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THE RATE OF CONVERGENCE ON SCHRÖDINGER OPERATOR

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ABSTRACT. Recently, Du, Guth and Li showed that the Schrödinger operator $e^{it\Delta}$ satisfies $\lim_{t\to 0} e^{it\Delta} f = f$ almost everywhere for all $f \in H^s(\mathbb{R}^2)$, provided that $s > 1/3$. In this paper, we discuss the rate of convergence on $e^{it\Delta}(f)$ by assuming more regularity on f. At $n = 2$, our result can be viewed as an application of the Du–Guth–Li theorem. We also address the same issue on the cases $n = 1$ and $n > 2$.

1. Introduction

Let Δ be the Laplace operator and $|\Delta|^{s/2}$ be the fractional Laplacian defined via the Fourier transform by

$$\widehat{|\Delta|^{s/2} f}(\xi) = |\xi|^s \widehat{f}(\xi),$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of f. The solution of the free Schrödinger equation

$$iu_t - \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}^n$$

is given by

$$e^{it\Delta} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{f}(\xi)\, d\xi.$$

If $f \in L^2(\mathbb{R}^n)$, the Plancherel theorem yields the $L^2(\mathbb{R}^n)$ norm convergence

$$\lim_{t\to 0} \| e^{it\Delta}(f) - f \|_{L^2(\mathbb{R}^n)} = 0.$$

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Concerning to the point convergence, it is not hard to see that there is an $L^2(\mathbb{R}^n)$ function f for which

$$\lim_{t\to 0} e^{it\Delta}(f)(x) = f(x)$$

fails in a set of positive measure. On the other hand, we always have the almost everywhere convergence (a.e. convergence) if f is sufficiently smooth, say $f \in H^s(\mathbb{R}^n)$ for a large $s > 0$, where H^s is the Sobolev space defined by

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} + \||\Delta|^{s/2} f\|_{L^2(\mathbb{R}^n)} < \infty\}.$$

In fact, using an interpolation on mix normed spaces and combining with the Sobolev imbedding theorem, we can obtain the inequality

$$\Big\|\sup_{t>0} |e^{it\Delta}(f)|\Big\|_{L^2(\mathbb{R}^n)} \preceq \|f\|_{H^s(\mathbb{R}^n)}$$

if $s > n/2$. It then yields the almost everywhere convergence of $e^{it\Delta}(f)$.

This observation raises to a natural question: Find the smallest s such that

$$\lim_{t\to 0} e^{it\Delta}(f)(x) = f(x) \quad \text{a.e.} \tag{1}$$

for all $f \in H^s(\mathbb{R}^n)$.

The above problem was originally posed by Carleson. In [3], Carleson proved that the convergence is true if $f \in H^s(\mathbb{R})$ and $s \geq 1/4$ by using methods of oscillatory integral. A few years later, Dahberg and Kenig in 1982 (see [4]) showed that (1) could fail for $s < 1/4$ in any dimension. Hence, the results by Carleson, Dahberg and Kenig include that $e^{it\Delta}(f)(x) \to f(x)$ a.e for $f \in H^s(\mathbb{R})$ if and only if $s \geq 1/4$, which completely solves the problem in the case $n = 1$.

When $n \geq 2$, the problem is much more involved, and it becomes one of the most interesting and challenging research topic in harmonic analysis. Among numerous research papers, in the following we list a few of them. Sjölin in 1987 (see [14]) and Vega in 1988 (see [18]) proved independently that (1) holds in any dimension if $s > 1/2$. Their result was improved recently by Bourgain in [1], in which Bourgain showed that (1) holds if $s > 1/2 - \frac{1}{4n}$. On the other hand, Luca and Rogers in [11], [12], Demete and Guo in [5] proved that $s \geq 1/2 - \frac{1}{n+2}$ does not guarantee the validity of the inequality in (1). Further, Bourgain in [2] improved this condition of necessity by giving counterexamples to show that (1) could fail if $s < 1/2 - \frac{1}{2n+2}$. Also, more research is focused on the case $n = 2$, see Moyua, Vargas, Vega [13], Tao, Vargas [17], Lee [10]. Particularly, in a most recent paper [6], Du, Larry and Li obtained the almost everywhere convergence of $e^{it\Delta}(f)(x)$ for $f \in H^s(\mathbb{R}^2)$ if $s > 1/3$. Combining their result and Bourgain's necessity criterion at $n = 2$, we know that, at $n = 2$, the Carleson's problem is completely solved except at the critical index $s = 1/3$. We notice that the proof on the celebrated theorem of Du, Guth and Li is very elegant and complicated, and it involves several new techniques and methods in analysis. For general n, Du–Guth–Li–Zhang in [7] improved the sufficient condition to $s>(n+1)/2(n+2)$. Very recently, Du–Zhang in [8] improved the sufficiently to the almost sharp range $s>n/2(n+1)$.

In this paper, we will not aim to pursue any improvement of known results in this topic. Instead of it, we will seek the convergence speed of $e^{it\Delta}(f)(x)$ as t goes to 0, if f has more regularity. Precisely, suppose that we have built $\lim_{t\to 0} e^{it\Delta}(f)(x)-f(x)=0$ almost everywhere for $f\in H^s(\mathbb{R}^n)$, whether or not we have, for $f\in H^{s+\delta}(\mathbb{R}^n)$ and $\delta\geq 0$,

$$e^{it\Delta}(f)(x)-f(x)=o\big(t^{\delta/2}\big)?$$

First, we will study this issue on the case of dimension $n=2$. As a non-trivial consequence of the theorem of Du–Guth–Li, we have the following result.

