = |\mathbf{n}|$, *then* $A_{\mathbf{n}}(\mathbf{i}, 0) = \delta_{i_1,n_1}\delta_{i_2,n_2} \cdots \delta_{i_d,n_d}$.

Proof. All of them follow from simple calculation and the fact that $\partial\rho/\partial\theta_j$ at $\theta = 0$ equals 0. □

$W_{\mathbf{n}}(t, X_t)$ is a martingale with respect to the filtration of the random walk. Coming back to the random walk (S_t, P_S^x), we have that

$$Y_{\mathbf{n}}(t) = E_S^x[W_{\mathbf{n}}(t, S_t - mt)\zeta_t] \tag{2.29}$$

is an $\mathcal{F}_t$-martingale, since ζ_t is an $\mathcal{H}_t$-martingale.

Proposition 2.1. *If* $|\mathbf{n}| \geq 1$, *then*

$$\lim_{t\to\infty} t^{-|\mathbf{n}|/2} Y_{\mathbf{n}}(t) = 0, \quad P\text{-}a.s.$$

Proof. We show that the martingale

$$Z_t \stackrel{\text{def}}{=} \sum_{s=1}^{t} s^{-|\mathbf{n}|/2}(Y_{\mathbf{n}}(s) - Y_{\mathbf{n}}(s-1))$$

remains L^2-bounded. This implies that Z_t converges a.s. and hence the Proposition 2.1 follows from Kronecker's lemma. We assume that $m = 0$ for simplicity.

$$\begin{aligned} E\Big[\big(Y_{\mathbf{n}}(t) - Y_{\mathbf{n}}(t-1)\big)^2\Big] &= E\Big[\big(E_S^x[W_{\mathbf{n}}(t,S)\zeta_t - W_{\mathbf{n}}(t-1,S)\zeta_{t-1}]\big)^2\Big] \\ &\leq 2E\Big[\big(E_S^x[W_{\mathbf{n}}(t,S)(\zeta_t - \zeta_{t-1})]\big)^2\Big] &\quad (2.30) \\ &\quad + 2E\Big[\big(E_S^x[(W_{\mathbf{n}}(t,S) - W_{\mathbf{n}}(t-1,S))\zeta_{t-1}]\big)^2\Big]. &\quad (2.31) \end{aligned}$$

Then, (2.31) is equal to 0 from the observation after the proof of Lemma 2.3. Moreover, we have from (2.1) that

$$\begin{aligned} &\text{the RHS of (2.30)} \\ &= 2E\Big[E_S^x[W_{\mathbf{n}}(t,S)(\zeta_t - \zeta_{t-1})]E_{\tilde{S}}^x[W_{\mathbf{n}}(t,\tilde{S})(\tilde{\zeta}_t - \tilde{\zeta}_{t-1})]\Big] \\ &= 2E_{S,\tilde{S}}^{x,x}\left[W_{\mathbf{n}}(t,S)W_{\mathbf{n}}(t,\tilde{S})e_{t-1}E\Big[\Big(\frac{A_{t,S_{t-1},S_t}}{a_{S_t - S_{t-1}}} - 1\Big)\Big(\frac{A_{t,\tilde{S}_{t-1},\tilde{S}_t}}{a_{\tilde{S}_t - \tilde{S}_{t-1}}} - 1\Big)\Big]\right] \\ &= 2E_{S,\tilde{S}}^{x,x}\Big[W_{\mathbf{n}}(t,S)^2(e_t - e_{t-1})\mathbf{1}_{\{S_t = \tilde{S}_t\}}\Big], \end{aligned}$$

where $\tilde{\zeta}_t$ is defined by (1.19) for the random walk $(\tilde{S}_t, P_{\tilde{S}}^x)$. It is easy to see that $W_{\mathbf{n}}(t,x)^2 \leq C_4|x|^{2|\mathbf{n}|} + C_5 t^{|\mathbf{n}|}$ from Lemma 2.3. From this, it is enough to estimate $E_{S,\tilde{S}}^{0,0}\Big[|S_t|^{2|\mathbf{n}|}e_t\mathbf{1}_{\{S_t=\tilde{S}_t\}}\Big]$ and $E_{S,\tilde{S}}^{0,0}\Big[|S_t|^{2|\mathbf{n}|}e_{t-1}\mathbf{1}_{\{S_t=\tilde{S}_t\}}\Big]$. We define $\chi_{t_1,\dots,t_k}$ by

