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ACTIONS OF MEASURED QUANTUM GROUPOIDS ON A FINITE BASIS

JONATHAN CRESPO

Abstract. In this article, we generalize to the case of measured quantum groupoids on a finite basis some important results concerning actions of locally compact quantum groups on C*-algebras (*Comm. Math. Phys.* **235** (2003) 139–167). Let $\mathcal{G}$ be a measured quantum groupoid on a finite basis. We prove that if $\mathcal{G}$ is regular, then any weakly continuous action of $\mathcal{G}$ on a C*-algebra is necessarily strongly continuous. Following (*K-Theory* **2** (1989) 683–721), we introduce and investigate a notion of $\mathcal{G}$-equivariant Hilbert C*-modules. By applying the previous results and a version of the Takesaki–Takai duality theorem obtained in (*Bull. Soc. Math. France* **145** (2017) 711–802) for actions of $\mathcal{G}$, we obtain a canonical equivariant Morita equivalence between a given $\mathcal{G}$-C*-algebra A and the double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$.

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Introduction

The notion of monoidal equivalence of compact quantum groups has been introduced by Bichon, De Rijdt and Vaes in [6]. Two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are said to be monoidally equivalent if their categories of representations are equivalent as monoidal C*-categories. They have proved that $\mathbb{G}_1$ and $\mathbb{G}_2$ are monoidally equivalent if and only if there exists a unital C*-algebra equipped with commuting continuous ergodic actions of full multiplicity of $\mathbb{G}_1$ on the left and of $\mathbb{G}_2$ on the right.

Many crucial results of the geometric theory of free discrete quantum groups rely on the monoidal equivalence of their dual compact quantum groups. Among the applications of monoidal equivalence to this theory, we mention the contributions to randow walks and their associated boundaries [26], [15], CCAP property and Haagerup property [14], the Baum–Connes conjecture and K-amenability [29], [28].

In his Ph.D. thesis [11], De Commer has extended the notion of monoidal equivalence to the locally compact case. Two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ (in the sense of Kustermans and Vaes [19]) are said to be monoidally equivalent if there exists a von Neumann algebra equipped with a left Galois action of $\mathbb{G}_1$ and a right Galois action of $\mathbb{G}_2$ that commute. He proved that this notion is completely encoded by a measured quantum groupoid (in the sense of Enock and Lesieur [17]) on the basis $\mathbb{C}^2$. Such a groupoid is called a colinking measured quantum groupoid.

The measured quantum groupoids have been introduced and studied by Lesieur and Enock (see [17], [20]). Roughly speaking, a measured quantum groupoid (in the sense of Enock–Lesieur) is an octuple $\mathcal{G} =$

$(N, M, \alpha, \beta, \Gamma, T, T', \nu)$, where N and M are von Neumann algebras (the basis N and M are the algebras of the groupoid corresponding respectively to the space of units and the total space for a classical groupoid), α and β are faithful normal $*$-homomorphisms from N and N^{o} (the opposite algebra) to M (corresponding to the source and target maps for a classical groupoid) with commuting ranges, Γ is a coproduct taking its values in a certain fiber product, ν is a normal semi-finite weight on N and T and T' are operator-valued weights satisfying some axioms.

In the case of a finite-dimensional basis N, the definition has been greatly simplified by De Commer [13], [11] and we will use this point of view in this article. Broadly speaking, we can take for ν the non-normalized Markov trace on the C*-algebra $N = \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$. The relative tensor product of Hilbert spaces (resp., the fiber product of von Neumann algebras) is replaced by the ordinary tensor product of Hilbert spaces (resp., von Neumann algebras). The coproduct takes its values in $M \otimes M$ but is no longer unital. In the following, these objects will be referred to as "measured quantum groupoids on a finite basis."

In [2], the authors introduce a notion of (strongly) continuous actions on C*-algebras of measured quantum groupoids on a finite basis. They extend the construction of the crossed product, the dual action and give a version of the Takesaki–Takai duality generalizing the Baaj–Skandalis duality theorem [3] in this setting.

If a colinking measured quantum groupoid $\mathcal{G}$, associated with a monoidal equivalence of two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, acts (strongly) continuously on a C*-algebra A, then A splits up as a direct sum $A = A_1 \oplus A_2$ of C*-algebras and the action of $\mathcal{G}$ on A restricts to an action of $\mathbb{G}_1$ (resp., $\mathbb{G}_2$) on A_1 (resp., A_2).

They also extend the induction procedure to the case of monoidally equivalent regular locally compact quantum groups. To any continuous action of $\mathbb{G}_1$ on a C*-algebra A_1, they associate canonically a C*-algebra A_2 endowed with a continuous action of $\mathbb{G}_2$. As important consequences of this construction, we mention the following:

- a one-to-one functorial correspondence between the continuous actions of the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, which generalizes the compact case [15] and the case of deformations by a 2-cocycle [21];
- a complete description of the continuous actions of colinking measured quantum groupoids;
- the equivalence of the categories $\mathsf{KK}_{\mathbb{G}_1}$ and $\mathsf{KK}_{\mathbb{G}_2}$, which generalizes to the regular locally compact case a result of Voigt [29].

The proofs of the above results rely crucially on the regularity of the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. They prove that the regularity of $\mathbb{G}_1$ and $\mathbb{G}_2$ is equivalent to the regularity in the sense of [16] (see also [24], [23]) of the associated

colinking measured quantum groupoid. By passing, this result solves some questions raised in [21] in the case of deformations by a 2-cocycle.

In this article, we generalize to the case of (semi-)regular measured quantum groupoid on a finite basis some important properties of (semi-)regular locally compact quantum groups [4], [1]. This work will give us enough formulas to generalize some crucial results of [5] concerning actions of (semi-)regular locally compact quantum groups.

More precisely, if $\mathcal{G}$ is a semi-regular measured quantum groupoid on a finite basis, then the space consisting of the continuous elements of any action of $\mathcal{G}$ is a C*-algebra. Moreover, if $\mathcal{G}$ is regular we prove that any weakly continuous action of $\mathcal{G}$ is necessarily continuous in the strong sense.

We introduce a notion of action of $\mathcal{G}$ on Hilbert C*-modules in line with the corresponding notion for quantum groups [3]. A $\mathcal{G}$-equivariant Hilbert C*-module is a Hilbert C*-module endowed with a continuous action (in a sense that will be specified). By using the previous result, if $\mathcal{G}$ is regular we prove that any action of $\mathcal{G}$ on a Hilbert C*-module is necessarily continuous. We are able to define the notion of $\mathcal{G}$-equivariant Morita equivalence of $\mathcal{G}$-C*-algebras. By applying a version of the Takesaki–Takai duality theorem obtained in [2], we prove that any $\mathcal{G}$-C*-algebra A is $\mathcal{G}$-equivariantly Morita equivalent to its double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ in a canonical way.

This article is organized as follows.

• *Chapter 1.* We recall the general conventions and notation used throughout this paper.
• *Chapter 2.* We make an overview of the theory of locally compact quantum groups (cf. [19] and [4]). We recall the construction of the Hopf C*-algebra associated with a locally compact quantum group and the notion of action of locally compact quantum groups in the C*-algebraic setting. We also recall the notion of equivariant Hilbert C*-modules (cf. [3]).
• *Chapter 3.* We make a very brief survey of the theory of measured quantum groupoid (cf. [20], [17]) and we recall the simplified definition in the case where the basis is finite-dimensional and the associated C*-algebraic structure provided by De Commer in [13], [11]. In the last section, we make an outline of the reflection technique across a Galois object provided by De Commer (cf. [11], [12]), the construction and the structure of the colinking measured quantum groupoid associated with monoidally equivalent locally compact quantum groups. We also recall the precise description of the C*-algebraic structure of colinking measured quantum groupoids (cf. [2]).
• *Chapter 4.* In this chapter, we make a review of the notions of regularity and semi-regularity for measured quantum groupoids on a finite basis (cf. [16], [24], [23], [2]) and we obtain new relations equivalent to the (semi-)regularity generalizing some results of Baaj and Skandalis [4], [1]. Given a (semi-)regular

measured quantum groupoid, we derive new relations that will play a crucial role in the subsequent chapters.

• *Chapter 5.* In the first section of this chapter, we recall the definitions and the main results of [2] concerning the notion of (strongly) continuous action of measured quantum groupoids on a finite basis on C*-algebras. We also recall the version of the Takesaki–Takai duality theorem obtained in [2] in this framework. The second section is dedicated to a brief overview of C*-algebras acted upon by a colinking measured quantum groupoid (cf. [2]). In the last section, we generalize to the setting of measured quantum groupoids on a finite basis the results of Baaj, Skandalis and Vaes [5] concerning the notion of weakly/strongly continuous action of (semi-)regular locally compact quantum groups.

• *Chapter 6.* We introduce the notion of action of measured quantum groupoid on a finite basis on Hilbert C*-module and we investigate in detail the case of a colinking measured quantum groupoid. In the last paragraph, we provide a direct approach of the induction procedure for equivariant Hilbert C*-modules equivalent to that obtained in [2]. In particular, if $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ is a colinking measured quantum groupoid associated with two monoidally equivalent regular locally compact groups $\mathbb{G}_1$ and $\mathbb{G}_2$ we obtain one-to-one correspondences between the actions of $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ on Hilbert C*-modules.

• *Chapter 7.* In this chapter, we introduce and discuss the notion of equivariant Morita equivalence. Given a $\mathcal{G}$-C*-algebra A, we prove that A and its double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ are $\mathcal{G}$-equivariantly Morita equivalent in a canonical way.

• *Chapter 8.* In the Appendix of this article, we have assembled a very brief review of the main notions and notation of the noncommutative measure and integration theory. We can also find some notation and important results used throughout this paper.

In a forthcoming article [10], we use the results of this paper to generalize those of Baaj and Skandalis concerning the equivariant Kasparov theory (cf. §6 [3] and 7.7 b) [4]).

1. Preliminary notation

We specify here some elementary notation and conventions used in this article. For more notation, we refer the reader to the Appendix and the index of this article.

• For all subset X of a normed vector space E, we denote $\langle X \rangle$ (resp., $[X]$) the linear span (resp., closed linear span) of X in E. If $X, Y \subset E$, we denote $XY := \{xy;\, x \in X, y \in Y\}$, where xy denotes the product/composition of x and y or the evaluation of x at y (when these operations make sense). If X is a subset of a $*$-algebra A, we denote by X^* the subset $\{x^*;\, x \in X\}$ of A.

• We denote by $\otimes$ the tensor product of Hilbert spaces, the tensor product of von Neumann algebras, the minimal tensor product of C*-algebras or the external tensor product of Hilbert C*-modules. We also denote by $\odot$ (resp., $\odot_A$) the algebraic tensor product over the field of complex numbers $\mathbb{C}$ (resp., an algebra A).
• Let A and B be C*-algebras. We denote by $\mathcal{M}(A)$ (resp., $\widetilde{A}$) the C*-algebra consisting of the multipliers of A (resp., the C*-algebra obtained from A by adjunction of a unit element). We denote by $\widetilde{\mathcal{M}}(A \otimes B)$ (or $\widetilde{\mathcal{M}}_B(A \otimes B)$ in case of ambiguity, §1 [3]) the B-relative multiplier algebra, *that is*, the C*-algebra consisting of the elements m of $\mathcal{M}(\widetilde{A} \otimes B)$ such that the relations $(\widetilde{A} \otimes B)m \subset A \otimes B$ and $m(\widetilde{A} \otimes B) \subset A \otimes B$ hold.

Let $\pi : A \to \mathcal{M}(B)$ be a (possibly degenerate) $*$-homomorphism. For all C*-algebra D, there exists a unique strictly continuous $*$-homomorphism $\pi \otimes \mathrm{id}_D : \widetilde{\mathcal{M}}(A \otimes D) \to \mathcal{M}(B \otimes D)$ satisfying the relation $(\pi \otimes \mathrm{id}_D)(x)(1_B \otimes d) = (\pi \otimes \mathrm{id}_D)(x(1_A \otimes d))$ for all $x \in \widetilde{\mathcal{M}}(A \otimes D)$ and $d \in D$. Indeed, denote by $\widetilde{\pi}$ the unital extension of π to $\widetilde{A}$. The non-degenerate $*$-homomorphism $\widetilde{\pi} \otimes \mathrm{id}_D : \widetilde{A} \otimes D \to \mathcal{M}(B \otimes D)$ uniquely extends to $\mathcal{M}(\widetilde{A} \otimes D)$. By restricting to $\widetilde{\mathcal{M}}(A \otimes D)$, we obtain the desired extension of $\pi \otimes \mathrm{id}_D$ (§1 [3]).
• If x and y are two elements of an algebra A, we denote by $[x, y]$ the commutator of x and y, *i.e.* the element of A defined by $[x, y] := xy - yx$.

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces (all inner products are assumed to be anti-linear in the first variable and linear in the second variable).
• We denote by $\mathcal{B}(\mathscr{H}, \mathscr{K})$ (resp., $\mathcal{K}(\mathscr{H}, \mathscr{K})$) the Banach space of bounded (resp., compact) linear operators from $\mathscr{H}$ to $\mathscr{K}$. For all $\xi \in \mathscr{K}$ and $\eta \in \mathscr{H}$, we denote by $\theta_{\xi,\eta} \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ the rank-one operator defined by $\theta_{\xi,\eta}(\zeta) := \langle \eta, \zeta \rangle \xi$ for all $\zeta \in \mathscr{H}$. We have the relation $\mathcal{K}(\mathscr{H}, \mathscr{K}) = [\theta_{\xi,\eta};\, \xi \in \mathscr{K}, \eta \in \mathscr{K}]$. Denote by $\mathcal{B}(\mathscr{H}) := \mathcal{B}(\mathscr{H}, \mathscr{H})$ (resp., $\mathcal{K}(\mathscr{H}) := \mathcal{K}(\mathscr{H}, \mathscr{H})$) the C*-algebra of bounded (resp., compact) linear operators on $\mathscr{H}$. Recall that $\mathcal{K}(\mathscr{H})$ is a closed two-sided ideal of $\mathcal{B}(\mathscr{H})$ and $\mathcal{B}(\mathscr{H}) = \mathcal{M}(\mathcal{K}(\mathscr{H}))$.
• We denote by $\Sigma_{\mathscr{K} \otimes \mathscr{H}}$ (or simply Σ) the flip map, that is to say the unitary operator $\mathscr{K} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{K}\,;\, \xi \otimes \eta \mapsto \eta \otimes \xi$.
• For $u \in \mathcal{B}(\mathscr{H})$, we denote by Ad_u the bounded operator on $\mathcal{B}(\mathscr{H})$ defined for all $x \in \mathcal{B}(\mathscr{H})$ by $\mathrm{Ad}_u(x) = uxu^*$.

In this article, we will use the notion of (right) Hilbert C*-module over a C*-algebra and their tensor products (internal and external). All the definitions and conventions are those of [18]. In particular, let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert C*-modules over a C*-algebra A.
• We denote by $\mathcal{L}(\mathscr{E}, \mathscr{F})$ the Banach space consisting of all adjointable operators from $\mathscr{E}$ to $\mathscr{F}$ and $\mathcal{L}(\mathscr{E})$ the C*-algebra $\mathcal{L}(\mathscr{E}, \mathscr{E})$.
• For $\xi \in \mathscr{F}$ and $\eta \in \mathscr{E}$, we denote by $\theta_{\xi,\eta}$ the elementary operator of $\mathcal{L}(\mathscr{E}, \mathscr{F})$ defined by $\theta_{\xi,\eta}(\zeta) := \xi \langle \eta, \zeta \rangle_A$ for all $\zeta \in \mathscr{E}$. Let $\mathcal{K}(\mathscr{E}, \mathscr{F}) := [\theta_{\xi,\eta};\, \xi \in \mathscr{F}, \eta \in \mathscr{E}]$

be the Banach space of "compact" adjointable operators. Denote by $\mathcal{K}(\mathscr{E})$ the C*-algebra $\mathcal{K}(\mathscr{E},\mathscr{E})$ consisting of the compact adjointable operators of $\mathcal{L}(\mathscr{E})$. Recall that $\mathcal{K}(\mathscr{E})$ is a closed two-sided ideal of $\mathcal{L}(\mathscr{E})$ and $\mathcal{L}(\mathscr{E}) = \mathcal{M}(\mathcal{K}(\mathscr{E}))$.
• Let $\mathscr{E}^* := \mathcal{K}(\mathscr{E}, A)$. We have $\mathscr{E}^* = \{T \in \mathcal{L}(\mathscr{E}, A);\ \exists \xi \in \mathscr{E}, \forall \eta \in \mathscr{E}, T(\eta) = \langle \xi, \eta \rangle_A\}$. We will identify $\mathscr{E} = \mathcal{K}(A, \mathscr{E}) \subset \mathcal{L}(A, \mathscr{E})$. Up to this identification, for $\xi \in \mathscr{E}$ the operator $\xi^* \in \mathscr{E}^*$ satisfies $\xi^*(\eta) = \langle \xi, \eta \rangle_A$ for all $\eta \in \mathscr{E}$. We recall that $\mathscr{E}^*$ is a Hilbert $\mathcal{K}(\mathscr{E})$-module for the inner product defined by $\langle T, T' \rangle_{\mathcal{K}(\mathscr{E})} := T^* \circ T'$ for $T, T' \in \mathscr{E}^*$ and the right action is defined by the composition of maps.

In this article, we will also use the leg numbering notation. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. We define the operators $T_{12}, T_{13}, T_{23} \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H})$ by setting $T_{12} := T \otimes 1$, $T_{23} := 1 \otimes T$ and $T_{13} := (\Sigma \otimes 1)(1 \otimes T)(\Sigma \otimes 1)$. We can generalize the leg numbering notation for operators acting on any tensor product of Hilbert spaces and for adjointable operators acting on any external tensor product of Hilbert C*-modules over possibly different C*-algebras.

2. Locally compact quantum groups

For the notation and conventions used in this article concerning the noncommutative integration theory and the canonical objects of the Tomita–Takesaki theory, we refer the reader to the Appendix of this article (cf. §A.1).

Definition 2.1 ([19]). A locally compact quantum group is a pair $\mathbb{G} = (\mathrm{L}^\infty(\mathbb{G}), \Delta)$, where $\mathrm{L}^\infty(\mathbb{G})$ is a von Neumann algebra and $\Delta : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \otimes \mathrm{L}^\infty(\mathbb{G})$ is a unital normal $*$-homomorphism satisfying the following conditions:

1. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;
2. there exist n.s.f. weights φ and ψ on $\mathrm{L}^\infty(\mathbb{G})$ such that:
   (a) φ is left invariant, *that is*, $\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \varphi(x)\omega(1)$, for all $\omega \in \mathrm{L}^\infty(\mathbb{G})_*^+$ and $x \in \mathfrak{M}_\varphi^+$,
   (b) ψ is right inveriant, *that is*, $\psi((\mathrm{id} \otimes \omega)\Delta(x)) = \psi(x)\omega(1)$, for all $\omega \in \mathrm{L}^\infty(\mathbb{G})_*^+$ and $x \in \mathfrak{M}_\psi^+$.

A left (resp., right) invariant n.s.f. weight on $\mathrm{L}^\infty(\mathbb{G})$ is called a left (resp., right) Haar weight on $\mathbb{G}$.

2.2. For a locally compact quantum group $\mathbb{G}$, there exists a unique left (resp., right) Haar weight on $\mathbb{G}$ up to a positive scalar [19]. Let us fix a locally compact quantum group $\mathbb{G} := (\mathrm{L}^\infty(\mathbb{G}), \Delta)$. Let us fix a left Haar weight φ on $\mathbb{G}$. Let $(\mathrm{L}^2(\mathbb{G}), \pi, \Lambda)$ be the G.N.S. construction for $(\mathrm{L}^\infty(\mathbb{G}), \varphi)$. The left regular representation of $\mathbb{G}$ is the multiplicative unitary [19], [4] $W \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G}))$ defined by

$$W^*\big(\Lambda(x) \otimes \Lambda(y)\big) = (\Lambda \otimes \Lambda)\big(\Delta(y)(x \otimes 1)\big), \quad \text{for all } x, y \in \mathfrak{N}_\varphi.$$

By identifying $\mathrm{L}^\infty(\mathbb{G})$ with its image by the G.N.S. representation π, we obtain:

- $\mathrm{L}^\infty(\mathbb{G})$ is the strong closure of the algebra $\{(\mathrm{id}\otimes\omega)(W);\, \omega\in\mathcal{B}(\mathrm{L}^2(\mathbb{G}))_*\}$;
- $\Delta(x)=W^*(1\otimes x)W$, for all $x\in\mathrm{L}^\infty(\mathbb{G})$.

2.3. The Hopf–von Neumann algebra $(\mathrm{L}^\infty(\mathbb{G}),\Delta)$ admits [19] a unitary antipode map $R_{\mathbb{G}}:\mathrm{L}^\infty(\mathbb{G})\to\mathrm{L}^\infty(\mathbb{G})$ and we can choose for right Haar weight on $\mathbb{G}$ the weight ψ defined by $\psi(x):=\varphi(R_{\mathbb{G}}(x))$, for all $x\in\mathrm{L}^\infty(\mathbb{G})_+$. The Connes cocycle derivative [8], [25] of ψ with respect to φ is given by

$$(D\psi:D\varphi)_t:=\nu^{\mathrm{i}t^2/2}d^{\mathrm{i}t},\quad \text{for all } t\in\mathbb{R},$$

where $\nu>0$ is the scaling constant of $\mathbb{G}$ and the operator $d\eta M$ is the modular element of $\mathbb{G}$ [19]. Let $\mathfrak{N}_\varphi^d:=\{x\in M;\, xd^{1/2}$ is bounded and its closure $\overline{xd^{1/2}}$ belongs to $\mathfrak{N}_\varphi\}$. The G.N.S. construction [25] for $(\mathrm{L}^\infty(\mathbb{G}),\psi)$ is given by $(\mathrm{L}^2(\mathbb{G}),\mathrm{id},\Lambda_\psi)$, where Λ_ψ is the closure of the map $\mathfrak{N}_\varphi^d\to\mathrm{L}^2(\mathbb{G});\, x\mapsto\Lambda(\overline{xd^{1/2}})$. We recall that $J_\psi=\nu^{\mathrm{i}/4}J$, where J denotes the modular conjugation for φ.

2.4. The right regular representation of the quantum group $\mathbb{G}$ is the multiplicative unitary $V\in\mathcal{B}(\mathrm{L}^2(\mathbb{G})\otimes\mathrm{L}^2(\mathbb{G}))$ defined by

$$V\big(\Lambda_\psi(x)\otimes\Lambda_\psi(y)\big)=(\Lambda_\psi\otimes\Lambda_\psi)\big(\Delta(x)(1\otimes y)\big),\quad \text{for all } x,y\in\mathfrak{N}_\psi.$$

Definition 2.5. The quantum group $\widehat{\mathbb{G}}$ dual of $\mathbb{G}$ is defined by the Hopf–von Neumann algebra $(\mathrm{L}^\infty(\widehat{\mathbb{G}}),\widehat{\Delta})$, where:

- $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is the strong closure of the algebra $\{(\mathrm{id}\otimes\omega)(V);\, \omega\in\mathcal{B}(\mathrm{L}^2(\mathbb{G})\}$;
- the coproduct $\widehat{\Delta}:\mathrm{L}^\infty(\widehat{\mathbb{G}})\to\mathrm{L}^\infty(\widehat{\mathbb{G}})\otimes\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is defined by $\widehat{\Delta}(x):=V^*(1\otimes x)V$ for all $x\in\mathrm{L}^\infty(\widehat{\mathbb{G}})$.

The quantum group $\widehat{\mathbb{G}}$ admits left and right Haar weights [19] and we can take the Hilbert space $\mathrm{L}^2(\mathbb{G})$ for G.N.S. space. We denote by $\widehat{J}$ the modular conjugation of the left Haar weight on $\widehat{\mathbb{G}}$.

2.1. Hopf C*-algebras associated with a quantum group. We associate [4], [19] with the quantum group $\mathbb{G}$ two Hopf C*-algebras (S,δ) and $(\widehat{S},\widehat{\delta})$ defined by:

- S (resp., $\widehat{S}$) is the norm closure of the algebra $\{(\omega\otimes\mathrm{id})(V);\, \omega\in\mathcal{B}(\mathrm{L}^2(\mathbb{G}))_*\}$ (resp., $\{(\mathrm{id}\otimes\omega)(V);\, \omega\in\mathcal{B}(\mathrm{L}^2(\mathbb{G}))_*\}$);
- the coproduct $\delta:S\to\mathcal{M}(S\otimes S)$ (resp., $\widehat{\delta}:\widehat{S}\to\mathcal{M}(\widehat{S}\otimes\widehat{S})$) is given by:

$$\delta(x):=V(x\otimes 1)V^*,\quad \text{for all } x\in S$$
$$\big(\text{resp., } \widehat{\delta}(x):=V^*(1\otimes x)V,\quad \text{for all } x\in\widehat{S}\big).$$

We call (S,δ) (resp., $(\widehat{S},\widehat{\delta})$) the Hopf C*-algebra (resp., dual Hopf C*-algebra) associated with $\mathbb{G}$. We can also denote by $\mathrm{C}_0(\mathbb{G}) := S$ the Hopf C*-algebra of $\mathbb{G}$. Note that $\mathrm{C}_0(\widehat{\mathbb{G}}) = \widehat{S}$.