THEOREM 1.1. *If $s>1/3$ and $0\leq\delta<2$, then for all $f\in H^{s+\delta}(\mathbb{R}^2)$*

$$e^{it\Delta}(f)(x)-f(x)=o\big(t^{\delta/2}\big),\quad a.e.\ as\ t\to 0.$$

It is easy to see that the main theorem in [6] is consistent with Theorem 1.1 if we take $\delta=0$. Also, to obtain the approximation saturation of the operator $e^{it\Delta}$, we have the following proposition for any dimension n.

PROPOSITION 1.1. *Let f be a Schwarz function. Then*

$$e^{it\Delta}(f)(x)-f(x)=o(t),\quad a.e.\ as\ t\to 0, \tag{2}$$

if and only if $f(x)=0$.

Proof. We prove the proposition using a contradiction argument. Assume (2) holds for a non-zero Schwarz function $f(x)$. The Fourier transform $\widehat{f}(\xi)$ must be a non-zero function on the frequency domain. Since

$$e^{it\Delta}(f)(x)-f(x)\approx\int_{\mathbb{R}^n}\big(e^{-it|\xi|^2}-1\big)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi,$$

by Taylor's expansion we have that

$$e^{it\Delta}(f)(x)-f(x)\approx t\int_{\mathbb{R}^n}|\xi|^2\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi+O\big(t^2\big).$$

By continuity, we have a ball $B_n(x_0,r)$ and a positive number c such that

$$\left|\int_{\mathbb{R}^n}|\xi|^2\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi\right|>c$$

for all $x\in B_n(x_0,r)$. It says that

$$\big|e^{it\Delta}(f)(x)-f(x)\big|\succeq t$$

in a set of positive measure, which contradicts to (2). □

Secondly, we discuss the case for $n = 1$. It is known that (1) fails for $s < 1/4$. On the other hand, if $n \geq 1/4$, Sjölin in [15] showed that (1) remains true if replacing $e^{it|\Delta|}$ by $e^{it|\Delta|^{\alpha/2}}$ for any $\alpha > 1$. We can build the corresponding result in this general case.

THEOREM 1.2. *If $s \geq 1/4$, $\alpha > 1$ and $0 \leqslant \delta < \alpha$, then for all $f \in H^{s+\delta}(\mathbb{R})$,*

$$e^{it|\Delta|^{\alpha/2}}(f)(x) - f(x) = o\big(t^{\delta/\alpha}\big), \quad a.e. \text{ as } t \to 0.$$

For general dimension n, we have the following result.

THEOREM 1.3. *If $s > n/2(n+1)$ and $0 \leq \delta < 2$, then for all $f \in H^{s+\delta}(\mathbb{R}^n)$,*

$$e^{it\Delta}(f)(x) - f(x) = o\big(t^{\delta/2}\big), \quad a.e. \text{ as } t \to 0.$$

This paper is organized as follows. In Section 2, we introduce some notations and known results that we will use in the proofs. The proofs of Theorem 1.1 and Theorem 1.2 are in Section 3 and Section 4, respectively. Finally, in Section 5, we address the problem for general dimension n.

2. Notations and standard results

In this section, we introduce some notations and standard results related to this context.

Throughout this paper, $C > 1$ and $c < 1$ denote positive constants, which might be different at each of their occurrences. We write $A \preceq B$ to mean that there exists C such that $A \leqslant CB$. Write $A \approx B$ if $A \preceq B$ and $B \preceq A$. We denote by p' the Hölder dual of $p \in [1, \infty]$, that is, $1/p + 1/p' = 1$. The space of all infinitely differentiable functions on $\mathbb{R}^n$ is denoted by $\mathcal{C}^\infty(\mathbb{R}^n)$. The space of $\mathcal{C}^\infty$ functions with compact support is denoted by $\mathcal{C}_0^\infty(\mathbb{R}^n)$. The Schwarz space $\mathcal{S}(\mathbb{R}^n)$ is the function space of all $\mathcal{C}^\infty$ functions f whose all derivatives are rapidly decreasing. We write the Lebesgue $L^p = L^p(\mathbb{R}^n)$ and the Sobolev space $H^s = H^s(\mathbb{R}^n)$ for simplify. Also, for the ball $B_n(x_0, r)$, we denote it by $B(x_0, r)$ if it does not cause any confusion. Denote the annulus $A(R)$ by $\{\xi \in \mathbb{R}^n : |\xi| \approx R\}$ for a positive R. For $f \in H^s(\mathbb{R}^n)$, by the Littlewood–Paley decomposition we may write

$$f = \sum_{k=0}^{\infty} f_k,$$

where the functions f_k satisfy

$$\operatorname{supp} \widehat{f_0} \subset \big\{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\big\}, \qquad \operatorname{supp} \widehat{f_k} \subset A\big(2^k\big), \qquad \|f_k\|_{L^2} \preceq 2^{-ks}\|f\|_{H^s}.$$

To obtain the convergence results of Schrödinger operator, we need to invoke the following lemmas.

LEMMA 2.1 ([6]). *When* $n=2$, *for any* $\epsilon>0$, *there exists a constant* C_ϵ *such that*

$$\Big\|\sup_{0<t\le R}\big|e^{it\Delta}(f)\big|\Big\|_{L^3(B_2(0,R))}\le C_\epsilon R^\epsilon\|f\|_{L^2(\mathbb{R}^2)}$$

holds for all $R\ge 1$ *and* f *with* $\operatorname{supp}\widehat{f}\subset A(1)$.

LEMMA 2.2 ([15]). *When* $n=1$, *for all* $f\in H^{\frac14}(\mathbb{R})$,

$$\Big\|\sup_{0<t\le 1}\big|e^{it\Delta}(f)\big|\Big\|_{L^2(\mathbb{R})}\preceq\|f\|_{H^{\frac14}(\mathbb{R})}.$$

We also recall the Milhlin multiplier theorem. For a convolution operator $T_m(f)$ defined by

$$\widehat{T_m(f)}(\xi)=m(\xi)\widehat{f}(\xi),$$

where m is the symbol (also called multiplier), if

$$\big\|T_m(f)\big\|_{L^q}\preceq\|f\|_{L^q}$$

then we say that m is an L^q multiplier.