$$\chi_{t_1,\dots,t_k} = \mathbf{1}_{\{\tau_1 = t_1, \tau_2 - \tau_1 = t_2, \dots, \tau_k - \tau_{k-1} = t_k\}},$$

where τ_j is given by (2.16). Since $|S_{\tau_k}| \leq |x| + \sum_{j=1}^{k} |S_{\tau_j} - S_{\tau_{j-1}}|$, we have that for $p > d$

$$\begin{aligned}
&E^{x,x}_{S,\tilde{S}}\left[|S_t|^p e_t \mathbf{1}_{\{S_t=\tilde{S}_t\}}\right]\\
&\quad\le \sum_{k=1}^{t} \sum_{\substack{t_1+\cdots+t_k=t,\\ t_i\ge 1}} E^{x,x}_{S,\tilde{S}}\left[|S_{\tau_k}|^p e_{\tau_k}\chi_{t_1,\dots,t_k}\right]\\
&\quad\le \sum_{k=1}^{t}(k+1)^{p-1} \sum_{\substack{t_1+\cdots+t_k=t,\\ t_i\ge 1}} E^{0,0}_{S,\tilde{S}}\left[\sum_{j=1}^{k}|S_{\tau_j}-S_{\tau_{j-1}}|^p e_{\tau_k}\chi_{t_1,\dots,t_k}\right]\\
&\qquad+\sum_{k=1}^{t}(k+1)^{p-1} \sum_{\substack{t_1+\cdots+t_k=t,\\ t_i\ge 1}} |x|^p E^{0,0}_{S,\tilde{S}}\left[e_{\tau_k}\chi_{t_1,\dots,t_k}\right].
\end{aligned}$$

Also, since $E^{x,x}_{S,\tilde{S}}\left[\sum_{j=1}^{k} e_{\tau_k}\chi_{t_1,\dots,t_k}\right] = a_{t_1}\cdots a_{t_k}$, it follows that

$$\begin{aligned}
\text{the last term on the RHS} &\le |x|^p \sum_{k=1}^{t}(k+1)^{p-1}C_1\alpha^k t^{-d/2}\\
&\le |x|^p C t^{-d/2}.
\end{aligned}$$

Moreover, from the Markov property and the shift invariance, we obtain the following inequality by using (2.20):

$$\begin{aligned}
&\sum_{\substack{t_1+\cdots+t_k=t,\\ t_i\ge 1}} E^{0,0}_{S,\tilde{S}}\left[|S_{\tau_j}-S_{\tau_{j-1}}|^p e_{\tau_j}\chi_{t_1,\dots,t_k}\right]\\
&\quad= \sum_{\substack{t_1+\cdots+t_k=t,\\ t_i\ge 1}} \left(\prod_{\substack{1\le m\le k,\\ m\ne j}} E^{0,0}_{S,\tilde{S}}[e_{t_m}\mathbf{1}_{\{\tau_1=t_m\}}]\right) E^{0,0}_{S,\tilde{S}}\left[|S_{t_j}|^p e_{t_j}\mathbf{1}_{\{\tau_1=t_j\}}\right]\\
&\quad= \sum_{s=1}^{t}\Bigg(\sum_{\substack{t_1+\cdots+t_{k-1}=s,\\ t_i\ge 1}} a_{t_1}\cdots a_{t_k}\Bigg) E^{0,0}_{S,\tilde{S}}\left[|S_{t-s}|^p e_{t-s}\mathbf{1}_{\{\tau_1=t-s\}}\right]\\
&\quad\le \sum_{s=1}^{t} C_1\alpha^{k-1}s^{-d/2}C(t-s)^{(p-d)/2}\\
&\quad\le C\alpha^{k-1}t^{(p-d)/2} \qquad\qquad \text{for all } 1\le j\le k,
\end{aligned}$$

where we have noted that $e_{t-s}\le \max w$ on the third line. We also have that

$$\begin{aligned}
\sum_{\substack{t_1+\cdots+t_k=t,\\ t_i\ge 1}} E^{0,0}_{S,\tilde{S}}\left[e_{\tau_k}\chi_{t_1\cdots t_k}\right] &= \sum_{\substack{t_1+\cdots+t_k=t,\\ t_i\ge 1}} a_{t_1}\cdots a_{t_k}\\
&\le C\alpha^k t^{-d/2}.
\end{aligned}$$

