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Non-existence of unbounded Fatou components of a meromorphic function

By

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Abstract

This paper is devoted to study of sufficient conditions under which a transcendental meromorphic function has no unbounded Fatou components and to extension of some results for entire functions to meromorphic functions. Actually, we shall mainly discuss non-existence of unbounded wandering domains of a meromorphic function. The case for a composition of finitely many meromorphic functions with at least one of them being transcendental can be also investigated in terms of the argument of this paper.

1. Introduction and main results

Let $\mathcal{M}$ be the family of all functions which are meromorphic in the complex plane $\mathbb{C}$ possibly outside one at most countable closed set and have ∞ as an essential singular point. For example, a composition of finitely many transcendental meromorphic functions is in $\mathcal{M}$. Here we mean a function meromprphic in $\mathbb{C}$ with only one essential singular point at ∞ by a transcendental meromorphic function. We shall study iterations of element in $\mathcal{M}$.

We denote the nth iteration of $f(z) \in \mathcal{M}$ by $f^n(z) = f(f^{n-1}(z)), n = 1, 2, \ldots$. Then $f^n(z)$ is well defined for all $z \in \mathbb{C}$ outside a (possible) countable closed set

$$E(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(E(f)),$$

here $E(f)$ is the set of all points at which $f(z)$ is not meromorphic. Define the Fatou set $F(f)$ of $f(z)$ as

$$F(f) = \{z \in \mathbb{C} : \{f^n(z)\} \text{ is well defined and normal in a neighborhood of } z\}$$

and $J(f) = \widehat{\mathbb{C}} \setminus F(f)$ is the Julia set of $f(z)$. $F(f)$ is open and $J(f)$ is closed, non-empty and perfect. It is well-known that both $F(f)$ and $J(f)$ are completely invariant under $f(z)$, that is, $z \in F(f)$ if and only if $f(z) \in F(f)$.

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And $F(f^n) = F(f)$ and $J(f^n) = J(f)$ for any positive integer n. We shall consider components of the Fatou set $F(f)$ and hence let U be a connected component of $F(f)$. Since $F(f)$ is completely invariant under f, $f^n(U)$ is contained in $F(f)$ and connected, so there exists a Fatou component U_n such that $f^n(U) \subseteq U_n$. If for some $n \geq 1, f^n(U) \subseteq U$, that is, $U_n = U$, then U is called a periodic component of $F(f)$ and such the smallest integer n is the period of periodic component U. In particular, a periodic component of period one is also called invariant. If for some n, U_n is periodic, but U is not periodic, then U is called pre-periodic; A periodic component U of period p must be one of the following five types: (i) attracting domain, that is, U contains a point a such that $f^p(a) = a$ and $|(f^p)'(a)| < 1$ and $f^{np}|_U \to a$ as $n \to \infty$; (ii) parabolic domain, that is, there exists a point $a \in \partial U$ such that $f^p(a) = a$ and $(f^p)'(a) = e^{2\pi i\alpha}$ for $\alpha \in \mathbb{Q}$ and $f^{np}|_U \to a$ as $n \to \infty$; (iii) Baker domain, that is, $f^{np}|_U \to a \in \partial U \cup \{\infty\}$ as $n \to \infty$ and $f^p(z)$ is not defined at $z = a$; (iv) Siegel disk, that is, U is simply connected and contains a point a such that $f^p(a) = a$ and $\phi \circ f^p \circ \phi^{-1}(z) = e^{2\pi i\alpha} z$ for some real irrational number α and a conformal mapping ϕ of U onto the unit disk with $\phi(a) = 0$; (v)Herman ring, that is, U is doubly connected and $\phi \circ f^p \circ \phi^{-1}(z) = e^{2\pi i\alpha} z$ for some real irrational number α and a conformal mapping ϕ of U onto $\{1 < |z| < r\}$. U is called wandering if it is neither periodic nor preperiodic, that is, $U_n \cap U_m = \emptyset$ for all $n \neq m$. For the basic knowledge of dynamics of a meromorphic function, the reader is referred to [5] and the book [15].

