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Levi conditions to the Gevrey well-posedness for hyperbolic operators of higher order

By

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Abstract

We consider a class of linear higher order hyperbolic equations with a single degenerate point. We give sufficient conditions in order for the Cauchy problem to be well-posed in Gevrey classes and in the C^∞ class.

1. Introduction

In this article we are concerned with the well-posedness in Gevrey classes of the Cauchy problem for linear hyperbolic equations of higher order with time dependent coefficients.

First of all, let us use the following notation: For $t \in \mathbb{R}$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, set $D_t = -\sqrt{-1}\dfrac{\partial}{\partial t}$, $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $D_x^\alpha = (-\sqrt{-1})^{|\alpha|} \times \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Also, for $m \geq 2$, denote the linear partial differential operators by

$$
\begin{aligned}
P(t, D_t, D_x) &= D_t^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(t) D_t^j D_x^\alpha, \\
L_k(t, D_t, D_x) &= \sum_{j+|\alpha|=k} a_{j,\alpha}(t) D_t^j D_x^\alpha \quad (k = 0, 1, \cdots, m-1), \\
L(t, D_t, D_x) &= \sum_{k=0}^{m-1} L_k(t, D_t, D_x),
\end{aligned}
$$

where $a_{j,\alpha}(t) \in C^m([0,T])$ is real analytic on $(0,T]$ for $j + |\alpha| = m$ and $a_{j,\alpha}(t) \in C^0([0,T])$ for $j + |\alpha| \leq m - 1$. We shall consider the homogeneous Cauchy problem

$$
(CP) \qquad \begin{cases} P(t, D_t, D_x)u(t,x) = L(t, D_t, D_x)u(t,x), \ (t,x) \in [0,T] \times \mathbb{R}^n, \\ D_t^j u(0,x) = u_j(x), \ x \in \mathbb{R}^n \quad (j = 0, 1, \cdots, m-1). \end{cases}
$$

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For convenience of the reader let us give the definitions of the Gevrey class and the well-posedness of the Cauchy problem (CP) in Gevrey class and the C^∞ class.

(1) For $s \geq 1$, the function $f \in C^\infty(\mathbb{R}^n)$ is said to belong to the *Gevrey class* γ^s with index s if for every compact set K in $\mathbb{R}^n$ there exists a constant $C_K > 0$ such that for all $\alpha \in \mathbb{Z}_+^n$

$$\sup_{x \in K} |D_x^\alpha f(x)| \leq C_K^{|\alpha|+1} |\alpha|!^s.$$

Then we stand for $f \in \gamma^s$.

(2) The Cauchy problem (CP) is said to be *well-posed in* γ^s if for any $u_j \in \gamma^s$ $(j = 0, \cdots, m-1)$ there exists a unique solution $u(t,x) \in C^m([0,T], \gamma^s)$ to (CP). Similarly, (CP) is C^∞ *well-posed* if for any $u_j \in C^\infty(\mathbb{R}^n)$ $(j = 0, \cdots, m-1)$ (CP) admits a unique solution $u(t,x) \in C^m([0,T], C^\infty(\mathbb{R}^n))$.

Throughout this article, we always impose the hyperbolicity condition as below: The principal symbol has only real-valued roots, that is, when it is decomposed into the factors in τ like

$$P(t,\tau,\xi) = \tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(t)\tau^j \xi^\alpha = \prod_{k=1}^m (\tau - \tau_k(t,\xi)),$$

every root τ_k is real-valued. $\tau_1, \cdots, \tau_m$ are called by *characteristic roots* of P. We know that this hyperbolicity condition is necessary for the well-posedness in γ^s of (CP) (see [13]). If P is *strictly hyperbolic*, namely its characteristic roots are distinct, then (CP) is well-posed in all Gevrey class $\cup_{s \geq 1} \gamma^s$ and $C^\infty(\mathbb{R}^n)$. On the other hand, if P is *weakly hyperbolic*, namely its characteristic roots are coincide, then (CP) is well-posed in γ^s for $1 < s < \dfrac{r}{r-1}$, where r is the greatest multiplicity of the characteristic roots (see [3]). In general, (CP) is not well-posed in γ^s for $s > \dfrac{r}{r-1}$ without additional conditions (see, e.g. [11]). We note that $\tau_1, \cdots, \tau_m$ are differentiable with respect to t due to Theorem 2 in [2].

For two functions $f, g : [0,T] \to \mathbb{R}$, we stand for $f \lesssim g$ if there is a constant $C > 0$ such that $f(t) \leq Cg(t)$ for all $t \in [0,T]$. Moreover we write $f \approx g$ whenever both $f \lesssim g$ and $g \lesssim f$.

Next we introduce m weight functions to describe our main results as follows. Let $\lambda_1(t), \cdots, \lambda_m(t) \in C^1([0,T])$ satisfy the following conditions:

(i) $\lambda_k(0) = \lambda_k'(0) = 0$, $\lambda_k'(t) > 0$ for $0 < t \leq T$,

(ii) $\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_m(t) \geq 0$ for $0 \leq t \leq T$,

(iii) $\lambda_j(t)|\xi| \lesssim |\tau_j(t,\xi) - \tau_k(t,\xi)|$ for $(t,\xi) \in [0,T] \times \mathbb{R}^n \setminus \{0\}$ if $j < k$,

(iv) $|\tau_k(t,\xi)| \lesssim \lambda_k(t)|\xi|$ for $(t,\xi) \in [0,T] \times \mathbb{R}^n \setminus \{0\}$,

(v) $\dfrac{\lambda_1'(t)}{\lambda_1(t)} \leq \dfrac{s}{s-1} \dfrac{\lambda_1(t)}{\Lambda_1(t)}$, for $0 < t \leq T$ and $s > 1$, where

$\Lambda_1(t) = \displaystyle\int_0^t \lambda_1(\tau)\, d\tau.$

Assumptions 1.1. *For some $s > 1$,*

$$(P) \qquad \frac{\Lambda_1(t)^{s/(s-1)}}{\lambda_1(t)} \sum_{j,k=1,j\neq k}^{m} \frac{|\tau_j'(t,\xi)| + |\tau_k'(t,\xi)|}{|\tau_j(t,\xi) - \tau_k(t,\xi)|} < \infty \quad \text{near} \quad t = 0,$$

where $\tau_j' = \dfrac{\partial \tau_j}{\partial t}$.

For $j + |\alpha| \leq m - 1$,

$$(L) \qquad |a_{j,\alpha}(t)| \leq C_{j,\alpha} \left(\prod_{k=1}^{|\alpha|} \lambda_k(t) \right) \left(\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}} \right)^{m-j-|\alpha|} \qquad (0 < t \leq T).$$

Theorem 1.1. *Assume that* (i) $\sim$ (v), (P) *and* (L) *are fulfilled. Then the Cauchy problem* (CP) *is well-posed in the Gevrey class* $\gamma^{\bar{s}}$ *provided* $1 < \bar{s} < s$.