LEMMA 2.3 (Milhlin multiplier theorem). *Suppose that* $m:\mathbb{R}^n\to\mathbb{C}$ *satisfies*

$$\big|\nabla^k m(\xi)\big|\preceq|\xi|^{-k},$$

for $0\le k\le\frac{n}{2}+1$. *Then* m *is an* L^p *multiplier for any* $1<p<\infty$.

LEMMA 2.4 (Van der Corput). *Assume that* $a<b$ *and set* $I=[a,b]$. *Let* $F\in C^\infty(I)$ *be real-valued and assume that* $\psi\in C^\infty(I)$.

(i) *Assume that* $|F'(x)|\ge\gamma>0$ *for* $x\in I$ *and that* F' *is monotonic on* I. *Then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right|\le C\frac{1}{\gamma}\left(\big|\psi(b)\big|+\int_a^b\big|\psi'(x)\big|\,dx\right),$$

where C *does not depend on* F, ψ *or* I.

(ii) *Assume that* $|F''(x)|\ge\gamma>0$ *for* $x\in I$. *Then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right|\le C\frac{1}{\gamma^{1/2}}\left(\big|\psi(b)\big|+\int_a^b\big|\psi'(x)\big|\,dx\right),$$

where C *does not depend on* F, ψ *or* I.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we will study an associated maximal operator

$$\mathfrak{R}^*(f)(x)=\sup_{0<t\le 1}\big|\mathfrak{R}_t(f)(x)\big|,$$

where

$$\mathfrak{R}_t(f)(x)=\frac{e^{it\Delta}(f)(x)-f(x)}{t^{\delta/2}}.$$

We then establish the following estimate for $\mathfrak{R}^*(f)$.

LEMMA 3.1. *If* $s > 1/3$ *and* $0 \le \delta < 2$, *then for all* $f \in H^{s+\delta}(\mathbb{R}^2)$,

$$\left\|\mathfrak{R}^*(f)\right\|_{L^3(B_2(0,1))} \preceq \|f\|_{H^{s+\delta}(\mathbb{R}^2)}.$$

Let us briefly describe how Lemma 3.1 implies Theorem 1.1. In fact, for any $M > 0$, by scaling we have

$$\begin{aligned}
&\Big\| \sup_{0<t\le M^2} \mathfrak{R}_t(f) \Big\|_{L^3(B_2(0,M))} \\
&\quad = \left\{ \int_{B_2(0,M)} \left| \sup_{0<t\le M^2} \int_{\mathbb{R}^2} \frac{e^{it|\xi|^2}-1}{t^{\delta/2}} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi \right|^3 dx \right\}^{1/3} \\
&\quad = M^{2/3-(2+\delta)} \left\{ \int_{B_2(0,1)} \left| \sup_{0<t\le 1} \int_{\mathbb{R}^2} \frac{e^{it|\xi|^2}-1}{t^{\delta/2}} \widehat{f}(\xi/M) e^{ix\cdot\xi}\, d\xi \right|^3 dx \right\}^{1/3} \\
&\quad \preceq M^{2/3-(2+\delta)} \left\{ \int_{\mathbb{R}^2} \left|\widehat{f}(\xi/M)\right|^2 \left(1+|\xi|^2\right)^{s+\delta} d\xi \right\}^{1/2} \\
&\quad \approx M^{s-1/3} \|f\|_{H^{s+\delta}(\mathbb{R}^2)}.
\end{aligned}$$

Now fix $\lambda > 0$. For any $\varepsilon > 0$, we choose a compactly supported C^∞ function g such that

$$\|f-g\|_{H^{s+\delta}(\mathbb{R}^2)} < \frac{\lambda \varepsilon^{1/3}}{2M^{s-1/3}}.$$

By Taylor's expansion we have that

$$\left| \frac{e^{it\Delta}(g)(x) - g(x)}{t^{\delta/2}} \right| \preceq t^{1-\delta/2} \int_{\mathbb{R}^n} |\xi|^2 \left|\widehat{g}(\xi)\right| d\xi.$$

This shows that for any $0 \le \delta < 2$,

$$\lim_{t\to 0} \frac{e^{it\Delta}(g)(x) - g(x)}{t^{\delta/2}} = 0$$

holds uniformly on $x \in \mathbb{R}^n$.

Hence, we may write

$$\begin{aligned}
&\left| \left\{ x \in B(0,M) : \limsup_{t\to 0} \left| \frac{e^{it\Delta}(f)(x) - f(x)}{t^{\delta/2}} \right| > \lambda \right\} \right| \\
&\quad \le \left| \left\{ x \in B(0,M) : \sup_{0<t\le 1} \left| \frac{e^{it\Delta}(f-g)(x) - (f-g)(x)}{t^{\delta/2}} \right| > \lambda/2 \right\} \right| \\
&\quad \preceq \left(\frac{2}{\lambda}\right)^3 \int_{B(0,M)} \left|\mathfrak{R}^*(f-g)(x)\right|^3 dx \\
&\quad \preceq M^{-1+3s} \left(\frac{2}{\lambda}\right)^3 \|f-g\|^3_{H^{s+\delta}(\mathbb{R}^2)} < \varepsilon.
\end{aligned}$$

Since M is arbitrary, we obtain Theorem 1.1.