If for a function $f \in \mathcal{M}$, $f^{-2}(E(f))$ contains at least three distinct points, then

$$J(f) = \overline{\bigcup_{n=1}^{\infty} f^{-n}(E(f))},$$

and what we should mention is that in any case, for every $n \geq 1$, $f^n(z)$ is analytic on $F(f)$. In particular, this result holds for a composition of finitely many meromorphic functions.

Our study in this paper relies on the Nevanlinna theory of value distribution. To the end, let us recall some basic concepts and notations in the theory. Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

and

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ is the number of poles of $f(z)$ in the disk $\{|z| \leq t\}$ counted with multiplicities, and

$$T(r, f) = m(r, f) + N(r, f)$$

which is known as the Nevanlinna characteristic function of $f(z)$. The quantity $\delta(\infty, f)$ is the Nevanlinna deficiency of f at ∞, defined by the following formula

$$\delta(\infty, f) = \liminf_{r\to\infty} \frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)}.$$

(See [7]). The growth order and lower order of $f(z)$ are defined respectively by

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\log\ T(r,f)}{\log\ r}$$

and

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log\ T(r,f)}{\log\ r}.$$

For $0 < \lambda(f) < +\infty$, define type of $f(z)$ by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\lambda(f)}}$$

and $f(z)$ is called maximal type if $\sigma(f) = +\infty$; mediem type if $0 < \sigma(f) < +\infty$; minimal type if $\sigma(f) = 0$.

I. N. Baker in 1984 raised a question of whether every component of $F(f)$ of a transcendental entire function $f(z)$ is bounded if its growth is sufficiently small. Following I. N. Baker's question, a number of papers gave out some sufficient conditions which confirm Baker's question for the case of entire functions.

In this paper, we take into account the question for the case of meromorphic functions. Zheng [17] made a discussion of non-existence of unbounded Fatou components of a meromorphic function and actually the method in [17] can be used to obtain the following

Theorem 1.1. *Let $f(z)$ be a function in $\mathcal{M}$. If we have*

$$\limsup_{r\to+\infty} \frac{L(r,f)}{r} = +\infty, \tag{1}$$

where $L(r,f) = \min\{|f(z)| : |z| = r\}$, then the Fatou set, $F(f)$, of f has no unbounded preperiodic or periodic components.

It is described by an example in Zheng [17] that the condition (1) is sharpen. Theorem 1.1 confirms that an entire function whose growth does not exceed order 1/2 and minimal type has no unbounded preperiodic or periodic components, whereas the result for the case of order less than 1/2 was proved in several papers, see [12] and [2]. Baker [3] shown by an example that the order 1/2 and minimal type is the best possible growth condition in terms of order. However, we do not know if the result is true for a meromorphic function with the growth not exceeding order 1/2 and minimal type.

In view of a well-known result that (1) is satisfied for a transcendental meromorphic function with lower order $\mu(f) < 1/2$ and $\delta(\infty, f) > 1 - \cos(\mu(f)\pi)$, Theorem 1.1 also confirms that such a meromorphic function has no unbounded preperiodic or periodic components. For a composition $g(z) = f_m \circ f_{m-1} \circ \cdots \circ f_1(z)$ of finitely many transcendental meromorphic functions $f_j(z)(j = 1, 2, \ldots, m; m \geq 1)$, from the method of [17] it follows that $F(g)$ has no unbounded periodic or preperiodic components if for each j, there exist a sequence of positive real numbers tending to infinity at which $L(r, f_j) > r$ and (1) holds for at least one f_{j_0}.

Therefore, the crucial point solving I. N. Baker's question is in discussion of non-existence of unbounded wandering domains of a meromorphic function. There are a series of results for the case of entire functions on which assumptions of order less than 1/2 and the certain regularity of the growth are imposed. Let $f(z)$ be a transcendental entire function with order $< 1/2$. Then every component of $F(f)$ is bounded, provided that one of the following statements holds:

(1) $\frac{\log M(2r,f)}{\log M(r,f)} \to c \geq 1$ as $r \to \infty$ (Stallard [13], 1993);

(2) $\frac{\varphi'(x)}{\varphi(x)} \geq \frac{c}{x}$, for all sufficiently large x, where $\varphi(x) = \log M(e^x, f)$ and $c > 1$ (Anderson and Hinkkanen [2], 1998);

(3) $\log M(r^m, f) \geq m^2 \log M(r, f)$ for each $m > 1$ and all sufficiently large r (Hua and Yang [10], 1999);

(4) $\mu(f) > 0$ (Wang [14], 2001).