NOTATION 2.1.1.

- Consider the unitary operator $U := \widehat{J}J \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}))$. Since $U = \nu^{\mathrm{i}/4}J\widehat{J}$, we have $U^* = \nu^{-\mathrm{i}/4}U$. In particular, $\mathrm{Ad}_U = \mathrm{Ad}_{U^*}$ on $\mathcal{B}(\mathrm{L}^2(\mathbb{G}))$.
- We have the following nondegenerate faithful representation of the C*-algebra S (resp., $\widehat{S}$):

$$\begin{aligned} L : S \to \mathcal{B}\big(\mathrm{L}^2(\mathbb{G})\big); &\qquad y \mapsto y; \\ R : S \to \mathcal{B}\big(\mathrm{L}^2(\mathbb{G})\big); &\qquad y \mapsto UL(y)U^* \\ \big(\text{resp., } \rho : \widehat{S} \to \mathcal{B}\big(\mathrm{L}^2(\mathbb{G})\big); &\qquad x \mapsto x; \\ \lambda : \widehat{S} \to \mathcal{B}\big(\mathrm{L}^2(\mathbb{G})\big); &\qquad x \mapsto U\rho(x)U^*\big). \end{aligned}$$

It follows from 2.15 [19] that $W = \Sigma(U \otimes 1)V(U^* \otimes 1)\Sigma$ and $[W_{12}, V_{23}] = 0$. The right regular representation of $\widehat{\mathbb{G}}$ is the multiplicative unitary $\widetilde{V} := \Sigma(1 \otimes U)V(1 \otimes U^*)\Sigma$.

NOTATION 2.1.2. Let $\mathscr{H}$ be a Hilbert space and $X \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. We denote by $\mathcal{C}(X)$ the norm closure of the subspace $\{(\mathrm{id} \otimes \omega)(\Sigma X);\, \omega \in \mathcal{B}(\mathscr{H})_*\}$ of $\mathcal{B}(\mathscr{H})$. If X is a multiplicative unitary, then $\{(\mathrm{id} \otimes \omega)(\Sigma X);\, \omega \in \mathcal{B}(\mathscr{H})_*\}$ is a subalgebra of $\mathcal{B}(\mathscr{H})$ [4].

DEFINITION 2.1.3 ([4], [1]). The quantum group $\mathbb{G}$ is said to be regular (resp., semi-regular), if $\mathcal{K}(\mathrm{L}^2(\mathbb{G})) = \mathcal{C}(V)$ (resp., $\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \subset \mathcal{C}(V)$).

Note that $\mathbb{G}$ is regular (resp., semi-regular) if, and only if, $\mathcal{K}(\mathrm{L}^2(\mathbb{G})) = \mathcal{C}(W)$ (resp., $\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \subset \mathcal{C}(W)$).

2.2. Continuous actions of locally compact quantum groups. We use the notation introduced in the previous paragraph. Let A be a C*-algebra.

DEFINITION 2.2.1.

1. An action of the quantum group $\mathbb{G}$ on A is a nondegenerate $*$-homomorphism $\delta_A : A \to \mathcal{M}(A \otimes S)$ satisfying $(\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A$.
2. An action δ_A of $\mathbb{G}$ on A is said to be strongly (resp., weakly) continuous if

$$\big[\delta_A(A)(1_A \otimes S)\big] = A \otimes S \qquad \big(\text{resp., } A = \big[(\mathrm{id}_A \otimes \omega)\delta_A(A);\, \omega \in \mathcal{B}\big(\mathrm{L}^2(\mathbb{G})\big)_*\big]\big).$$

3. A $\mathbb{G}$-C*-algebra is a pair (A, δ_A), where A is a C*-algebra and $\delta_A : A \to \mathcal{M}(A \otimes S)$ is a strongly continuous action of $\mathbb{G}$ on A.

If $\mathbb{G}$ is regular, any weakly continuous action of $\mathbb{G}$ is necessarily continuous in the strong sense, cf. 5.8 [5].

NOTATION 2.2.2. Let $\delta_A : A \to \mathcal{M}(A \otimes S)$ (resp., $\delta_A : A \to \mathcal{M}(A \otimes \widehat{S})$) be a strongly continuous action of $\mathbb{G}$ (resp., $\widehat{\mathbb{G}}$) on the C*-algebra A. Denote by π_L (resp., $\widehat{\pi}_\lambda$) the representation of A on the Hilbert A-module $A \otimes \mathrm{L}^2(\mathbb{G})$ defined by $\pi_L := (\mathrm{id}_A \otimes L)\delta_A$ (resp., $\widehat{\pi}_\lambda := (\mathrm{id}_A \otimes \lambda)\delta_A$).

DEFINITION 2.2.3 (*cf.* 7.1 [4]). Let (A, δ_A) be a $\mathbb{G}$-C*-algebra (resp., $\widehat{\mathbb{G}}$-C*-algebra). We call (reduced) crossed product of A by $\mathbb{G}$ (resp., $\widehat{\mathbb{G}}$), the C*-subalgebra $A \rtimes \mathbb{G}$ (resp., $A \rtimes \widehat{\mathbb{G}}$) of $\mathcal{L}(A \otimes \mathrm{L}^2(\mathbb{G}))$ generated by the products $\pi_L(a)(1_A \otimes \rho(x))$ (resp., $\widehat{\pi}_\lambda(a)(1_A \otimes L(x))$) for $a \in A$ and $x \in \widehat{S}$ (resp., $x \in S$).

The crossed product $A \rtimes \mathbb{G}$ (resp., $A \rtimes \widehat{\mathbb{G}}$) is endowed with a strongly continuous action of $\widehat{\mathbb{G}}$ (resp., $\mathbb{G}$), cf. 7.3 [4]. If $\mathbb{G}$ is regular, then the Takesaki–Takai duality extends to this setting, cf. 7.5 [4].

DEFINITION 2.2.4. Let A and B be two C*-algebras. Let $\delta_A : A \to \mathcal{M}(A \otimes S)$ and $\delta_B : B \to \mathcal{M}(B \otimes S)$ be two actions of $\mathbb{G}$ on A and B respectively. A nondegenerate $*$-homomorphism $f : A \to \mathcal{M}(B)$ is said to be $\mathbb{G}$-equivariant if $(f \otimes \mathrm{id}_S)\delta_A = \delta_B \circ f$. We denote by $\mathsf{Alg}_{\mathbb{G}}$ the category whose objects are the $\mathbb{G}$-C*-algebras and the morphisms are the $\mathbb{G}$-equivariant nondegenerate $*$-homomorphisms.

2.3. Equivariant Hilbert C*-modules and bimodules.

Preliminaries. In this paragraph, we briefly recall some classical notationand elementary facts concerning Hilbert C*-modules. Let A be a C*-algebra and $\mathscr{E}$ a Hilbert A-module.

NOTATION 2.3.1. Let us consider the following maps:

- $\iota_A : A \to \mathcal{K}(\mathscr{E} \oplus A)$, the $*$-homomorphism given by $\iota_A(a)(\xi \oplus b) = 0 \oplus ab$ for all $a, b \in A$ and $\xi \in \mathscr{E}$;
- $\iota_{\mathscr{E}} : \mathscr{E} \to \mathcal{K}(\mathscr{E} \oplus A)$, the bounded linear map given by $\iota_{\mathscr{E}}(\xi)(\eta \oplus a) = \xi a \oplus 0$ for all $a \in A$ and $\xi, \eta \in \mathscr{E}$;
- $\iota_{\mathscr{E}^*} : \mathscr{E}^* \to \mathcal{K}(\mathscr{E} \oplus A)$, the bounded linear map given by $\iota_{\mathscr{E}^*}(\xi^*)(\eta \oplus a) = 0 \oplus \xi^*\eta$ for all $\xi, \eta \in \mathscr{E}$ and $a \in A$;
- $\iota_{\mathcal{K}(\mathscr{E})} : \mathcal{K}(\mathscr{E}) \to \mathcal{K}(\mathscr{E} \oplus A)$, the $*$-homomorphism given by $\iota_{\mathcal{K}(\mathscr{E})}(k)(\eta \oplus a) = k\eta \oplus 0$ for all $k \in \mathcal{K}(\mathscr{E})$, $\eta \in \mathscr{E}$ and $a \in A$.

The result below follows from straightforward computations.

PROPOSITION 2.3.2. *We have the following statements*:

1. $\iota_{\mathscr{E}}(\xi a) = \iota_{\mathscr{E}}(\xi)\iota_A(a)$ *and* $\iota_A(a)\iota_{\mathscr{E}^*}(\xi^*) = \iota_{\mathscr{E}^*}(a\xi^*)$ *for all* $\xi \in \mathscr{E}$ *and* $a \in A$;
2. $\iota_{\mathscr{E}^*}(\xi^*) = \iota_{\mathscr{E}}(\xi)^*$ *and* $\iota_{\mathcal{K}(\mathscr{E})}(\theta_{\xi,\eta}) = \iota_{\mathscr{E}}(\xi)\iota_{\mathscr{E}}(\eta)^*$ *for all* $\xi, \eta \in \mathscr{E}$;
3. $\iota_{\mathscr{E}}(\xi)^*\iota_{\mathscr{E}}(\eta) = \iota_A(\langle \xi, \eta \rangle)$ *for all* $\xi, \eta \in \mathscr{E}$;
4. $\mathcal{K}(\mathscr{E} \oplus A)$ *is the C*-algebra generated by the set* $\iota_A(A) \cup \iota_{\mathscr{E}}(\mathscr{E})$.

REMARKS 2.3.3.

1. For $a \in A$, $\xi \in \mathscr{E}$ and $k \in \mathcal{K}(\mathscr{E})$, the operators $\iota_A(a)$, $\iota_{\mathscr{E}}(\xi)$, $\iota_{\mathscr{E}^*}(\xi^*)$ and $\iota_{\mathcal{K}(\mathscr{E})}(k)$ can be represented by 2-by-2 matrices acting on the Hilbert A-module $\mathscr{E} \oplus A$ as follows:
$$\iota_A(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}; \qquad \iota_{\mathscr{E}}(\xi) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix};$$
$$\iota_{\mathscr{E}^*}(\xi^*) = \begin{pmatrix} 0 & 0 \\ \xi^* & 0 \end{pmatrix}; \qquad \iota_{\mathcal{K}(\mathscr{E})}(k) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}.$$
Moreover, any operator $x \in \mathcal{K}(\mathscr{E} \oplus A)$ can be written in a unique way as follows:
$$x = \begin{pmatrix} k & \xi \\ \eta^* & a \end{pmatrix}, \quad \text{with } k \in \mathcal{K}(\mathscr{E}), \xi, \eta \in \mathscr{E} \text{ and } a \in A.$$
2. Note that ι_A and $\iota_{\mathcal{K}(\mathscr{E})}$ extend uniquely to strictly/$*$-strongly continuous unital $*$-homomorphisms $\iota_A : \mathcal{M}(A) \to \mathcal{L}(\mathscr{E} \oplus A)$ and $\iota_{\mathcal{K}(\mathscr{E})} : \mathcal{L}(\mathscr{E}) \to \mathcal{L}(\mathscr{E} \oplus A)$. Besides, we have $\iota_A(m)(\xi \oplus a) = 0 \oplus ma$ and $\iota_{\mathcal{K}(\mathscr{E})}(T)(\xi \oplus a) = T\xi \oplus 0$ for all $m \in \mathcal{M}(A)$, $T \in \mathcal{L}(\mathscr{E})$, $\xi \in \mathscr{E}$ and $a \in A$.
3. $\iota_{\mathscr{E}^*}$ admits an extension to a bounded linear map $\iota_{\mathscr{E}^*} : \mathcal{L}(\mathscr{E}, A) \to \mathcal{L}(\mathscr{E} \oplus A)$ in a straightforward way. Similarly, up to the identification $\mathscr{E} = \mathcal{K}(A, \mathscr{E})$, we can also extend $\iota_{\mathscr{E}}$ to a bounded linear map $\iota_{\mathscr{E}} : \mathcal{L}(A, \mathscr{E}) \to \mathcal{L}(\mathscr{E} \oplus A)$.
4. As in 1, we can represent the operators $\iota_A(m)$, $\iota_{\mathcal{K}(\mathscr{E})}(T)$, $\iota_{\mathscr{E}^*}(S)$ and $\iota_{\mathscr{E}}(S^*)$, for $m \in \mathcal{M}(A)$, $T \in \mathcal{L}(\mathscr{E})$ and $S \in \mathcal{L}(A, \mathscr{E})$, by 2-by-2 matrices. Moreover, any operator $x \in \mathcal{L}(\mathscr{E} \oplus A)$ can be written in a unique way as follows:
$$x = \begin{pmatrix} T & S' \\ S^* & m \end{pmatrix}, \quad \text{with } T \in \mathcal{L}(\mathscr{E}), S, S' \in \mathcal{L}(A, \mathscr{E}) \text{ and } m \in \mathcal{M}(A).$$

By using the matrix notations described above, we derive easily the following useful technical lemma.

LEMMA 2.3.4. *Let $x \in \mathcal{L}(\mathscr{E} \oplus A)$ (resp., $x \in \mathcal{K}(\mathscr{E} \oplus A)$). We have*:

1. *$x \in \iota_{\mathscr{E}}(\mathcal{L}(A, \mathscr{E}))$ (resp., $\iota_{\mathscr{E}}(\mathscr{E})$) if, and only if, $x\iota_{\mathscr{E}}(\xi) = 0$ for all $\xi \in \mathscr{E}$ and $\iota_A(a)x = 0$ for all $a \in A$; in that case, we have $\iota_A(m)x = 0$ for all $m \in \mathcal{M}(A)$;*
2. *$x \in \iota_{\mathcal{K}(\mathscr{E})}(\mathcal{L}(\mathscr{E}))$ (resp., $\iota_{\mathcal{K}(\mathscr{E})}(\mathcal{K}(\mathscr{E}))$) if, and only if, $x\iota_A(a) = 0$ and $\iota_A(a)x = 0$ for all $a \in A$; in that case, we have $x\iota_A(m) = \iota_A(m)x = 0$ for all $m \in \mathcal{M}(A)$.*

Let us recall the notion of relative multiplier module, cf. 2.1 [3].

DEFINITION 2.3.5. Let A and B be two C*-algebras and let $\mathscr{E}$ be a Hilbert C*-module over A. Up to the identification $\mathscr{E} \otimes B = \mathcal{K}(A \otimes B, \mathscr{E} \otimes B)$, we define $\widetilde{\mathcal{M}}(\mathscr{E} \otimes B)$ (or $\widetilde{\mathcal{M}}_B(\mathscr{E} \otimes B)$ in case of ambiguity) to be the following subspace of $\mathcal{L}(A \otimes B, \mathscr{E} \otimes B)$:
$$\big\{T \in \mathcal{L}(A \otimes B, \mathscr{E} \otimes B);\, \forall x \in B, (1_{\mathscr{E}} \otimes x)T \in \mathscr{E} \otimes B \text{ and} T(1_A \otimes x) \in \mathscr{E} \otimes B\big\}.$$

Note that $\widetilde{\mathcal{M}}(\mathscr{E}\otimes B)$ is a Hilbert C*-module over $\widetilde{\mathcal{M}}(A\otimes B)$, whose $\widetilde{\mathcal{M}}(A\otimes B)$-valued inner product is given by

$$\langle\xi,\eta\rangle := \xi^*\circ\eta, \quad \text{for all } \xi,\eta\in\widetilde{\mathcal{M}}(\mathscr{E}\otimes B)\subset\mathcal{L}(A\otimes B,\mathscr{E}\otimes B).$$

Note also that we have $\mathcal{K}(\widetilde{\mathcal{M}}(\mathscr{E}\otimes B))\subset\widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E})\otimes B)$.

Proposition-Definition 2.3.6. *Let $B\subset\mathcal{B}(\mathscr{K})$ be a C*-algebra of operators on a Hilbert space $\mathscr{K}$. For all $T\in\mathcal{L}(A\otimes B,\mathscr{E}\otimes B)$ and $\omega\in\mathcal{B}(\mathscr{K})_*$, there exists a unique $(\mathrm{id}_{\mathscr{E}}\otimes\omega)(T)\in\mathcal{L}(A,\mathscr{E})$ such that*

$$\iota_{\mathscr{E}}(\mathrm{id}_{\mathscr{E}}\otimes\omega)(T) = (\mathrm{id}_{\mathcal{K}(\mathscr{E}\oplus A)}\otimes\omega)\big(\iota_{\mathscr{E}\otimes B}(T)\big)\in\mathcal{L}(\mathscr{E}\oplus A),$$

where $\iota_{\mathscr{E}\otimes B}:\mathcal{L}(A\otimes B,\mathscr{E}\otimes B)\to\mathcal{L}((\mathscr{E}\otimes B)\oplus(A\otimes B))=\mathcal{M}(\mathcal{K}(\mathscr{E}\oplus A)\otimes B)$. If $B\subset\mathcal{B}(\mathscr{K})$ is nondegenerate and $T\in\widetilde{\mathcal{M}}(\mathscr{E}\otimes B)$, then we have $(\mathrm{id}_{\mathscr{E}}\otimes\omega)(T)\in\mathscr{E}$.

Proof. This is a direct consequence of Lemma 2.3.4(1) and the fact that $\iota_{\mathscr{E}\otimes B}(T)\in\widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}\oplus A)\otimes B)$ if $T\in\widetilde{\mathcal{M}}(\mathscr{E}\otimes B)$. □

Notion of equivariant Hilbert C-module.* In this paragraph, we recall the notion of equivariant Hilbert C*-module through the three equivalent pictures developed in §2 [3]. Let us fix a $\mathbb{G}$-C*-algebra (A,δ_A) and a Hilbert A-module $\mathscr{E}$.

Definition 2.3.7. An action of the locally compact quantum group $\mathbb{G}$ on $\mathscr{E}$ is a linear map $\delta_{\mathscr{E}}:\mathscr{E}\to\widetilde{\mathcal{M}}(\mathscr{E}\otimes S)$ such that:

1. $\delta_{\mathscr{E}}(\xi)\delta_A(a)=\delta_{\mathscr{E}}(\xi a)$ and $\delta_A(\langle\xi,\eta\rangle)=\langle\delta_{\mathscr{E}}(\xi),\delta_{\mathscr{E}}(\eta)\rangle$, for all $a\in A$ and $\xi,\eta\in\mathscr{E}$;
2. $[\delta_{\mathscr{E}}(\mathscr{E})(A\otimes S)]=\mathscr{E}\otimes S$;
3. the linear maps $\delta_{\mathscr{E}}\otimes\mathrm{id}_S$ and $\mathrm{id}_{\mathscr{E}}\otimes\delta$ extend to linear maps from $\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)$ to $\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S)$ and we have $(\delta_{\mathscr{E}}\otimes\mathrm{id}_S)\delta_{\mathscr{E}}=(\mathrm{id}_{\mathscr{E}}\otimes\delta)\delta_{\mathscr{E}}$.

An action $\delta_{\mathscr{E}}$ of $\mathbb{G}$ on $\mathscr{E}$ is said to be continuous if we have $[(1_{\mathscr{E}}\otimes S)\delta_{\mathscr{E}}(\mathscr{E})]=\mathscr{E}\otimes S$. A $\mathbb{G}$-equivariant Hilbert A-module is a Hilbert A-module $\mathscr{E}$ endowed with a continuous action $\delta_{\mathscr{E}}:\mathscr{E}\to\widetilde{\mathcal{M}}(\mathscr{E}\otimes S)$ of $\mathbb{G}$ on $\mathscr{E}$.

2.3.8. The datum of a continuous action of $\mathbb{G}$ on $\mathscr{E}$ is equivalent to that of a continuous action $\delta_{\mathcal{K}(\mathscr{E}\oplus A)}:\mathcal{K}(\mathscr{E}\oplus A)\to\mathcal{M}(\mathcal{K}(\mathscr{E}\oplus A)\otimes S)$ of $\mathbb{G}$ on the linking C*-algebra $\mathcal{K}(\mathscr{E}\oplus A)$ such that the $*$-homomorphism $\iota_A:A\to\mathcal{K}(\mathscr{E}\oplus A)$ is $\mathbb{G}$-equivariant, cf. 2.7 [3].

2.3.9. If $\delta_{\mathscr{E}}$ is an action of $\mathbb{G}$ on $\mathscr{E}$, we have the unitary operator $\mathscr{V}\in\mathcal{L}(\mathscr{E}\otimes_{\delta_A}(A\otimes S),\mathscr{E}\otimes S)$ defined by $\mathscr{V}(\xi\otimes_{\delta_A}x):=\delta_{\mathscr{E}}(\xi)x$ for all $\xi\in\mathscr{E}$ and $x\in A\otimes S$. This unitary satisfies the relation

$$\begin{aligned}&(\mathscr{V}\otimes_{\mathbb{C}}\mathrm{id}_S)(\mathscr{V}\otimes_{\delta_A\otimes\mathrm{id}_S}1)\\&\quad=\mathscr{V}\otimes_{\mathrm{id}_A\otimes\delta}1\quad\text{in }\mathcal{L}\big(\mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S),\mathscr{E}\otimes S\otimes S\big),\end{aligned}$$

where $\delta_A^2 := (\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A$; cf. 2.3 and 2.4 (a) [3] for the details. Conversely, if there exists a unitary operator $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ satisfying the above relation and the fact that $\mathscr{V} T_\xi \in \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ for all $\xi \in \mathscr{E}$, where $T_\xi \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes_{\delta_A} (A \otimes S))$ is defined by $T_\xi(x) := \xi \otimes_{\delta_A} x$ for all $x \in A \otimes S$, then the map $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$; $\xi \mapsto \mathscr{V} T_\xi$ is an action of $\mathbb{G}$ on $\mathscr{E}$; cf. 2.4 (b) [3].

2.3.10. An action of $\mathbb{G}$ on $\mathscr{E}$ determines an action $\delta_{\mathcal{K}(\mathscr{E})} : \mathcal{K}(\mathscr{E}) \to \widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}) \otimes S)$ of $\mathbb{G}$ on $\mathcal{K}(\mathscr{E})$ defined by $\delta_{\mathcal{K}(\mathscr{E})}(k) = \mathscr{V}(k \otimes_{\delta_A} 1)\mathscr{V}^*$ for all $k \in \mathcal{K}(\mathscr{E})$, where $\mathscr{V}$ is the unitary operator associated to the action; cf. 2.8 [3]. Moreover, if $\mathscr{E}$ is a $\mathbb{G}$-equivariant Hilbert module, then $\mathcal{K}(\mathscr{E})$ turns into a $\mathbb{G}$-C*-algebra.

3. Measured quantum groupoids

For reminders concerning the relative tensor product of Hilbert spaces and the fiber product of von Neumann algebras, we refer the reader to the Appendix of this article (cf. §A.2).

Definition 3.1 (*cf.* 3.7 [17], 4.1 [20]). We call a measured quantum groupoid an octuple $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$, where:

- M and N are von Neumann algebras;
- $\Gamma : M \to M \,{}_\beta\!\star_\alpha\, M$ is a faithful normal unital $*$-homomorphism, called the coproduct;
- $\alpha : N \to M$ and $\beta : N^{\mathrm{o}} \to M$ are faithful normal unital $*$-homormorphisms, called the range and source maps of $\mathcal{G}$;
- $T : M_+ \to \alpha(N)_+^{\mathrm{ext}}$ and $T' : M_+ \to \beta(N^{\mathrm{o}})_+^{\mathrm{ext}}$ are n.s.f. operator-valued weights;
- ν is a n.s.f. weight on N;

such that the following conditions are satisfied:

1. $[\alpha(n'), \beta(n^{\mathrm{o}})] = 0$, for all $n, n' \in N$;
2. $\Gamma(\alpha(n)) = \alpha(n) \,{}_\beta\!\otimes_\alpha\, 1$ and $\Gamma(\beta(n^{\mathrm{o}})) = 1 \,{}_\beta\!\otimes_\alpha\, \beta(n^{\mathrm{o}})$, for all $n \in N$;
3. Γ is coassociative, *i.e.* $(\Gamma \,{}_\beta\!\star_\alpha\, \mathrm{id})\Gamma = (\mathrm{id} \,{}_\beta\!\star_\alpha\, \Gamma)\Gamma$;
4. the n.s.f. weights φ and ψ on M given by $\varphi = \nu \circ \alpha^{-1} \circ T$ and $\psi = \nu \circ \beta^{-1} \circ T'$ satisfy:
   - $\forall x \in \mathfrak{M}_T^+, T(x) = (\mathrm{id} \,{}_\beta\!\star_\alpha\, \varphi)\Gamma(x), \forall x \in \mathfrak{M}_{T'}^+, T'(x) = (\psi \,{}_\beta\!\star_\alpha\, \mathrm{id})\Gamma(x)$,
   - σ_t^φ and σ_s^ψ commute for all $s, t \in \mathbb{R}$.

Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid. We denote by $(\mathscr{H}, \pi, \Lambda)$ the G.N.S. construction for (M, φ) where $\varphi := \nu \circ \alpha^{-1} \circ T$. Let $(\sigma_t)_{t \in \mathbb{R}}$, ∇ and J be respectively the modular automorphism group, the modular operator and the modular conjugation for φ. In the following, we identify M with its image by π in $\mathcal{B}(\mathscr{H})$.

- We have a coinvolutive $*$-antiautomomorphism $R_{\mathcal{G}} : M \to M$ such that $R_{\mathcal{G}}^2 = \mathrm{id}_M$ (cf. 3.8 [17]).

From now on, we will assume that $T' = R_{\mathcal{G}} \circ T \circ R_{\mathcal{G}}$ and then also $\psi = \varphi \circ R_{\mathcal{G}}$.

- There exist self-adjoint positive nonsingular operators λ and d respectively affiliated to $\mathcal{Z}(M)$ and M such that $(D\psi : D\varphi)_t = \lambda^{\mathrm{i}t^2/2} d^{\mathrm{i}t}$ for all $t \in \mathbb{R}$. The operators λ and d are respectively called the scaling operator and the modular operator of $\mathcal{G}$.
- The G.N.S. construction for (M, ψ) is given by $(\mathscr{H}, \pi_\psi, \Lambda_\psi)$, where: Λ_ψ is the closure of the operator which sends any element $x \in M$ such that $xd^{1/2}$ is closable and its closure $\overline{xd^{1/2}} \in \mathfrak{N}_\varphi$ to $\Lambda_\varphi(\overline{xd^{1/2}})$; $\pi_\psi : M \to \mathcal{B}(\mathscr{H})$ is given by the formula $\pi_\psi(a)\Lambda_\psi(x) = \Lambda_\psi(ax)$.
- The modular conjugation J_ψ for ψ is given by $J_\psi = \lambda^{\mathrm{i}/4} J$.
- We will denote by $\mathscr{W}_{\mathcal{G}} : \mathscr{H}\ {}_\beta\!\otimes_\alpha \mathscr{H} \to \mathscr{H}\ {}_\alpha\!\otimes_{\widehat{\beta}} \mathscr{H}$ the pseudo-multiplicative unitary of $\mathcal{G}$ (cf. 3.3.4 [27], 3.6 [17]).

Proposition-Definition 3.2 (cf. 3.10 [17]). *We define the (Pontryagin) dual of $\mathcal{G}$ to be the measured quantum groupoid $\widehat{\mathcal{G}} := (N, \widehat{M}, \alpha, \widehat{\beta}, \widehat{\Gamma}, \widehat{T}, \widehat{R} \circ \widehat{T} \circ \widehat{R}, \nu)$, where:*

- *$\widehat{M}$ is the von Neumann algebra generated by $\{(\omega \star \mathrm{id})(\mathscr{W}_{\mathcal{G}}); \omega \in \mathcal{B}(\mathscr{H})_*\} \subset \mathcal{B}(\mathscr{H})$;*
- *$\widehat{\beta} : N^{\mathrm{o}} \to \widehat{M}$ is given by $\widehat{\beta}(n^{\mathrm{o}}) := J\alpha(n^*)J$ for all $n \in N$;*
- *$\widehat{\Gamma} : \widehat{M} \to \widehat{M}\ {}_{\widehat{\beta}}\!\star_\alpha \widehat{M}$ is given for all $x \in \widehat{M}$ by $\widehat{\Gamma}(x) := \sigma_{\alpha\widehat{\beta}} \mathscr{W}_{\mathcal{G}}(x\ {}_\beta\!\otimes_\alpha 1) \mathscr{W}_{\mathcal{G}}^* \sigma_{\widehat{\beta}\alpha}$;*
- *there exists a unique n.s.f. weight $\widehat{\varphi}$ on $\widehat{M}$ whose G.N.S. construction is $(\mathscr{H}, \mathrm{id}, \Lambda_{\widehat{\varphi}})$, where $\Lambda_{\widehat{\varphi}}$ is the closure of the operator $(\omega \star \mathrm{id})(\mathscr{W}_{\mathcal{G}}) \mapsto a_\varphi(\omega)$ defined for normal linear forms ω in a dense subspace of $\mathscr{I}_\varphi = \{\omega \in \mathcal{B}(\mathscr{H})_*; \exists k \in \mathbb{R}_+, \forall x \in \mathfrak{N}_\varphi, |\omega(x^*)|^2 \leqslant k\varphi(x^*x)\}$ and $a_\varphi(\omega) \in \mathscr{H}$ satisfies $\omega(x^*) = \langle \Lambda_\varphi(x), a_\varphi(\omega) \rangle$ for all $x \in \mathfrak{N}_\varphi$;*
- *$\widehat{T}$ is the unique n.s.f. operator-valued weight from $\widehat{M}$ to $\alpha(N)$ such that $\widehat{\varphi} = \nu \circ \alpha^{-1} \circ \widehat{T}$ and $\widehat{T}' = R_{\widehat{\mathcal{G}}} \circ \widehat{T} \circ R_{\widehat{\mathcal{G}}}$, where $R_{\widehat{\mathcal{G}}} : \widehat{M} \to \widehat{M}$ is given by $R_{\widehat{\mathcal{G}}}(x) := Jx^*J$ for all $x \in \widehat{M}$.*

The pseudo-multiplicative unitary $\mathscr{W}_{\widehat{\mathcal{G}}}$ of $\widehat{\mathcal{G}}$ is given by $\mathscr{W}_{\widehat{\mathcal{G}}} = \sigma_{\beta\alpha} \mathscr{W}_{\mathcal{G}}^* \sigma_{\widehat{\beta}\alpha}$.

We will denote by $\widehat{J}$ the modular conjugation for $\widehat{\varphi}$. Note that the scaling operator of $\widehat{\mathcal{G}}$ is λ^{-1}. In particular, we have $\lambda^{\mathrm{i}t} \in \mathcal{Z}(M) \cap \mathcal{Z}(\widehat{M})$ for all $t \in \mathbb{R}$.

- Let $\widehat{\alpha}(n) := J\beta(n^{\mathrm{o}})^*J = \widehat{J}\widehat{\beta}(n^{\mathrm{o}})^*\widehat{J}$ for $n \in N$. We recall the following relations (cf. 3.11 (v) [17]): $M \cap \widehat{M} = \alpha(N)$, $M \cap \widehat{M}' = \beta(N^{\mathrm{o}})$, $M' \cap \widehat{M} = \widehat{\beta}(N^{\mathrm{o}})$ and $M' \cap \widehat{M}' = \widehat{\alpha}(N)$.
- Let $U := \widehat{J}J \in \mathcal{B}(\mathscr{H})$. Then, $U^* = \lambda^{-i/4}U$ and $U^2 = \lambda^{i/4}$ (cf. 3.11 (iv) [17]). In particular, U is unitary. We have $\widehat{\alpha}(n) = U\alpha(n)U^*$ and $\widehat{\beta}(n^{\mathrm{o}}) =$

$U\beta(n^{\mathrm{o}})U^*$ for all $n \in N$. Since $\lambda^{-\mathrm{i}/4} \in \mathcal{Z}(M)$, we also have $\widehat{\alpha}(n) = U^*\alpha(n)U$ and $\widehat{\beta}(n^{\mathrm{o}}) = U^*\beta(n^{\mathrm{o}})U$ for all $n \in N$.

Proposition-Definition 3.3 (cf. 3.12 [17]).

- *The octuple* $(N^{\mathrm{o}}, M, \beta, \alpha, \varsigma_{\beta\alpha} \circ \Gamma, R_{\mathcal{G}} \circ T \circ R_{\mathcal{G}}, T, \nu^{\mathrm{o}})$ *is a measured quantum groupoid denoted by* $\mathcal{G}^{\mathrm{o}}$ *and called the opposite of* $\mathcal{G}$. *The pseudo-multiplicative unitary of* $\mathcal{G}^{\mathrm{o}}$ *is given by* $\mathscr{W}_{\mathcal{G}^{\mathrm{o}}} = ({}_{\beta}\widehat{J}_{\alpha} \otimes_{\widehat{\alpha}} \widehat{J}_{\widehat{\beta}})\mathscr{W}_{\mathcal{G}}({}_{\beta}\widehat{J}_{\alpha} \otimes_{\alpha} \widehat{J}_{\beta})$.
- *Let* $C_M : M \to M'$ *be the canonical* $*$*-antihomomorphism given by* $C_M(x) := Jx^*J$ *for all* $x \in M$. *Let us define*:

$$\Gamma^{\mathrm{c}} := (C_M \,{}_{\beta}\star_{\alpha}\, C_M) \circ \Gamma \circ C_M^{-1};$$
$$R_{\mathcal{G}}^{\mathrm{c}} := C_M \circ R_{\mathcal{G}} \circ C_M^{-1};$$
$$T^{\mathrm{c}} = C_M \circ T \circ C_M^{-1}.$$

Then the octuple $(N^{\mathrm{o}}, M', \widehat{\beta}, \widehat{\alpha}, \Gamma^{\mathrm{c}}, T^{\mathrm{c}}, R_{\mathcal{G}}^{\mathrm{c}} T^{\mathrm{c}} R_{\mathcal{G}}^{\mathrm{c}}, \nu^{\mathrm{o}})$ *is a measured quantum groupoid denoted by* $\mathcal{G}^{\mathrm{c}}$ *and called the commutant of* $\mathcal{G}$. *The pseudo-multiplicative unitary* $\mathscr{W}_{\mathcal{G}^{\mathrm{c}}}$ *of* $\mathcal{G}^{\mathrm{c}}$ *is given by* $\mathscr{W}_{\mathcal{G}^{\mathrm{c}}} = ({}_{\widehat{\beta}}J_{\alpha} \otimes_{\alpha} J_{\widehat{\beta}})\mathscr{W}_{\mathcal{G}}({}_{\beta}J_{\widehat{\alpha}} \otimes_{\alpha} J_{\widehat{\beta}})$.

Notation 3.4. For a given measured quantum groupoid $\mathcal{G}$, we will need the following pseudo-multiplicative unitaries:

$$\mathscr{V} := \mathscr{W}_{\mathcal{G}}; \qquad \mathscr{V} := \mathscr{W}_{\widehat{(\mathcal{G}^{\mathrm{o}})}} = \mathscr{W}_{(\widehat{\mathcal{G}})^{\mathrm{c}}}; \qquad \widetilde{\mathscr{V}} := \mathscr{W}_{(\mathcal{G}^{\mathrm{o}})^{\mathrm{c}}}.$$

Convention 3.5. Henceforth, we will refer to $(\widehat{\mathcal{G}})^{\mathrm{c}}$ instead of $\widehat{\mathcal{G}}$ as the dual of $\mathcal{G}$ since this groupoid is better suited for right actions of $\mathcal{G}$. We have

$$(\widehat{\mathcal{G}})^{\mathrm{c}} = \big(N^{\mathrm{o}}, \widehat{M'}, \beta, \widehat{\alpha}, \widehat{\Gamma}^{\mathrm{c}}, \widehat{T}^{\mathrm{c}}, \widehat{T}^{\mathrm{c}\prime}, \nu^{\mathrm{o}}\big),$$

where the coproduct and the operator-valued weights are given by:

- $\widehat{\Gamma}^{\mathrm{c}}(x) = (\mathscr{W}_{(\widehat{\mathcal{G}})^{\mathrm{c}}})^*(1 \,{}_{\beta}\otimes_{\alpha}\, x)\mathscr{W}_{(\widehat{\mathcal{G}})^{\mathrm{c}}}$, for all $x \in \widehat{M'}$;
- $\widehat{T}^{\mathrm{c}} = C_{\widehat{M}} \circ \widehat{T} \circ C_{\widehat{M}}^{-1}$, where $C_{\widehat{M}} : \widehat{M} \to \widehat{M'}$; $x \mapsto \widehat{J}x^*\widehat{J}$;
- $\widehat{T}^{\mathrm{c}\prime} = R_{(\widehat{\mathcal{G}})^{\mathrm{c}}} \circ \widehat{T}^{\mathrm{c}} \circ R_{(\widehat{\mathcal{G}})^{\mathrm{c}}}$.

Note also that the commutant weight $\widehat{\varphi}^{\mathrm{c}} := \nu^{\mathrm{o}} \circ \beta^{-1} \circ \widehat{T}^{\mathrm{c}}$ derived from the weight $\widehat{\varphi}$ is left invariant for the coproduct $\widehat{\Gamma}^{\mathrm{c}}$. In the following, we will simply denote by $\widehat{\mathcal{G}}$ the dual groupoid of $\mathcal{G}$ (since no ambiguity will arise with the Pontryagin dual). Note that the bidual groupoid is $(\mathcal{G}^{\mathrm{o}})^{\mathrm{c}} = (\mathcal{G}^{\mathrm{c}})^{\mathrm{o}}$.

3.1. Case where the basis is finite-dimensional. In [11], De Commer provides an equivalent definition of a measured quantum groupoid on a finite basis. This definition is far more tractable since it allows us to circumvent the use of relative tensor products and fiber products.

In the following, we fix a finite-dimensional C*-algebra $N := \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$ endowed with the nonnormalized Markov trace $\epsilon := \bigoplus_{1 \leqslant l \leqslant k} n_l \cdot \mathrm{Tr}_l$, where Tr_l denotes the nonnormalized trace on $\mathrm{M}_{n_l}(\mathbb{C})$.

We refer to §A.2 of the Appendix for the definitions of $v_{\beta\alpha}$ and $q_{\beta\alpha}$. Let us a fix a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \epsilon)$. We have a unital normal $*$-isomorphism $M\,{}_\beta\!\star_\alpha M \to q_{\beta\alpha}(M \otimes M)q_{\beta\alpha}$; $x \mapsto v^*_{\beta\alpha} x v_{\beta\alpha}$ (cf. Proposition-Definition A.2.14). Let $\Delta : M \to M \otimes M$ be the (nonnecessarily unital) faithful normal $*$-homomorphism given by $\Delta(x) = v^*_{\beta\alpha}\Gamma(x)v_{\beta\alpha}$ for all $x \in M$. We have $\Delta(1) = q_{\beta\alpha}$. This has led De Commer to the following equivalent definition of a measured quantum groupoid on a finite basis.

Definition 3.1.1 (*cf.* 11.1.2 [11]). A measured quantum groupoid on the finite-dimensional basis N is an octuple $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$, where:

- M is a von Neumann algebra, $\alpha : N \to M$ and $\beta : N^{\mathrm{o}} \to M$ are unital faithful normal $*$-homomorphisms;
- $\Delta : M \to M \otimes M$ is a faithful normal $*$-homomorphism;
- $T : M_+ \to \alpha(N)^{\mathrm{ext}}_+$ and $T' : M_+ \to \beta(N^{\mathrm{o}})^{\mathrm{ext}}_+$ are n.s.f. operator-valued weights;

such that the following conditions are satisfied:

1. $[\alpha(n'), \beta(n^{\mathrm{o}})] = 0$, for all $n, n' \in N$;
2. $\Delta(1) = q_{\beta\alpha}$;
3. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;
4. $\Delta(\alpha(n)) = \Delta(1)(\alpha(n) \otimes 1)$ and $\Delta(\beta(n^{\mathrm{o}})) = \Delta(1)(1 \otimes \beta(n^{\mathrm{o}}))$, for all $n \in N$;
5. the n.s.f. weights φ and ψ on M given by $\varphi := \epsilon \circ \alpha^{-1} \circ T$ and $\psi := \epsilon \circ \beta^{-1} \circ T'$ satisfy:
$$T(x) = (\mathrm{id} \otimes \varphi)\Delta(x) \quad \text{for all } x \in \mathfrak{M}^+_T,$$
$$T'(x) = (\psi \otimes \mathrm{id})\Delta(x), \quad \text{for all } x \in \mathfrak{M}^+_{T'};$$
6. $\sigma^T_t \circ \beta = \beta$ and $\sigma^{T'}_t \circ \alpha = \alpha$, for all $t \in \mathbb{R}$.

Let us fix a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$.

Notation 3.1.2. Let us consider the injective bounded linear map
$$\iota^{\beta}_{\widehat{\alpha}\alpha} : \mathcal{B}(\mathscr{H}\,{}_{\widehat{\alpha}}\!\otimes_\beta \mathscr{H}, \mathscr{H}\,{}_\beta\!\otimes_\alpha \mathscr{H}) \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H}); \qquad X \mapsto v^*_{\beta\alpha} X v_{\widehat{\alpha}\beta}.$$
Similarly, we also define $\iota^{\alpha}_{\beta\widehat{\beta}}$ and $\iota^{\widehat{\alpha}}_{\widehat{\beta}\beta}$. Let
$$V := \iota^{\beta}_{\widehat{\alpha}\alpha}(\mathscr{V}), \qquad W := \iota^{\alpha}_{\beta\widehat{\beta}}(\widehat{\mathscr{V}}) \quad \text{and} \quad \widetilde{V} := \iota^{\widehat{\alpha}}_{\widehat{\beta}\beta}(\widetilde{\mathscr{V}}),$$
where $\mathscr{V} = \mathscr{W}_{\widehat{\mathcal{G}}}$, $\widehat{\mathscr{V}} = \mathscr{W}_{\mathcal{G}}$ and $\widetilde{\mathscr{V}} = \mathscr{W}_{(\mathcal{G}^{\mathrm{o}})^{\mathrm{c}}}$ (cf. Notation 3.4).

In what follows, we recall the main properties satisfied by V, W, and $\widetilde{V}$. The proof of the results below are derived from the properties satisfied by the pseudo-multiplicative unitaries $\mathscr{V}$, $\widehat{\mathscr{V}}$, and $\widetilde{\mathscr{V}}$ (cf. [17], §11 [11], and §2 [2]).

Proposition 3.1.3 (cf. 3.11 (iii), 3.12 (v), (vi) [17], 2.2 [2]). *The operators V, W and $\widetilde{V}$ are multiplicative partial isometries acting on $\mathscr{H} \otimes \mathscr{H}$ such that*:

1. $W = \Sigma(U \otimes 1)V(U^* \otimes 1)\Sigma$, $\widetilde{V} = \Sigma(1 \otimes U)V(1 \otimes U^*)\Sigma = (U \otimes U)W(U^* \otimes U^*)$;
2. $V^* = (J \otimes \widehat{J})V(J \otimes \widehat{J})$, $W^* = (\widehat{J} \otimes J)W(\widehat{J} \otimes J)$;
3. *the initial and final projections are given by*

$$V^*V = q_{\widehat{\alpha}\beta} = \widetilde{V}\widetilde{V}^*, \qquad W^*W = q_{\beta\alpha} = VV^*,$$
$$WW^* = q_{\alpha\widehat{\beta}} \quad \textit{and} \quad \widetilde{V}^*\widetilde{V} = q_{\widehat{\beta}\widehat{\alpha}}.$$

Proposition 3.1.4 (cf. 3.8, 3.12 [17]).

1. *The von Neumann algebra* M (*resp.*, $\widehat{M}$) *is the weak closure of* $\{(\mathrm{id} \otimes \omega)(W);\ \omega \in \mathcal{B}(\mathscr{H})_*\}$ (*resp.*, $\{(\omega \otimes \mathrm{id})(W);\ \omega \in \mathcal{B}(\mathscr{H})_*\}$).
2. *We have* $W \in M \otimes \widehat{M}$, $V \in \widehat{M}' \otimes M$, *and* $\widetilde{V} \in M' \otimes \widehat{M}'$. *In particular, we have the commutation relations* $[W_{12}, V_{23}] = 0$ *and* $[V_{12}, \widetilde{V}_{23}] = 0$.
3. *The coproduct* $\Delta : M \to M \otimes M$ *of* $\mathcal{G}$ (*resp.*, $\widehat{\Delta} : \widehat{M}' \to \widehat{M}' \otimes \widehat{M}'$ *of* $\widehat{\mathcal{G}}$) *satisfies*

$$\Delta(x) = V(x \otimes 1)V^* = W^*(1 \otimes x)W, \quad \textit{for all } x \in M$$
$$\big(\textit{resp.}, \ \widehat{\Delta}(x) = V^*(1 \otimes x)V = \widetilde{V}(x \otimes 1)\widetilde{V}^*, \quad \textit{for all } x \in \widehat{M}'\big).$$

Proposition 3.1.5 (cf. 3.2. (i), 3.6. (ii) [17] and 11.1.2 [11]). *For all* $n \in N$, *we have*:

1. $[V, \alpha(n) \otimes 1] = 0$, $[V, \widehat{\beta}(n^{\mathrm{o}}) \otimes 1] = 0$, $[V, 1 \otimes \widehat{\alpha}(n)] = 0$, $[V, 1 \otimes \widehat{\beta}(n^{\mathrm{o}})] = 0$;
2. $V(1 \otimes \alpha(n)) = (\widehat{\alpha}(n) \otimes 1)V$, $V(\beta(n^{\mathrm{o}}) \otimes 1) = (1 \otimes \beta(n^{\mathrm{o}}))V$;
3. $[W, \widehat{\beta}(n^{\mathrm{o}}) \otimes 1] = 0$, $[W, \widehat{\alpha}(n) \otimes 1] = 0$, $[W, 1 \otimes \beta(n^{\mathrm{o}})] = 0$, $[W, 1 \otimes \widehat{\alpha}(n)] = 0$;
4. $W(1 \otimes \widehat{\beta}(n^{\mathrm{o}})) = (\beta(n^{\mathrm{o}}) \otimes 1)W$, $W(\alpha(n) \otimes 1) = (1 \otimes \alpha(n))W$;
5. $[\widetilde{V}, \alpha(n) \otimes 1] = 0$, $[\widetilde{V}, \beta(n^{\mathrm{o}}) \otimes 1] = 0$, $[\widetilde{V}, 1 \otimes \alpha(n)] = 0$, $[\widetilde{V}, 1 \otimes \widehat{\beta}(n^{\mathrm{o}})] = 0$;
6. $\widetilde{V}(1 \otimes \beta(n^{\mathrm{o}})) = (\widehat{\beta}(n^{\mathrm{o}}) \otimes 1)\widetilde{V}$, $\widetilde{V}(\widehat{\alpha}(n) \otimes 1) = (1 \otimes \widehat{\alpha}(n))\widetilde{V}$.

Proposition 3.1.6 (cf. 11.1.4 [11]). *For all* $n \in N$, *we have*:

1. $W(\beta(n^{\mathrm{o}}) \otimes 1) = W(1 \otimes \alpha(n))$, $(1 \otimes \widehat{\beta}(n^{\mathrm{o}}))W = (\alpha(n) \otimes 1)W$;
2. $V(1 \otimes \beta(n^{\mathrm{o}})) = V(\widehat{\alpha}(n) \otimes 1)$, $(1 \otimes \alpha(n))V = (\beta(n^{\mathrm{o}}) \otimes 1)V$;
3. $\widetilde{V}(\widehat{\beta}(n^{\mathrm{o}}) \otimes 1) = \widetilde{V}(1 \otimes \widehat{\alpha}(n))$, $(1 \otimes \beta(n^{\mathrm{o}}))\widetilde{V} = (\widehat{\alpha}(n) \otimes 1)\widetilde{V}$.

3.2. Weak Hopf C*-algebras associated with a measured quantum groupoid on a finite basis. We recall the definitions and the main results concerning the weak Hopf C*-algebras associated with a measured quantum groupoid on a finite basis, cf. §11.2 [11] (with different notation and conventions; cf. §2.3 [2]). Let us fix a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$ on the finite-dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$.