We start to work with Lemma 3.1. By the Littlewood–Paley decomposition

$$f=\sum_{k=0}^{\infty} f_k,$$

where $\widehat{f_0}$ is supported in $B(0,1)$ and $\widehat{f_k}$ is supported in $A(2^k)$ for $k\geq 1$, and they satisfy

$$\|f_k\|_{L^2(\mathbb{R}^2)} \preceq 2^{-ks}\|f\|_{H^s(\mathbb{R}^2)}.$$

By the Minkowski inequality, we have

$$\big\|\mathfrak{R}^*(f)\big\|_{L^3(B_2(0,1))} \leq \sum_{k=0}^{\infty}\big\|\mathfrak{R}^*(f_k)\big\|_{L^3(B_2(0,1))}.$$

Changing variables we see

$$\begin{aligned}&\big\|\mathfrak{R}^*(f_k)\big\|_{L^3(B_2(0,1))}\\ &\quad= 2^{2k}\left(\int_{B_2(0,1)}\left|\sup_{0<t\leq 1}\int_{\mathbb{R}^2}\frac{e^{it2^{2k}|\xi|^2}-1}{t^{\delta/2}}\widehat{f_k}\big(2^k\xi\big)e^{i2^k x\cdot\xi}\,d\xi\right|^3 dx\right)^{1/3}\\ &\quad= 2^{4k/3}2^{k\delta}\left(\int_{B_2(0,2^k)}\left|\sup_{0<t\leq 2^{2k}}\int_{\mathbb{R}^2}\frac{e^{it|\xi|^2}-1}{t^{\delta/2}}\widehat{f_k}\big(2^k\xi\big)e^{ix\cdot\xi}\,d\xi\right|^3 dx\right)^{1/3}.\end{aligned}$$

Note

$$\begin{aligned}\big\|\widehat{f_k}\big(2^k\xi\big)\big\|_{L^2(\mathbb{R}^2)} &= \left(\int_{\mathbb{R}^2}\big|\widehat{f_k}\big(2^k\xi\big)\big|^2\,d\xi\right)^{1/2}\\ &= 2^{-k}\|f_k\|_{L^2(\mathbb{R}^2)} \preceq 2^{-k(1+s+\delta)}\|f\|_{H^{s+\delta}(\mathbb{R}^2)},\end{aligned}$$

where $\widehat{f_k}(2^k\xi)$ is supported on $A(1)$.

Thus, if we can prove that, for all f with $\operatorname{supp}\widehat{f}\subset A(1)$ and any $\epsilon>0$,

$$\Big\|\sup_{0<t\leq R}\big|\mathfrak{R}_t(f)\big|\Big\|_{L^3(B_R)} \preceq R^{\epsilon}\|f\|_{L^2}$$

holds for all $R\geq 1$, then for $s=1/3+2\epsilon$,

$$\begin{aligned}\big\|\mathfrak{R}^*(f_k)\big\|_{L^3(B_2(0,1))} &\preceq 2^k 2^{-2k/3}2^{-ks}2^{2k\epsilon}\|f\|_{H^{s+\delta}(\mathbb{R}^2)}\\ &\preceq 2^{-k\epsilon}\|f\|_{H^{s+\delta}(\mathbb{R}^2)}.\end{aligned}$$

It yields

$$\big\|\mathfrak{R}^*(f)\big\|_{L^3(B_2(0,1))} \preceq \sum_{k=0}^{\infty}2^{-k\epsilon}\|f\|_{H^{s+\delta}(\mathbb{R}^2)} \preceq \|f\|_{H^{s+\delta}(\mathbb{R}^2)}.$$

The above discussion suggests that, to prove Lemma 3.1, it suffices to show the following lemma.

LEMMA 3.2. *If $0 \leqslant \delta < 2$, then for all f with* $\operatorname{supp}\widehat{f} \subset A(1)$ *we have that*

$$\Big\| \sup_{0<t\leq R} \big|\mathfrak{R}_t(f)\big| \Big\|_{L^3(B_2(0,R))} \preceq R^{\epsilon} \|f\|_{L^2(\mathbb{R}^2)}$$

holds for all $R \geq 1$.

Proof. We write

$$\begin{aligned}\Big\| \sup_{0<t\leq R} \big|\mathfrak{R}_t(f)\big| \Big\|_{L^3(B_2(0,R))} \\ \leq \Big\| \sup_{0<t\leq 1} \big|\mathfrak{R}_t(f)\big| \Big\|_{L^3(B_2(0,R))} + \Big\| \sup_{1<t\leq R} \big|\mathfrak{R}_t(f)\big| \Big\|_{L^3(B_2(0,R))}.\end{aligned}$$

Using the same argument as we treat $\mathfrak{R}^*(f_0)$, we have

$$\Big\| \sup_{0<t\leq 1} \big|\mathfrak{R}_t(f)\big| \Big\|_{L^3(B_2(0,R))} \preceq \sum_{j=0}^{\infty} \frac{j^2}{j!} \|f\|_{L^2(\mathbb{R}^2)} \preceq \|f\|_{L^2(\mathbb{R}^2)}.$$

Finally,

$$\sup_{1<t\leq R} \big|\mathfrak{R}_t(f)(x)\big| \leq \big|e^{it\Delta} f(x)\big| + \big|f(x)\big|.$$

Thus Lemma 2.1 yields

$$\Big\| \sup_{1<t\leq R} \big|e^{it\Delta} f\big| \Big\|_{L^3(B_2(0,R))} \preceq R^{\epsilon} \|f\|_{L^2},$$

and Bernstein's theorem gives

$$\|f\|_{L^3(B_2(0,R))} \preceq \|f\|_{L^2}.$$

The proof of Lemma 3.2 is completed. □

4. Proof of Theorem 1.2

Without loss of generality, we may study only the case $s = 1/4$, since the proof for $s > 1/4$ is the same and it is simper. As the proof for the case $n = 2$, to prove Theorem 1.2, it suffices to show the following lemma.