A straightforward calculation deduces that an entire function satisfying the Stallard's condition with $c > 1$ must be of lower order at least $\log c/\log 2$. However, an entire function with $0 < \mu(f) \leq \lambda(f) < \infty$ must satisfy the Hua and Yang's condition for m with $\mu(f)m > \lambda(f)$. In fact, choosing $\varepsilon > 0$ with $(\mu - \varepsilon)m > \lambda + 2\varepsilon$, we have for sufficiently large $r > 0$

$$\log M(r^m, f) > (r^m)^{\mu-\varepsilon} > r^\varepsilon r^{\lambda+\varepsilon} \geq r^\varepsilon \log M(r, f). \tag{2}$$

What we should mention is that by modifying a little the proof given in [10], Hua and Yang's condition for sufficiently large m instead of each $m > 1$ suffices to confirm their result to be true.

Zheng and Wang [18] in 2004 proved the following

Theorem 1.2. *Let $f(z)$ be a transcendental entire function. If there exists a $d > 1$ such that for all sufficiently large $r > 0$ we can find a $\tilde{r} \in [r, r^d]$ satisfying*

$$\log L(\tilde{r}, f) \geq d \log M(r, f), \tag{3}$$

then every component of $F(f)$ is bounded.

A discussion of the case of composition of a number of entire functions was made in [18]. In 2005, Hinkkanen [8] also gave out a weaker condition than (3), that is, the coefficient "d" before $\log M(r, f)$ is replaced by "$d(1 - (\log r)^{-\delta})$" with $\delta > 0$. For more material about this subject, the reader is referred to the excellent survey recently written by Hinkkanen [9].

In this paper, in view of the Nevanlinna theory of a meromorphic function, we consider the case of a meromorphic function and our main result is the following.

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function. Assume that for $D > d > 1$ and for all the sufficiently large r, there exists a $t \in [r, r^d]$ satisfying*

$$\log L(t, f) > DT(r, f). \tag{4}$$

Then $F(f)$ has no unbounded components.

Actually, the assumption in Theorem 1.3 is also a sufficient condition of existence of buried points of the Julia set of a meromorphic function with at least one pole which is not the form $f(z) = a + (z-a)^{-p}e^{g(z)}$. For such a meromorphic function, $J(f) = \overline{\bigcup_{j=0}^{\infty} f^{-j}(\infty)}$ and from Theorem 1.3, the Fatou set $F(f)$ has only bounded components and hence ∞ cannot stand on boundary of a Fatou component. This implies that ∞ is a buried point of $f(z)$ and therefore so are all prepoles.

As a consequence of Theorem 1.3, we have

Theorem 1.4. *Let $f(z)$ be a transcendental meromorphic function and such that for some $\alpha \in (0, 1)$ and $D > d > 1$ and all the sufficiently large r, there exists a $t \in [r, r^d]$ satisfying*

$$\log L(t, f) > \alpha T(r, f), \tag{5}$$

and

$$T(r^d, f) \geq DT(r, f). \tag{6}$$

Then $F(f)$ has no unbounded components.

As a consequence of Theorem 1.4, we have the following

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function with*

$$\delta(\infty, f) > 1 - \cos(\pi\lambda(f))$$

and $0 < \mu(f) \leq \lambda(f) < 1/2$. Then $F(f)$ has no unbounded components.

In particular, Wang's result can be deduced from Theorem 1.5, for any transcendental entire function has the Nevanlinna deficiency one at ∞, that is to say, $\delta(\infty, f) = 1$.