Hence,

$$E^{x,x}_{S,\tilde S}\Big[|S_t|^p e_t \mathbf{1}_{\{S_t=\tilde S_t\}}\Big] \le Ct^{(p-d)/2}.$$

Thus, for $|\mathbf{n}| \ge 1$ it follows from Hölder's inequality that

$$\begin{aligned} &E^{x,x}_{S,\tilde S}\Big[|S_t|^{2|\mathbf{n}|} e_t \mathbf{1}_{\{S_t=\tilde S_t\}}\Big] \\ &\le \left(E^{x,x}_{S,\tilde S}\Big[|S_t|^{2p|\mathbf{n}|} e_t \mathbf{1}_{\{S_t=\tilde S_t\}}\Big]\right)^{1/p}\left(E^{x,x}_{S,\tilde S}\Big[e_t \mathbf{1}_{\{S_t=\tilde S_t\}}\Big]\right)^{1/q} \\ &\le Ct^{|\mathbf{n}|-d/2p}t^{-d/2q} = Ct^{|\mathbf{n}|-d/2}, \end{aligned}$$

where p,q are the positive numbers satisfying that $1/p+1/q=1$ and $2p|\mathbf{n}|>d$.

Also, we have that $E^{x,x}_{S,\tilde S}[|S_t|^{2\mathbf{n}} e_{t-1}\mathbf{1}_{\{S_t=\tilde S_t\}}] \le Ct^{|\mathbf{n}|-d/2}$ by a similar argument. Therefore, we obtain

$$\begin{aligned} \sup_{t\ge1} E\Big[\sum_{s=1}^t s^{-|\mathbf{n}|/2}\big(Y_{\mathbf{n}}(t)-Y_{\mathbf{n}}(t-1)\big)\Big]^2 &= \sup_{t\ge1}\sum_{s=1}^t s^{-|\mathbf{n}|} E\Big[\big(Y_{\mathbf{n}}(t)-Y_{\mathbf{n}}(t-1)\big)^2\Big] \\ &\le \sum_{t\ge1} Ct^{-d/2} < \infty, \end{aligned}$$

and hence the proof is completed. □

Since we have proved Proposition 2.1, we can show Theorem 2.1. From Fuglede [6], it is enough to show the following proposition instead of (2.11).

Proposition 2.2. *Suppose that $d \ge 3$ and (2.6). For all $\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{N}^d$,*

$$\lim_{t\to\infty}\sum_{x\in\mathbb{Z}^d} N_{0,x} E^x_S\left[\Big(\frac{S_t-mt}{\sqrt t}\Big)^{\mathbf{n}}\zeta_t\right] = |\overline{N}_\infty|\int_{\mathbb{R}^d} x^{\mathbf{n}} d\nu(x), \quad P\text{-a.s.} \tag{2.32}$$

Proof. It is sufficient to show the claim for the case in $N_{0,x}=\delta_x$. By induction, it follows from Lemma 2.3 (a), (c), and Proposition 2.1 that for all $\mathbf{n}\in\mathbb{N}^d$

$$\sup_{t\ge1}\Big|E^0_S\Big[\Big(\frac{S_t-mt}{\sqrt t}\Big)^{\mathbf{n}}\zeta_t\Big]\Big| < \infty, \quad P\text{-a.s.} \tag{2.33}$$

To see this, we divide $Y_{\mathbf{n}}(t)$ into three parts as follows:

$$\begin{aligned} Y^1_{\mathbf{n}}(t) &= t^{|\mathbf{n}|/2}E^0_S\Big[\Big(\frac{S_t-mt}{\sqrt t}\Big)^{\mathbf{n}}\zeta_t\Big] \\ Y^2_{\mathbf{n}}(t) &= t^{|\mathbf{n}|/2}E^0_S\Big[\sum_{|\mathbf{i}|+2j=|\mathbf{n}|,j\ge1} A_{\mathbf{n}}(\mathbf{i},j)\Big(\frac{S_t-mt}{\sqrt t}\Big)^{\mathbf{i}}\zeta_t\Big] \\ Y^3_{\mathbf{n}}(t) &= E^0_S\Big[\sum_{|\mathbf{i}|+2j<|\mathbf{n}|} t^{|\mathbf{i}|/2+j}A_{\mathbf{n}}(\mathbf{i},j)\Big(\frac{S_t-mt}{\sqrt t}\Big)^{\mathbf{i}}\zeta_t\Big]. \end{aligned} \tag{2.34}$$