LEMMA 4.1. *If $\alpha > 1$ and $0 \leqslant \delta < \alpha$, then for all $f \in H^{\frac{1}{4}+\delta}(\mathbb{R})$,*

$$\bigg\| \sup_{0<t\leq 1} \bigg| \frac{e^{it|\Delta|^{\alpha/2}}(f) - f}{t^{\delta/\alpha}} \bigg| \bigg\|_{L^2(B_1(0,1))} \preceq \|f\|_{H^{\frac{1}{4}+\delta}(\mathbb{R})}.$$

Proof. Take two cutoff functions Ψ and Φ that $\Psi + \Phi = 1$, where $\Psi \in S$ satisfying $\Psi(\xi) = 1$ if $|\xi| \leq 1$ and $\Psi(\xi) = 0$ if $|\xi| \geq 2$. Hence

$$\begin{aligned}\frac{e^{it|\Delta|^{\alpha/2}}(f)(x) - f(x)}{t^{\delta/\alpha}} &= \int_{\mathbb{R}} \frac{e^{it|\xi|^{\alpha}} - 1}{t^{\delta/\alpha}} \widehat{f}(\xi) e^{i\xi x}\, d\xi \\ &= \int_{\mathbb{R}} \frac{e^{it|\xi|^{\alpha}} - 1}{|t^{1/\alpha}\xi|^{\delta}} \big(|\xi|^{\delta} \widehat{f}(\xi)\big) \Psi\big(\big|t^{1/\alpha}\xi\big|\big) e^{i\xi x}\, d\xi\end{aligned}$$

$$\begin{aligned}
&+\int_{\mathbb{R}} \frac{e^{it|\xi|^\alpha}-1}{|t^{1/\alpha}\xi|^\delta}\big(|\xi|^\delta \widehat{f}(\xi)\big)\Phi\big(|t^{1/\alpha}\xi|\big)e^{i\xi x}\,d\xi \\
&=\int_{\mathbb{R}} \frac{e^{it|\xi|^\alpha}-1}{|t^{1/\alpha}\xi|^\delta}\widehat{g}(\xi)\Psi\big(|t^{1/\alpha}\xi|\big)e^{i\xi x}\,d\xi \\
&\quad+\int_{\mathbb{R}} \frac{e^{it|\xi|^\alpha}-1}{|t^{1/\alpha}\xi|^\delta}\widehat{g}(\xi)\Phi\big(|t^{1/\alpha}\xi|\big)e^{i\xi x}\,d\xi \\
&=S_1^{\alpha,\delta}g(x,t)+S_2^{\alpha,\delta}g(x,t),
\end{aligned}$$

where $g(x)=(I_{-\delta}f)(x)$ and $I_{-\delta}$ means the fractional Laplacian.

First, we work with $S_1^{\alpha,\delta}g(x,t)$. Since

$$\sup_{0<t\le 1}\big|S_1^{\alpha,\delta}g(x,t)\big|\le \sup_{t>0}\big|g*K_t^1(x)\big|,$$

where

$$K_t^1(x)=\frac{1}{t^{1/\alpha}}K^1\left(\frac{x}{t^{1/\alpha}}\right)$$

and

$$K^1(x)=\int_{\mathbb{R}}\frac{e^{i|\xi|^\alpha}-1}{|\xi|^\delta}\Psi\big(|\xi|\big)e^{i\xi x}\,d\xi=\int_{\mathbb{R}}\frac{i|\xi|^\alpha+w(\xi)}{|\xi|^\delta}\Psi\big(|\xi|\big)e^{i\xi x}\,d\xi.$$

Here $w(\xi)$ satisfies

$$\big(\partial^\beta w\big)(\xi)=O\big(|\xi|^{2\alpha-\beta}\big)$$

for any $\beta\in N$ by Taylor's expansion. Using integration by parts (see [9]), we get the bound

$$\big|K^1(x)\big|\preceq\frac{1}{(1+|x|)^{1+\alpha-\delta}},$$

due to $\alpha>\delta$. Hence,

$$\sup_{0<t\le 1}\big|S_1^{\alpha,\delta}g(x,t)\big|\preceq M(g)(x)=M\big(I_{-\delta}(f)\big)(x),$$

where M is the Hardy–Littlewood maximal function.

Hence,

$$\Big\|\sup_{0<t\le 1}\big|S_1^{\alpha,\delta}g(x,t)\big|\Big\|_{L^2(B_1(0,1))}\preceq\big\|M\big(I_{-\delta}(f)\big)\big\|_{L^2}\preceq\|f\|_{H^\delta}\preceq\|f\|_{H^{\frac14+\delta}}.$$

Now we deal with $S_2^{\alpha,\delta}g(x,t)$,

$$\begin{aligned}
S_2^{\alpha,\delta}g(x,t)&=\int_{\mathbb{R}}\frac{e^{it|\xi|^\alpha}-1}{|t^{1/\alpha}\xi|^\delta}\widehat{g}(\xi)\Phi\big(|t^{1/\alpha}\xi|\big)e^{i\xi x}\,d\xi \\
&=\int_{\mathbb{R}}\frac{e^{it|\xi|^\alpha}}{|t^{1/\alpha}\xi|^\delta}\widehat{g}(\xi)\Phi\big(|t^{1/\alpha}\xi|\big)e^{i\xi x}\,d\xi \\
&\quad-\int_{\mathbb{R}}\frac{1}{|t^{1/\alpha}\xi|^\delta}\widehat{g}(\xi)\Phi\big(|t^{1/\alpha}\xi|\big)e^{i\xi x}\,d\xi \\
&=S_3^{\alpha,\delta}g(x,t)-S_4^{\alpha,\delta}g(x,t).
\end{aligned}$$

We first consider $S_4^{\alpha,\delta}g(x,t)$,

$$\sup_{0<t\leq 1}\left|S_4^{\alpha,\delta}g(x,t)\right| \leq \sup_{t>0}\left|g * K_t^2(x)\right|,$$

where

$$K_t^2(x) = \frac{1}{t^{1/\alpha}}K^2\left(\frac{x}{t^{1/\alpha}}\right)$$

and

$$K^2(x) = \int_{\mathbb{R}} \frac{\Phi(|\xi|)}{|\xi|^\delta} e^{i\xi x}\, d\xi.$$