2. The proof of theorems

To prove Theorems, we need some preliminary results. First preliminary result will be established by using the hyperbolic metric and it has independent significance. To the end, let us recall some properties on the hyperbolic metric, see ([1], [4]), etc. An open set W in $\mathbb{C}$ is called hyperbolic if $\mathbb{C} \setminus W$ contains at least two points (Notice that ∞ has been put outside W). Let U be a hyperbolic domains in $\mathbb{C}$. $\lambda_U(z)$ is the density of the hyperbolic metric on U and $\rho_U(z_1, z_2)$ stands for the hyperbolic distance between z_1 and z_2 in U, i.e.

$$\rho_U(z_1, z_2) = \inf_{\gamma \in U} \int_\gamma \lambda_U(z)|dz|,$$

where γ is a Jordan curve connecting z_1 and z_2 in U. For a hyperbolic open set W, the hyperbolic density $\lambda_W(z)$ of W is the hyperbolic density for each component of W. Then we convent that the hyperbolic distance between two points which are in disjoint components equals to ∞ and the hyperbolic distance of two points a and b in one component U equals to $\rho_W(a, b) = \rho_U(a, b)$. For a fixed point $a \notin W$, introduce a domain constant

$$C_W(a) = \inf\{|z - a|\lambda_W(z) : z \in W\}.$$

If U is simply-connected and $d(z, \partial U)$ is an Euclidean distance between $z \in U$ and ∂U, then for any $z \in U$,

$$\frac{1}{2d(z, \partial U)} \le \lambda_U(z) \le \frac{2}{d(z, \partial U)}. \tag{7}$$

The inequality follows from the Koebe Quarter Theorem. Let $f : U \to V$ be analytic, where both U and V are hyperbolic domains. By the principle of hyperbolic metric, we have

$$\rho_V(f(z_1), f(z_2)) \le \rho_U(z_1, z_2), \text{ for } z_1, z_2 \in U. \tag{8}$$

In particular, if $U \subset V$, then $\lambda_V(z) \le \lambda_U(z)$ for $z \in U$.

Lemma 2.1 (cf. Zheng [15]). *Let U be a hyperbolic domain and $f(z)$ a function such that each $f^n(z)$ is analytic in U and $\bigcup_{n=0}^\infty f^n(U) \subseteq W$. If for some fixed point $a \notin W$, $C_W(a) > 0$ and $f^n|_U \to \infty$, then for any compact subset K of U there exists a positive constant $M = M(K)$ such that for all sufficiently large n, we have*

$$M^{-1}|f^n(z)| \le |f^n(w)| \le M|f^n(z)| \text{ for } z, w \in K. \tag{9}$$

Proof. Under the assumption of Lemma 2.1, we obtain

$$\lambda_W(z) \ge \frac{C_W(a)}{|z - a|} \ge \frac{C_W(a)}{|z| + |a|}. \tag{10}$$

It follows that

$$
\begin{aligned}
\rho_{f^n(U)}(f^n(z), f^n(w)) &\geq \rho_W(f^n(z), f^n(w)) \\
&\geq C_W(a)\left|\int_{|f^n(z)|}^{|f^n(w)|} \frac{dr}{r+|a|}\right| \\
&= C_W(a)\left|\log\frac{|f^n(z)|+|a|}{|f^n(w)|+|a|}\right|.
\end{aligned} \tag{11}
$$

Set $A = \max\{\rho_U(z,w) : z,\ w \in K\}$. Clearly $A \in (0, +\infty)$. From (8), we have

$$\rho_{f^n(U)}(f^n(z), f^n(w)) \leq \rho_U(z,w) \leq A. \tag{12}$$

Therefore, combining (11) and (12) gives

$$(|f^n(w)|+|a|)e^{-A/C_W(a)} \leq |f^n(z)|+|a| \leq (|f^n(w)|+|a|)e^{A/C_W(a)}. \tag{13}$$

The above inequality together with the condition $f^n|_U \to \infty$ deduces that $f^n(z)$ uniformly converges to infinity on K as $n \to \infty$. Then there exists a positive integer N such that for $n \geq N$ and $z \in K$, we have $|f^n(z)| \geq |a|$ and furthermore in view of (13), we have

$$\frac{1}{2}|f^n(w)|e^{-A/C_W(a)} \leq |f^n(z)| \leq 2|f^n(w)|e^{A/C_W(a)}.$$

This is (9) with $M = 2e^{A/C_W(a)}$. We completes the proof of Lemma 2.1. □

The following result comes from Lemma 4 of Zheng [17] (also see Theorem 1.6.7 of [15]).