Remark 1.1. Colombini-Orrú in [6] treated the case of $a_{j,\alpha}(t) \equiv 0$ for $j + |\alpha| \leq m - 1$ under the finitely degeneracy assumption, that is, either

$$\sum_{k=1}^{m} |D_t^k a_{0,\alpha}(0)| > 0 \quad \text{for} \quad |\alpha| = m$$

or

$$|D_t a_{1,\alpha}(0)| + \cdots + |D_t^{([m/2])} a_{1,\alpha}(0)| > 0 \quad \text{for} \quad |\alpha| = m - 1.$$

They showed that (CP) is (uniformly) C^∞ well-posed if and only if

$$t^2 \sum_{j,k=1,j\neq k}^{m} \frac{|\tau_j'(t,\xi)|^2 + |\tau_k'(t,\xi)|^2}{|\tau_j(t,\xi) - \tau_k(t,\xi)|^2} < \infty \quad \text{near} \quad t = 0.$$

Our assumption (P) is naturally motivated by this condition because for the choice of $\lambda_1(t) = t^\ell$ $(\ell > 1)$

$$\frac{\Lambda_1(t)^{s/(s-1)}}{\lambda_1(t)} = (\ell + 1)^{s/(1-s)} t^{1+(\ell+1)/(s-1)} \to \frac{t}{\ell + 1}$$

as $s \to \infty$. (P) was already proposed in [4].

Remark 1.2. The assumption (P) is also rewritten in terms of the estimates for the coefficents. That is, under (ii) and (iii), (P) is equivalent to the conditon

$$|D_t^k a_{j,\alpha}(t)| \leq C_{j,\alpha} \left(\prod_{i=1}^{|\alpha|} \lambda_i(t) \right) \left(\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}} \right)^k$$

for $j + |\alpha| = m$ (in fact, $|\alpha| \geq 1$) and $k = 0, 1$ (cf. Lemma 1.2 in [12]). In case of $\lambda_1(t) = \lambda_2(t) = \cdots = \lambda_m(t)$, Theorem 1.1 essentially follows from Theorem 5.2 in [16], where some of their conditions are different from (v) (see also §2.2 in [17]).

Our other concern here is a relation with the Levi condition introduced in [7] by Colombini and Taglialatela. So let us recall their assumptions and conditions as follows.

Assumptions 1.2 (Colombini-Taglialatela [7])**.**
(a) *There exist some real numbers* $r_m \geq \cdots \geq r_2 \geq r_1 \geq 0$ *satisfying*

$$|\tau_j(t,\xi) - \tau_k(t,\xi)| \approx t^{r_j}|\xi| \quad \textit{if} \quad j < k.$$

(b) *There is a constant* $s_m \geq 0$ *such that*

$$t^{1+s_m}|\tau_j'(t,\xi)| \lesssim |\tau_j(t,\xi) - \tau_k(t,\xi)| \quad \textit{if} \quad j \neq k.$$

(c) *For each* $j \in \{1, \cdots, m-1\}$, $k \in \{1, 2, \cdots, j+1\}$ *there is a constant* $s_j \geq 0$ *such that*

$$t^{s_j}|L_j(t, \tau_k(t,\xi), \xi)| \lesssim \prod_{k \neq \ell = 1}^{j+1} |\tau_k(t,\xi) - \tau_\ell(t,\xi)|.$$

The conditions (a), (b) show the finite degeneracy of the characteristic roots of P. (c) is a Levi condition by Colombini and Taglialatela, which is our second main subject here. We would like to propose a natural variant of (c) in the case including the infinitely degenerate characteristic roots. Let us extend Assumptions 1.1, 1.2 to the following.

Assumptions 1.3.
(A)

$$|\tau_j(t,\xi) - \tau_k(t,\xi)| \approx \lambda_j(t)|\xi| \quad \textit{if} \quad j < k.$$

(B) *For some* $s > 1$

$$\frac{\Lambda_1(t)^{s/(s-1)}}{\lambda_1(t)}|\tau_j'(t,\xi)| \lesssim |\tau_j(t,\xi) - \tau_k(t,\xi)| \quad \textit{if} \quad j \neq k.$$

(C)

$$\left(\frac{\Lambda_1(t)^{s/(s-1)}}{\lambda_1(t)}\right)^{m-j} |L_j(t, \tau_k(t,\xi), \xi)| \lesssim \prod_{k \neq \ell = 1}^{j+1} |\tau_k(t,\xi) - \tau_\ell(t,\xi)|$$

for some $s > 1$ *and every* $j \in \{1, \cdots, m-1\}$, $k \in \{1, 2, \cdots, j+1\}$

Remark 1.3. (i) $\sim$ (iv) imply (A). (B) is equivalent to (P). Under (i) $\sim$ (iv), (L) implies (C) (see §4). As mentioned in Remark 1.1 of [7], we note that (C) is equivalent to

$$\left(\frac{\Lambda_1(t)^{s/(s-1)}}{\lambda_1(t)}\right)^{m-j} |L_j(t, \tau_k(t,\xi), \xi)| \lesssim |\partial_\tau^{m-j} P(t, \tau_k(t,\xi), \xi)|$$

for some $s > 1$ and every $j \in \{1, \cdots, m-1\}$, $k \in \{1, 2, \cdots, j+1\}$.

Theorem 1.2. *Assume that* (i) $\sim$ (v), (B) *and* (C) *are fulfilled. Then the Cauchy problem* (CP) *is well-posed in the Gevrey class* $\gamma^{\bar{s}}$ *provided* $1 < \bar{s} < s$. *Moreover, if* $s = \infty$ *in* (v), (B) *and* (C), *then* (CP) *is also* C^∞ *well-posed.*

Remark 1.4. In [7], they imposed that $a_{j,\alpha}(t) \in C^m([0,T])$ for $j+|\alpha| = m$. In opposition to [7], we assume the analyticity of $a_{j,\alpha}(t)$ on $(0,T]$ for $j + |\alpha| = m$. This stronger assumption leads us to the finiteness of the zero-point sets of the derivatives $\tau_k'(t,\xi)$, which helps us to estimate the lower order terms L_j in the non-hyperbolicity influential region of the cotangent space (see §3.2). This is an essential difference from [7], which is a key point in our way. By Remark 1.3 it suffices to prove Theorem 1.2. We do not use any reduction of the equation to a first order system as in [5], [10], [12] and [16].

Remark 1.5. Though we must restrict degenerate points to be isolated, there is a nice way allowing accumulatively degenerate points (see [8] in detail). Besides, the differential geometers Han, Hong and Lin applied the theory of degenerate hyperbolic equations to the existence of locally isometric embedding of surfaces with nonpositive Gauss curvature (see [6] in the references of [9]).