Notation 3.2.1. With the notation of §3.1, we denote by S (resp., $\widehat{S}$) the norm closure of the subalgebra

$$\big\{(\omega \otimes \mathrm{id})(V);\ \omega \in \mathcal{B}(\mathscr{H})_*\big\} \qquad \big(\text{resp.}, \big\{(\mathrm{id} \otimes \omega)(V);\ \omega \in \mathcal{B}(\mathscr{H})_*\big\}\big).$$

According to §11.2 [11], we have the following statements:

- the Banach space S (resp., $\widehat{S}$) is a nondegenerate C*-subalgebra of $\mathcal{B}(\mathscr{H})$, weakly dense in M (resp., $\widehat{M'}$);
- the C*-algebra S (resp., $\widehat{S}$) is endowed with the faithful nondegenerate $*$-representations:

$$L : S \to \mathcal{B}(\mathscr{H}); \qquad x \mapsto x; \qquad R : S \to \mathcal{B}(\mathscr{H}); \qquad x \mapsto UL(x)U^*$$
$$\big(\text{resp., } \rho : \widehat{S} \to \mathcal{B}(\mathscr{H}); \qquad x \mapsto x; \qquad \lambda : \widehat{S} \to \mathcal{B}(\mathscr{H}); \qquad x \mapsto U\rho(x)U^*\big);$$

- $\alpha(N) \subset \mathcal{M}(S)$, $\beta(N^{\mathrm{o}}) \subset \mathcal{M}(S)$, $\beta(N^{\mathrm{o}}) \subset \mathcal{M}(\widehat{S})$ and $\widehat{\alpha}(N) \subset \mathcal{M}(\widehat{S})$;
- $V \in \mathcal{M}(\widehat{S} \otimes S)$, $W \in \mathcal{M}(S \otimes \lambda(\widehat{S}))$ and $\widetilde{V} \in \mathcal{M}(R(S) \otimes \widehat{S})$;
- Δ (resp., $\widehat{\Delta}$) restricts to a strictly continuous $*$-homomorphism $\delta : S \to \mathcal{M}(S \otimes S)$ (resp., $\widehat{\delta} : \widehat{S} \to \mathcal{M}(\widehat{S} \otimes \widehat{S})$), which uniquely extends to a strictly continuous $*$-homomorphism $\delta : \mathcal{M}(S) \to \mathcal{M}(S \otimes S)$ (resp., $\widehat{\delta} : \mathcal{M}(\widehat{S}) \to \mathcal{M}(\widehat{S} \otimes \widehat{S})$) satisfying $\delta(1_S) = q_{\beta\alpha}$ (resp., $\widehat{\delta}(1_{\widehat{S}}) = q_{\widehat{\alpha}\beta}$);
- δ (resp., $\widehat{\delta}$) is coassociative and satisfies $[\delta(S)(1_S \otimes S)] = \delta(1_S)(S \otimes S) = [\delta(S)(S \otimes 1_S)]$ (resp., $[\widehat{\delta}(\widehat{S})(1_{\widehat{S}} \otimes \widehat{S})] = \widehat{\delta}(1_{\widehat{S}})(\widehat{S} \otimes \widehat{S}) = [\widehat{\delta}(\widehat{S})(\widehat{S} \otimes 1_{\widehat{S}})]$);
- the unital faithful $*$-homomorphisms $\alpha : N \to \mathcal{M}(S)$ and $\beta : N^{\mathrm{o}} \to \mathcal{M}(S)$ satisfy

$$\delta\big(\alpha(n)\big) = \delta(1_S)\big(\alpha(n) \otimes 1_S\big) \quad \text{and}$$
$$\delta\big(\beta\big(n^{\mathrm{o}}\big)\big) = \delta(1_S)\big(1_S \otimes \beta\big(n^{\mathrm{o}}\big)\big), \quad \text{for all } n \in N;$$

- the unital faithful $*$-homomorphisms $\beta : N^{\mathrm{o}} \to \mathcal{M}(\widehat{S})$ and $\widehat{\alpha} : N \to \mathcal{M}(\widehat{S})$ satisfy

$$\widehat{\delta}\big(\beta\big(n^{\mathrm{o}}\big)\big) = \widehat{\delta}(1_{\widehat{S}})\big(\beta\big(n^{\mathrm{o}}\big) \otimes 1_{\widehat{S}}\big) \quad \text{and}$$
$$\widehat{\delta}\big(\widehat{\alpha}(n)\big) = \widehat{\delta}(1_{\widehat{S}})\big(1_{\widehat{S}} \otimes \widehat{\alpha}(n)\big), \quad \text{for all } n \in N.$$

Definition 3.2.2. With the above notation, we call the pair (S, δ) (resp., $(\widehat{S}, \widehat{\delta})$) the weak Hopf C*-algebra (resp., dual weak Hopf C*-algebra) associated with the measured quantum groupoid $\mathcal{G}$.

Remark 3.2.3. With the notations of the above definition, the pair $(\widehat{S}, \widehat{\delta})$ is the weak Hopf C*-algebra of $\widehat{\mathcal{G}}$ while its dual weak Hopf C*-algebra is the pair $(R(S), \delta_R)$, where $R(S) = USU^*$ and the coproduct δ_R is given by $\delta_R(y) := \widetilde{V}^*(1 \otimes y)\widetilde{V}$ for all $y \in R(S)$.

3.3. Measured quantum groupoid associated with a monoidal equivalence. We will recall the construction of the measured quantum groupoid associated with a monoidal equivalence between two locally compact quantum groups provided by De Commer [11], [12]. First of all, we will need to recall the definitions and the crucial results of De Commer [11], [12].

DEFINITION 3.3.1. Let $\mathbb{G}$ be a locally compact quantum group. A right (resp., left) Galois action of $\mathbb{G}$ on a von Neumann algebra N is an ergodic integrable right (resp., left) action $\alpha_N : N \to N \otimes \mathrm{L}^\infty(\mathbb{G})$ (resp., $\gamma_N : N \to \mathrm{L}^\infty(\mathbb{G}) \otimes N$) such that the crossed product $N \rtimes_{\alpha_N} \mathbb{G}$ (resp., $\mathbb{G}_{\gamma_N} \ltimes N$) is a type I factor. Then the pair (N, α_N) (resp., (N, γ_N)) is called a right (resp., left) Galois object for $\mathbb{G}$.

Let $\mathbb{G}$ be a locally compact quantum group and let us fix a right Galois object (N, α_N) for $\mathbb{G}$. In his thesis, De Commer was able to build a locally compact quantum group $\mathbb{H}$ equipped with a left Galois action γ_N on N commuting with α_N, *i.e.* $(\mathrm{id} \otimes \alpha_N)\gamma_N = (\gamma_N \otimes \mathrm{id})\alpha_N$. This construction is called the *reflection technique* and $\mathbb{H}$ is called the *reflected locally compact quantum group across* (N, α_N).

In a canonical way, he was also able to associate a right Galois object (O, α_O) for $\mathbb{H}$ and a left Galois action $\gamma_O : O \to \mathrm{L}^\infty(\mathbb{G}) \otimes O$ of $\mathbb{G}$ on O commuting with α_O. Finally, De Commer has built a measured quantum groupoid

$$\mathcal{G}_{\mathbb{H},\mathbb{G}} = \big(\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon\big),$$

where: $M = \mathrm{L}^\infty(\mathbb{H}) \oplus N \oplus O \oplus \mathrm{L}^\infty(\mathbb{G})$; $\Delta : M \to M \otimes M$ is made up of the coactions and coproducts of the constituents of M; the operator-valued weights T and T' are given by the invariants weights; the nonnormalized Markov trace ϵ on $\mathbb{C}^2$ is simply given by $\epsilon(a, b) = a + b$ for all $(a, b) \in \mathbb{C}^2$. Moreover, the source and target maps α and β have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$.

Conversely, if $\mathcal{G} = (\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon)$ is a measured quantum groupoid whose source and target maps have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$, then $\mathcal{G}$ is of the form $\mathcal{G}_{\mathbb{H},\mathbb{G}}$ in a unique way, where $\mathbb{H}$ and $\mathbb{G}$ are locally compact quantum groups canonically associated with $\mathcal{G}$.

In what follows, we fix a measured quantum groupoid $\mathcal{G} = (\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon)$ whose source and target maps have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$. It is worth noticing that for such a groupoid we have the following.

LEMMA 3.3.2 (cf. 2.21 [2]). *$\widehat{\alpha} = \beta$ and $\widehat{\beta} = \alpha$.*

Following the notation introduced in [11], we recall the precise description of the left and right regular representations W and V of $\mathcal{G}$ introduced in the previous section. We identify M with its image by π in $\mathcal{B}(\mathscr{H})$, where $(\mathscr{H}, \pi, \Lambda)$ is the G.N.S. construction for M endowed with the n.s.f. weight $\varphi = \epsilon \circ \alpha^{-1} \circ T$. We also consider the n.s.f. weight $\psi = \epsilon \circ \beta^{-1} \circ T'$. Denote by $(\varepsilon_1, \varepsilon_2)$ the standard basis of the vector space $\mathbb{C}^2$.

NOTATION 3.3.3. Let us introduce some useful notation and make some remarks concerning them.

- For $i, j = 1, 2$, we define the following nonzero central self-adjoint projection of M:

$$p_{ij} := \alpha(\varepsilon_i)\beta(\varepsilon_j).$$

It follows from $\beta(\varepsilon_1)+\beta(\varepsilon_2)=1_M$ and $\alpha(\varepsilon_1)+\alpha(\varepsilon_2)=1_M$ that

$$\alpha(\varepsilon_i)=p_{i1}+p_{i2} \quad \text{and} \quad \beta(\varepsilon_j)=p_{1j}+p_{2j}, \quad \text{for all } i,j=1,2.$$

- We have

$$\Delta(1)=\alpha(\varepsilon_1)\otimes\beta(\varepsilon_1)+\alpha(\varepsilon_2)\otimes\beta(\varepsilon_2) \quad \text{and}$$
$$\widehat{\Delta}(1)=\beta(\varepsilon_1)\otimes\beta(\varepsilon_1)+\beta(\varepsilon_2)\otimes\beta(\varepsilon_2)$$

since $\widehat{\alpha}=\beta$.
- Let $M_{ij}:=p_{ij}M$, for $i,j=1,2$. Then M_{ij} is a nonzero von Neumann subalgebra of M.
- Let $\mathscr{H}_{ij}:=p_{ij}\mathscr{H}$, for $i,j=1,2$. Then $\mathscr{H}_{ij}$ is a nonzero Hilbert subspace of $\mathscr{H}$ for all $i,j=1,2$.
- Let $\varphi_{ij}:=\varphi\restriction_{(M_{ij})_+}$ and $\psi_{ij}:=\psi\restriction_{(M_{ij})_+}$, for $i,j=1,2$. Then φ_{ij} and ψ_{ij} are n.s.f. weights on M_{ij}.
- For all $i,j,k=1,2$, we denote by $\Delta_{ij}^k: M_{ij}\to M_{ik}\otimes M_{kj}$ the unital normal $*$-homomorphism given by

$$\Delta_{ij}^k(x):=(p_{ik}\otimes p_{kj})\Delta(x), \quad \text{for all } x\in M_{ij}.$$

- We have $Jp_{kl}=p_{kl}J$, $\widehat{J}p_{kl}=p_{lk}\widehat{J}$ and $Up_{kl}=p_{lk}U$ for all $k,l=1,2$. We define the anti-unitaries $J_{kl}:\mathscr{H}_{kl}\to\mathscr{H}_{kl}$, $\widehat{J}_{kl}:\mathscr{H}_{kl}\to\mathscr{H}_{lk}$ and the unitary $U_{kl}:\mathscr{H}_{kl}\to\mathscr{H}_{lk}$ by setting $J_{kl}=p_{kl}Jp_{kl}$, $\widehat{J}_{kl}=p_{lk}\widehat{J}p_{kl}$ and $U_{kl}=p_{lk}Up_{kl}=\widehat{J}_{kl}J_{kl}$.
- For $i,j,k,l=1,2$, let $\Sigma_{ij\otimes kl}:=\Sigma_{\mathscr{H}_{ij}\otimes\mathscr{H}_{kl}}:\mathscr{H}_{ij}\otimes\mathscr{H}_{kl}\to\mathscr{H}_{kl}\otimes\mathscr{H}_{ij}$.

We readily obtain:

$$M=\bigoplus_{i,j=1,2}M_{ij}; \qquad \mathscr{H}=\bigoplus_{i,j=1,2}\mathscr{H}_{ij};$$
$$\Delta(p_{ij})=p_{i1}\otimes p_{1j}+p_{i2}\otimes p_{2j}, \quad \text{for all } i,j=1,2.$$

Note that in terms of the parts Δ_{ij}^k of Δ, the coassociativity condition reads as follows:

$$\big(\Delta_{ik}^l\otimes\mathrm{id}_{M_{kj}}\big)\Delta_{ij}^k=\big(\mathrm{id}_{M_{il}}\otimes\Delta_{lj}^k\big)\Delta_{ij}^l, \quad \text{for all } i,j,k,l=1,2.$$

The G.N.S. representation for (M_{ij},φ_{ij}) is obtained by restriction of the G.N.S. representation of (M,φ) to M_{ij}. In particular, the G.N.S. space $\mathscr{H}_{\varphi_{ij}}$ is identified with $\mathscr{H}_{ij}$.

PROPOSITION 3.3.4. *For all $i,j,k,l=1,2$, we have*:

$$(p_{ij}\otimes 1)V(p_{kl}\otimes 1)=\delta_k^i(p_{ij}\otimes p_{jl})V(p_{il}\otimes p_{jl});$$
$$(1\otimes p_{ij})W(1\otimes p_{kl})=\delta_j^l(p_{ik}\otimes p_{ij})W(p_{ik}\otimes p_{kj});$$
$$(1\otimes p_{ji})\widetilde{V}(1\otimes p_{lk})=\delta_j^l(p_{ki}\otimes p_{ji})\widetilde{V}(p_{ki}\otimes p_{jk}).$$

NOTATION 3.3.5. The operators V, W, and $\widetilde{V}$ each splits up into eight unitaries

$$V^i_{jl} : \mathscr{H}_{il} \otimes \mathscr{H}_{jl} \to \mathscr{H}_{ij} \otimes \mathscr{H}_{jl},$$
$$W^j_{ik} : \mathscr{H}_{ik} \otimes \mathscr{H}_{kj} \to \mathscr{H}_{ik} \otimes \mathscr{H}_{ij} \quad \text{and}$$
$$\widetilde{V}^j_{ki} : \mathscr{H}_{ki} \otimes \mathscr{H}_{jk} \to \mathscr{H}_{ki} \otimes \mathscr{H}_{ji}$$

for $i,j,k,l = 1,2$, given by $V^i_{jl} = (p_{ij} \otimes p_{jl})V(p_{il} \otimes p_{jl})$, $W^j_{ik} = (p_{ik} \otimes p_{ij})W(p_{ik} \otimes p_{kj})$ and $\widetilde{V}^j_{ki} = (p_{ki} \otimes p_{ji})\widetilde{V}(p_{ki} \otimes p_{jk})$.

Let $i,j,k,l,l' = 1,2$. These unitaries are related to each other by the following relations (cf. Proposition 3.1.3):

$$W^j_{ik} = \Sigma_{ij\otimes ik}(U_{ji} \otimes 1)V^j_{ik}\big(U^*_{jk} \otimes 1\big)\Sigma_{ik\otimes kj};$$
$$\widetilde{V}^j_{ki} = \Sigma_{ji\otimes ki}(1 \otimes U_{ik})V^j_{ik}\big(\otimes U^*_{ik}\big)\Sigma_{ki\otimes jk};$$
$$\widetilde{V}^j_{ki} = (U_{ik} \otimes U_{ij})W^j_{ik}\big(U^*_{ik} \otimes U^*_{kj}\big).$$

Furthermore, we also have:

$$\big(V^i_{jl}\big)^* = (J_{il} \otimes \widehat{J}_{lj})V^i_{lj}(J_{ij} \otimes \widehat{J}_{jl}) \quad \text{and}$$
$$\big(W^l_{ik}\big)^* = (\widehat{J}_{ki} \otimes J_{kj})W^j_{ki}(\widehat{J}_{ik} \otimes J_{ij}).$$

Moreover, these unitaries satisfy the following pentagonal equations:

$$\big(V^i_{jk}\big)_{12}\big(V^i_{kl}\big)_{13}\big(V^j_{kl}\big)_{23} = \big(V^j_{kl}\big)_{23}\big(V^i_{jl}\big)_{12};$$
$$\big(W^k_{ij}\big)_{12}\big(W^l_{ij}\big)_{13}\big(W^l_{jk}\big)_{23} = \big(W^l_{ik}\big)_{23}\big(W^k_{ij}\big)_{12};$$
$$\big(\widetilde{V}^k_{ji}\big)_{12}\big(\widetilde{V}^l_{ji}\big)_{13}\big(\widetilde{V}^l_{kj}\big)_{23} = \big(\widetilde{V}^l_{ki}\big)_{23}\big(\widetilde{V}^k_{ji}\big)_{12}.$$

We also have the following commutation relations:

$$\big(V^l_{kj}\big)_{23}\big(W^j_{ll'}\big)_{12} = \big(W^k_{ll'}\big)_{12}\big(V^{l'}_{kj}\big)_{23};$$
$$\big(V^l_{ki}\big)_{12}\big(\widetilde{V}^j_{ki}\big)_{23} = \big(\widetilde{V}^j_{ki}\big)_{23}\big(V^l_{ki}\big)_{12}.$$

Furthermore, we have

$$\Delta^k_{ij}(x) = \big(W^j_{ik}\big)^*(1 \otimes x)W^j_{ik} = V^i_{kj}(x \otimes 1)\big(V^i_{kj}\big)^*, \quad \text{for all } x \in M_{ij}.$$

Note that for all $\omega \in \mathcal{B}(\mathscr{H})_*$ we have:

$$(\mathrm{id} \otimes p_{jl}\omega p_{jl})\big(V^i_{jl}\big) = p_{ij}(\mathrm{id} \otimes \omega)(V)p_{il};$$
$$(p_{ik}\omega p_{ik} \otimes \mathrm{id})\big(W^j_{ik}\big) = p_{ij}(\omega \otimes \mathrm{id})(W)p_{kj};$$
$$(p_{ki}\omega p_{ki} \otimes \mathrm{id})\big(\widetilde{V}^j_{ki}\big) = p_{ji}(\mathrm{id} \otimes \omega)(\widetilde{V})p_{jk}.$$

PROPOSITION 3.3.6. *Let $i,j = 1,2$ such that $i \neq j$. With the notation of Notation 3.3.3, we have*:

1. $\mathbb{G}_i := (M_{ii}, \Delta^i_{ii}, \varphi_{ii}, \psi_{ii})$ *is a locally compact quantum group whose left (resp., right) regular representation is* W^i_{ii} *(resp.,* V^i_{ii}*);*

2. $(M_{ij}, \Delta_{ij}^{j})$ *is a right Galois object for* $\mathbb{G}_j$ *whose canonical implementation is* V_{jj}^{i};
3. $(M_{ij}, \Delta_{ij}^{i})$ *is a left Galois object for* $\mathbb{G}_i$ *whose canonical implementation is* W_{ii}^{j};
4. *the actions* Δ_{ij}^{j} *and* Δ_{ij}^{i} *on* M_{ij} *commute*;
5. *the Galois isometry associated with the right Galois object* $(M_{ij}, \Delta_{ij}^{j})$ *for* $\mathbb{G}_j$ (*cf.* 6.4.1, 6.4.2 [11]) *is the unitary* $\Sigma_{ij\otimes jj}(W_{ij}^{j})^{*}\Sigma_{ij\otimes ij}$.

Definition 3.3.7. A measured quantum groupoid $(\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon)$ such that the source and target maps have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$ will be denoted by $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$, where $\mathbb{G}_i = (M_{ii}, \Delta_{ii}^{i}, \varphi_{ii}, \psi_{ii})$ (cf. Proposition 3.3.6) and will be called a colinking measured quantum groupoid.

Definition 3.3.8. Let $\mathbb{G}$ and $\mathbb{H}$ be two locally compact quantum groups. We say that $\mathbb{G}$ and $\mathbb{H}$ are monoidally equivalent if there exists a colinking measured quantum groupoid $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ between two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ such that $\mathbb{H}$ (resp., $\mathbb{G}$) is isomorphic to $\mathbb{G}_1$ (resp., $\mathbb{G}_2$).

Let (S, δ) be the weak Hopf C*-algebra associated with $\mathcal{G}$. Note that

$$p_{ij} = \alpha(\varepsilon_i)\beta(\varepsilon_j) \in \mathcal{Z}\big(\mathcal{M}(S)\big), \quad \text{for all } i, j = 1, 2.$$

Notation 3.3.9. Let us recall the notation below (cf. 2.26 [2]).

1. Let $S_{ij} := p_{ij}S$, for $i, j = 1, 2$. Then, S_{ij} is a C*-algebra (actually a closed two-sided ideal) of S weakly dense in M_{ij}.
2. For $i, j, k = 1, 2$, let $\iota_{ij}^{k} : \mathcal{M}(S_{ik} \otimes S_{kj}) \to \mathcal{M}(S \otimes S)$ be the unique strictly continuous extension of the inclusion map $S_{ik} \otimes S_{kj} \subset S \otimes S$ satisfying $\iota_{ij}^{k}(1_{S_{ik}\otimes S_{kj}}) = p_{ik} \otimes p_{kj}$.
3. Let $\delta_{ij}^{k} : S_{ij} \to \mathcal{M}(S_{ik} \otimes S_{kj})$ be the unique $*$-homomorphism such that

$$\iota_{ij}^{k} \circ \delta_{ij}^{k}(x) = (p_{ik} \otimes p_{kj})\delta(x), \quad \text{for all } x \in S_{ij}.$$

With these notation, we have the following.

Proposition 3.3.10 (cf. 7.4.13, 7.4.14 [11], 2.27 [2]). *Let* $i, j, k, l = 1, 2$.

1. $(\delta_{ik}^{l} \otimes \mathrm{id}_{S_{kj}})\delta_{ij}^{k} = (\mathrm{id}_{S_{il}} \otimes \delta_{lj}^{k})\delta_{ij}^{l}$.
2. $\delta_{ij}^{k}(x) = (W_{ik}^{j})^{*}(1_{\mathscr{H}_{ik}} \otimes x)W_{ik}^{j} = V_{kj}^{i}(x \otimes 1_{\mathscr{H}_{kj}})(V_{kj}^{i})^{*}$, *for all* $x \in S_{ij}$.
3. $[\delta_{ij}^{k}(S_{ij})(1_{S_{ik}} \otimes S_{kj})] = S_{ik} \otimes S_{kj} = [\delta_{ij}^{k}(S_{ij})(S_{ik} \otimes 1_{S_{kj}})]$. *In particular, we have*

$$S_{kj} = \big[(\mathrm{id}_{S_{ik}} \otimes \omega)\delta_{ij}^{k}(x);\, x \in S_{ij}, \omega \in \mathcal{B}(\mathscr{H}_{kj})_{*}\big].$$

4. *The pair* $(S_{jj}, \delta_{jj}^{j})$ *is the Hopf C*-algebra associated with* $\mathbb{G}_j$.

4. Contributions to the notions of semi-regularity and regularity

The notion of regular measured quantum groupoid has been introduced in [16] and studied in the compact case. Note that this notion has been generalized in the setting of pseudo-multiplicative unitaries; cf. [24], [23]. The notion of semi-regular measured quantum groupoid has been introduced in [2], [9], where the notions of regularity and semi-regularity have been studied in the case of a finite-dimensional basis.

In this chapter, we fix a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$ on the finite-dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$ and we use all the notation introduced in §3.1, §3.2. In the Appendix (cf. Definition A.2.1), for any $\xi \in \mathscr{H}$ we have given the definition of the operator

$$R_\xi^\alpha \in \mathcal{B}(\mathscr{H}_\epsilon, \mathscr{H}) \qquad \big(\text{resp., } L_\xi^\beta \in \mathcal{B}(\mathscr{H}_\epsilon, \mathscr{H})\big)$$

and the definition of the weakly dense ideal of $\alpha(N)'$ (resp., $\beta(N^{\mathrm{o}})'$)

$$\mathcal{K}_\alpha := \big[R_\xi^\alpha \big(R_\eta^\alpha\big)^*; \xi, \eta \in \mathscr{H}\big] \qquad \big(\text{resp., } \mathcal{K}_\beta := \big[L_\xi^\beta \big(L_\eta^\beta\big)^*; \xi, \eta \in \mathscr{H}\big]\big).$$

Note that $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ are C*-subalgebras of $\mathcal{K} := \mathcal{K}(\mathscr{H})$.

We first recall the following important consequence of the irreducibility (cf. 2.13 [2]) of $\mathcal{G}$.

Proposition 4.1 (cf. 2.15 [2]). *The Banach spaces* $[S\widehat{S}]$ *and* $\mathcal{C}(V)$ (*cf.* 2.1.2) *are C*-algebras and we have* $[S\widehat{S}] = U\mathcal{C}(V)U^*$.