Lemma 2.2. *Let $U \subset \mathbb{C}$ be a hyperbolic domain without isolated boundary points and let $f(z)$ be analytic on U without fixed points. If $f(U) \subset U$ and $f^n|_U \to \infty (n \to \infty)$, then for any compact subset K of U, we have some $M = M(K) > 1$ such that for all sufficiently large n,* (9) *holds for $z, w \in K$.*

In terms of Lemma 2.1 and Lemma 2.2, we establish the following, which is of independent significance.

Theorem 2.1. *Let $f(z)$ be a function in $\mathcal{M}$. If $F(f)$ contains an unbounded component, then for any compact subset K in a component of $F(f)$ with $f^n|_K \to \infty$ as $n \to \infty$, we have a positive constant $M = M(K) > 1$ such that for all sufficiently large n,* (9) *holds for $z, w \in K$.*

Proof. Assume that K is contained in a component U of $F(f)$. If $J(f)$ has one unbounded component in $\mathbb{C}$, then we can find a subset Γ of $J(f)$ such that $W = \mathbb{C} \setminus \Gamma$ is simply-connected. In view of (7), for each $a \in \Gamma$, $C_W(a) \geq \frac{1}{2}$. Then in view of Lemma 2.1 we get $M = M(K)$ such that (9) holds by noting that $\bigcup_{n=0}^{\infty} f^n(U) \subset W$.

Now assume that $J(f)$ only has bounded components in $\mathbb{C}$ and thus $F(f)$ has only one unbounded component denoted by V. If $\bigcup_{n=0}^{\infty} f^n(U)$ does not intersect V, then in view of the fact that the boundary of V has only bounded components we can choose a path L in V tending to ∞ such that $\bigcup_{n=0}^{\infty} f^n(U) \subset W = \mathbb{C} \setminus L$. Thus as we did above, the result of Theorem 2.1 follows.

Next let us consider the case when $U \subseteq \bigcup_{n=0}^{\infty} f^{-n}(V)$. If V is wandering, then for some $m > 1$, $\bigcup_{n=m}^{\infty} f^n(U)$ does not intersect V and therefore we can also prove Theorem 2.1 in this case; If V is preperiodic or periodic, then there exist $m \geq 0$ and $N > 0$ such that $f^N(f^m(U)) \subseteq f^m(U)$ and further for $0 \leq j \leq N-1$, we have

$$f^N(f^{m+j}(U)) = f^{N+j}(f^m(U)) \subseteq f^{m+j}(U).$$

Let U_{m+j} $(0 \leq j \leq N-1)$ be the Fatou component containing $f^{m+j}(U)$ and then $f^N(U_{m+j}) \subseteq U_{m+j}$. It is clear that U_{m+j} has no isolated boundary points. Recalling the definition of functions in $\mathcal{M}$, we know that ∞ is an essential singular point of f and hence $\infty \in J(f)$. Since $f^n|_U \to \infty$ as $n \to \infty$, U_{m+j} does not contain any fixed-points. Since $\infty \notin U_{m+j}$, U_{m+j} is a hyperbolic domain in $\mathbb{C}$. An application of Lemma 2.2 to f^N and U_{m+j} yields that there exists a positive constant $M_j = M_j(K)$ such that for each j and sufficiently large n, we have

$$M_j^{-1}|f^{nN}(f^{m+j}(w))| \leq |f^{nN}(f^{m+j}(z))| \leq M_j|f^{nN}(f^{m+j}(w))|,$$

$z, w \in K$. Since every positive integer $Q > m$ has the form $Q = nN + m + j$ for some n and j with $0 \leq j \leq N-1$, we have (9) for all sufficiently large n and $M = \max\{M_j : 0 \leq j \leq N-1\}$.