2. Algebraic preliminaries

In this section we prepare some algebraic lemmata to establish the energy estimates to the solution of the Cauchy problem (CP) in weighted Sobolev spaces. They are almost the same except one as those in §3 of [7], so their proofs are omitted which rely on induction and elementary properties of symmetric functions. We refer to [7] for their details.

For any smooth function $v(t,\xi)$ of t on $[0,T]$ with parameter $\xi \in \mathbb{R}^n \setminus \{0\}$, let us put a series of quantities for v as follows:

$$
\begin{aligned}
[v]_0^2 &= |v|^2, \\
[v]_1^2 &= \sum_{j=1}^{m} |F_j v|^2, \\
[v]_2^2 &= \sum_{1 \le j < k \le m} |F_{jk} v|^2, \\
&\vdots \\
[v]_j^2 &= \sum_{1 \le k_1 < \cdots < k_j \le m} |F_{k_1 \cdots k_j} v|^2 \quad (j = 1, 2, \cdots, m)
\end{aligned}
$$

where

$$
\begin{aligned}
F_j v &= D_t v - \tau_j v, \\
F_{jk} v &= D_t^2 v - (\tau_j + \tau_k) D_t v + \tau_j \tau_k v
\end{aligned}
$$

for every $j, k \in \{1, 2, \cdots, m\}$ with $j \neq k$, in general, $F_{k_1 \cdots k_j}$ is the differential operator with symbol

$$F_{k_1\cdots k_j}(t,\tau,\xi)=\prod_{\ell=1}^{j}\left(\tau-\tau_{k_\ell}(t,\xi)\right).$$

We note that by Viète theorem

$$F_{k_1\cdots k_j}(t,D_t,\xi)=\sum_{h=0}^{j}\sigma_h(\tau_{k_1},\cdots,\tau_{k_j})D_t^{j-h},$$

where $\sigma_h(t_1,\cdots,t_j)$ is the elementary symmetric polynomial of degree h, that is,

$$\sigma_h(t_1,\cdots,t_j)=\sum_{1\le k_1<\cdots<k_h\le j}t_{k_1}\cdots t_{k_h}$$

for $h\ge 1$ and $\sigma_0(t_1,\cdots,t_j)=1$.

The first one of our lemmata is relevant to a suitable estimate for $F_{k_1\cdots k_j}$ by $[v]_*$.

Lemma 2.1 (Lemma 3.1 in [7]). *Let $p\in\{1,\cdots,m-2\}$, $\ell_1,\cdots,\ell_p\in\{1,\cdots,m\}$ with $\ell_j\ne\ell_k$ if $j\ne k$. Also, let $k_1,\cdots,k_q\in\{1,\cdots,m\}\backslash\{\ell_1,\cdots,\ell_p\}$ with $q\in\{2,\cdots,m-p\}$ and $k_1<\cdots<k_q$. Then we have*

$$|F_{\ell_1\cdots\ell_p}v|\lesssim\frac{[v]_{p+q-1}}{\prod_{j=1}^{q-1}|\tau_{k_j}-\tau_{k_{j+1}}|}. \tag{2.1}$$

Next we introduce several special operators to represent a good decomposition of the lower order terms $L_j(t,D_t,\xi)$ according to [7].

Let $\mathcal{F}_0=1$, $\mathcal{F}_1=F_m$, $\mathcal{F}_2=F_{m,m-1}$, $\cdots$, $\mathcal{F}_m=F_{m,m-1,\cdots,1}$. Then as a corollary of Lemma 2.1 and (A), we get

$$|\mathcal{F}_k v|\lesssim\frac{[v]_{k+h}}{\prod_{\ell=1}^{h}|\tau_\ell-\tau_{\ell+1}|}\approx\frac{[v]_{k+h}}{\lambda_1\cdots\lambda_h|\xi|^h} \tag{2.2}$$

for $k\in\{1,2,\cdots,m-2\}$ and $h\in\{1,\cdots,m-k-1\}$.

Let $M(X)$ be a polynomial of degree j in one variable X. Further we recursively define the Newton divided difference operators by

$$\begin{aligned}
\Delta_0[M](X_0)&=M(X_0),\\
\Delta_1[M](X_0,X_1)&=\frac{\Delta_0[M](X_0)-\Delta_0[M](X_1)}{X_0-X_1}=\frac{M(X_0)-M(X_1)}{X_0-X_1},\\
&\ \ \vdots\\
\Delta_k[M](X_0,\cdots,X_k)&\\
&=\frac{\Delta_{k-1}[M](X_0,\cdots,X_{k-2},X_{k-1})-\Delta_{k-1}[M](X_0,\cdots,X_{k-2},X_k)}{X_{k-1}-X_k}
\end{aligned}$$

for $k\in\{0,1,\cdots,j\}$. Then we know the symmetry of the polynomial $\Delta_k[M]$, but we will not later use it and its degree is rather important for our argument.

Lemma 2.2 (Proposition 3.1 in [7]). *If $M(X)$ is a polynomial of degree j, then $\Delta_k[M](X_0, \cdots, X_k)$ is a symmetric polynomial of degree $j-k$ in $X_0, \cdots, X_k$.*

Now we can regard a differential operator $M(t, D_t, \xi)$ as its symbol $M(t, \tau, \xi)$ which is a polynomial in τ with parameters t, ξ. Hence we may construct the functions $\Delta_k[\sigma(M)]$ of τ as before, where $\sigma(M)$ means the symbol of M. Then we require the estimate of $\Delta_k[L_j](\tau_{j+1-k}, \cdots, \tau_{j+1})$ for all $j \in \{0, 1, \cdots, m-1\}$ and $k \in \{0, \cdots, j\}$ to evaluate the lower order part L.

Lemma 2.3. *If L_j satisfies (C), then for any $j_0 < j_1 < \cdots < j_k$ we have*

(2.3)
$$\left(\frac{\Lambda_1(t)^{s/(s-1)}}{\lambda_1(t)}\right)^{m-j} |\Delta_k[L_j](\tau_{j_0}, \cdots, \tau_{j_k})| \lesssim \prod_{\ell \in \{1, \cdots, j+1\} \setminus \{j_0, \cdots, j_k\}} |\tau_{j_0} - \tau_\ell|,$$

in particular, from (A)

$$|\Delta_k[L_j](\tau_{j+1-k}, \cdots, \tau_{j+1})| \lesssim \lambda_1 \cdots \lambda_{j-k} \left(\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}}\right)^{m-j} |\xi|^{j-k} \tag{2.4}$$

for $t \in (0, T]$.

The proof is similarly done by induction on k as in that of Lemma 3.2 in [7], so we omit it. The next proposition gives us a good decomposition of L_j.