Definition 4.2 (*cf.* 4.7 [16], 2.37 [2]). The groupoid $\mathcal{G}$ is said to be semi-regular (resp., regular) if we have $\mathcal{K}_\beta \subset \mathcal{C}(V)$ (resp., $\mathcal{K}_\beta = \mathcal{C}(V)$).

Proposition 4.3 (cf. 2.8 [2], 3.2.8 [9]). *The following statements are equivalent*:

(i) $\mathcal{G}$ *is semi-regular* (*resp.*, *regular*), *that is*, $\mathcal{K}_\alpha \subset \mathcal{C}(W)$ (*resp.*, $\mathcal{K}_\alpha = \mathcal{C}(W)$);
(ii) $\widehat{\mathcal{G}}$ *is semi-regular* (*resp.*, *regular*), *that is*, $\mathcal{K}_\beta \subset \mathcal{C}(V)$ (*resp.*, $\mathcal{K}_\beta = \mathcal{C}(V)$);
(iii) $(\mathcal{G}^{\mathrm{o}})^{\mathrm{c}}$ *is semi-regular* (*resp.*, *regular*), *that is*, $\mathcal{K}_{\widehat{\alpha}} \subset \mathcal{C}(\widetilde{V})$ (*resp.*, $\mathcal{K}_{\widehat{\alpha}} = \mathcal{C}(\widetilde{V})$).

Proposition 4.4 (cf. 2.8 [2], 3.2.9 [9]). *The following statements are equivalent*:

(i) $\mathcal{G}$ *is semi-regular* (*resp.*, *regular*);
(ii) $\mathcal{K}_{\widehat{\beta}} \subset [\widehat{S}S]$ (*resp.*, $\mathcal{K}_{\widehat{\beta}} = [\widehat{S}S]$);
(iii) $\mathcal{K}_\alpha \subset [R(S)\widehat{S}]$ (*resp.*, $\mathcal{K}_\alpha = [R(S)\widehat{S}]$);
(iv) $\mathcal{K}_{\widehat{\alpha}} \subset [S\lambda(\widehat{S})]$ (*resp.*, $\mathcal{K}_{\widehat{\alpha}} = [S\lambda(\widehat{S})]$).

In particular, if $\mathcal{G}$ *is regular we have* $[\widehat{S}S] \subset \mathcal{K}$, $[R(S)\widehat{S}] \subset \mathcal{K}$ *and* $[S\lambda(\widehat{S})] \subset \mathcal{K}$ (*and also* $\mathcal{C}(V) \subset \mathcal{K}$, $\mathcal{C}(W) \subset \mathcal{K}$ *and* $\mathcal{C}(\widetilde{V}) \subset \mathcal{K}$).

The semi-regularity and the regularity of colinking measured quantum groupoids have been treated in detail in §2.5 [2].

THEOREM 4.5 (cf. 2.45 [2]). *Let $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ be a colinking measured quantum groupoid associated with two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. The groupoid $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ is semi-regular (resp., regular) if, and only if, $\mathbb{G}_1$ and $\mathbb{G}_2$ are semi-regular (resp., regular).*

In the following, we use the multi-index notation introduced in the Appendix of this article (cf. Notations A.2.21, A.2.22, and Remarks A.2.23) with $\gamma := \alpha$ and $\pi := \beta$.

LEMMA 4.6. *For all $\xi, \eta \in \mathscr{H}$, we have*

$$R^{\alpha}_{\xi}\big(R^{\alpha}_{\eta}\big)^{*} = \sum_{I\in\mathscr{I}} n_I^{-1}\cdot e_I\theta_{\xi,\eta}e_{\overline{I}} \quad \textit{and} \quad L^{\beta}_{\xi}\big(L^{\beta}_{\eta}\big)^{*} = \sum_{I\in\mathscr{I}} n_I^{-1}\cdot f_I\theta_{\xi,\eta}f_{\overline{I}}.$$

Proof. For $\eta,\zeta\in\mathscr{H}$, let $X_{\eta,\zeta}\in N$ be defined by $X_{\eta,\zeta} := \sum_{I\in\mathscr{I}} n_I^{-1}\langle e_I\eta,\zeta\rangle\cdot\varepsilon_I$. For all $x\in N$ and $\eta\in\mathscr{H}$, we have $R^{\alpha}_{\eta}\Lambda_\epsilon(x) = \alpha(x)\eta = \sum_{I\in\mathscr{I}} x_I\cdot e_I\eta$. Let $\eta,\zeta\in\mathscr{H}$, and $I\in\mathscr{I}$. $\langle (R^{\alpha}_{\eta})^{*}\zeta, \Lambda_\epsilon(\varepsilon_I)\rangle = \langle\zeta, e_I\eta\rangle$. By disjunction elimination, we prove that $\epsilon(\varepsilon_{JI}) = \delta^{J}_{\overline{I}} n_I$ for all $I,J\in\mathscr{I}$. On the other hand, we have

$$\begin{aligned}
\big\langle \Lambda_\epsilon(X_{\eta,\zeta}), \Lambda_\epsilon(\varepsilon_I)\big\rangle &= \epsilon\big(X^{*}_{\eta,\zeta}\varepsilon_I\big)\\
&= \epsilon\Big(\sum_{J\in\mathscr{I}} n_J^{-1}\overline{\langle e_{\overline{J}}\eta,\zeta\rangle}\cdot\varepsilon_{JI}\Big)\\
&= \sum_{J\in\mathscr{I}} n_J^{-1}\langle\zeta, e_{\overline{J}}\eta\rangle\epsilon(\varepsilon_{JI})\\
&= \langle\zeta, e_I\eta\rangle.
\end{aligned}$$

Hence, $\langle (R^{\alpha}_{\eta})^{*}\zeta,\Lambda_\epsilon(\varepsilon_I)\rangle = \langle\Lambda_\epsilon(X_{\eta,\zeta}),\Lambda_\epsilon(\varepsilon_I)\rangle$. Hence, $(R^{\alpha}_{\eta})^{*}\zeta = \Lambda_\epsilon(X_{\eta,\zeta})$ for all $\eta,\zeta\in\mathscr{H}$. Let $\xi,\eta\in\mathscr{H}$. For all $\zeta\in\mathscr{H}$, we have

$$\begin{aligned}
R^{\alpha}_{\xi}\big(R^{\alpha}_{\eta}\big)^{*}\zeta = R^{\alpha}_{\xi}\Lambda_\epsilon(X_{\eta,\zeta}) &= \sum_{I\in\mathscr{I}}(X_{\eta,\zeta})_I\cdot e_I\xi\\
&= \sum_{I\in\mathscr{I}} n_I^{-1}\langle\eta, e_{\overline{I}}\zeta\rangle\cdot e_I\xi\\
&= \sum_{I\in\mathscr{I}} n_I^{-1}\theta_{e_I\xi,e_I\eta}(\zeta).
\end{aligned}$$

Hence, $R^{\alpha}_{\xi}(R^{\alpha}_{\eta})^{*} = \sum_{I\in\mathscr{I}} n_I^{-1}\cdot e_I\theta_{\xi,\eta}e_{\overline{I}}$. The second formula is proved in a similar way. □

We refer to Proposition-Definition A.2.15 and Proposition A.2.18 for the definition of the operators q_α, q_β and $q_{\widehat{\alpha}}$. Propositions 4.7, 4.8, and 4.9 below have to be compared with their corresponding statements in the quantum group case; cf. 3.2 b), 3.6 b), and 3.6 d) [4].

PROPOSITION 4.7. *The following statements are equivalent*:

(i) *$\mathcal{G}$ is regular (resp., semi-regular)*;

(ii) $[(\mathcal{K}\otimes 1)W(1\otimes\mathcal{K})] = [(\mathcal{K}\otimes 1)q_\alpha(1\otimes\mathcal{K})]$ (*resp.*, $\supset [(\mathcal{K}\otimes 1)q_\alpha(1\otimes\mathcal{K})]$);
(iii) $[(\mathcal{K}\otimes 1)V(1\otimes\mathcal{K})] = [(\mathcal{K}\otimes 1)q_\beta(1\otimes\mathcal{K})]$ (*resp.*, $\supset [(\mathcal{K}\otimes 1)q_\beta(1\otimes\mathcal{K})]$);
(iv) $[(\mathcal{K}\otimes 1)\widetilde{V}(1\otimes\mathcal{K})] = [(\mathcal{K}\otimes 1)q_{\widehat{\alpha}}(1\otimes\mathcal{K})]$ (*resp.*, $\supset [(\mathcal{K}\otimes 1)q_{\widehat{\alpha}}(1\otimes\mathcal{K})]$).

Proof. It is known that $\mathcal{G}$ is regular (resp., semi-regular) if, and only if, $\widehat{\mathcal{G}}$ is regular (resp., semi-regular). Therefore, it suffices to prove that (i) is equivalent to (ii). We have

$$\big[(\mathcal{K}\otimes 1)W(1\otimes\mathcal{K})\big] = \Sigma\big(\mathcal{C}(W)\otimes\mathcal{K}\big) \quad \text{(cf. 3.1 [4])}$$

($\mathcal{C}(W)$ is a C*-algebra regardless of the regularity of $\mathcal{G}$). Note that $\mathcal{G}$ is regular (resp., semi-regular) if, and only if, $\Sigma(\mathcal{C}(W)\otimes\mathcal{K}) = \Sigma(\mathcal{K}_\alpha\otimes\mathcal{K})$ (resp., $\Sigma(\mathcal{C}(W)\otimes\mathcal{K}) \supset \Sigma(\mathcal{K}_\alpha\otimes\mathcal{K})$). Let $\xi,\eta,\zeta,\chi\in\mathscr{H}$. We have $e_I\theta_{\xi,\eta}e_{\overline{I}}\otimes\theta_{\zeta,\chi} = \theta_{e_I\xi,e_I\eta}\otimes\theta_{\zeta,\chi} = \theta_{e_I\xi\otimes\zeta,e_I\eta\otimes\chi}$, for all $I\in\mathscr{I}$. Hence, $\Sigma(e_I\theta_{\xi,\eta}e_{\overline{I}}\otimes\theta_{\zeta,\chi}) = \theta_{\zeta\otimes e_I\xi,e_I\eta\otimes\chi} = \theta_{\zeta,e_I\eta}\otimes\theta_{e_I\xi,\chi} = \theta_{\zeta,\eta}e_{\overline{I}}\otimes e_I\theta_{\xi,\chi}$ for all $I\in\mathscr{I}$. By Lemma 4.6, we obtain

$$\Sigma\big(R^\alpha_\xi\big(R^\alpha_\eta\big)^*\otimes\theta_{\zeta,\chi}\big) = (\theta_{\zeta,\eta}\otimes 1)q_\alpha(1\otimes\theta_{\xi,\chi}).$$

Hence, $\Sigma(\mathcal{K}_\alpha\otimes\mathcal{K}) = [(\mathcal{K}\otimes 1)q_\alpha(1\otimes\mathcal{K})]$ and the equivalence ((i) $\Leftrightarrow$ (ii)) is proved. □

PROPOSITION 4.8. *If $\mathcal{G}$ is regular* (*resp.*, *semi-regular*), *we have*:

1. $[(S\otimes 1)W(1\otimes\mathcal{K})] = [(S\otimes 1)q_\alpha(1\otimes\mathcal{K})]$ (*resp.*, $\supset [(S\otimes 1)q_\alpha(1\otimes\mathcal{K})]$);
2. $[(\mathcal{K}\otimes 1)V(1\otimes S)] = [(\mathcal{K}\otimes 1)q_\beta(1\otimes S)]$ (*resp.*, $\supset [(\mathcal{K}\otimes 1)q_\beta(1\otimes S)]$);
3. $[(R(S)\otimes 1)\widetilde{V}(1\otimes\mathcal{K})] = [(R(S)\otimes 1)q_{\widehat{\alpha}}(1\otimes\mathcal{K})]$ (*resp.*, $\supset [(R(S)\otimes 1)q_{\widehat{\alpha}}(1\otimes\mathcal{K})]$).

Proof. Assume that $\mathcal{G}$ is regular (resp., semi-regular). Let us prove the first statement. The others will be obtained by using similar arguments. Let $a,b\in\mathcal{K}$, $\omega\in\mathcal{B}(\mathscr{H})_*$ and $y = (\mathrm{id}\otimes a\omega)(W)$. We have

$$\begin{aligned}(y\otimes 1)W(1\otimes b) &= (\mathrm{id}\otimes\omega\otimes\mathrm{id})\big(W_{12}W_{13}(1\otimes a\otimes b)\big)\\ &= (\mathrm{id}\otimes\omega\otimes\mathrm{id})\big(W_{23}W_{12}\big(1\otimes W^*(a\otimes b)\big)\big).\end{aligned}$$

However, $W^*(\mathcal{K}\otimes\mathcal{K}) = q_{\beta\alpha}(\mathcal{K}\otimes\mathcal{K})$. Moreover, since $[W, 1\otimes\beta(n^{\mathrm{o}})] = 0$ for all $n\in N$, we have $[W_{12}, q_{\beta\alpha,23}] = 0$. Hence, $W_{23}W_{12}q_{\beta\alpha,23} = W_{23}W_{12}$. We obtain (cf. Proposition 4.7)

$$\begin{aligned}&\big[(S\otimes 1)W(1\otimes\mathcal{K})\big]\\ &\quad= \big[(\mathrm{id}\otimes a\omega\otimes\mathrm{id})\big(W_{23}W_{12}(1\otimes 1\otimes b)\big);\, \omega\in\mathcal{B}(\mathscr{H})_*, a,, b\in\mathcal{K}\big]\\ &\quad= \big[(\mathrm{id}\otimes\omega a\otimes\mathrm{id})\big(W_{23}W_{12}(1\otimes 1\otimes b)\big);\, \omega\in\mathcal{B}(\mathscr{H})_*, a, b\in\mathcal{K}\big]\\ &\quad= \big[(\mathrm{id}\otimes\omega\otimes\mathrm{id})\big(\big((a\otimes 1)W(1\otimes b)\big)_{23}W_{12}\big);\, \omega\in\mathcal{B}(\mathscr{H})_*, a, b\in\mathcal{K}\big]\\ &\quad= \big[(\mathrm{id}\otimes\omega\otimes\mathrm{id})\big(\big((a\otimes 1)q_\alpha(1\otimes b)\big)_{23}W_{12}\big);\, \omega\in\mathcal{B}(\mathscr{H})_*, a, b\in\mathcal{K}\big]\\ &\quad\big(\text{resp., } \supset \big[(\mathrm{id}\otimes\omega\otimes\mathrm{id})\big(\big((a\otimes 1)q_\alpha(1\otimes b)\big)_{23}W_{12}\big);\, \omega\in\mathcal{B}(\mathscr{H})_*, a, b\in\mathcal{K}\big]\big).\end{aligned}$$

However, for all $\omega \in \mathcal{B}(\mathscr{H})_*$ and $a, b \in \mathcal{K}$ we have

$$\begin{aligned}&(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(\big((a \otimes 1)q_\alpha(1 \otimes b)\big)_{23} W_{12}\big)\\ &\quad = (\mathrm{id} \otimes \omega a \otimes \mathrm{id})\big(q_{\alpha,23} W_{12}(1 \otimes 1 \otimes b)\big).\end{aligned}$$

Since $(1 \otimes \alpha(n))W = W(\alpha(n) \otimes 1)$ for all $n \in N$, we have $q_{\alpha,23} W_{12} = W_{12} q_{\alpha,13}$. Hence,

$$\begin{aligned}&(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(\big((a \otimes 1)q_\alpha(1 \otimes b)\big)_{23} W_{12}\big)\\ &\quad = \big((\mathrm{id} \otimes \omega a)(W) \otimes 1\big) q_\alpha(1 \otimes b)\end{aligned}$$

and the result is proved. □

Proposition 4.9. *If $\mathcal{G}$ is regular (resp., semi-regular), then we have*:

1. $[(S \otimes 1)W(1 \otimes \lambda(\widehat{S}))] = [(S \otimes 1)q_\alpha(1 \otimes \lambda(\widehat{S}))]$ *(resp.,* $\supset [(S \otimes 1)q_\alpha(1 \otimes \lambda(\widehat{S}))]$*)*;
2. $[(\widehat{S} \otimes 1)V(1 \otimes S)] = [(\widehat{S} \otimes 1)q_\beta(1 \otimes S)]$ *(resp.,* $\supset [(\widehat{S} \otimes 1)q_\beta(1 \otimes S)]$*)*;
3. $[(R(S) \otimes 1)\widetilde{V}(1 \otimes \widehat{S})] = [(R(S) \otimes 1)q_{\widehat{\alpha}}(1 \otimes \widehat{S})]$ *(resp.,* $\supset [(R(S) \otimes 1)q_{\widehat{\alpha}}(1 \otimes \widehat{S})]$*)*.

In particular, we have $[(S \otimes 1)W(1 \otimes \lambda(\widehat{S}))] \subset S \otimes \lambda(\widehat{S})$, $[(\widehat{S} \otimes 1)V(1 \otimes S)] \subset \widehat{S} \otimes S$ *and* $[(R(S) \otimes 1)\widetilde{V}(1 \otimes \widehat{S})] \subset R(S) \otimes \widehat{S}$.

Proof. We have the pentagonal equation $V_{12}V_{13} = V_{23}V_{12}V_{23}^*$. Since $V \in \mathcal{M}(\widehat{S} \otimes S)$ is a partial isometry, we have $V^*(\widehat{S} \otimes S) = q_{\widehat{\alpha}\beta}(\widehat{S} \otimes S)$. Since $[V, 1 \otimes \widehat{\alpha}(n)] = 0$ for all $n \in N$, we have $[V_{12}, q_{\widehat{\alpha}\beta,23}] = 0$. Hence, $V_{23}V_{12}q_{\widehat{\alpha}\beta,23} = V_{23}V_{12}$. Hence,

$$\begin{aligned}&\big[(\widehat{S} \otimes 1)V(1 \otimes S)\big]\\ &\quad = \big[\big((\mathrm{id} \otimes \omega)(V) \otimes 1\big)V(1 \otimes y);\ \omega \in \mathcal{B}(\mathscr{H})_*, y \in S\big]\\ &\quad = \big[(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(V_{12}V_{13}(1 \otimes 1 \otimes y)\big);\ \omega \in \mathcal{B}(\mathscr{H})_*, y \in S\big]\\ &\quad = \big[(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(V_{23}V_{12}V_{23}^*(1 \otimes x \otimes y)\big);\ \omega \in \mathcal{B}(\mathscr{H})_*, y \in S, x \in \widehat{S}\big]\\ &\quad = \big[(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(V_{23}V_{12}(1 \otimes x \otimes y)\big);\ \omega \in \mathcal{B}(\mathscr{H})_*, x \in \widehat{S}, y \in S\big]\\ &\quad = \big[(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(V_{23}(1 \otimes 1 \otimes y)V_{12}\big);\ \omega \in \mathcal{B}(\mathscr{H})_*, y \in S\big]\\ &\quad = \big[(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(\big((a \otimes 1)V(1 \otimes y)\big)_{23} V_{12}\big);\ \omega \in \mathcal{B}(\mathscr{H})_*, a \in \mathcal{K}, y \in S\big].\end{aligned}$$

Let $X := [(\mathrm{id} \otimes \omega \otimes \mathrm{id})(((a \otimes 1)q_\beta(1 \otimes y))_{23}V_{12});\ \omega \in \mathcal{B}(\mathscr{H})_*, a \in \mathcal{K}, y \in S]$. Since $\mathcal{G}$ is regular (resp., semi-regular), it follows from Proposition 4.8 that

$$\big[(\widehat{S} \otimes 1)V(1 \otimes S)\big] = X \qquad \big(\text{resp., } \big[(\widehat{S} \otimes 1)V(1 \otimes S)\big] \supset X\big).$$

However, since $(1 \otimes \beta(n^{\mathrm{o}}))V = V(\beta(n^{\mathrm{o}}) \otimes 1)$ for all $n \in N$, we have

$$\begin{aligned}X &= \big[(\mathrm{id} \otimes \omega a \otimes \mathrm{id})\big(q_{\beta,23}V_{12}(1 \otimes 1 \otimes y)\big);\ \omega \in \mathcal{B}(\mathscr{H})_*, a \in \mathcal{K}, y \in S\big]\\ &= \big[(\mathrm{id} \otimes \omega \otimes \mathrm{id})\big(V_{12}q_{\beta,13}(1 \otimes 1 \otimes y)\big);\ \omega \in \mathcal{B}(\mathscr{H})_*, y \in S\big]\end{aligned}$$

$$= \big[\big((\mathrm{id}\otimes\omega)(V)\otimes 1\big)q_\beta(1\otimes y);\, \omega\in\mathcal{B}(\mathscr{H})_*, y\in S\big]$$
$$= \big[(\widehat{S}\otimes 1)q_\beta(1\otimes S)\big].$$

The second statement is proved and the third one follows by applying it to $\widehat{\mathcal{G}}$. We obtain the first statement by combining the third one with the formulas $W=(U^*\otimes U^*)\widetilde{V}(U\otimes U)$ and $\widehat{\alpha}=\mathrm{Ad}_U\circ\alpha$. Finally, the last statement follows from the inclusions $\beta(N^{\mathrm{o}})\subset\mathcal{M}(S)$, $\widehat{\beta}(N^{\mathrm{o}})\subset\mathcal{M}(\widehat{S})$, $\alpha(N)\subset\mathcal{M}(S)$ and $\widehat{\alpha}(N)\subset\mathcal{M}(\widehat{S})$. □

In the result below, we refer again to Proposition-Definition A.2.15 and Proposition A.2.18 for the definition of the operators $q_{\beta\widehat{\beta}}$, $q_{\widehat{\alpha}\alpha}$ and $q_{\widehat{\beta}\beta}$.

Corollary 4.10. *If $\mathcal{G}$ is regular (resp., semi-regular), then we have*:

1. $[(1\otimes\lambda(\widehat{S}))W(S\otimes 1)]=[(1\otimes\lambda(\widehat{S}))q_{\beta\widehat{\beta}}(S\otimes 1)]$ *(resp.,* $\supset[(1\otimes\lambda(\widehat{S}))q_{\beta\widehat{\beta}}(S\otimes 1)]$*)*;
2. $[(1\otimes S)V(\widehat{S}\otimes 1)]=[(1\otimes S)q_{\widehat{\alpha}\alpha}(\widehat{S}\otimes 1)]$ *(resp.,* $\supset[(1\otimes S)q_{\widehat{\alpha}\alpha}(\widehat{S}\otimes 1)]$*)*;
3. $[(1\otimes\widehat{S})\widetilde{V}(R(S)\otimes 1)]=[(1\otimes\widehat{S})q_{\widehat{\beta}\beta}(R(S)\otimes 1)]$ *(resp.,* $\supset[(1\otimes\widehat{S})q_{\widehat{\beta}\beta}(R(S)\otimes 1)]$*)*.

If $\mathcal{G}$ is regular, then we have $[(1\otimes\lambda(\widehat{S}))W(S\otimes 1)]\subset S\otimes\lambda(\widehat{S})$, $[(1\otimes S)V(\widehat{S}\otimes 1)]\subset\widehat{S}\otimes S$ *and* $[(1\otimes\widehat{S})\widetilde{V}(R(S)\otimes 1)]\subset R(S)\otimes\widehat{S}$.

Proof. This is a direct consequence of Proposition 4.9 and the formulas $\widehat{\beta}=\mathrm{Ad}_U\circ\beta$, $\widehat{\alpha}=\mathrm{Ad}_U\circ\alpha$, $W=\Sigma(U\otimes 1)V(U^*\otimes 1)\Sigma$ and $\widetilde{V}=\Sigma(1\otimes U)V(1\otimes U^*)\Sigma$. The second statement follows from the inclusions $\beta(N^{\mathrm{o}})\subset\mathcal{M}(S)$, $\widehat{\beta}(N^{\mathrm{o}})\subset\mathcal{M}(\widehat{S})$, $\alpha(N)\subset\mathcal{M}(S)$ and $\widehat{\alpha}(N)\subset\mathcal{M}(\widehat{S})$. □

5. Measured quantum groupoids on a finite basis in action

5.1. Continuous actions, crossed product, and biduality. In this section, we fix a measured quantum groupoid $\mathcal{G}=(N,M,\alpha,\beta,\Delta,T,T',\epsilon)$ on the finite-dimensional basis $N=\bigoplus_{1\leqslant l\leqslant k}\mathrm{M}_{n_l}(\mathbb{C})$ and we use all the notation introduced in §3.1, §3.2. In the following, we recall the definitions, notations and results of §3.1, §3.2.1, §3.2.2, and §3.3.1 [2] (see also [9] Chapter 4).