Using the same argument in [9], we have that for any integer $L > 0$,

$$\left|K^2(x)\right| \preceq \begin{cases} |x|^{\delta-1} & |x| \leq 2, \\ |x|^L & |x| > 2. \end{cases}$$

We can assume $\delta > 0$ (when $\delta = 0$, it is the classical convergence result, see [3]), then

$$\sup_{0<t\leq 1}\left|S_4^{\alpha,\delta}g(x,t)\right| \preceq M(g)(x) = M\big(I_{-\delta}(f)\big)(x).$$

By the same method as we treat $S_1^{\alpha,\delta}g(x,t)$,

$$\left\|\sup_{0<t\leq 1}\left|S_4^{\alpha,\delta}g(x,t)\right|\right\|_{L^2(B_1(0,1))} \preceq \|f\|_{H^{\frac{1}{4}+\delta}}.$$

As to the term $S_3^{\alpha,\delta}g(x,t)$, we will prove

$$\left\|\sup_{0<t\leq 1}\left|\int_{\mathbb{R}} e^{ix\xi+it|\xi|^\alpha} \frac{\Phi(|t^{1/\alpha}\xi|)}{|t^{1/\alpha}\xi|^\delta}\widehat{g}(\xi)\, d\xi\right|\right\|_{L^2(B_1(0,1))} \preceq \|g\|_{H^{\frac{1}{4}}}.$$

In fact, we prove the following stronger version,

$$\left\|\sup_{0<t\leq 1}\left|\int_{\mathbb{R}} e^{ix\xi+it|\xi|^\alpha} \frac{\Phi(|t^{1/\alpha}\xi|)}{|t^{1/\alpha}\xi|^\delta}\widehat{g}(\xi)\, d\xi\right|\right\|_{L^4} \preceq \|g\|_{H^{\frac{1}{4}}},$$

i.e.

$$\left\|\sup_{0<t\leq 1}\left|\int_{\mathbb{R}} e^{ix\xi+it|\xi|^\alpha} \frac{\Phi(|t^{1/\alpha}\xi|)}{|t^{1/\alpha}\xi|^\delta}|\xi|^{-\frac{1}{4}}\widehat{g}(\xi)\, d\xi\right|\right\|_{L^4} \preceq \|g\|_{L^2}.$$

Set

$$R(g)(x) = \int_{\mathbb{R}} e^{ix\xi+it(x)|\xi|^\alpha} \frac{\Phi(|t(x)^{1/\alpha}\xi|)}{|t(x)^{1/\alpha}\xi|^\delta}|\xi|^{-\frac{1}{4}}\widehat{g}(\xi)\, d\xi,$$

then our aim is to prove

$$\big\|R(g)\big\|_{L^4} \preceq \|g\|_{L^2}. \tag{3}$$

Let $\rho \in C_0^\infty(\mathbb{R})$ be real-valued and assume that $\rho(x) = 1$, $|x| \leq 1$, and $\rho(x) = 0$, $|x| \geq 2$. We set $\rho_N(x) = \rho(x/N)$ and

$$R_N(g)(x) = \rho_N(x)\int_{\mathbb{R}} e^{ix\xi+it(x)|\xi|^\alpha} \frac{\Phi(|t(x)^{1/\alpha}\xi|)}{|t(x)^{1/\alpha}\xi|^\delta}|\xi|^{-\frac{1}{4}}\rho_N(\xi)\widehat{g}(\xi)\, d\xi.$$

By the dominated convergence theorem, we only need to prove

$$\big\|R_N(g)\big\|_{L^4} \preceq \|g\|_{L^2}, \tag{4}$$

which is equivalent to show

$$\big\|R_N^*(g)\big\|_{L^2} \preceq \|g\|_{L^{\frac{4}{3}}}, \tag{5}$$

where R_N^* denotes the adjoint of R_N.

We have

$$\big\|R_N^*(g)\big\|_{L^2}^2 = \int_{\mathbb{R}} \big|R_N^*(g)(x)\big|^2\,dx \preceq \int_{\mathbb{R}}\int_{\mathbb{R}} \big|K_N(y,z)\big|\big|g(y)\big|\big|g(z)\big|\,dy\,dz,$$

where

$$K_N(y,z) = \rho_N(y)\rho_N(z)\int_{\mathbb{R}} \frac{\Phi(|t(y)^{1/\alpha}\xi|)}{|t(y)^{1/\alpha}\xi|^{\delta}}\,\frac{\Phi(|t(z)^{1/\alpha}\xi|)}{|t(z)^{1/\alpha}\xi|^{\delta}} \times e^{i[(y-z)\xi+(t(z)-t(y))|\xi|^{\alpha}]}\mu_N(\xi)|\xi|^{-\frac{1}{2}}\,d\xi$$

and $\mu=\rho^2$, $\mu_N(\xi)=\mu(\xi/N)$.

Now we claim that

$$\big|K_N(y,z)\big| \preceq |y-z|^{-\frac{1}{2}}. \tag{6}$$

If (6) holds, we have

$$\begin{aligned}\big\|R_N^*(g)\big\|_{L^2}^2 &\preceq \int_{\mathbb{R}}\int_{\mathbb{R}} |y-z|^{-\frac{1}{2}}\big|g(y)\big|\big|g(z)\big|\,dy\,dz\\ &\preceq \int_{\mathbb{R}} \big|g(y)\big| I_{1/2}\big(|g|\big)(y)\,dy \le \|g\|_{L^{\frac{4}{3}}}\big\|I_{1/2}\big(|g|\big)\big\|_{L^4}\\ &\preceq \|g\|_{L^{\frac{4}{3}}}^2.\end{aligned}$$

□

Now we prove the formula in (6). The following lemma is an analogue of Lemma 1 in [16], with a slight difference.