Lemma 2.4 (Proposition 3.2 in [7]). *If $M_j(t, D_t, \xi)$ is a differential operator of order j, then M_j is expressed by*

$$M_j = \sum_{k=0}^{j} \Delta_k[M_j](\tau_{m-k}, \cdots, \tau_m)\mathcal{F}_k. \tag{2.5}$$

In addition, we need to estimate the time derivatives of $F_{k_1, \cdots, k_j}$ to obtain the energy estimates. They are generally shown in the following form.

Lemma 2.5 (Lemma 3.3 in [7]). *Let $j \in \{1, \cdots, m-1\}$ and $k_1, \cdots, k_j \in \{1, 2, \cdots, m\}$. For any $k' \in \{1, 2, \cdots, m\} \setminus \{k_1, \cdots, k_j\}$ we have*

(2.6)
$$\partial_t |F_{k_1, \cdots, k_j} v|^2 \lesssim |F_{k_1, \cdots, k_j, k'} v||F_{k_1, \cdots, k_j} v| + \sum_{\ell=1}^{j} |\tau'_{k_\ell}||F_{k_1, \cdots, \widehat{k_\ell}, \cdots, k_j} v||F_{k_1, \cdots, k_j} v|.$$

3. Energy estimates in two zones

In this section we shall prove Theorem 1.2. First of all, by partial Fourier transform with respect to space variables x, the equation is reduced to the ordinary differential equation in t with parameter $\xi \in \mathbb{R}^n$:

$$P(t, D_t, \xi)v(t, \xi) = L(t, D_t, \xi)v(t, \xi), \tag{3.1}$$

where $v(t,\xi) = \dfrac{1}{(2\pi)^{n/2}} \displaystyle\int_{\mathbb{R}^n} e^{-\sqrt{-1}x\cdot\xi} u(t,x)\,dx$ in the Schwartz class $\mathcal{S}$. It is enough to consider the equation (3.1) when $|\xi| \geq 1$ and $s > 1$. Then for any given $\xi \neq 0$ there is an unique point $t_\xi > 0$ with $\Lambda_1(t_\xi)^s|\xi|^{s-1} = 1$ due to the monotonicity of Λ_1. It is evident that $t_\xi \to 0$ as $|\xi| \to \infty$. We may also assume that $t_\xi \in (0,T)$ by taking $|\xi|$ large, if necessary. That point t_ξ was used in [4] and [10]. So we separate the interval $[0,T]$ into the two zones: $[0,T] = Z_{pd} \cup Z_{hyp}$, where $Z_{pd} = [0,t_\xi]$ and $Z_{hyp} = [t_\xi, T]$. We follow [17] and call them *pseudodifferential zone* and *hyperbolic zone* respectively. Henceforth we shall derive proper energy estimates in each zone, but they are mutually equivalent up to the growth of some powers of $|\xi|$.

3.1. Estimate in the hyperbolic zone

In this zone we shall derive a fine energy estimate to reflect our Levi condition (C) precisely from Lemma 2.3. Our argument here is surely analogous as in [4], [6], [7], but will much clarifies its essence. It is better to refer to §2 in [7] for understanding our strategy briefly.

For $t \in [t_\xi, T]$ and $|\xi| \geq 1$, we introduce the following energy function

$$E_1(t,\xi) = \sum_{j=0}^{m-1} \left(\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}} \right)^{2(m-1-j)} [v]_j^2.$$

From now on we shall lead to the estimate for $E_1(t)$ by Gronwall's inequality. To do so, differentiating E_1 with respect to t, we obtain

$$E_1'(t) = \sum_{j=0}^{m-1} \left\{ [v]_j^2 \partial_t \left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}} \right)^{2(m-1-j)} + \left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}} \right)^{2(m-1-j)} \partial_t [v]_j^2 \right\},$$

where due to (v) the first terms can be evaluated as follows:

$$\begin{aligned} &[v]_j^2 \partial_t \left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}} \right)^{2(m-1-j)} \\ &= 2(m-1-j) \left(\frac{\lambda_1'}{\lambda_1} - \frac{s}{s-1} \frac{\lambda_1}{\Lambda_1} \right) \left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}} \right)^{2(m-1-j)} [v]_j^2 \\ &\leq 0. \end{aligned}$$

Meanwhile, by use of Lemma 2.5 we see that

$$\begin{aligned} \sum_{j=0}^{m-1} \left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}} \right)^{2(m-1-j)} \partial_t [v]_j^2 \lesssim \sum_{j=0}^{m-1} \left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}} \right)^{2(m-1-j)} [v]_{j+1} [v]_j \\ + \sum_{j=0}^{m-1} \left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}} \right)^{2(m-1-j)} \sum_{1 \leq k_1 < \cdots < k_j \leq m} \sum_{\ell=1}^{j} |\tau_{k_\ell}'| |F_{k_1,\cdots,\widehat{k_\ell},\cdots,k_j} v| [v]_j, \end{aligned}$$

where $\left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}}\right)^{2(m-1-j)}[v]_{j+1}[v]_j \lesssim \left(1+\frac{\lambda_1}{\Lambda_1^{s/(s-1)}}\right)E_1$ for $j=0,\cdots,m-2$. On the other hand, by (3.1)

$$[v]_m = |Pv| = |Lv| \le \sum_{j=0}^{m-1} |L_j v|,$$

where it follows from Lemma 2.4 that

$$|L_j v| \le \sum_{k=0}^{j} |\Delta_k[L_j](\tau_{m-k},\cdots,\tau_m)||\mathcal{F}_k v|.$$

Now we may claim that (A) and (C) imply

$$|\Delta_k[L_j](\tau_{m-k},\cdots,\tau_m)||\mathcal{F}_k v| \lesssim \frac{\lambda_1}{\Lambda_1^{s/(s-1)}}\sqrt{E_1} \tag{3.1.1}$$

for $j\in\{1,2,\cdots,m-1\}$, $k\in\{0,\cdots,j-1\}$ and $t\in(0,T]$. Let us show (3.1.1) as below.

In the beginning, let us recall from (2.2)

$$|\mathcal{F}_k v| \lesssim \frac{[v]_{k+h}}{\lambda_1\cdots\lambda_h|\xi|^h}$$

for any $h\in\{1,\cdots,m-k-1\}$. While, in view of Lemma 2.3, (A) and (ii)

$$\begin{aligned}|\Delta_k[L_j](\tau_{m-k},\cdots,\tau_m)| &\lesssim \left(\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}}\right)^{m-j}\prod_{\ell=1}^{j-k}|\tau_{m-k}-\tau_\ell| \\ &\approx \lambda_1\cdots\lambda_{j-k}|\xi|^{j-k}\left(\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}}\right)^{m-j}.\end{aligned}$$

Therefore, by the choice of $h=j-k$, we can verify that

$$|\Delta_k[L_j](\tau_{m-k},\cdots,\tau_m)||\mathcal{F}_k v| \lesssim \left(\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}}\right)^{m-j}[v]_j \lesssim \frac{\lambda_1}{\Lambda_1^{s/(s-1)}}\sqrt{E_1}.$$

Moreover, by means of Lemma 2.1 and (B)

$$|\tau'_{k_\ell}||F_{k_1,\cdots,\widehat{k_\ell},\cdots,k_j}v| \lesssim \frac{|\tau'_{k_\ell}|}{|\tau_{k_\ell}-\tau_{k'}|}[v]_j \lesssim \frac{\lambda_1}{\Lambda_1^{s/(s-1)}}[v]_j$$

for $k'\in\{1,\cdots,m\}\setminus\{k_1,\cdots,k_j\}$.