Then, we can write

$$E_S^0\Big[\Big(\frac{S_t - mt}{\sqrt{t}}\Big)^{\mathbf{n}}\zeta_t\Big] = t^{-d/2}Y_{\mathbf{n}}^1(t) \tag{2.35}$$
$$= t^{-d/2}(Y_{\mathbf{n}} - Y_{\mathbf{n}}^2 - Y_{\mathbf{n}}^3).$$

We suppose that (2.33) holds for $\mathbf{n} \in \mathbb{N}^d$ with $|\mathbf{n}| \le k$. From Proposition 2.1, we have $\sup_{t\ge1} t^{-|\mathbf{n}|/2}\big|Y_{\mathbf{n}}(t)\big| < \infty$ P-a.s. for all $\mathbf{n} \in \mathbb{N}^d$. It is easy to check that for $\mathbf{n} \in \mathbb{N}^d$ with $|\mathbf{n}| = k+1$,

$$\sup_{t\ge1} t^{-|\mathbf{n}|/2}\big|Y_{\mathbf{n}}^2(t)\big| < \infty \quad \text{and} \quad \sup_{t\ge1} t^{-|\mathbf{n}|/2}\big|Y_{\mathbf{n}}^3(t)\big| < \infty \quad P\text{-a.s.}$$

Thus, (2.33) holds for all $\mathbf{n} \in \mathbb{N}^d$. Therefore, we conclude that

$$\lim_{t\to\infty} t^{-|\mathbf{n}|/2}Y_{\mathbf{n}}^3(t) = 0, \quad P\text{-a.s.} \tag{2.36}$$

and hence from Lemma 2.3 and Proposition 2.1 that for $|\mathbf{n}| \ge 1$

$$\lim_{t\to\infty} t^{-|\mathbf{n}|/2}\Big(Y_{\mathbf{n}}^1(t) + Y_{\mathbf{n}}^2(t)\Big) = 0, \quad P\text{-a.s.} \tag{2.37}$$

On the other hand, let Z be an $\mathbb{R}^d$-valued random variable with density ν. Then, it can be seen that $\rho_1(\theta)$ is a polynomial of degree 2, where $\rho_1(\theta)$ is given by (2.27) for $\xi_1 = Z$. Moreover, we have that for $|\mathbf{n}| \ge 1$,

$$0 = \Big(\frac{\partial}{\partial\theta}\Big)^{\mathbf{n}} E[\exp(\theta\cdot Z - \rho_1(\theta))]$$
$$= E\Big[\sum_{|\mathbf{i}|+2j\le 2|\mathbf{n}|} A'_{\mathbf{n}}(\mathbf{i}, j)Z^{\mathbf{i}}\Big]$$

where $A'_{\mathbf{n}}(\mathbf{i}, j)$ is defined by (2.28). From Lemma 2.3, $A'_{\mathbf{n}}(\mathbf{i}, j)$ coincides with $A_{\mathbf{n}}(\mathbf{i}, j)$ for $(\mathbf{i}, j)$ with $|\mathbf{i}| + 2j = 2|\mathbf{n}|$, and hence we can write for $|\mathbf{n}| \ge 1$

$$E\Big[Z^{\mathbf{n}} + \sum_{|\mathbf{i}|+2j=|\mathbf{n}|, j\ge1} A_{\mathbf{n}}(\mathbf{i}, j)Z^{\mathbf{i}}\Big] = 0. \tag{2.38}$$

We know $\lim_{t\to\infty} E_S^0[\zeta_t] = |\overline{N}_\infty|$ for $|\mathbf{n}| = 0$. If (2.32) holds for all $\mathbf{n} \in \mathbb{N}^d$ with $|\mathbf{n}| \le k$, then we have that for all $\mathbf{n} \in \mathbb{N}^d$ with $|\mathbf{n}| = k+1$,

$$\lim_{t\to\infty} t^{-|\mathbf{n}|/2}Y_{\mathbf{n}}^2(t) = |\overline{N}_\infty| E\Big[\sum_{\substack{|\mathbf{i}|+2j=|\mathbf{n}|,\\ j\ge1}} A_{\mathbf{n}}(\mathbf{i}, j)Z^{\mathbf{i}}\Big], \quad P\text{-a.s.} \tag{2.39}$$