Lemma 4.2. *Assume that $\alpha>1$, $0\le\delta<\alpha$, $b\in(0,1]$, $d\in(0,1]$, $1/2\le s<1$ and $\mu\in C_0^{\infty}(\mathbb{R})$. Then*

$$\left|\int_{\mathbb{R}} e^{ix\xi+it|\xi|^{\alpha}}\frac{\Phi(|b^{1/\alpha}\xi|)}{|b^{1/\alpha}\xi|^{\delta}}\,\frac{\Phi(|d^{1/\alpha}\xi|)}{|d^{1/\alpha}\xi|^{\delta}}|\xi|^{-s}\mu\left(\frac{\xi}{N}\right)d\xi\right| \le C\frac{1}{|x|^{1-s}}$$

for $x\in\mathbb{R}$, $t\in\mathbb{R}$ and $N=1,2,3,\ldots$ Here the constant C may depend on s, δ, and α but not on b, d, x, t or N.

Proof. We only need to discuss the case for $\xi>0$. Letting I denote the integral in the lemma, we have

$$\begin{aligned}I &= \int_{\mathbb{R}} e^{ix\xi+it\xi^{\alpha}}\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\,\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\mu\left(\frac{\xi}{N}\right)d\xi\\ &= \int_0^{|x|^{-1}} e^{ix\xi+it\xi^{\alpha}}\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\,\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\mu\left(\frac{\xi}{N}\right)d\xi\end{aligned}$$

$$+\int_{|x|^{-1}}^{\infty} e^{ix\xi+it\xi^{\alpha}}\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\mu\left(\frac{\xi}{N}\right)d\xi$$
$$=A+B.$$

Noting the support of Φ, we have $b^{1/\alpha}\xi\geq 1$ and $d^{1/\alpha}\xi\geq 1$, then

$$|A|\leq C\int_0^{|x|^{-1}}\xi^{-s}\,d\xi=C|x|^{s-1}.$$

As to B, we assume first $|x|^{\alpha}\leq t/2$. Set $F(\xi)=x\xi+t\xi^{\alpha}$, then

$$F'(\xi)=x+t\alpha\xi^{\alpha-1}=x\left(1+\frac{t}{x}\alpha\xi^{\alpha-1}\right).$$

For $\xi\geq|x|^{-1}$ we have

$$\left|\frac{t}{x}\alpha\xi^{\alpha-1}\right|\geq 2\alpha|x|^{\alpha}\frac{1}{|x|}|x|^{1-\alpha}=2\alpha>2,$$

so F' is monotonic and $|F'(\xi)|\geq|x|$. Let

$$\psi(\xi)=\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\mu\left(\frac{\xi}{N}\right).$$

Since $\xi\geq|x|^{-1}$, we have

$$\left|\psi(\xi)\right|\leq C|x|^{s}.$$

We also have

$$\begin{aligned}\psi'(\xi)&=b^{\frac{1}{\alpha}}\frac{\Phi'(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\mu\left(\frac{\xi}{N}\right)\\&\quad-\delta b^{\frac{1}{\alpha}}\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta+1}}\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\mu\left(\frac{\xi}{N}\right)\\&\quad+d^{\frac{1}{\alpha}}\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\frac{\Phi'(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\mu\left(\frac{\xi}{N}\right)\\&\quad-\delta d^{\frac{1}{\alpha}}\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta+1}}\xi^{-s}\mu\left(\frac{\xi}{N}\right)\\&\quad+\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s}\frac{1}{N}\mu'\left(\frac{\xi}{N}\right)\\&\quad-s\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\xi^{-s-1}\mu\left(\frac{\xi}{N}\right).\end{aligned}$$

Then

$$\begin{aligned}&\int_{|x|^{-1}}^{\infty}\left|\psi'(\xi)\right|d\xi\\&\quad\leq C|x|^{s}\int_{|x|^{-1}}^{\infty}b^{\frac{1}{\alpha}}\frac{\Phi'(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\,d\xi+C\int_{|x|^{-1}}^{\infty}\frac{\Phi(b^{1/\alpha}\xi)}{(b^{1/\alpha}\xi)^{\delta}}\xi^{-s-1}\,d\xi\end{aligned}$$

$$
\begin{aligned}
&+ C|x|^s \int_{|x|^{-1}}^{\infty} d^{\frac{1}{\alpha}} \frac{\Phi'(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}}\, d\xi + C\int_{|x|^{-1}}^{\infty} \frac{\Phi(d^{1/\alpha}\xi)}{(d^{1/\alpha}\xi)^{\delta}} \xi^{-s-1}\, d\xi \\
&+ C|x|^s \int_{|x|^{-1}}^{\infty} \frac{1}{N}\left|\mu'\left(\frac{\xi}{N}\right)\right| d\xi + C\int_{|x|^{-1}}^{\infty} \xi^{-s-1}\, d\xi \\
\leq{}& C|x|^s \int_0^{\infty} \frac{\Phi'(\xi)}{\xi^{\delta}}\, d\xi + C\int_{|x|^{-1}}^{\infty} \xi^{-s-1}\, d\xi \\
&+ C|x|^s \int_0^{\infty} \left|\mu'(\xi)\right| d\xi + C\int_{|x|^{-1}}^{\infty} \xi^{-s-1}\, d\xi \\
\leq{}& C|x|^s.
\end{aligned}
$$

Using Lemma 2.4,

$$
\left|\int_{|x|^{-1}}^{\infty} e^{iF(\xi)}\psi(\xi)\, d\xi\right| \leq C\frac{1}{|x|}|x|^s = C|x|^{s-1}.
$$