Up to now, summing up these estimates, we can deduce that

$$E_1'(t) \lesssim \left(1+\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}}\right)E_1(t)$$

for every $t \in [t_\xi, T]$. Thus, thanks to Gronwall's inequality and

$$\int_{t_\xi}^{T} \left(1 + \frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}}\right) dt \le T + (s-1)\Lambda_1(t_\xi)^{1/(1-s)} = T + (s-1)|\xi|^{1/s},$$

we have the required energy estimate in Z_{hyp}

$$E_1(t_2,\xi) \lesssim E_1(t_1,\xi) \exp\left(C_{s,T}|\xi|^{1/s} + 1\right) \tag{3.1.2}$$

for any $t_1, t_2 \in [t_\xi, T]$.

3.2. Estimate in the pseudodifferential zone

In this zone we employ a refined energy function, completely different from that in §4.1 of [7], which correctly reflects the degeneracy of the characteristic roots.

Fot $t \in [0, t_\xi]$ and $|\xi| \ge 1$, let us define the energy function

$$E_2(t,\xi) = \sum_{j=0}^{m-1} (\lambda_1(t)|\xi| + h)^{2(m-1-j)}[v]_j^2,$$

where

$$h = \begin{cases} 1 + \max\limits_{k=1,\cdots,m} \sup\limits_{t\in[0,t_\xi],|\xi|\ge 1} |\tau_k(t,\xi)| & \text{if } \max\limits_{k=1,\cdots,m} \sup\limits_{t\in[0,t_\xi],|\xi|\ge 1} |\tau_k(t,\xi)| < \infty, \\ 1 + \left(\max\limits_{k=1,\cdots,m} C_k\right)\lambda_1(t)|\xi| & \text{otherwise} \end{cases}$$

and the constants $C_k > 1$ are taken like

$$\tau_k(t,\xi) + C_k\lambda_k(t)|\xi| \ge 0$$

for all $t \in [0, t_\xi]$ and $|\xi| \ge 1$ on account of (iv).

We first get by differentiating E_2 in t

$$E_2'(t) = 2\sum_{j=0}^{m-2}(m-1-j)\frac{\lambda_1'|\xi| + h'}{\lambda_1|\xi| + h}(\lambda_1|\xi| + h)^{2(m-1-j)}[v]_j^2 + \sum_{j=0}^{m-1}(\lambda_1|\xi| + h)^{2(m-1-j)}\partial_t[v]_j^2,$$

where from Lemma 2.5

$$\sum_{j=0}^{m-1}(\lambda_1|\xi| + h)^{2(m-1-j)}\partial_t[v]_j^2 \lesssim \sum_{j=0}^{m-1}(\lambda_1|\xi| + h)^{2(m-1-j)}[v]_{j+1}[v]_j + \sum_{j=0}^{m-1}(\lambda_1|\xi| + h)^{2(m-1-j)} \sum_{1\le k_1<\cdots<k_j\le m}\sum_{\ell=1}^{j}|\tau_{k_\ell}'||F_{k_1,\cdots,\widehat{k_\ell},\cdots,k_j}v|[v]_j,$$

where $(\lambda_1|\xi|+h)^{2(m-1-j)}[v]_{j+1}[v]_j \lesssim (\lambda_1|\xi|+h)E_2$ for $j=0,\cdots,m-2$. Meanwhile, by (3.1)

$$[v]_m = |Pv| = |Lv| \le \sum_{j=0}^{m-1} |L_j v|,$$

where it also holds from Lemma 2.4 that

$$|L_j v| \le \sum_{k=0}^{j} |\Delta_k[L_j](\tau_{m-k},\cdots,\tau_m)||\mathcal{F}_k v|.$$

Now, in view of Lemma 2.2, (iv) and (ii) we know

$$|\Delta_k[L_j](\tau_{m-k},\cdots,\tau_m)| \lesssim \sum_{h_0+\cdots+h_k=j-k} |\tau_m|^{h_0}\cdots|\tau_{m-k}|^{h_k} \lesssim (\lambda_1|\xi|)^{j-k},$$

so that

$$|L_j v| \lesssim (\lambda_1|\xi|+1)\sum_{k=0}^{j}(\lambda_1|\xi|+1)^{j-k-1}|\mathcal{F}_k v| \lesssim (\lambda_1|\xi|+h)\sqrt{E_2}.$$

Further, if $t_\xi \le |\xi|^{1/s-1}$, then due to Bronstein's Lemma $|\tau_k'| \lesssim |\xi|$ (see Theorem 2 in [2])

$$\begin{aligned}(\lambda_1|\xi|+h)^{2(m-1-j)}|\tau_{k_\ell}'||F_{k_1,\cdots,\widehat{k}_\ell,\cdots,k_j}v|[v]_j &\lesssim |\xi|(\lambda_1|\xi|+h)^{2(m-1-j)}[v]_{j-1}[v]_j\\ &\lesssim |\xi|E_2.\end{aligned}$$

Hence $\displaystyle\int_0^{t_\xi}|\xi|\,dt \le \int_0^{|\xi|^{1/s-1}}|\xi|\,dt \le |\xi|^{1/s}$ when $t_\xi \le |\xi|^{1/s-1}$.

So it suffices to examine the case $t_\xi > |\xi|^{1/s-1}$ as well. Then we divide the interval $[0,t_\xi]$ into the two subintervals like

$$[0,t_\xi] = [0,|\xi|^{1/s-1}] \cup [|\xi|^{1/s-1},t_\xi].$$

Firstly, in $[0,|\xi|^{1/s-1}]$, as above

$$(\lambda_1|\xi|+h)^{2(m-1-j)}|\tau_{k_\ell}'||F_{k_1,\cdots,\widehat{k}_\ell,\cdots,k_j}v|[v]_j \lesssim |\xi|E_2,$$

where $\displaystyle\int_0^{|\xi|^{1/s-1}}|\xi|\,dt \le |\xi|^{1/s}$. Next, since $\tau_k(t,\xi)$ are distinct in $[|\xi|^{1/s-1},t_\xi]$ by (iii), we have $\partial_\tau P(t,\tau_k,\xi) \ne 0$ for any $t \in [|\xi|^{1/s-1},t_\xi]$. Now, let us verify that $\tau_k(t,\xi)$ is real analytic in $t \in [|\xi|^{1/s-1},t_\xi]$ for every fixed $\xi \in \mathbb{R}^n \setminus \{0\}$.