From this, (2.37), and Proposition 2.1, it follows that the RHS of (2.35) converges to

$$-|\overline{N}_\infty| E\Big[\sum_{\substack{|\mathbf{i}|+2j=|\mathbf{n}|,\\ j\ge1}} A_{\mathbf{n}}(\mathbf{i}, j)Z^{\mathbf{i}}\Big],$$

almost surely as $t \nearrow \infty$, so that (2.32) holds for $\mathbf{n} \in \mathbb{N}^d$ with $|\mathbf{n}| = k+1$ from (2.38). Therefore, we complete the proof of Proposition 2.2 and Theorem 2.1. □

Remark 6. One might expect that the same statement as Theorem 2.1 holds for the dual process. However the next example shows that the uniform square integrability of $|\overline{M}_t|$ is not sufficient for the (non-degenerate) central limit theorem of the dual process. We can construct the counterexample as follows.

We set $M_{0,x} = \delta_x$. Let $\{e_{t,y}; (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$ defined by the random vectors with $P(e_{t,y} = e) = \frac{1}{2d}$ for $e \in \mathbb{Z}^d$ with $|e| = 1$. We give $A_{t,y,x}$ by

$$A_{t,y,x} = \mathbf{1}_{\{e_{t,x}=y-x\}}.$$

Then, it is easy to see that $|M_t| = 1$ a.s. and therefore $|M_t|$ is uniformly square integrable. Moreover, we have that there is some $x_t \in \mathbb{Z}^d$ for each $t \in \mathbb{N}$ and $M_{t,y} = \delta_{x_t}$. Thus, the central limit theorem does not hold for this dual process. Under some non-degeneracy condition, we can prove the central limit theorem for the dual process.

Proposition 2.3. *Suppose $d \geq 3$ and (2.6). In addition, we assume the non-degeneracy,*

$$w'(0,0,y,\tilde{y}) \neq 0, \quad \textit{for some } y \neq \tilde{y},$$

where $w'(x,\tilde{x},y,\tilde{y})$ is defined by

$$w'(x,\tilde{x},y,\tilde{y}) = \begin{cases} E\big[\frac{B_{1,x,y}B_{1,\tilde{x},\tilde{y}}}{b_{y-x}b_{\tilde{y}-\tilde{x}}}\big] = \Big(E\big[\frac{B_{1,0,y-x}B_{1,0,\tilde{y}-x}}{b_{y-x}b_{\tilde{y}-x}}\big]\Big)^{\delta_{x,\tilde{x}}} & \textit{if } b_{y-x}b_{\tilde{y}-\tilde{x}} \neq 0, \\ 0 & \textit{if } b_{y-x}b_{\tilde{y}-\tilde{x}} = 0. \end{cases}$$

Then, we have that for all $f \in C_b(\mathbb{R}^d)$

$$\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\Big(\frac{x - m't}{\sqrt{t}}\Big)\overline{M}_{t,x} = |\overline{M}_\infty| \int_{\mathbb{R}^d} f(x)d\nu'(x), \quad P\text{-a.s.},$$

where $C_b(\mathbb{R}^d)$ stands for the set of bounded continuous functions on $\mathbb{R}^d$,

$$m' = (m'_1, \ldots, m'_d) = \sum_{x\in\mathbb{Z}^d} xb_x/|b|,$$

and ν' is the Gaussian measure with

$$\int_{\mathbb{R}^d} x_i d\nu'(x) = 0, \ \int_{\mathbb{R}^d} x_i x_j d\nu' = \sum_{x\in\mathbb{Z}^d} (x_i - m'_i)(x_j - m'_j)b_x/|b|, \ i,j = 1,\ldots,d.$$