5.1.1. *Notion of actions of measured quantum groupoids on a finite basis.*

Lemma 5.1.1. *Let A and B be two C^*-algebras, $f:A\to\mathcal{M}(B)$ a $*$-homomorphism and $e\in\mathcal{M}(B)$. The following statements are equivalent*:

(i) *there exists an approximate unit $(u_\lambda)_\lambda$ of A such that $f(u_\lambda)\to e$ with respect to the strict topology*;
(ii) *f extends to a strictly continuous $*$-homomorphism $f:\mathcal{M}(A)\to\mathcal{M}(B)$, necessarily unique, such that $f(1_A)=e$*;
(iii) $[f(A)B]=eB$.

In that case, e is a self-adjoint projection, for all approximate unit $(v_\mu)_\mu$ of A we have $f(v_\mu) \to e$ with respect to the strict topology and $[Bf(A)] = Be$.

DEFINITION 5.1.2. An action of $\mathcal{G}$ on a C*-algebra A is a pair (β_A, δ_A) consisting of a nondegenerate $*$-homomorphism $\beta_A : N^{\mathrm{o}} \to \mathcal{M}(A)$ and a faithful $*$-homomorphism $\delta_A : A \to \mathcal{M}(A \otimes S)$ such that:

1. δ_A extends to a strictly continuous $*$-homomorphism $\delta_A : \mathcal{M}(A) \to \mathcal{M}(A \otimes S)$ such that $\delta_A(1_A) = q_{\beta_A \alpha}$ (cf. Proposition-Definition A.2.19);
2. $(\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A$;
3. $\delta_A(\beta_A(n^{\mathrm{o}})) = q_{\beta_A \alpha}(1_A \otimes \beta(n^{\mathrm{o}}))$, for all $n \in N$.

We say that the action (β_A, δ_A) is strongly continuous if we have

$$\big[\delta_A(A)(1_A \otimes S)\big] = q_{\beta_A \alpha}(A \otimes S).$$

If that case, we say that the triple (A, β_A, δ_A) is a $\mathcal{G}$-C*-algebra.

REMARKS 5.1.3.

- By Lemma 5.1.1, the condition 1 is equivalent to requiring that for some (and then any) approximate unit (u_λ) of A, we have $\delta_A(u_\lambda) \to q_{\beta_A \alpha}$ with respect to the strict topology of $\mathcal{M}(A \otimes S)$. It is also equivalent to $[\delta_A(A)(A \otimes S)] = q_{\beta_A \alpha}(A \otimes S)$.
- Condition 1 implies that the $*$-homomorphisms $\delta_A \otimes \mathrm{id}_S$ and $\mathrm{id}_A \otimes \delta$ extend uniquely to strictly continuous $*$-homomorphisms from $\mathcal{M}(A \otimes S)$ to $\mathcal{M}(A \otimes S \otimes S)$ such that $(\delta_A \otimes \mathrm{id}_S)(1_{A \otimes S}) = q_{\beta_A \alpha, 12}$ and $(\mathrm{id}_A \otimes \delta)(1_{A \otimes S}) = q_{\beta\alpha, 23}$. In particular, condition 2 does make sense and we denote by $\delta_A^2 := (\delta_A \otimes \mathrm{id}_S)\delta_A : A \to \mathcal{M}(A \otimes S \otimes S)$ the iterated coaction map.

EXAMPLES 5.1.4. Let us give two basic examples.

- (S, β, δ) is a $\mathcal{G}$-C*-algebra.
- Let $\beta_{N^\circ} := \mathrm{id}_{N^\circ}$. Let $\delta_{N^\circ} : N^{\mathrm{o}} \to \mathcal{M}(N^{\mathrm{o}} \otimes S)$ be the faithful unital $*$-homomorphism given by $\delta_{N^\circ}(n^{\mathrm{o}}) := q_{\beta_{N^\circ} \alpha}(1_{N^\circ} \otimes \beta(n^{\mathrm{o}}))$ for all $n \in N$. Then the pair $(\beta_{N^\circ}, \delta_{N^\circ})$ is an action of $\mathcal{G}$ on N^{o} called the trivial action.

PROPOSITION 5.1.5. *Let (δ_A, β_A) be an action of $\mathcal{G}$ on A. We have the following statements*:

1. *the iterated coaction map δ_A^2 extends uniquely to a strictly continuous $*$-homomorphism $\delta_A^2 : \mathcal{M}(A) \to \mathcal{M}(A \otimes S \otimes S)$ such that $\delta_A^2(1_A) = q_{\beta_A \alpha, 12} q_{\beta\alpha, 23}$; moreover, we have $\delta_A^2(m) = (\delta_A \otimes \mathrm{id}_S)\delta_A(m) = (\mathrm{id}_A \otimes \delta)\delta_A(m)$ for all $m \in \mathcal{M}(A)$;*
2. *for all $n \in N$, we have $\delta_A(\beta_A(n^{\mathrm{o}})) = (1_A \otimes \beta(n^{\mathrm{o}}))q_{\beta_A \alpha}$;*
3. *if (β_A, δ_A) is strongly continuous, then we have $[(1_A \otimes S)\delta_A(A)] = (A \otimes S)q_{\beta_A \alpha}$.*

Let us provide a more explicit definition of what an action of the dual measured quantum groupoid $\widehat{\mathcal{G}}$ on a C*-algebra B is.

DEFINITION 5.1.6. An action of $\widehat{\mathcal{G}}$ on a C*-algebra B is a pair (α_B, δ_B) consisting of a nondegenerate $*$-homomorphism $\alpha_B : N \to \mathcal{M}(B)$ and a faithful $*$-homomorphism $\delta_B : B \to \mathcal{M}(B \otimes \widehat{S})$ such that:

1. δ_B extends to a strictly continuous $*$-homomorphism $\delta_B : \mathcal{M}(B) \to \mathcal{M}(B \otimes \widehat{S})$ such that $\delta_B(1_B) = q_{\alpha_B\beta}$ (cf. Proposition-Definition A.2.19);
2. $(\delta_B \otimes \mathrm{id}_{\widehat{S}})\delta_B = (\mathrm{id}_B \otimes \widehat{\delta})\delta_B$;
3. $\delta_B(\alpha_B(n)) = q_{\alpha_B\beta}(1_B \otimes \widehat{\alpha}(n))$, for all $n \in N$.

We say that the action (α_B, δ_B) is strongly continuous if we have

$$\big[\delta_B(B)(1_B \otimes \widehat{S})\big] = q_{\alpha_B\beta}(B \otimes \widehat{S}).$$

If (δ_B, α_B) is a strongly continuous action of $\widehat{\mathcal{G}}$ on B, we say that the triple (B, α_B, δ_B) is a $\widehat{\mathcal{G}}$-C*-algebra.

REMARKS 5.1.7. As for actions of $\mathcal{G}$, we have:

- the condition 1 is equivalent to requiring that for some (and then any) approximate unit $(u_\lambda)_\lambda$ of B we have $\delta_B(u_\lambda) \to q_{\alpha_B\beta}$ with respect to the strict topology, which is also equivalent to the relation $[\delta_B(B)(B \otimes \widehat{S})] = q_{\alpha_B\beta}(B \otimes \widehat{S})$;
- the $*$-homomorphisms $\mathrm{id}_B \otimes \widehat{\delta}$ and $\delta_B \otimes \mathrm{id}_{\widehat{S}}$ extend uniquely to strictly continuous $*$-homomorphisms from $\mathcal{M}(B \otimes \widehat{S})$ to $\mathcal{M}(B \otimes \widehat{S} \otimes \widehat{S})$ such that $(\mathrm{id}_B \otimes \widehat{\delta})(1_{B\otimes\widehat{S}}) = q_{\widehat{\alpha}\beta,23}$ and $(\delta_B \otimes \mathrm{id}_{\widehat{S}})(1_{B\otimes\widehat{S}}) = q_{\alpha_B\beta,12}$. In particular, condition 2 does make sense and we denote by $\delta_B^2 := (\delta_B \otimes \mathrm{id}_{\widehat{S}})\delta_B : B \to \mathcal{M}(B \otimes \widehat{S} \otimes \widehat{S})$ the iterated coaction map.

EXAMPLES 5.1.8. Let us give two basic examples:

- $(\widehat{S}, \widehat{\alpha}, \widehat{\delta})$ is a $\widehat{\mathcal{G}}$-C*-algebra;
- Let $\alpha_N := \mathrm{id}_N$ and $\delta_N : N \to \mathcal{M}(N \otimes \widehat{S})$; $n \mapsto q_{\alpha_N\beta}(1_N \otimes \widehat{\alpha}(n))$; then the pair (α_N, δ_N) is an action of $\widehat{\mathcal{G}}$ on N called the trivial action.

PROPOSITION 5.1.9. *Let (α_B, δ_B) be an action of $\widehat{\mathcal{G}}$ on B. We have the following statements*:

1. *the iterated coaction map δ_B^2 extends uniquely to a strictly continuous $*$-homomorphism $\delta_B^2 : \mathcal{M}(B) \to \mathcal{M}(B \otimes \widehat{S} \otimes \widehat{S})$ such that $\delta_B^2(1_B) = q_{\alpha_B\beta,12}q_{\widehat{\alpha}\beta,23}$; moreover, we have $\delta_B^2(m) = (\delta_B \otimes \mathrm{id}_{\widehat{S}})\delta_B(m) = (\mathrm{id}_B \otimes \widehat{\delta})\delta_B(m)$ for all $m \in \mathcal{M}(B)$;*
2. *for all $n \in N$, we have $\delta_B(\alpha_B(n)) = (1_B \otimes \widehat{\alpha}(n))q_{\alpha_B\beta}$;*
3. *if (α_B, δ_B) is strongly continuous, then we have $[(1_B \otimes \widehat{S})\delta_B(B)] = (B \otimes \widehat{S})q_{\alpha_B\beta}$.*

DEFINITION 5.1.10. For $i = 1, 2$, let A_i (resp., B_i) be a C*-algebra. For $i = 1, 2$, let $(\beta_{A_i}, \delta_{A_i})$ (resp., $(\alpha_{B_i}, \delta_{B_i})$) be an action of $\mathcal{G}$ (resp., $\widehat{\mathcal{G}}$) on A_i (resp., B_i). A nondegenerate $*$-homomorphism $f : A_1 \to \mathcal{M}(A_2)$

(resp., $f : B_1 \to \mathcal{M}(B_2)$) is said to be $\mathcal{G}$-equivariant (resp., $\widehat{\mathcal{G}}$-equivariant) if $(f \otimes \mathrm{id}_S)\delta_{A_1} = \delta_{A_2} \circ f$ and $f \circ \beta_{A_1} = \beta_{A_2}$ (resp., $(f \otimes \mathrm{id}_{\widehat{S}})\delta_{B_1} = \delta_{B_2} \circ f$ and $f \circ \alpha_{B_1} = \alpha_{B_2}$).

Remark 5.1.11. With the notation and hypotheses of Definition 5.1.10, if f satisfies the relation $(f \otimes \mathrm{id}_S)\delta_{A_1} = \delta_{A_2} \circ f$ (resp., $(f \otimes \mathrm{id}_{\widehat{S}})\delta_{B_1} = \delta_{B_2} \circ f$), then f satisfies necessarily the relation $f \circ \beta_{A_1} = \beta_{A_2}$ (resp., $f \circ \alpha_{B_1} = \alpha_{B_2}$), *that is*, f is $\mathcal{G}$-equivariant (resp., $\widehat{\mathcal{G}}$-equivariant). Indeed, let $n \in N$. For all $a \in A_1$ and $x \in A_2$, we have

$$\begin{aligned}\delta_{A_2}\big(f\big(\beta_{A_1}\big(n^{\mathrm{o}}\big)\big)f(a)x\big) &= (f \otimes \mathrm{id}_S)\delta_{A_1}\big(\beta_{A_1}\big(n^{\mathrm{o}}\big)a\big)\delta_{A_2}(x)\\ &= \big(1_{A_2} \otimes \beta\big(n^{\mathrm{o}}\big)\big)(f \otimes \mathrm{id}_S)\delta_{A_1}(a)\delta_{A_2}(x)\\ &= \big(1_{A_2} \otimes \beta\big(n^{\mathrm{o}}\big)\big)\delta_{A_2}\big(f(a)x\big)\\ &= \delta_{A_2}\big(\beta_{A_2}\big(n^{\mathrm{o}}\big)f(a)x\big).\end{aligned}$$

Hence, $f(\beta_{A_1}(n^{\mathrm{o}}))f(a)x = \beta_{A_2}(n^{\mathrm{o}})f(a)x$ for all $a \in A_1$ and $x \in A_2$ since δ_{A_2} is faithful. Hence, we have $f(\beta_{A_1}(n^{\mathrm{o}})) = \beta_{A_2}(n^{\mathrm{o}})$ since f is nondegenerate.

Notation 5.1.12. We denote by $\mathsf{Alg}_{\mathcal{G}}$ the category whose objects are the $\mathcal{G}$-C*-algebras and whose set of arrows between $\mathcal{G}$-C*-algebras is the set of $\mathcal{G}$-equivariant nondegenerate $*$-homomorphisms.

5.1.2. *Crossed product and dual action.* Let us fix a strongly continuous action (β_A, δ_A) of $\mathcal{G}$ on a C*-algebra A.

Notation 5.1.13. The $*$-representation

$$\pi_L := (\mathrm{id}_A \otimes L) \circ \delta_A : A \to \mathcal{L}(A \otimes \mathscr{H})$$

of A on the Hilbert A-module $A \otimes \mathscr{H}$ extends uniquely to a strictly/$*$-strongly continuous faithful $*$-representation $\pi_L : \mathcal{M}(A) \to \mathcal{L}(A \otimes \mathscr{H})$ such that $\pi_L(1_A) = q_{\beta_A\alpha}$. Moreover, we have $\pi_L(m) = \pi_L(m)q_{\beta_A\alpha} = q_{\beta_A\alpha}\pi_L(m)$ for all $m \in \mathcal{M}(A)$. Consider the Hilbert A-module

$$\mathcal{E}_{A,L} := q_{\beta_A\alpha}(A \otimes \mathscr{H}).$$

By restricting π_L, we obtain a strictly/$*$-strongly continuous faithful unital $*$-representation

$$\pi : \mathcal{M}(A) \to \mathcal{L}(\mathcal{E}_{A,L}); \qquad m \mapsto \pi_L(m)\restriction_{\mathcal{E}_{A,L}}.$$

We have $[1_A \otimes T, q_{\beta_A\alpha}] = 0$ for all $T \in \mathcal{M}(\widehat{S})$. We then obtain a strictly/$*$-strongly continuous unital $*$-representation

$$\widehat{\theta} : \mathcal{M}(\widehat{S}) \to \mathcal{L}(\mathcal{E}_{A,L}); \qquad T \mapsto (1_A \otimes T)\restriction_{\mathcal{E}_{A,L}}.$$

Note that if β_A is faithful, then so is $\widehat{\theta}$.

PROPOSITION-DEFINITION 5.1.14. *The norm closed subspace of* $\mathcal{L}(\mathcal{E}_{A,L})$ *spanned by the products of the form* $\pi(a)\widehat{\theta}(x)$ *for* $a \in A$ *and* $x \in \widehat{S}$ *is a* C*-*subalgebra called the* (*reduced*) *crossed product of* A *by the strongly continuous action* (β_A, δ_A) *of* $\mathcal{G}$ *and denoted by* $A \rtimes \mathcal{G}$.

In particular, the morphism π (resp., $\widehat{\theta}$) defines a faithful unital $*$-homomorphism (resp., unital $*$-homomorphism) $\pi : \mathcal{M}(A) \to \mathcal{M}(A \rtimes \mathcal{G})$ (resp., $\widehat{\theta} : \widehat{S} \to \mathcal{M}(A \rtimes \mathcal{G})$).

Since $[\widetilde{V}, \alpha(n) \otimes 1] = 0$, we have $[\widetilde{V}_{23}, q_{\beta_A \alpha, 12}] = 0$. The operator $\widetilde{V}_{23} \in \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$ restricts to a partial isometry

$$X := \widetilde{V}_{23}{}_{\restriction \mathcal{E}_{A,L} \otimes \mathscr{H}} \in \mathcal{L}(\mathcal{E}_{A,L} \otimes \mathscr{H}),$$

whose initial and final projections are $X^*X = q_{\widehat{\beta}\alpha, 23}{}_{\restriction \mathcal{E}_{A,L} \otimes \mathscr{H}}$ and $XX^* = q_{\widehat{\alpha}\beta, 23}{}_{\restriction \mathcal{E}_{A,L} \otimes \mathscr{H}}$.

PROPOSITION-DEFINITION 5.1.15. *Let*

$$\delta_{A \rtimes \mathcal{G}} : A \rtimes \mathcal{G} \to \mathcal{L}(\mathcal{E}_{A,L} \otimes \mathscr{H}) \quad \text{and} \quad \alpha_{A \rtimes \mathcal{G}} : N \to \mathcal{M}(A \rtimes \mathcal{G})$$

be the linear maps defined by:

- $\delta_{A \rtimes \mathcal{G}}(b) := X(b \otimes 1)X^*$, *for all* $b \in A \rtimes \mathcal{G}$;
- $\alpha_{A \rtimes \mathcal{G}}(n) := \widehat{\theta}(\widehat{\alpha}(n)) = (1_A \otimes \widehat{\alpha}(n))_{\restriction \mathcal{E}_{A,L}}$, *for all* $n \in N$.

Then, $\delta_{A \rtimes \mathcal{G}}$ *is a faithful* $*$*-homomorphism and* $\alpha_{A \rtimes \mathcal{G}}$ *is a nondegenerate* $*$*-homomorphism. Moreover, we have the following statements*:

1. $\delta_{A \rtimes \mathcal{G}}(\pi(a)\widehat{\theta}(x)) = (\pi(a) \otimes 1_{\widehat{S}})(\widehat{\theta} \otimes \mathrm{id}_{\widehat{S}})\widehat{\delta}(x)$, *for all* $a \in A$ *and* $x \in \widehat{S}$; *in particular,* $\delta_{A \rtimes \mathcal{G}}$ *takes its values in* $\mathcal{M}((A \rtimes \mathcal{G}) \otimes \widehat{S})$;
2. $\alpha_{A \rtimes \mathcal{G}}(n)\pi(a)\widehat{\theta}(x) = \pi(a)\widehat{\theta}(\widehat{\alpha}(n)x)$ *and* $\pi(a)\widehat{\theta}(x)\alpha_{A \rtimes \mathcal{G}}(n) = \pi(a)\widehat{\theta}(x\widehat{\alpha}(n))$ *for all* $n \in N$, $a \in A$ *and* $x \in \widehat{S}$.

PROPOSITION-DEFINITION 5.1.16. *With the notation of Proposition-Definition* 5.1.15, *the pair* $(\alpha_{A \rtimes \mathcal{G}}, \delta_{A \rtimes \mathcal{G}})$ *is a strongly continuous action of* $\widehat{\mathcal{G}}$ *on* $A \rtimes \mathcal{G}$ *called the dual action of* (β_A, δ_A).

In a similar way, we define the crossed product of a C*-algebra B by a strongly continuous action (α_B, δ_B) of the dual measured quantum groupoid $\widehat{\mathcal{G}}$.

NOTATION 5.1.17. The $*$-representation

$$\widehat{\pi}_\lambda := (\mathrm{id}_B \otimes \lambda) \circ \delta_B : B \to \mathcal{L}(B \otimes \mathscr{H})$$

of B on the Hilbert B-module $B \otimes \mathscr{H}$ extends uniquely to a strictly/$*$-strongly continuous faithful $*$-representation $\widehat{\pi}_\lambda : \mathcal{M}(B) \to \mathcal{L}(B \otimes \mathscr{H})$ such that $\widehat{\pi}_\lambda(1_B) = q_{\alpha_B \widehat{\beta}}$. Moreover, we have $\widehat{\pi}_\lambda(m) = \widehat{\pi}_\lambda(m) q_{\alpha_B \widehat{\beta}} = q_{\alpha_B \widehat{\beta}} \widehat{\pi}_\lambda(m)$, for all $m \in \mathcal{M}(B)$. Consider the Hilbert B-module

$$\mathcal{E}_{B,\lambda} := q_{\alpha_B \widehat{\beta}}(B \otimes \mathscr{H}).$$

By restricting $\widehat{\pi}_\lambda$, we obtain a strictly/$*$-strongly continuous faithful unital $*$-representation

$$\widehat{\pi} : \mathcal{M}(B) \to \mathcal{L}(\mathcal{E}_{B,\lambda}); \qquad m \mapsto \widehat{\pi}_\lambda(m)\restriction_{\mathcal{E}_{B,\lambda}}.$$

We have $[1_B \otimes T, q_{\alpha_B \widehat{\beta}}] = 0$ for all $T \in \mathcal{M}(S)$. We then obtain a strictly/$*$-strongly continuous unital $*$-representation

$$\theta : \mathcal{M}(S) \to \mathcal{L}(\mathcal{E}_{B,\lambda}); \qquad T \mapsto (1_B \otimes T)\restriction_{\mathcal{E}_{B,\lambda}}.$$

Note that if α_B is faithful, then so is θ.

Proposition-Definition 5.1.18. *The norm closed subspace of $\mathcal{L}(\mathcal{E}_{B,\lambda})$ spanned by the products of the form $\widehat{\pi}(b)\theta(x)$ for $b \in B$ and $x \in S$ is a C^*-subalgebra called the (reduced) crossed product of B by the strongly continuous action (α_B, δ_B) of $\widehat{\mathcal{G}}$ and denoted by $B \rtimes \widehat{\mathcal{G}}$.*

In particular, the morphism $\widehat{\pi}$ (resp., θ) defines a faithful unital $*$-homomorphism (resp., unital $*$-homomorphism) $\widehat{\pi} : \mathcal{M}(B) \to \mathcal{M}(B \rtimes \widehat{\mathcal{G}})$ (resp., $\theta : S \to \mathcal{M}(B \rtimes \widehat{\mathcal{G}})$).

Since $[V, \beta(n^{\mathrm{o}}) \otimes 1] = 0$, we have $[V_{23}, q_{\alpha_B \beta, 12}] = 0$. The operator $V_{23} \in \mathcal{L}(B \otimes \mathscr{H} \otimes \mathscr{H})$ restricts to a partial isometry

$$Y := V_{23}\restriction_{\mathcal{E}_{B,\lambda} \otimes \mathscr{H}} \in \mathcal{L}(\mathcal{E}_{B,\lambda} \otimes \mathscr{H}),$$

whose initial and final projections are $Y^*Y = q_{\widehat{\alpha}\beta, 23}\restriction_{\mathcal{E}_{B,\lambda} \otimes \mathscr{H}}$ and $YY^* = q_{\beta\alpha, 23}\restriction_{\mathcal{E}_{B,\lambda} \otimes \mathscr{H}}$.

Proposition-Definition 5.1.19. *Let*

$$\delta_{B \rtimes \widehat{\mathcal{G}}} : B \rtimes \widehat{\mathcal{G}} \to \mathcal{L}(\mathcal{E}_{B,\lambda} \otimes \mathscr{H}) \quad \text{and} \quad \beta_{B \rtimes \widehat{\mathcal{G}}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E}_{B,\lambda})$$

be the linear maps defined by:

- $\delta_{B \rtimes \widehat{\mathcal{G}}}(c) := Y(c \otimes 1_{\mathscr{H}})Y^*$, *for all* $c \in B \rtimes \widehat{\mathcal{G}}$;
- $\beta_{B \rtimes \widehat{\mathcal{G}}}(n^{\mathrm{o}}) := \theta(\beta(n^{\mathrm{o}})) = (1_B \otimes \beta(n^{\mathrm{o}}))\restriction_{\mathcal{E}_{B,\lambda} \otimes \mathscr{H}}$, *for all* $n \in N$.

Then, $\delta_{B \rtimes \widehat{\mathcal{G}}}$ is a faithful $*$-homomorphism and $\beta_{B \rtimes \widehat{\mathcal{G}}}$ is a nondegenerate $*$-homomorphism. Moreover, we have the following statements:

1. $\delta_{B \rtimes \widehat{\mathcal{G}}}(\widehat{\pi}(b)\theta(x)) = (\widehat{\pi}(b) \otimes 1_S)(\theta \otimes \mathrm{id}_S)\delta(x)$, *for all* $b \in B$ *and* $x \in S$; *in particular*, $\delta_{B \rtimes \widehat{\mathcal{G}}}$ *takes its values in* $\mathcal{M}((B \rtimes \widehat{\mathcal{G}}) \otimes S)$;
2. $\beta_{B \rtimes \widehat{\mathcal{G}}}(n^{\mathrm{o}})\widehat{\pi}(b)\theta(x) = \widehat{\pi}(b)\theta(\beta(n^{\mathrm{o}})x)$ *and* $\widehat{\pi}(b)\theta(x)\beta_{B \rtimes \widehat{\mathcal{G}}}(n^{\mathrm{o}}) = \widehat{\pi}(b) \times \theta(x\beta(n^{\mathrm{o}}))$ *for all* $n \in N$, $b \in B$, *and* $x \in S$.