Indeed, if we consider $P(t,\tau,\xi)$ as a holomorphic function of τ on a neighborhood U_k of τ_k, then by the argument principle there exists some neighborhood V_k of 0 such that

$$\tau(t,\xi) = \frac{1}{2\pi\sqrt{-1}}\oint_{\Gamma_k}\zeta\frac{\partial_\tau P(t,\zeta,\xi)}{P(t,\zeta,\xi)-P(t,\tau,\xi)}\,d\zeta$$

in U_k, where Γ_k is a small closed Jordan curve in U_k enclosing τ_k, but excluding τ_j for all $j \neq k$ (see, e.g. §5.2 in [1]). In particular,

$$\tau_k(t,\xi) = \frac{1}{2\pi\sqrt{-1}} \oint_{\Gamma_k} \zeta \frac{\partial_\tau P(t,\zeta,\xi)}{P(t,\zeta,\xi)}\, d\zeta$$

is real analytic in $t \in [|\xi|^{1/s-1}, t_\xi]$ owing to the analyticity of $a_{j,\alpha}(t)$ on $(0,T]$ for all $j + |\alpha| = m$. Because of the implicit function theorem

$$\tau_k'(t,\xi) = -\frac{\partial_t P(t,\tau_k(t,\xi),\xi)}{\partial_\tau P(t,\tau_k(t,\xi),\xi)}$$

for $t > 0$. Consequently, the zero-point set of τ_k'

$$Z_k(\xi) = \{\, t \in [|\xi|^{1/s-1}, t_\xi] \mid \tau_k'(t,\xi) = 0 \,\}$$

is finite for every fixed $\xi \in \mathbb{R}^n \setminus \{0\}$. Also, for every ξ_0 on the unit sphere S^{n-1} there is a neighborhood W_{ξ_0} of ξ_0 in S^{n-1} such that the supremum of the cardinalities of $Z_k(\xi)$ does not depend on $\xi \in W_{\xi_0}$. In fact, let $|\xi_0| = 1$, $t_0 = 1$, $Z_k(\xi_0) = \{t_1, \cdots, t_N\}$, $t_{N+1} = t_{\xi_0}$ and

$$B = \{\, \zeta \in \mathbb{C} \mid t_0/2 < \operatorname{Re}\zeta < t_{N+1} + 1, |\operatorname{Im}\zeta| < \varepsilon_0 \,\}.$$

Then, by the unicity theorem, $X_k = \{\zeta \in B | \tau_k'(\zeta,\xi_0) = 0\}$ is finite and so we can define the positive constant

$$b_k = \begin{cases} \min\{\, |\operatorname{Im}\zeta| \mid \zeta \in I_k \,\}/2 & \text{if} \quad I_k \neq \emptyset, \\ \varepsilon_0/2 & \text{otherwise}, \end{cases}$$

where $I_k = X_k \cap \{\zeta \in \mathbb{C} | \operatorname{Im}\zeta \neq 0\}$. Next, put $0 < \delta_j < (t_{j+1} - t_j)/2$ and

$$B_j = \{\, \zeta \in B \mid t_j + \delta_j \leq \operatorname{Re}\zeta \leq t_{j+1} - \delta_j, |\operatorname{Im}\zeta| \leq b_k \,\}$$

for $j \in \{0, 1, \cdots, N\}$. Then $\tau_k'(\zeta,\xi_0) \neq 0$ for all $\zeta \in \cup_{j=0}^N B_j$, more precisely,

$$\inf_{\zeta \in \cup_{j=0}^N B_j} |\tau_k'(\zeta,\xi_0)| > 0$$

due to the continuity of τ_k' in t. Hence, taking $\varepsilon > 0$ like

$$0 < \varepsilon < \inf_{\zeta \in B_j} |\tau_k'(\zeta,\xi_0)|,$$

in view of the continuity of τ_k' with respect to ξ uniformly in $\zeta \in B_j$, we have some neighborhood $W_j(\delta_j)$ of ξ_0 in S^{n-1} such that

$$|\tau_k'(\zeta,\xi) - \tau_k'(\zeta,\xi_0)| < \varepsilon < |\tau_k'(\zeta,\xi_0)|$$

is valid provided $\zeta \in B_j$ and $\xi \in W_j$. Thus it follows from Rouché's theorem that $Z_k(\xi) \subset Z_k(\xi_0)$ for any $\xi \in \cup_{\delta>0} \cup_{j=0}^N W_j(\delta)$. Consequently, we can find the resulting W_{ξ_0} as

$$W_{\xi_0} = \cup_{\delta>0} \cup_{j=0}^N W_j(\delta).$$

So that the supremum of the cardinalities of $Z_k(\xi)$ is independent of $\xi \in S^{n-1}$. Moreover, by the homogeneity of $\partial_t P$ and $\partial_\tau P$ in (τ, ξ), the supremum of the cardinalities of $Z_k(\xi)$ provided $\xi \in \mathbb{R}^n \setminus \{0\}$ is finite.

Up to now, to estimate the remaining terms including τ_k' for the case $t_\xi > |\xi|^{1/s-1}$, we observe that from (iv), (ii) and (i)

$$\begin{aligned}
&(\lambda_1|\xi| + h)^{2(m-1-j)}|\tau'_{k_\ell}||F_{k_1,\cdots,\widehat{k_\ell},\cdots,k_j}v|[v]_j \\
&\quad\le \frac{|\tau'_{k_\ell}|}{\tau_{k_\ell}+h}(\tau_{k_\ell}+h)(\lambda_1|\xi|+h)^{2(m-1-j)}[v]_{j-1}[v]_j \\
&\quad\lesssim \frac{|\tau'_{k_\ell}|}{\tau_{k_\ell}+h}(\lambda_{k_\ell}|\xi|+h)(\lambda_1|\xi|+h)^{2(m-1-j)}[v]_{j-1}[v]_j \\
&\quad\le \frac{|\tau'_{k_\ell}|}{\tau_{k_\ell}+h}(\lambda_1|\xi|+h)^{2(m-j)-1}[v]_{j-1}[v]_j \\
&\quad\lesssim \frac{|\tau'_{k_\ell}|}{\tau_{k_\ell}+h}E_2
\end{aligned}$$

for $t \in [|\xi|^{1/s-1}, t_\xi]$. It now follows from the finiteness of $Z_{k_\ell}(\xi)$ and (iv) that