Thus we complete the proof of Theorem 2.1. □

The second preliminary result comes from the Poisson formula.

Lemma 2.3. *Let $f(z)$ be meromorphic on $\{|z| \leq 3R\}$. Then there exists a $r \in (R, 2R)$ such that on $|z| = r$, we have*

$$\log^+ |f(z)| \leq KT(3R, f). \tag{14}$$

where $K(\leq 24)$ is a universal constant, that is, it is independent of R, r and f.

Proof. Set $D = \{|z| \leq \frac{5}{2}R\}$. We denote by $G_D(\zeta, z)$ the Green function of D, that is,

$$G_D(\zeta, z) = \log \left| \frac{(2.5R)^2 - \bar{z}\zeta}{2.5R(\zeta - z)} \right|, \ z, \zeta \in D.$$

A simple calculation implies that for z with $|z| \leq 2R$, we have

$$G_D(\zeta, z) \leq \log \frac{5R}{|\zeta - z|}$$

and for $\zeta = 2.5Re^{i\theta}$ and $r = |z| \leq 2R$,

$$\frac{\partial}{\partial \vec{n}} G_D(\zeta, z) ds = \mathrm{Re} \frac{2.5Re^{i\theta} + z}{2.5Re^{i\theta} - z} d\theta \leq \frac{2.5R + r}{2.5R - r} d\theta \leq 9 d\theta.$$

In view of the Poisson formula, we have

$$\begin{aligned} \log|f(z)| &= \frac{1}{2\pi} \int_{\partial D} \log|f(\zeta)| \frac{\partial}{\partial \vec{n}} G_D(\zeta, z) ds \\ &\quad - \sum_{a_n \in D} G_D(a_n, z) + \sum_{b_n \in D} G_D(b_n, z) \\ &\leq 9m(2.5R, f) + \sum_{b_n \in D} \log \frac{5R}{|b_n - z|}, \end{aligned}$$

where a_n is a zero and b_n a pole of $f(z)$ in D counted according to their multiplicities. According to the definition of $N(r, f)$, we have

$$\begin{aligned} n(2.5R, f) &\leq \left(\log \frac{6}{5} \right)^{-1} \int_{2.5R}^{3R} \frac{n(t, f)}{t} dt \\ &\leq 6N(3R, f). \end{aligned}$$

From the Boutroux-Cartan Theorem (see [6]) it follows that

$$\prod_{n=1}^{N} |z - b_n| \geq \left(\frac{R}{2e} \right)^N, \quad N = n(2.5R, f),$$

for all $z \in \mathbb{C}$ outside at most N disks (γ) the total sum of whose diameters does not exceed $R/2$. Therefore there exists a $r \in [R, 2R]$ such that $\{|z| = r\} \cap (\gamma) = \emptyset$ and then on the circle $|z| = r$, we have

$$\log^+ |f(z)| \leq 9m(2.5R, f) + N \log 10e < 24T(3R, f).$$

Thus we complete the proof of Lemma 2.3. □

Proof of Theorem 1.3. From Lemma 2.3, for all $r > e$, there exists an $r_1 \in (r, 2r)$ such that for z with $|z| = r_1$, we have

$$\log|f(z)| \leq KT(3r, f), \tag{15}$$

where K is a positive constant independent of f and r. Take a positive integer m such that $D^{m-2} > Kd^m = KH,\ H = d^m$.

Suppose that f has an unbounded Fatou component, say U. Take a large positive number R_0 such that for all $r \geq R_0$, (4) holds. Assume that U intersects $|z| = R_0$, otherwise we magnify R_0. Take a point z_0 in $U \cap \{|z| = R_0\}$. Draw a curve $\gamma \in U$ from z_0 to $U \cap \{|z| = R_0^H\}$ such that $\gamma \subset \{|z| < R_0^H\}$ except the end point of γ.