Proposition-Definition 5.1.20. *With the notation of Proposition-Definition 5.1.19, the pair $(\beta_{B \rtimes \widehat{\mathcal{G}}}, \delta_{B \rtimes \widehat{\mathcal{G}}})$ is a strongly continuous action of $\mathcal{G}$ on $B \rtimes \widehat{\mathcal{G}}$ called the dual action of (α_B, δ_B).*

5.1.3. *Takesaki–Takai duality.* Let (β_A, δ_A) be a strongly continuous action of the groupoid $\mathcal{G}$ on a C*-algebra A.

NOTATION 5.1.21. The $*$-representation

$$\pi_R := (\mathrm{id}_A \otimes R) \circ \delta_A : A \to \mathcal{L}(A \otimes \mathscr{H})$$

of A on the Hilbert A-module $A \otimes \mathscr{H}$ extends uniquely to a strictly/$*$-strongly continuous faithful $*$-representation $\pi_R : \mathcal{M}(A) \to \mathcal{L}(A \otimes \mathscr{H})$ satisfying $\pi_R(m) = (\mathrm{id}_A \otimes R)\delta_A(m)$ for all $m \in \mathcal{M}(A)$ and $\pi_R(1_A) = q_{\beta_A\widehat{\alpha}}$. Consider the Hilbert A-module

$$\mathcal{E}_{A,R} := q_{\beta_A\widehat{\alpha}}(A \otimes \mathscr{H}).$$

We recall that the Banach space

$$D := \big[\pi_R(a)\big(1_A \otimes \lambda(x)L(y)\big);\, a \in A, x \in \widehat{S}, y \in S\big]$$

is a C*-subalgebra of $\mathcal{L}(A \otimes \mathscr{H})$ such that $dq_{\beta_A\widehat{\alpha}} = d = dq_{\beta_A\widehat{\alpha}}$ for all $d \in D$. Moreover, we have $D(A \otimes \mathscr{H}) = \mathcal{E}_{A,R}$. We also recall that there exists a unique strictly/$*$-strongly continuous faithful $*$-representation $j_D : \mathcal{M}(D) \to \mathcal{L}(A \otimes \mathscr{H})$ extending the inclusion map $D \subset \mathcal{L}(A \otimes \mathscr{H})$ such that $j_D(1_D) = q_{\beta_A\widehat{\alpha}}$.

PROPOSITION 5.1.22. *There exists a unique $*$-isomorphism $\phi : (A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to D$ such that $\phi(\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)) = \pi_R(a)(1_A \otimes \lambda(x)L(y))$ for all $a \in A$, $x \in \widehat{S}$ and $y \in S$.*

NOTATION 5.1.23. We denote $\mathcal{K} := \mathcal{K}(\mathscr{H})$ for short. Let $\delta_0 : A \otimes \mathcal{K} \to \mathcal{M}(A \otimes \mathcal{K} \otimes S)$ be the $*$-homomorphism defined by $\delta_0(a \otimes k) = \delta_A(a)_{13}(1_A \otimes k \otimes 1_S)$ for all $a \in A$ and $k \in \mathcal{K}$. The morphism δ_0 extends uniquely to a strictly continuous $*$-homomorphism $\delta_0 : \mathcal{M}(A \otimes \mathcal{K}) \to \mathcal{M}(A \otimes \mathcal{K} \otimes S)$ such that $\delta_0(1_{A\otimes\mathcal{K}}) = q_{\beta_A\alpha,13}$. Let $\mathcal{V} \in \mathcal{L}(\mathscr{H} \otimes S)$ be the unique partial isometry such that $(\mathrm{id}_{\mathcal{K}} \otimes L)(\mathcal{V}) = V$.

THEOREM 5.1.24. *There exists a unique strongly continuous action (β_D, δ_D) of $\mathcal{G}$ on the C*-algebra $D = [\pi_R(a)(1_A \otimes \lambda(x)L(y));\, a \in A, x \in \widehat{S}, y \in S]$ defined by the relations*:

$$(j_D \otimes \mathrm{id}_S)\delta_D(d) = \mathcal{V}_{23}\delta_0(d)\mathcal{V}_{23}^*, \quad d \in D;$$
$$j_D\big(\beta_D\big(n^{\mathrm{o}}\big)\big) = q_{\beta_A\widehat{\alpha}}\big(1_A \otimes \beta\big(n^{\mathrm{o}}\big)\big), \quad n \in N.$$

Moreover, the canonical $*$-isomorphism $\phi : (A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to D$ (cf. Proposition 5.1.22) is $\mathcal{G}$-equivariant. If the groupoid $\mathcal{G}$ is regular, then we have $D = q_{\beta_A\widehat{\alpha}}(A \otimes \mathcal{K})q_{\beta_A\widehat{\alpha}}$.

The $\mathcal{G}$-C*-algebra D defined above will be referred to as the bidual $\mathcal{G}$-C*-algebra of A.

5.2. Case of a colinking measured quantum groupoid. In this section, we fix a colinking measured quantum groupoid $\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ associated with two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. We follow all the notation recalled in §3.3 concerning the objects associated with $\mathcal{G}$.

In the following, we recall the notation and the main results of §3.2.3 [2] concerning the equivalent description of the $\mathcal{G}$-C*-algebras in terms of $\mathbb{G}_1$-C*-algebras and $\mathbb{G}_2$-C*-algebras. Let us fix a $\mathcal{G}$-C*-algebra (A, β_A, δ_A).

NOTATION 5.2.1.

- The morphism $\beta_A : \mathbb{C}^2 \to \mathcal{M}(A)$ is central. Let $q_j := \beta_A(\varepsilon_j)$ for $j = 1, 2$. Then, q_j is a central self-adjoint projection of $\mathcal{M}(A)$ and $q_1 + q_2 = 1_A$. Let $A_j := q_j A$ for $j = 1, 2$. For $j = 1, 2$, A_j is a C*-subalgebra (actually a closed two-sided ideal) of A and we have $A = A_1 \oplus A_2$.
- For $j, k = 1, 2$, let $\pi_j^k : \mathcal{M}(A_k \otimes S_{kj}) \to \mathcal{M}(A \otimes S)$ be the unique strictly continuous extension of the inclusion $A_k \otimes S_{kj} \subset A \otimes S$ such that $\pi_j^k(1_{A_k \otimes S_{kj}}) = q_k \otimes p_{kj}$.

In case of ambiguity, we will denote $\pi_{A,j}^k$ and $q_{A,j}$ instead of π_j^k and q_j.

PROPOSITION 5.2.2. *For all $j, k = 1, 2$, there exists a unique faithful non-degenerate $*$-homomorphism*

$$\delta_{A_j}^k : A_j \to \mathcal{M}(A_k \otimes S_{kj})$$

such that for all $x \in A_j$, we have

$$\begin{aligned}\pi_j^k \circ \delta_{A_j}^k(x) &= (q_k \otimes p_{kj})\delta_A(x) = (q_k \otimes 1_S)\delta_A(x) \\ &= \big(1_A \otimes \alpha(\varepsilon_k)\big)\delta_A(x) = (1_A \otimes p_{kj})\delta_A(x).\end{aligned}$$

Moreover, we have:

1. $\delta_A(a) = \sum_{k,j=1,2} \pi_j^k \circ \delta_{A_j}^k(q_j a)$, *for all* $a \in A$;
2. $(\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}})\delta_{A_j}^k = (\mathrm{id}_{A_l} \otimes \delta_{lj}^k)\delta_{A_j}^l$, *for all* $j, k, l = 1, 2$;
3. $[\delta_{A_j}^k(A_j)(1_{A_k} \otimes S_{kj})] = A_k \otimes S_{kj}$, *for all* $j, k = 1, 2$; *in particular, we have*

$$A_k = \big[(\mathrm{id}_{A_k} \otimes \omega)\delta_{A_j}^k(a);\, a \in A_j, \omega \in \mathcal{B}(\mathscr{H}_{kj})_*\big];$$

4. $\delta_{A_j}^j : A_j \to \mathcal{M}(A_j \otimes S_{jj})$ *is a strongly continuous action of* $\mathbb{G}_j$ *on* A_j.

From this concrete description of $\mathcal{G}$-C*-algebras, we can also give a convenient description of the $\mathcal{G}$-equivariant $*$-homomorphisms. With the above notation, we have the result below.

PROPOSITION 5.2.3. *Let A and B be two $\mathcal{G}$-C*-algebras. For $k = 1, 2$, let $\iota_k : \mathcal{M}(B_k) \to \mathcal{M}(B)$ be the unique strictly continuous extension of the inclusion map $B_k \subset B$ such that $\iota_k(1_{B_k}) = q_{B,k}$.*

1. *Let $f : A \to \mathcal{M}(B)$ be a non-degenerate $\mathcal{G}$-equivariant $*$-homomorphism. Then, for all $j = 1, 2$, there exists a unique nondegenerate $*$-homomorphism $f_j : A_j \to \mathcal{M}(B_j)$ such that for $k = 1, 2$ we have*

$$(f_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{A_j} = \delta^k_{B_j} \circ f_j. \tag{5.1}$$

Moreover, we have $f(a) = \iota_1 \circ f_1(aq_{A,1}) + \iota_2 \circ f_2(aq_{A,2})$ for all $a \in A$.

2. *Conversely, for $j = 1, 2$ let $f_j : A_j \to \mathcal{M}(B_j)$ be a nondegenerate $*$-homomorphism such that (5.1) holds for all $j, k = 1, 2$. Then, the map $f : A \to \mathcal{M}(B)$, defined for all $a \in A$ by*

$$f(a) := \iota_1 \circ f_1(aq_{A,1}) + \iota_2 \circ f_2(aq_{A,2}),$$

is a nondegenerate $\mathcal{G}$-equivariant $*$-homomorphism.

The above results show that for $j = 1, 2$ we have a functor

$$\mathsf{Alg}_{\mathcal{G}} \to \mathsf{Alg}_{\mathbb{G}_j}; \qquad (A, \beta_A, \delta_A) \mapsto \big(A_j, \delta^j_{A_j}\big).$$

In §4 [2], it has been proved that if $\mathcal{G}$ is regular (cf. Theorem 4.5), then $(A, \delta_A, \beta_A) \to (A_1, \delta^1_{A_1})$ is an equivalence of categories. Moreover, the authors build explicitly the inverse functor $(A_1, \delta_{A_1}) \to (A, \beta_A, \delta_A)$. More precisely, to any $\mathbb{G}_1$-C*-algebra (A_1, δ_{A_1}) they associate a $\mathbb{G}_2$-C*-algebra (A_2, δ_{A_2}) in a canonical way. Then the C*-algebra $A := A_1 \oplus A_2$ can be equipped with a strongly continuous action (β_A, δ_A) of the groupoid $\mathcal{G}$. This allowed them to build the inverse functor $(A_1, \delta_{A_1}) \to (A, \beta_A, \delta_A)$. The equivalence of categories $(A_1, \delta_{A_1}) \to (A_2, \delta_{A_2})$ generalizes the correspondence of actions for monoidally equivalent *compact* quantum groups of De Rijdt and Vander Vennet [15]. We bring to the reader's attention that an induction procedure has been developed by De Commer in the von Neumann algebraic setting (cf. §8 [11]).

In the following, we recall the notations and the main results of §4 [2]. We assume that the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are regular.

NOTATION 5.2.4. Let $\delta_{A_1} : A_1 \to \mathcal{M}(A_1 \otimes S_{11})$ be a continuous action of $\mathbb{G}_1$ on a C*-algebra A_1. Let us denote:

$$\delta^1_{A_1} := \delta_{A_1}, \qquad \delta^{(2)}_{A_1} := \big(\mathrm{id}_{A_1} \otimes \delta^2_{11}\big)\delta_{A_1} : A_1 \to \mathcal{M}(A_1 \otimes S_{12} \otimes S_{21}).$$

Then, $\delta^{(2)}_{A_1}$ is a faithful nondegenerate $*$-homomorphism. In the following, we will identify S_{21} with a C*-subalgebra of $\mathcal{B}(\mathscr{H}_{21})$. Let

$$\begin{aligned}\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1) := \big[(\mathrm{id}_{A_1 \otimes S_{12}} \otimes \omega)\delta^{(2)}_{A_1}(a);\, a \in A_1, \omega \in \mathcal{B}(\mathscr{H}_{21})_*\big] \\ \subset \mathcal{M}(A_1 \otimes S_{12}).\end{aligned}$$

PROPOSITION 5.2.5. *The Banach space $A_2 := \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1) \subset \mathcal{M}(A_1 \otimes S_{12})$ is a C*-algebra. Moreover, we have:*

1. *$[A_2(1_{A_1} \otimes S_{12})] = A_1 \otimes S_{12} = [(1_{A_1} \otimes S_{12})A_2]$; in particular, $A_2 \subset \mathcal{M}(A_1 \otimes S_{12})$ defines a faithful nondegenerate $*$-homomorphism and $\mathcal{M}(A_2) \subset \mathcal{M}(A_1 \otimes S_{12})$;*

2. *let* $\delta_{A_2} := (\mathrm{id}_{A_1} \otimes \delta_{12}^2)\restriction_{A_2}$, *we have* $\delta_{A_2}(A_2) \subset \mathcal{M}(A_2 \otimes S_{22})$ *and* δ_{A_2} *is a continuous action of* $\mathbb{G}_2$ *on* A_2;
3. *the correspondence* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} : \mathsf{Alg}_{\mathbb{G}_1} \to \mathsf{Alg}_{\mathbb{G}_2}$ *is functorial.*

By exchanging the roles of the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, we obtain *mutatis mutandis* a functor $\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1} : \mathsf{Alg}_{\mathbb{G}_2} \to \mathsf{Alg}_{\mathbb{G}_1}$.

Proposition 5.2.6. *Let* $j,k = 1,2$ *with* $j \neq k$. *Let* (A_j, δ_{A_j}) *be a* $\mathbb{G}_j$*-C*-algebra. Let*

$$A_k := \mathrm{Ind}_{\mathbb{G}_j}^{\mathbb{G}_k}(A_j) \subset \mathcal{M}(A_j \otimes S_{jk}) \quad \textit{and}$$
$$C := \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(A_k) \subset \mathcal{M}(A_k \otimes S_{kj})$$

endowed with the continuous actions $\delta_{A_k} := (\mathrm{id}_{A_j} \otimes \delta_{jk}^k)\restriction_{A_k}$ *and* $\delta_C := (\mathrm{id}_{A_k} \otimes \delta_{kj}^j)\restriction_C$ *respectively. Then we have*:

1. $C \subset \mathcal{M}(A_k \otimes S_{kj}) \subset \mathcal{M}(A_j \otimes S_{jk} \otimes S_{kj})$ *and* $C = \delta_{A_j}^{(k)}(A_j)$;
2. $\pi_j : A_j \to C;\ a \mapsto \delta_{A_j}^{(k)}(a) := (\mathrm{id}_{A_j} \otimes \delta_{jj}^k)\delta_{A_j}(a)$ *is a* $\mathbb{G}_j$*-equivariant* $*$*-isomorphism*;
3. $\delta_{A_j}^k : A_j \to \mathcal{M}(A_k \otimes S_{kj});\ a \mapsto \delta_{A_j}^{(k)}(a) := (\mathrm{id}_{A_j} \otimes \delta_{jj}^k)\delta_{A_j}(a)$ *is a faithful nondegenerate* $*$*-homomorphism.*

The above result shows that the functors $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}$ and $\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}$ are inverse of each other.

Notation 5.2.7. Let (B_1, δ_{B_1}) be a $\mathbb{G}_1$-C*-algebra. Let (B_2, δ_{B_2}) be the induced $\mathbb{G}_2$-C*-algebra, that is to say $B_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1)$ and $\delta_{B_2} = (\mathrm{id}_{B_1} \otimes \delta_{12}^2)\restriction_{B_2}$. In virtue of Proposition 5.2.6, we have four $*$-homomorphisms:

$$\delta_{B_j}^k : B_j \to \mathcal{M}(B_k \otimes S_{kj}), \quad j,k = 1,2.$$

Let us give a precise description of them. We denote $\delta_{B_1}^1 := \delta_{B_1}$ and $\delta_{B_2}^2 := \delta_{B_2}$. The $*$-homomorphism $\delta_{B_1}^2 : B_1 \to \mathcal{M}(B_2 \otimes S_{21})$ is given by

$$b \in B_1 \mapsto \delta_{B_1}^2(b) := \delta_{B_1}^{(2)}(b)$$
$$\in \mathcal{M}(B_2 \otimes S_{21}) \quad \big(\text{with } \delta_{B_1}^{(2)}(b) := \big(\mathrm{id}_{B_1} \otimes \delta_{11}^2\big)\delta_{B_1}^1(b), \text{for } b \in B_1\big)$$

whereas the $*$-homomorphism $\delta_{B_2}^1 : B_2 \to \mathcal{M}(B_1 \otimes S_{12})$ is defined by the relation

$$(\pi_1 \otimes \mathrm{id}_{S_{12}})\delta_{B_2}^1(b) = \delta_{B_2}^{(1)}(b) \quad \text{for } b \in B_2,$$

where $\delta_{B_2}^{(1)} := (\mathrm{id}_{B_2} \otimes \delta_{22}^1)\delta_{B_2}^2$ and $\pi_1 : B_1 \to \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(B_2);\ b \mapsto \delta_{B_1}^{(2)}(b)$ (cf. Proposition 5.2.6(2)).

Proposition 5.2.8. *Let* (A, β_A, δ_A) *be a* $\mathcal{G}$*-C*-algebra. Let* $j,k = 1,2$ *with* $j \neq k$. *With the notation of Proposition* 5.2.2, *let*

$$(\widetilde{A}_j, \delta_{\widetilde{A}_j}) := \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}\big(A_k, \delta_{A_k}^k\big).$$

If $x \in A_j$, then we have $\delta^k_{A_j}(x) \in \widetilde{A}_j \subset \mathcal{M}(A_k \otimes S_{kj})$ and the map $\widetilde{\pi}_j : A_j \to \widetilde{A}_j$; $x \mapsto \delta^k_{A_j}(x)$ is a $\mathbb{G}_j$-equivariant $*$-isomorphism.

PROPOSITION 5.2.9. *Let (B_1, δ_{B_1}) be a $\mathbb{G}_1$-C^*-algebra. Let $B_2 = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(B_1)$ be the induced $\mathbb{G}_2$-C^*-algebra. Let $B := B_1 \oplus B_2$. For $j, k = 1, 2$ with $j \neq k$, let $\pi^k_j : \mathcal{M}(B_k \otimes S_{kj}) \to \mathcal{M}(B \otimes S)$ be the strictly continuous $*$-homomorphism extending the canonical injection $B_k \otimes S_{kj} \to B \otimes S$ and $\delta^k_{B_j} : B_j \to \mathcal{M}(B_k \otimes S_{kj})$ the $*$-homomorphisms defined in Notation 5.2.7. Let $\beta_B : \mathbb{C}^2 \to \mathcal{M}(B)$ and $\delta_B : B \to \mathcal{M}(B \otimes S)$ be the $*$-homomorphisms defined by:*

$$\beta_B(\lambda, \mu) := \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad (\lambda, \mu) \in \mathbb{C}^2;$$

$$\delta_B(b) := \sum_{k,j=1,2} \pi^k_j \circ \delta^k_{B_j}(b_j), \quad b = (b_1, b_2) \in B.$$

Therefore, we have:

1. *(β_B, δ_B) is a strongly continuous action of $\mathcal{G}$ on B;*
2. *the correspondence $\mathsf{Alg}_{\mathbb{G}_1} \to \mathsf{Alg}_{\mathcal{G}}$; $(B_1, \delta_{B_1}) \mapsto (B, \beta_B, \delta_B)$ is functorial;*
3. *the functors $\mathsf{Alg}_{\mathbb{G}_1} \to \mathsf{Alg}_{\mathcal{G}}$ and $\mathsf{Alg}_{\mathcal{G}} \to \mathsf{Alg}_{\mathbb{G}_1}$ are inverse of each other.*

5.3. Actions of (semi-)regular measured quantum groupoids. In this section, we fix a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$ on a finite-dimensional basis $N := \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$ endowed with the nonnormalized Markov trace $\epsilon = \bigoplus_{1 \leqslant l \leqslant k} n_l \cdot \mathrm{Tr}_l$ and we use all the notation introduced in §3.1.

We begin this section by a characterization of the regularity (resp., semi-regularity) of $\mathcal{G}$ in terms of the action of $\mathcal{G}$ on itself (cf. Examples 5.1.4), which generalizes 2.6 [5] to the setting of measured quantum groupoids on a finite basis.

PROPOSITION 5.3.1. *Let $S \rtimes \mathcal{G}$ be the crossed product of S by the strongly continuous action (β, δ) of $\mathcal{G}$. Then we have a canonical $*$-isomorphism $S \rtimes \mathcal{G} \simeq [S\widehat{S}]$. In particular, $\mathcal{G}$ is regular (resp., semi-regular) if, and only if, we have $\mathcal{K}_{\widehat{\beta}} = S \rtimes \mathcal{G}$ (resp., $\mathcal{K}_{\widehat{\beta}} \subset S \rtimes \mathcal{G}$).*

Proof. Let us identify $\mathcal{L}(\mathcal{E}_{S,L}) = \{T \in \mathcal{L}(S \otimes \mathscr{H}); Tq_{\beta\alpha} = T = q_{\beta\alpha}T\}$. Let us denote by $j_{S \rtimes \mathcal{G}} : S \rtimes \mathcal{G} \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$, the faithful $*$-representation defined by $j_{S \rtimes \mathcal{G}}(u) = (L \otimes \mathrm{id}_{\mathcal{K}})(u)$ for all $u \in S \rtimes \mathcal{G} \subset \mathcal{L}(\mathcal{E}_{S,L})$. Let $\pi : S \to \mathcal{M}(S \rtimes \mathcal{G})$ and $\widehat{\theta} : \widehat{S} \to \mathcal{M}(S \rtimes \mathcal{G})$ be the canonical morphisms (cf. Notation 5.1.13). We claim that there exists a unique $*$-isomorphism $\phi : S \rtimes \mathcal{G} \to [S\widehat{S}]$ such that

$$\phi\big(\pi(s)\widehat{\theta}(x)\big) = L(s)\rho(x), \quad \text{for all } s \in S \text{ and } x \in \widehat{S}.$$

Let $s \in S$ and $x \in \widehat{S}$. Since $W \in M \otimes \widehat{M}$ and $\rho(\widehat{S}) \subset \widehat{M}'$, we have

$$\begin{aligned} j_{S\rtimes\mathcal{G}}\big(\pi(s)\widehat{\theta}(x)\big) &= (L\otimes L)\delta(s)\big(1\otimes\rho(x)\big) \\ &= W^*\big(1\otimes L(s)\big)W\big(1\otimes\rho(x)\big) \\ &= W^*\big(1\otimes L(s)\rho(x)\big)W. \end{aligned}$$

Let $C := \mathrm{im}(j_{S\rtimes\mathcal{G}}) = \{W^*(1\otimes z)W;\, z \in [S\widehat{S}]\}$. The representation $j_{S\rtimes\mathcal{G}}$ induces a $*$-isomorphism $\psi : S \rtimes \mathcal{G} \to C$. Since $WW^* = q_{\alpha\widehat{\beta}}$ and $[1\otimes z, q_{\alpha\widehat{\beta}}] = 0$ for all $z \in [S\widehat{S}]$, the map

$$\chi : [S\widehat{S}] \to C; \qquad z \to W^*(1\otimes z)W$$

is a $*$-homomorphism satisfying $W\chi(z)W^* = q_{\alpha\widehat{\beta}}(1\otimes z)$ for all $z \in [S\widehat{S}]$. Let $\omega \in \mathcal{B}(\mathscr{H})_*$ such that $\omega\circ\alpha = \epsilon$. We have

$$(\omega\otimes\mathrm{id})\big(W\chi(z)W^*\big) = z, \quad \text{for all } z \in [S\widehat{S}].$$

Hence, χ is a $*$-isomorphism. Hence, $\phi := \chi^{-1}\circ\psi : S\rtimes\mathcal{G} \to [S\widehat{S}];\ \pi(s)\widehat{\theta}(x) \mapsto L(s)\widehat{\theta}(x)$ is a $*$-isomorphism. The second statement of the proposition follows from Proposition 4.4. □

Proposition 5.3.4 and Theorem 5.3.6 are the generalizations of 5.7 and 5.8 of [5] to measured quantum groupoids on a finite basis.

Notation 5.3.2. Let (β_A, δ_A) be an action of $\mathcal{G}$ on a C*-algebra A. With the notation of Notations A.2.21 and A.2.22, let $e_I := \alpha(\varepsilon_I)$ and $q_I := \beta_A(\varepsilon_I)$ for all $I \in \mathscr{I}$.