It remains to estimate the case for $|x|^{\alpha} > t/2$. We decompose the integral to

$$
\int_{|x|^{-1}}^{\infty} e^{iF}\psi\, d\xi = \int_{I_1} e^{iF}\psi\, d\xi + \int_{I_2} e^{iF}\psi\, d\xi + \int_{I_3} e^{iF}\psi\, d\xi = B_1 + B_2 + B_3,
$$

where

$$
\begin{aligned}
I_1 &= \left\{\xi \geq |x|^{-1} | \xi \leq l\left(\frac{t}{|x|}\right)^{1/(1-\alpha)}\right\}, \\
I_2 &= \left\{\xi \geq |x|^{-1} | l\left(\frac{t}{|x|}\right)^{1/(1-\alpha)} \leq \xi \leq K\left(\frac{t}{|x|}\right)^{1/(1-\alpha)}\right\},
\end{aligned}
$$

and

$$
I_3 = \left\{\xi \geq |x|^{-1} | \xi \geq K\left(\frac{t}{|x|}\right)^{1/(1-\alpha)}\right\}.
$$

Here $l > 0$ is a small number and K is a large number.

When $\xi \in I_1$,

$$
t\alpha\xi^{\alpha-1} \leq t\alpha l^{\alpha-1}\frac{|x|}{t} = \alpha l^{\alpha-1}|x| \leq \frac{|x|}{2},
$$

then we have that

$$
\left|F'(\xi)\right| \geq \frac{|x|}{2},
$$

which together with Lemma 2.4 implies

$$
|B_1| \leq C\frac{1}{|x|}|x|^s = C|x|^{s-1}.
$$

When $\xi \in I_3$,

$$
t\alpha\xi^{\alpha-1} \geq t\alpha K^{\alpha-1}\frac{|x|}{t} = \alpha K^{\alpha-1}|x| \geq 2|x|,
$$

then we have that

$$\left|F'(\xi)\right| \geq |x|,$$

which together with Lemma 2.4 implies

$$|B_3| \leq C\frac{1}{|x|}|x|^s = C|x|^{s-1}.$$

As to B_2, we have

$$F''(\xi) = \alpha(\alpha-1)t\xi^{\alpha-2}.$$

When $\xi \in I_2$, there exists a positive number c to have

$$F''(\xi) \geq ct\left(\frac{t}{|x|}\right)^{(\alpha-2)/(1-\alpha)} = ct^{1/(\alpha-1)}|x|^{(\alpha-2)/(\alpha-1)}.$$

By the same method as we treat B, we see

$$\left|\psi(\xi)\right| \leq C\left(\frac{t}{|x|}\right)^{s/(\alpha-1)},$$

and

$$\int_{I_2}\left|\psi'(\xi)\right| d\xi \leq C\left(\frac{t}{|x|}\right)^{s/(\alpha-1)}.$$

Using Lemma 2.4,

$$\begin{aligned}
|B_2| &\leq Ct^{-1/(2(\alpha-1))}|x|^{-(\alpha-2)/(2(\alpha-1))}\left(\frac{|x|}{t}\right)^{s/(\alpha-1)}\\
&= Ct^{(2s-1)/(2(\alpha-1))}|x|^{(2-\alpha-2s)/(2(\alpha-1))}\\
&\leq C|x|^{(2s-1)\alpha/(2(\alpha-1))}|x|^{(2-\alpha-2s)/(2(\alpha-1))}\\
&= C|x|^{s-1},
\end{aligned}$$

due to $s \geq 1/2$ and $|x|^\alpha > t/2$. □

5. Case for general dimension

We firstly quote the almost sharp result from Du–Zhang in [8].

LEMMA 5.1 ([8]). *When $n \geq 1$, for any $\epsilon > 0$, there exists a constant C_ϵ such that*

$$\Big\|\sup_{0<t\leq R}\left|e^{it\Delta}(f)\right|\Big\|_{L^2(B_n(0,R))} \leq C_\epsilon R^{\frac{n}{2(n+1)}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}$$

holds for all $R \geq 1$ and f with $\operatorname{supp}\widehat{f} \subset A(1)$.

Via the same argument in Section 3 and Lemma 5.1, we can prove Theorem 1.3. We write down the useful lemma and omit the details of the proof.

LEMMA 5.2. *If $s > n/2(n+1)$ and $0 \leq \delta < 2$, then for all $f \in H^{s+\delta}(\mathbb{R}^n)$,*

$$\left\|\mathfrak{R}^*(f)\right\|_{L^2(B_n(0,1))} \preceq \|f\|_{H^{s+\delta}(\mathbb{R}^n)}.$$

LEMMA 5.3. *If* $0 \le \delta < 2$, *then for all* f *with* $\operatorname{supp} \widehat{f} \subset A(1)$ *we have that*

$$\Big\| \sup_{0<t\le R} \big| \mathfrak{R}_t(f) \big| \Big\|_{L^2(B_n(0,R))} \preceq R^{\frac{n}{2(n+1)}+\epsilon} \|f\|_{L^2(\mathbb{R}^n)}$$

holds for all $R \ge 1$.

We are grateful to an anonymous referee for valuable suggestions on an earlier version of this manuscript. Particularly, his suggestion helps us to formulate Theorem 1.3 in the current version. Also, based on a famous result by Bourgain [2], the referee's comment leads us to believe that Theorem 1.3 is sharp in the sense of the following statement.

For $\delta \in (0, 2)$, it fails to have, for arbitrary $\epsilon > 0$,

$$\lim_{t \to 0} \big(e^{it\Delta} f(x) - f(x) \big) / t^{\frac{\delta}{2}+\epsilon} = 0, \quad \text{a.e.}$$

whenever $f \in H^{n/2(n+1)+\delta}(\mathbb{R}^n)$.

We are not able to prove this statement, so we post it as an unsolved problem.

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