Then there exists a $z_1 \in \gamma \cap \{R_0 \leq |z| \leq 2R_0\}$ such that $\log |f(z_1)| \leq KT(3R_0, f)$. And there exists a $r_1 \in (R_0^{d^{m-1}}, R_0^H)$ such that

$$\log L(r_1, f) \geq T(R_0^{d^{m-1}}, f) \geq D^{m-2}T(R_0^d, f) > KHT(3R_0, f), \tag{16}$$

on $|z| = r_1$. Set $R_1 = \exp(KT(3R_0, f))$. Then

$$f(\gamma) \cap \{|z| < R_1\} \neq \emptyset \text{ and } f(\gamma) \cap \{|z| > R_1^H\} \neq \emptyset. \tag{17}$$

By the same argument as above, we have a $z_2 \in f(\gamma) \cap \{R_1 \leq |z| \leq 2R_1\}$ such that $\log |f(z_2)| \leq KT(3R_1, f)$ and a $r_2 \in (R_1^{d^{m-1}}, R_1^H)$ such that

$$\log L(r_2, f) \geq HKT(3R_1, f), \quad \text{on } |z| = r_2.$$

Set $R_2 = \exp(KT(3R_1, f))$. Then since the circle $\{|z| = r_2\}$ intersects $f(\gamma)$, we have

$$f^2(\gamma) \cap \{|z| < R_2\} \neq \emptyset \text{ and } f^2(\gamma) \cap \{|z| > R_2^H\} \neq \emptyset. \tag{18}$$

Define $R_n = \exp(KT(3R_{n-1}, f))$ inductively. Then for each $n > 0$ we always have

$$f^n(\gamma) \cap \{|z| < R_n\} \neq \emptyset$$

and

$$f^n(\gamma) \cap \{|z| \geq R_n^H\} \neq \emptyset.$$

Thus there is two points $z_n,\ w_n \in \gamma$ such that

$$|f^n(z_n)| > R_n^H > |f^n(w_n)|^H. \tag{19}$$

Combining (19) and Theorem 2.1 gives

$$|f^n(w_n)|^H < |f^n(z_n)| \leq M|f^n(w_n)|, \tag{20}$$

for some positive number $M = M(\gamma)$. This is impossible as $n \to \infty$, because $H > 1$ and $|f^n(z_n)| > R_n^H \to +\infty$ as $n \to +\infty$ and further $|f^n(w_n)| \to +\infty$ as $n \to +\infty$.

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. For $\alpha > 0$, there exists a natural number k such that $D^{k-1}\alpha \geq d^{k+1}$. Set $h = d^k$. In view of (5) and (6), for all $r \geq R_0$, we have a $t \in (r^{d^{k-1}}, r^h)$ such that

$$\begin{aligned} \log L(t, f) &\geq \alpha T(r^{d^{k-1}}, f) \geq \alpha D^{k-1}T(r, f) \\ &\geq d^{k+1}T(r, f), \quad \text{on } |z| = t. \end{aligned} \tag{21}$$

Then the condition (4) is satisfied with h in place of d and $D = dh$ and so Theorem 1.4 follows from Theorem 1.3. □

To prove Theorem 1.5, we need the following result, which was proved by Gol'dberg and Sokolovskaya [11].

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function with $\delta(\infty, f) > 1 - \cos(\pi\lambda(f))$ and $\lambda(f) < 1/2$. Then*

$$\underline{\log \mathrm{dens}} E > 0,$$

where $E = \{r > 0 : \log L(r, f) > \alpha T(r, f)\}$ for some positive α.

In fact, Lemma 2.4 asserts that for sufficiently large $r > 0$, we can find a $t \in [r, r^d]$ for some $d > 1$ with

$$\log L(t, f) > \alpha T(r, f).$$

For a function $f(z)$ with $0 < \mu(f) \leq \lambda(f) < +\infty$, we easily see that

$$\lim_{r\to\infty} \frac{T(r^d, f)}{T(r, f)} = \infty$$

for d with $d\mu(f) > \lambda(f)$.

Therefore Theorem 1.5 follows immediately from Theorem 1.4. □

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