Outline of the proof. From a similar argument to the case of LSE, there exists the limit

$$e'_\infty = \lim_{t\to\infty} e'_t \quad P\text{-a.s.}$$

where e'_t is defined by replacing w with w' in (2.5). It follows from the Markov property that

$$E^{0,0}_{S,\tilde{S}}\Big[e'_\infty\Big] = \sum_{k=0}^{\infty} \Big(E^{0,0}_{S,\tilde{S}}\Big[e'_\tau : \tau < \infty\Big]\Big)^k E^{0,0}_{S,\tilde{S}}\Big[w'(0,0,S_1,\tilde{S}_1) : \tau = \infty\Big] < \infty$$

where τ is given by $\tau = \inf\{t \geq 1 : S_t = \tilde{S}_t\}$. Therefore, we have from non-degeneracy that

$$E^{0,0}_{S,\tilde{S}}\Big[e_\tau : \tau < \infty\Big] < 1.$$

Then, we can show that similar results to Lemma 2.2 and Proposition 2.1. □

We get an interesting result for DRWOBP as follows.

Corollary 2.1. *Suppose $d \geq 3$. Let $(M_{t,y})$ be the distribution of the directed random walk on oriented bond percolation (DRWOBP) defined by (1.14). We set*

$$\mu_t(x) = \frac{M_{t,x}}{|M_t|}\mathbf{1}\{|M_t| > 0\}.$$

If p is close to 1, then the central limit theorem holds in the sense that for all $f \in C_b(\mathbb{R}^d)$,

$$\lim_{t\to\infty}\sum_{x\in\mathbb{Z}^d} f\Big(\frac{x}{\sqrt{t}}\Big)\mu_t(x) = \int_{\mathbb{R}^d} f(x)d\nu(x) \quad P\text{-a.s. on } \{|\overline{M}_\infty| > 0\},$$

where ν is the Gaussian measure with mean 0 and covariance matrix $1/d$ times the identity matrix.

Remark 7. $\mu_t(x)$ is the distribution of the random walk conditioned on the set {random walk survives at time t}.

Proof. It is sufficient to show that $w'(0,0,y,\tilde{y})\pi_d < 1$, since

$$E^{0,0}_{S,\tilde{S}}[e_\tau : \tau < \infty] \leq \max_{y,\tilde{y}\in\mathbb{Z}^d} w'(0,0,y,\tilde{y})\pi_d.$$

In the typical case $p = 1$, we have that $w'(0,0,y,\tilde{y}) = \mathbf{1}_{|y|,|\tilde{y}|=1}$. Also, it follows from (1.15) and (1.16) that $w'(0,0,y,\tilde{y})$ is continuous in $p \in (0,1]$. Thus, if p is close to 1, then $w'(0,0,y,\tilde{y})\pi_d < 1$. □

3. Appendix: Meeting probability of two random walks

In this section, we follow [17] to estimate the probability with which independent copies of a certain random walk meet. Let $S = (S_t)_{t\in\mathbb{N}}$ be a random walk on $\mathbb{Z}^d$ such that

$$P(S_1 = x) = \begin{cases} \frac{p}{2d} & \text{if } |x| = 1, \\ q & \text{if } x = 0, \end{cases}$$

where $p + q = 1$, $p > 0$. We remark that for BCPP, p and q are replaced by $p/(p+q)$ and $q/(p+q)$, respectively. We define $\rho(x)$ by

$$\rho(x) = P^x_S(S_t = 0, \text{ for some } t \geq 1). \tag{3.1}$$

First, we give the representation of $\rho(x)$ by using the return probability of the simple random walk.

Lemma 3.1.

$$\rho(x) = \begin{cases} q + (1-q)\rho_S(0) & \text{if } x = 0, \\ \rho_S(x) & \text{if } x \neq 0, \end{cases}$$

where ρ_S *is defined by (*3.1*) for the simple random walk.*

Proof. It is easy to check that

$$\overline{S}_t = S_{J_t} \text{ is a simple random walk,}$$

where J_t is the "t-th jump time"defined by

$$J_t = \inf\{s \geq 1 : \sum_{u=1}^{s} \mathbf{1}_{S_{u-1} \neq S_u} = t\}.$$

Also, we have $\{S_t\}_{t \geq J_1} = \{\overline{S}_t\}_{t \geq 1}$. Therefore, it follows that for $x \neq 0$

$$\rho(x) = P_S^x(\{S_t\}_{t \geq J_1} \ni 0) = P_S^x(\{\overline{S}_t\}_{t \geq 1} \ni 0) = \rho_S(x),$$

and for $x = 0$

$$\begin{aligned} \rho(0) &= P_S^0(S_1 = 0) + P_S^0(|S_1| = 1,\ P_S^{S_1}(S_t = 0,\ \text{for some } t \geq 1)) \\ &= q + (1-q)\rho_S(e_1) \\ &= q + (1-q)\rho_S(0), \end{aligned}$$

where $e_1 = (1, 0, \ldots, 0)$. □

Let $(\tilde{S}_t)_{t \geq 1}$ be an independent copy of $(S_t)_{t \geq 1}$. We define $\pi(x)$ by

$$\pi(x) = P_{S,\tilde{S}}^{x,0}(S_t = \tilde{S}_t,\ \text{for some } t \geq 1).$$