$$\left| \int_{|\xi|^{1/s-1}}^{t_\xi} \frac{|\tau'_{k_\ell}(t,\xi)|}{\tau_{k_\ell}(t,\xi)+h}\,dt \right| \lesssim \log(|\xi|+1) = o(|\xi|^\varepsilon)$$

as $|\xi| \to \infty$ for any $\varepsilon > 0$. Thus, noting that

$$\begin{aligned}
&\int_0^{t_\xi} \frac{\lambda_1'(t)|\xi|+h'}{\lambda_1(t)|\xi|+h}\,dt = \log(\lambda_1(t_\xi)|\xi|+h) = o(|\xi|^\varepsilon), \\
&\int_0^{t_\xi} (\lambda_1(t)|\xi|+h)\,dt \lesssim \Lambda_1(t_\xi)|\xi| + T = |\xi|^{1/s} + T
\end{aligned}$$

and Gronwall's inequality, we can obtain the required energy estimate in Z_{pd}

$$E_2(t_2,\xi) \lesssim E_2(t_1,\xi)\exp\left(C_{s,T}|\xi|^{1/s}+1\right) \tag{3.2.1}$$

for any $t_1, t_2 \in [0, t_\xi]$.

Next we shall show the equivalence between $E_1(t,\xi)$ and $E_2(t,\xi)$ in the standard Sobolev spaces. At first, from the definition of t_ξ

$$E_1(t,\xi) \le E_2(t,\xi)$$

holds for any $t \in [t_\xi, T]$. Conversely, since $\lambda_1(t)/\Lambda_1(t)^{s/(s-1)}$ is monotonically decreasing in $(0,T]$ by (v), we get

$$\frac{\lambda_1(t)}{\Lambda_1(t)^{s/(s-1)}} \ge \frac{\lambda_1(T)}{\Lambda_1(T)^{s/(s-1)}} > 0,$$

consequently

$$E_2(t,\xi) \lesssim (\lambda_1(t)|\xi|+h)^{2(m-1)}E_1(t,\xi)$$

for every $t \in (0, T]$. In particular, for any $t \in [t_\xi, T]$

$$E_2(t,\xi) \lesssim (|\xi|+1)^{2(m-1)} E_1(t,\xi).$$

Hence we enjoy by virtue of (3.1.2)

$$E_2(t_2,\xi) \lesssim E_2(t_1,\xi)(|\xi|+1)^{2(m-1)} \exp\left(C_{s,T}|\xi|^{1/s}+1\right) \tag{3.2.2}$$

for any $t_1, t_2 \in [t_\xi, T]$.

In addition, if we put the standard energy function in Sobolev spaces

$$E_3(t,\xi) = \sum_{j=0}^{m-1} |\xi|^{2(m-1-j)} |v^{(j)}|^2,$$

then it is easy to check that

$$\sum_{j=0}^{m-1} [v]_j^2 \lesssim |\xi|^M E_3(t,\xi)$$

in $[0, T]$ for some constant $M \geq 0$, only dependent on m. Conversely, as can be seen in §4.2 in [7], by Lemma 2.4

$$|v^{(j)}| \leq \sum_{k=0}^{j} |\Delta_k[\tau^j](\tau_{m-k}, \cdots, \tau_m)||\mathcal{F}_k v|,$$

where in aid of Lemma 2.2, (iv) and (ii)

$$\begin{aligned} |\Delta_k[\tau^j](\tau_{m-k}, \cdots, \tau_m)| &\lesssim \sum_{h_0+\cdots+h_k=j-k} |\tau_m|^{h_0} \cdots |\tau_{m-k}|^{h_k} \\ &\lesssim \begin{cases} \lambda_1 |\xi|^{j-k} & \text{if} \quad k<j, \\ 1 & \text{if} \quad k=j, \end{cases} \end{aligned}$$

in particular, for $t = 0$ or $k = j$

$$|v^{(j)}| \lesssim |\mathcal{F}_j v| \leq [v]_j.$$

On the other hand, again from (2.2) with $h = 1$

$$|\mathcal{F}_k v| \lesssim \frac{[v]_{k+1}}{\lambda_1|\xi|}$$

for any $t \in (0, T]$ and $k \in \{1, \cdots, j-1\}$. Therefore

$$|\xi|^{m-1-j}|v^{(j)}| \lesssim \sum_{k=0}^{j} |\xi|^{m-1-k} [v]_k$$

for all $j \in \{0, \cdots, m-1\}$, which lead us to the inverse inequality

$$E_3(t,\xi) \lesssim |\xi|^{2(m-1)} \sum_{j=0}^{m-1} [v]_j^2$$

in $[0,T]$. Because $1 \leq \lambda_1(t)|\xi| + h \lesssim |\xi| + 1$, $E_2(t,\xi)$ is also equivalent to $E_3(t,\xi)$ up to some polynomial of $|\xi|$ for any $t \in [0,T]$. That is, we can finally deduce that there are some constants $M_1, M_2 \geq 0$ satisfying

$$|\xi|^{M_1} E_3(t,\xi) \lesssim E_2(t,\xi) \lesssim |\xi|^{M_2} E_3(t,\xi) \tag{3.2.3}$$

for every $t \in [0,T]$ and $|\xi| \geq 1$. Thus we have proved the conclusion of Theorem 1.2 in the standard way thanks to from (3.2.1) through (3.2.3).

Besides, if $s = \infty$, then it is just enough to change the definition of t_ξ by

$$\Lambda_1(t_\xi)|\xi| = \log|\xi|$$

for $|\xi| \geq 1$. For the details, see, e.g. [17].

4. Some remarks and examples

In this section we shall give some examples and consider relations among (L), (c) and (C).

First of all, let us examine the relation between (L) and (C). We can easily verify it as below.

Proposition 4.1. *Under* (i) $\sim$ (iv), *(L) implies (C).*

Proof. Let us denote

$$L_j(t,\tau,\xi) = \sum_{i+|\alpha|=j} a_{i,\alpha}(t)\tau^i\xi^\alpha$$

for each $j \in \{0, 1, \cdots, m-1\}$. When (L) holds, we get

$$|a_{i,\alpha}||\tau_k|^i|\xi|^{|\alpha|} \lesssim |\tau_k|^i(\lambda_1|\xi|)\cdots(\lambda_{|\alpha|}|\xi|)\left(\frac{\lambda_1}{\Lambda_1^{s/(s-1)}}\right)^{m-j}.$$