Lemma 3.2.

$$\pi(x) \leq \begin{cases} q + (1-q)\rho_S(0) & \text{if } x = 0, \\ \frac{1-\pi(0)}{1-q}\frac{\rho_S(x)}{1-\rho_S(0)} & \text{if } x \neq 0. \end{cases}$$

Moreover, if $x \neq 0$*, then*

$$\pi(x) \leq C_3(1/d) \quad \text{as } d \nearrow \infty.$$

Proof. It follows from a property of the random walk that

$$\begin{aligned} \frac{1}{1-\pi(0)} &= \sum_{t=0}^{\infty} P_{S,\tilde{S}}^{0,0}(S_t - \tilde{S}_t = 0) = \sum_{t=0}^{\infty} P_S^0(S_{2t} = 0) \\ &\leq \sum_{t=0}^{\infty} P_S^0(S_t = 0) = \frac{1}{1-\rho(0)}. \end{aligned}$$

Therefore, we have from Lemma 3.1 that $\pi(0) \leq \rho(0) \leq q + (1-q)\rho_S(0)$. Also, for $x \neq 0$, we obtain that

$$\begin{aligned}\frac{\pi(x)}{1-\pi(0)} &= \sum_{t=0}^{\infty} P^{x,0}_{S,\tilde{S}}(S_t - \tilde{S}_t = 0) = \sum_{t-0}^{\infty} P^x_S(S_{2t} = 0)\\ &\leq \sum_{t=0}^{\infty} P^x_S(S_t = 0) = \frac{\rho(x)}{1-\rho(0)} \leq \frac{\rho_S(x)}{(1-q)(1-\rho_S(0))}.\end{aligned}$$

The last statement follows from the fact that $\rho_S(x) \leq C(1/d)$ as $d \nearrow \infty$. □

Now, we verify (2.26). (2.24) and Lemma 3.2 (where p and q are replaced by $p/(p+q)$ and $q/(p+q)$ respectively) implies that

$$\eta \leq \frac{1}{p+q} + \frac{(p+q)(1-\pi(0))}{p} \frac{\rho_S(0)}{1-\rho_S(0)} \Big(1 - \frac{1}{2d}\Big(\frac{p}{p+q}\Big)^2 - \Big(\frac{q}{p+q}\Big)^2\Big).$$

Thus, it is sufficient to check that the right hand side is smaller than 1 for $p = q = 1$. We have that $1 - \pi(0) = 1/G$, where G is the expectation of the number of meeting of two random walks starting from the origin. Since we know that for $d = 3$, $\rho_S(0) = .3405\cdots$, and that for $d \geq 4$, $\rho_S(0) \leq .1932\cdots$ (see,[8]), we can choose $0 < p, q \leq 1$ so that $\eta < 1$. Here, we use the equation

$$\begin{aligned}G &= \sum_{k=0}^{\infty} P^0_S(S_{2t} = 0)\\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \Big(\frac{q}{p+q}\Big)^{2k} \binom{2n}{2k} \Big(\frac{p}{p+q}\Big)^{2n-2k} p_{2k},\\ &\geq \sum_{n=0}^{\infty} \Big(\frac{p}{p+q}\Big)^{2n} + \sum_{n=1}^{\infty} n(2n-1)\Big(\frac{q}{p+q}\Big)^2 \Big(\frac{p}{p+q}\Big)^{2n-2} p_2\end{aligned}$$

where p_{2n} is the probability of a simple random walk visiting 0 at time $2n$. If we set $p = q = 1$, then

$$G \geq \begin{cases} \frac{4}{3} + \frac{14}{81} = 1.5061\cdots & \text{if } d = 3,\\ \frac{4}{3} & \text{if } d \geq 4.\end{cases}$$

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