If $|\alpha| < k$, then from (iii) and (iv)

$$|\tau_k|^i \lesssim |\tau_k - \tau_{k+1}|\cdots|\tau_k - \tau_{k+i}|,$$

meanwhile, by (A)

$$(\lambda_1|\xi|)\cdots(\lambda_{|\alpha|}|\xi|) \approx |\tau_k - \tau_1|\cdots|\tau_k - \tau_{|\alpha|}|,$$

which lead to (C). Also, if $|\alpha| \geq k$, then we look that

$$\begin{aligned} &(\lambda_1|\xi|)\cdots(\lambda_{k-1}|\xi|)(\lambda_k|\xi|)\cdots(\lambda_{|\alpha|}|\xi|) \\ \lesssim &(\lambda_1|\xi|)\cdots(\lambda_{k-1}|\xi|)\overbrace{(\lambda_k|\xi|)\cdots(\lambda_k|\xi|)}^{|\alpha|-k+1\text{ times}} \\ \approx &|\tau_k - \tau_1|\cdots|\tau_k - \tau_{k-1}||\tau_k - \tau_{k+1}|\cdots|\tau_k - \tau_{|\alpha|+1}| \end{aligned}$$

due to (ii) and (A). On the other hand, by use of (iii) and (iv)

$$|\tau_k|^i \lesssim |\tau_k - \tau_{|\alpha|+2}| \cdots |\tau_k - \tau_{|\alpha|+i+1}|.$$

Therefore (L) means (C) for any case. □

Remark 4.1. Conversely, does (C) imply (L)? It is false in general, from the similar observation as above because τ_k cannot be dominated by $\lambda_{k+1}|\xi|, \cdots, \lambda_{|\alpha|}|\xi|$ when $|\alpha| > k$. In particular, (C) implies (L) only if $\lambda_1 = \cdots = \lambda_m$.

Next let us consider the case that all characteristic roots are finitely degenerate at only $t = 0$ to compare Theorem 1.2 with Theorem 1.1 in [7]. Namely, suppose that there are some constants $r_m \geq \cdots \geq r_2 \geq r_1 \geq 0$ fulfilling

$$|\tau_j(t,\xi) - \tau_k(t,\xi)| \approx t^{r_j}|\xi| \quad \text{if} \quad j < k.$$

Then we can define the weight functions by

$$\lambda_1 = t^{r_1}, \cdots, \lambda_m = t^{r_m}.$$

As for (B), since

$$\frac{\Lambda_1^{s/(s-1)}}{\lambda_1} \approx t^{(s+r_1)/(s-1)},$$

(b) is satisfied for

$$s \leq \begin{cases} \infty & \text{if} \quad s_m = 0, \\ 1 + \dfrac{r_1+1}{s_m} & \text{otherwise.} \end{cases} \tag{4.1}$$

Also, as to (C), because

$$\left(\frac{\Lambda_1^{s/(s-1)}}{\lambda_1}\right)^{m-j} \approx t^{(m-j)(s+r_1)/(s-1)},$$

(c) is verified for

$$s \leq \begin{cases} \infty & \text{if} \quad s_j \leq m-j \quad \text{for all} \quad j \in \{1,\cdots,m-1\}, \\ 1 + \min I & \text{otherwise,} \end{cases} \tag{4.2}$$

where $I = \left\{ \dfrac{(m-j)(r_1+1)}{s_j - (m-j)} \;\middle|\; s_j > m-j \text{ for some } j \in \{1,\cdots,m-1\} \right\}$.
Hence, applying Theorem 1.2 with (4.1) and (4.2), we have the following result on the well-posedness of (CP).

Corollary 4.1. *Suppose that (a), (b) and (c). Then (CP) is the Gevrey well-posed in $\gamma^{\bar{s}}$ of (CP) provided $1 < \bar{s} < s$ with*

$$s \le \begin{cases} \infty \quad \textit{if} \quad s_m = 0 \quad \textit{and} \quad s_j \le m-j \quad \textit{for all} \quad j \in \{1, \cdots, m-1\}, \\ 1 + \dfrac{r_1+1}{s_m} \quad \textit{if} \quad s_m > 0 \quad \textit{and} \quad s_j \le m-j \quad \textit{for all} \quad j \in \{1, \cdots, m-1\}, \\ 1 + \min I \quad \textit{if} \quad s_m = 0 \quad \textit{and} \quad s_j > m-j \quad \textit{for some} \quad j \in \{1, \cdots, m-1\}, \\ 1 + \min \left\{ \dfrac{r_1+1}{s_m}, \ \min I \right\} \quad \textit{otherwise.} \end{cases}$$

Unfortunately, Corollary 4.1 is a partial result of Theorem 1.1 in [7] except the case $s_j \le m-j$ for all $j \in \{1, \cdots, m\}$. However, as emphasized in §1, we can adapt Theorem 1.1 and Theorem 1.2 to infinitely degenerate characteristic roots as well.

Example 4.1 (Ivrii's Example in [11], [14])**.** Let us consider the case $m = 2$, $n = 1$, $a_{0,2}(t) = t^{2k}$ and $a_{0,1}(t) = t^{\ell}$ with $0 \le \ell < k-1$. Then $\tau_1, \tau_2 = \pm t^k \xi$. So we may choose $\lambda_1(t) = \lambda_2(t) = t^k$, consequently $\Lambda_1(t) = \dfrac{t^{k+1}}{k+1}$. In this case (i) $\sim$ (v) and (P) are automatically satisfied because of $k > 1$ and $s > 1$. As well, by

$$\frac{\lambda_1(t)^2}{\Lambda_1(t)^{s/(s-1)}} \approx t^{((k-1)s-2k)/(s-1)},$$

(L) is equivalent to the condition

$$\ell \ge \frac{(k-1)s-2k}{s-1},$$

that is,

$$s \le \frac{2k-\ell}{k-\ell-1} \tag{4.3}$$

is just the same range which tells us a necessary and sufficient condition for the the Gevrey well-posedness in $\gamma^{\bar{s}}$ of (CP) provided $1 < \bar{s} < s$, given by Ivrii in [11]. The same sufficiency for the Gevrey well-posedness is also obtained for the general dimensional case $n \ge 1$ by Shinkai and Taniguchi in [14]. We should remark that (4.3) is not necessary for the Gevrey well-posedness in γ^s when $n \ge 2$.

Example 4.2 (Yagdjian's Example 2 in [15])**.** He especially treated (CP) for an infinitely degenerate case $m = 2$, $n = 1$, $a_{0,2}(t) = \exp(-2t^{-r})$, $a_{0,1}(t) = t^q \exp(-bt^{-r})$ with b, $r > 0$ and proved the Gevrey well-posedness in γ^s for

$$s < \begin{cases} \frac{2-b}{1-b} & \text{if} \quad b < 1, \\ \infty & \text{if} \quad b \ge 1, \end{cases}$$

which is just the same one got by (L) since

$$\frac{\lambda_1(t)^2}{\Lambda_1(t)^{s/(s-1)}} \approx t^{(r+1)s/(1-s)} \exp\left(-\frac{s-2}{s-1}t^{-r}\right)$$

for $\lambda_1(t) = \exp(-t^{-r})$ near $t = 0$.

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