Lemma 5.3.3. *Let (β_A, δ_A) be an action of $\mathcal{G}$ on a C*-algebra A. With the notation of Notation 5.3.2, we have*:

1. $q_{\beta_A\alpha} = \sum_{I\in\mathscr{I}} n_I^{-1} q_I \otimes e_{\overline{I}}$;
2. $(q_I\otimes 1_S)\delta_A(a) = (1_A\otimes e_I)\delta_A(a)$, *for all* $a \in A$ *and* $I \in \mathscr{I}$;
3. $\delta_A(a)(q_I\otimes 1_S) = \delta_A(a)(1_A\otimes e_I)$, *for all* $a \in A$ *and* $I \in \mathscr{I}$.

Proof. Statement 1 is just restatement of Proposition A.2.18. By a straightforward computation, we verify that $(q_I\otimes 1_S)q_{\beta_A\alpha} = (1_A\otimes e_I)q_{\beta_A\alpha}$ for all $I \in \mathscr{I}$. Statement 2 then follows from the fact that $\delta_A(1_A) = q_{\beta_A\alpha}$. The last statement follows from the second one by taking the adjoint. □

Proposition 5.3.4. *Let (β_A, δ_A) be an action of $\mathcal{G}$ on a C*-algebra A. If $\mathcal{G}$ is semi-regular, the Banach space* $[(\mathrm{id}_A\otimes\omega)\delta_A(a);\, a \in A, \omega \in \mathcal{B}(\mathscr{H})_*] \subset \mathcal{M}(A)$ *is a C*-algebra.*

Proof. Let us denote $T := [(\mathrm{id}_A\otimes\omega)\delta_A(a);\, a \in A, \omega \in \mathcal{B}(\mathscr{H})_*]$. For all $a \in A$ and $\omega \in \mathcal{B}(\mathscr{H})_*$, we have $(\mathrm{id}_A\otimes\omega)(\delta_A(a))^* = (\mathrm{id}_A\otimes\overline{\omega})\delta_A(a^*)$. Hence, $T^* \subset T$.

Let us prove that $TT \subset T$. Let $\omega, \phi \in \mathcal{B}(\mathscr{H})_*$, $a, b \in A$ and $x, y \in \mathcal{K}$. We have

$$\begin{aligned}&(\mathrm{id}_A \otimes y\omega)\delta_A(a)(\mathrm{id}_A \otimes \phi x)\delta_A(b)\\ &\quad = (\mathrm{id}_A \otimes \phi \otimes \omega)\big(\delta_A(a)_{13}(1_A \otimes x \otimes y)\delta_A(b)_{12}\big).\end{aligned}$$

By Lemma 5.3.3(1), (2), we have

$$\begin{aligned}&\delta_A(a)_{13}(1_A \otimes x \otimes y)\delta_A(b)_{12}\\ &\quad = \sum_{I \in \mathscr{I}} n_I^{-1}\delta_A(a)_{13}(1_A \otimes x \otimes e_{\overline{I}}y)(q_I \otimes 1 \otimes 1)\delta_A(b)_{12}\\ &\quad = \sum_{I \in \mathscr{I}} n_I^{-1}\delta_A(a)_{13}(1_A \otimes xe_I \otimes e_{\overline{I}}y)\delta_A(b)_{12}\\ &\quad = \delta_A(a)_{13}\big((x \otimes 1)q_\alpha(1 \otimes y)\big)_{23}\delta_A(b)_{12}.\end{aligned}$$

It follows from Proposition 4.7 that $(\mathrm{id}_A \otimes y\omega)\delta_A(a)(\mathrm{id}_A \otimes \phi x)\delta_A(b)$ is the norm limit of finite sums of elements of the form

$$\begin{aligned}c &:= (\mathrm{id}_A \otimes \phi \otimes \omega)\big(\delta_A(a)_{13}\big((x' \otimes 1)W\big(1 \otimes y'\big)\big)_{23}\delta_A(b)_{12}\big)\\ &= \big(\mathrm{id}_A \otimes \phi x' \otimes y'\omega\big)\big(\delta_A(a)_{13}W_{23}\delta_A(b)_{12}\big),\end{aligned}$$

where $x', y' \in \mathcal{K}$. By combining the following formulas;

$$\delta_A^2(a) = W_{23}^*\delta_A(a)_{13}W_{23}; \qquad WW^* = q_{\alpha\widehat{\beta}};$$
$$\big[\delta_A(a)_{13}, q_{\alpha\widehat{\beta},23}\big] = 0 \quad \big(\text{since } \widehat{\beta}\big(N^{\mathrm{o}}\big) \subset M'\big);$$

we obtain $\delta_A(a)_{13}W_{23} = W_{23}\delta_A^2(a)$. Hence, we have $c = (\mathrm{id}_A \otimes \psi)(\delta_A^2(a)_{13} \times \delta_A(b)_{12})$, where $\psi := (\phi x' \otimes y\omega)W \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})_*$. Therefore, c is the norm limit of finite sums of elements of the form $(\mathrm{id}_A \otimes \phi' \otimes \omega')(\delta_A^2(a)_{13}\delta_A(b)_{12}) = (\mathrm{id} \otimes \phi')\delta_A((\mathrm{id}_A \otimes \omega')\delta_A(a)b)$, where $\phi', \omega' \in \mathcal{B}(\mathscr{H})_*$. Hence, $(\mathrm{id}_A \otimes y\omega) \times \delta_A(a)(\mathrm{id}_A \otimes \phi x)\delta_A(b) \in T$. □

Definition 5.3.5. Let (β_A, δ_A) be an action of $\mathcal{G}$ on a C*-algebra A. We say that (β_A, δ_A) is weakly continuous if we have $A = [(\mathrm{id}_A \otimes \omega)\delta_A(a);\, a \in A, \omega \in \mathcal{B}(\mathscr{H})_*]$.

Note that any strongly continuous action (β_A, δ_A) of $\mathcal{G}$ on a C*-algebra A is necessarily continuous in the weak sense. Indeed, if (β_A, δ_A) is strongly continuous we have the inclusion $\delta_A(A)(1_A \otimes S) \subset A \otimes S$. Hence, $[(\mathrm{id}_A \otimes \omega)\delta_A(a);\, a \in A, \omega \in \mathcal{B}(\mathscr{H})_*] \subset A$ since $S \subset \mathcal{B}(\mathscr{H})$ is nondegenerate. Conversely, let $\omega \in \mathcal{B}(\mathscr{H})_*$ such that $\omega \circ \alpha = \epsilon$. We have $(\mathrm{id}_A \otimes \omega)(q_{\beta_A\alpha}) = 1_A$. By writing $\omega = y\omega'$ for some $\omega' \in \mathcal{B}(\mathscr{H})_*$ and $y \in S$, we obtain $(\mathrm{id}_A \otimes \omega')(q_{\beta_A\alpha}(a \otimes y)) = a$ for all $a \in A$.

Theorem 5.3.6. *If the groupoid $\mathcal{G}$ is regular, then any weakly continuous action of $\mathcal{G}$ is strongly continuous.*

Proof. Let us fix an action (β_A, δ_A) of $\mathcal{G}$ on a C*-algebra A. Let us assume that (β_A, δ_A) is weakly continuous. Since $W \in \mathcal{M}(S \otimes \mathcal{K})$ is a partial isometry such that $W^*W = q_{\beta\alpha}$, we have $(S \otimes \mathcal{K})W = (S \otimes \mathcal{K})q_{\beta\alpha}$. We recall that $\delta_A(a)_{13}W_{23} = W_{23}\delta_A^2(a)$ for all $a \in A$ (cf. proof of Proposition 5.3.4). By Proposition 5.1.5(1), we have $q_{\beta\alpha,23}\delta_A^2(a) = \delta_A^2(a)$ for all $a \in A$. By combining the assertions of the above discussion with Lemma 5.3.3(3) and Proposition 4.8(1), we have

$$
\begin{aligned}
&(A \otimes S)q_{\beta_A\alpha}\\
&\quad = \big[\big((\mathrm{id}_A \otimes \omega)\delta_A(a) \otimes y\big)q_{\beta_A\alpha};\, a \in A, y \in S, \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(\mathrm{id}_A \otimes \mathrm{id}_S \otimes x\omega)\big(\delta_A(a)_{13}(1_A \otimes y \otimes 1_S)q_{\beta_A\alpha,12}\big);\, a \in A, x \in \mathcal{K}, y \in S,\\
&\qquad \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(\mathrm{id}_A \otimes \mathrm{id}_S \otimes \omega)\big(\delta_A(a)_{13}\big((y \otimes 1_{\mathcal{K}})q_\alpha(1_S \otimes x)\big)_{23}\big);\, a \in A, x \in \mathcal{K}, y \in S,\\
&\qquad \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(\mathrm{id}_A \otimes \mathrm{id}_S \otimes \omega)\big(\delta_A(a)_{13}\big((y \otimes 1_{\mathcal{K}})W(1_S \otimes x)\big)_{23}\big);\, a \in A, x \in \mathcal{K}, y \in S,\\
&\qquad \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(\mathrm{id}_A \otimes \mathrm{id}_S \otimes \omega)\big((1_A \otimes y \otimes 1)\delta_A(a)_{13}W_{23}\big);\, a \in A, x \in \mathcal{K}, y \in S,\\
&\qquad \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(\mathrm{id}_A \otimes \mathrm{id}_S \otimes \omega)\big(\big(1_A \otimes (y \otimes x)W\big)\delta_A^2(a)\big);\, a \in A, x \in \mathcal{K}, y \in S,\\
&\qquad \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(\mathrm{id}_A \otimes \mathrm{id}_S \otimes \omega)\big((1_A \otimes y \otimes x)\delta_A^2(a)\big);\, a \in A, x \in \mathcal{K}, y \in S, \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(1_A \otimes y)\delta_A\big((\mathrm{id}_A \otimes \omega x)\delta_A(a)\big);\, a \in A, x \in \mathcal{K}, y \in S, \omega \in \mathcal{B}(\mathscr{H})_*\big]\\
&\quad = \big[(1_A \otimes S)\delta_A(A)\big]. \qquad \square
\end{aligned}
$$

As a first application, we have the result below.

PROPOSITION 5.3.7. *If the groupoid $\mathcal{G}$ is regular, then the trivial action of $\mathcal{G}$ on N^{o} (cf. Examples 5.1.4) is strongly continuous and there exists a unique $\widehat{\mathcal{G}}$-equivariant $*$-isomorphism $\tau : N^{\mathrm{o}} \rtimes \mathcal{G} \to \widehat{S}$ such that $\tau(\pi(n^{\mathrm{o}})\widehat{\theta}(x)) = \beta(n^{\mathrm{o}})x$ for all $n \in N$ and $x \in \widehat{S}$.*

Proof. In this proof, we use the notation of Notation 5.3.2 with $A := N^{\mathrm{o}}$. In this case, we have $q_I = \varepsilon_I^{\mathrm{o}}$ for all $I \in \mathscr{I}$. According to Theorem 5.3.6, it suffices to show that the trivial action is weakly continuous. Since the C*-algebra N is finite-dimensional, it amounts to proving that $\varepsilon_I^{\mathrm{o}} \in \langle (\mathrm{id}_{N^{\mathrm{o}}} \otimes \omega)\delta_{N^{\mathrm{o}}}(n^{\mathrm{o}});\, n \in N, \omega \in \mathcal{B}(\mathscr{H})_*\rangle$ for all $I \in \mathscr{I}$. Let $I \in \mathscr{I}$. For all $n' \in N$, there exists $\omega \in \mathcal{B}(\mathscr{H})_*$ such that $\omega(\alpha(n)) = \epsilon(n'n)$ for all $n \in N$ (extension of normal linear forms). In particular, there exists $\omega \in \mathcal{B}(\mathscr{H})_*$ such that $\omega(\alpha(\varepsilon_J)) = n_I\delta_J^I$ for all $J \in \mathscr{I}$. By a straightforward computation, we have $\varepsilon_I^{\mathrm{o}} = (\mathrm{id}_{N^{\mathrm{o}}} \otimes \omega)\delta_{N^{\mathrm{o}}}(1_{N^{\mathrm{o}}})$

and the weak continuity of the trivial action is then proved since N^{o} is unital. Since N^{o} is unital, we have $\widehat{\theta}(x) = \pi(1_{N^{\mathrm{o}}})\widehat{\theta}(x) \in N^{\mathrm{o}} \rtimes \mathcal{G}$ for all $x \in \widehat{S}$. Moreover, we have $\pi(n^{\mathrm{o}})\widehat{\theta}(x) = \widehat{\theta}(\beta(n^{\mathrm{o}})x)$ for all $n \in N$ and $x \in \widehat{S}$. Hence, the morphism $\widehat{\theta}$ induces a $*$-isomorphism $\widehat{S} \simeq N^{\mathrm{o}} \rtimes \mathcal{G}$. The equivariance is easily obtained from the definitions. □

6. Notion of equivariant Hilbert C*-modules

6.1. Actions of measured quantum groupoids on Hilbert C*-modules. In this section, we introduce a notion of $\mathcal{G}$-equivariant Hilbert C*-module for a measured quantum groupoid $\mathcal{G}$ on a finite basis in the spirit of [3]. We fix a measured quantum groupoid $\mathcal{G}$ on a finite-dimensional basis $N = \bigoplus_{1\leqslant l\leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$ endowed with the nonnormalized Markov trace $\epsilon = \bigoplus_{1\leqslant l\leqslant k} n_l \cdot \mathrm{Tr}_l$. We use all the notation introduced in §3.1, §3.2. Let us fix a $\mathcal{G}$-C*-algebra A.

The three pictures. Following §2 [3], an action of $\mathcal{G}$ on a Hilbert A-module $\mathscr{E}$ will be defined by three equivalent data:

- a pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ consisting of a $*$-homomorphism $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ and a linear map $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$; cf. Definition 6.1.1;
- a pair $(\beta_{\mathscr{E}}, \mathscr{V}_{\mathscr{E}})$ consisting of a $*$-homomorphism $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ and an isometry $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$; cf. Definition 6.1.4;
- an action (β_J, δ_J) of $\mathcal{G}$ on $J := \mathcal{K}(\mathscr{E} \oplus A)$; cf. Definition 6.1.8;

satisfying some conditions.

We have the following unitary equivalences of Hilbert modules:

$$\begin{aligned} A \otimes_{\delta_A} (A \otimes S) &\to q_{\beta_A\alpha}(A \otimes S);\\ a \otimes_{\delta_A} x &\mapsto \delta_A(a)x; \end{aligned} \tag{6.1}$$

$$\begin{aligned} (A \otimes S) \otimes_{\delta_A\otimes \mathrm{id}_S} (A \otimes S \otimes S) &\to q_{\beta_A\alpha,12}(A \otimes S \otimes S);\\ x \otimes_{\delta_A\otimes \mathrm{id}_S} y &\mapsto (\delta_A \otimes \mathrm{id}_S)(x)y; \end{aligned} \tag{6.2}$$

$$\begin{aligned} (A \otimes S) \otimes_{\mathrm{id}_A\otimes\delta} (A \otimes S \otimes S) &\to q_{\beta\alpha,23}(A \otimes S \otimes S);\\ x \otimes_{\mathrm{id}_A\otimes\delta} y &\mapsto (\mathrm{id}_A \otimes \delta)(x)y. \end{aligned} \tag{6.3}$$

In the following, we fix a Hilbert A-module $\mathscr{E}$. We will apply the usual identifications $\mathcal{M}(A \otimes S) = \mathcal{L}(A \otimes S)$, $\mathcal{K}(\mathscr{E}) \otimes S = \mathcal{K}(\mathscr{E} \otimes S)$ and $\mathcal{M}(\mathcal{K}(\mathscr{E}) \otimes S) = \mathcal{L}(\mathscr{E} \otimes S)$.

Definition 6.1.1. An action of $\mathcal{G}$ on the Hilbert A-module $\mathscr{E}$ is a pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$, where $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ is a nondegenerate $*$-homomorphism and $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ is a linear map such that:

1. for all $a \in A$ and $\xi, \eta \in \mathscr{E}$, we have
$$\delta_{\mathscr{E}}(\xi a) = \delta_{\mathscr{E}}(\xi)\delta_A(a) \quad \text{and} \quad \big\langle \delta_{\mathscr{E}}(\xi), \delta_{\mathscr{E}}(\eta)\big\rangle = \delta_A\big(\langle \xi, \eta\rangle\big);$$
2. $[\delta_{\mathscr{E}}(\mathscr{E})(A \otimes S)] = q_{\beta_{\mathscr{E}}\alpha}(\mathscr{E} \otimes S)$;

3. for all $\xi \in \mathscr{E}$ and $n \in N$, we have $\delta_{\mathscr{E}}(\beta_{\mathscr{E}}(n^{\mathrm{o}})\xi) = (1_{\mathscr{E}} \otimes \beta(n^{\mathrm{o}}))\delta_{\mathscr{E}}(\xi)$;
4. the linear maps $\delta_{\mathscr{E}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathscr{E}} \otimes \delta$ extend to linear maps from $\mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$ to $\mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$ and we have

$$(\delta_{\mathscr{E}} \otimes \mathrm{id}_S)\delta_{\mathscr{E}}(\xi) = (\mathrm{id}_{\mathscr{E}} \otimes \delta)\delta_{\mathscr{E}}(\xi) \in \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S), \quad \text{for all } \xi \in \mathscr{E}.$$

Remarks 6.1.2.

- If the second formula of the condition 1 holds, then $\delta_{\mathscr{E}}$ is isometric (cf. [4], Remarks A.3.2(1)).
- If the condition 1 holds, then the condition 2 is equivalent to:

$$\big[\delta_{\mathscr{E}}(\mathscr{E})(1_A \otimes S)\big] = q_{\beta_{\mathscr{E}}\alpha}(\mathscr{E} \otimes S).$$

  Indeed, if $(u_\lambda)_\lambda$ is an approximate unit of A we have

$$\delta_{\mathscr{E}}(\xi) = \lim_\lambda \delta_{\mathscr{E}}(\xi u_\lambda) = \lim_\lambda \delta_{\mathscr{E}}(\xi)\delta_A(u_\lambda) = \delta_{\mathscr{E}}(\xi) q_{\beta_A\alpha}, \quad \text{for all } \xi \in \mathscr{E}.$$

  By strong continuity of the action (β_A, δ_A), the condition 1 of Definition 6.1.1 and the equality $\mathscr{E}A = \mathscr{E}$, we then have $[\delta_{\mathscr{E}}(\mathscr{E})(A \otimes S)] = [\delta_{\mathscr{E}}(\mathscr{E})(1_A \otimes S)]$ and the equivalence follows.
- Note that we have $q_{\beta_{\mathscr{E}}\alpha}\delta_{\mathscr{E}}(\xi) = \delta_{\mathscr{E}}(\xi) = \delta_{\mathscr{E}}(\xi) q_{\beta_A\alpha}$ for all $\xi \in \mathscr{E}$.
- We will prove (cf. Remarks 6.1.7) that if $\delta_{\mathscr{E}}$ satisfies the conditions 1 and 2 of Definition 6.1.1, then the extensions of $\delta_{\mathscr{E}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathscr{E}} \otimes \delta$ always exist and satisfy the formulas:

$$(\mathrm{id}_{\mathscr{E}} \otimes \delta)(T)(\mathrm{id}_A \otimes \delta)(x) = (\mathrm{id}_{\mathscr{E}} \otimes \delta)(Tx);$$
$$(\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(T)(\delta_A \otimes \mathrm{id}_S)(x) = (\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(Tx);$$

  for all $x \in A \otimes S$ and $T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$.

Notation 6.1.3. For $\xi \in \mathscr{E}$, let us denote by $T_\xi \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes_{\delta_A} (A \otimes S))$ the operator defined by

$$T_\xi(x) := \xi \otimes_{\delta_A} x, \quad \text{for all } x \in A \otimes S.$$

We have $T_\xi^*(\eta \otimes_{\delta_A} y) = \delta_A(\langle \xi, \eta \rangle) y$ for all $\eta \in \mathscr{E}$ and $y \in A \otimes S$. In particular, we have $T_\xi^* T_\eta = \delta_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathscr{E}$.

Definition 6.1.4. Let $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ be an isometry and $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ a nondegenerate $*$-homomorphism such that:

1. $\mathscr{V}\mathscr{V}^* = q_{\beta_{\mathscr{E}}\alpha}$;
2. $\mathscr{V}(\beta_{\mathscr{E}}(n^{\mathrm{o}}) \otimes_{\delta_A} 1) = (1_{\mathscr{E}} \otimes \beta(n^{\mathrm{o}}))\mathscr{V}$, for all $n \in N$.

Then, $\mathscr{V}$ is said to be admissible if we further have:

3. $\mathscr{V} T_\xi \in \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$, for all $\xi \in \mathscr{E}$;
4. $(\mathscr{V} \otimes_{\mathbb{C}} \mathrm{id}_S)(\mathscr{V} \otimes_{\delta_A \otimes \mathrm{id}_S} 1) = \mathscr{V} \otimes_{\mathrm{id}_A \otimes \delta} 1 \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A^2} (A \otimes S \otimes S), \mathscr{E} \otimes S \otimes S)$.

The fourth statement in the previous definition makes sense since we have used the canonical identifications thereafter. By combining the associativity of the internal tensor product with the unitary equivalences (6.2) and (6.3), we obtain the following unitary equivalences of Hilbert $A \otimes S$-modules:

$$\begin{aligned}(6.4)\qquad \big(\mathscr{E} \otimes_{\delta_A} (A \otimes S)\big) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S) &\to \mathscr{E} \otimes_{\delta_A^2} (A \otimes S \otimes S),\\ (\xi \otimes_{\delta_A} x) \otimes_{\delta_A \otimes \mathrm{id}_S} y &\mapsto \xi \otimes_{\delta_A^2} (\delta_A \otimes \mathrm{id}_S)(x)y;\end{aligned}$$

$$\begin{aligned}(6.5)\qquad \big(\mathscr{E} \otimes_{\delta_A} (A \otimes S)\big) \otimes_{\mathrm{id}_A \otimes \delta} (A \otimes S \otimes S) &\to \mathscr{E} \otimes_{\delta_A^2} (A \otimes S \otimes S),\\ (\xi \otimes_{\delta_A} x) \otimes_{\mathrm{id}_A \otimes \delta} y &\mapsto \xi \otimes_{\delta_A^2} (\mathrm{id}_A \otimes \delta)(x)y.\end{aligned}$$

We also have the following:

$$\begin{aligned}(6.6)\qquad (\mathscr{E} \otimes S) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S) &\to \big(\mathscr{E} \otimes_{\delta_A} (A \otimes S)\big) \otimes S,\\ (\xi \otimes s) \otimes_{\delta_A \otimes \mathrm{id}_S} (x \otimes t) &\mapsto (\xi \otimes_{\delta_A} x) \otimes st;\end{aligned}$$

$$\begin{aligned}(6.7)\qquad (\mathscr{E} \otimes S) \otimes_{\mathrm{id}_A \otimes \delta} (A \otimes S \otimes S) &\to q_{\beta\alpha,23}(\mathscr{E} \otimes S \otimes S) \subset \mathscr{E} \otimes S \otimes S,\\ \xi \otimes_{\mathrm{id}_A \otimes \delta} y &\mapsto (\mathrm{id}_{\mathscr{E}} \otimes \delta)(\xi)y.\end{aligned}$$

In particular, $\mathscr{V} \otimes_{\delta_A \otimes \mathrm{id}_S} 1 \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A^2} (A \otimes S \otimes S), (\mathscr{E} \otimes S) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S))$ (6.4) and $\mathscr{V} \otimes_{\mathbb{C}} \mathrm{id}_S \in \mathcal{L}((\mathscr{E} \otimes S) \otimes_{\delta_A \otimes \mathrm{id}_S} (A \otimes S \otimes S), \mathscr{E} \otimes S \otimes S)$ (6.6).

The next result provides an equivalence of Definitions 6.1.1 and 6.1.4.

Proposition 6.1.5.

(a) *Let $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ be a linear map and $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ a nondegenerate $*$-homomorphism which satisfy the conditions* 1, 2, *and* 3 *of Definition* 6.1.1. *Then there exists a unique isometry $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ such that $\delta_{\mathscr{E}}(\xi) = \mathscr{V} T_\xi$ for all $\xi \in \mathscr{E}$. Moreover, the pair $(\beta_{\mathscr{E}}, \mathscr{V})$ satisfies the conditions* 1, 2, *and* 3 *of Definition* 6.1.4.

(b) *Conversely, let $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ be an isometry and $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ a nondegenerate $*$-homomorphism, which satisfy the conditions* 1, 2, *and* 3 *of Definition* 6.1.4. *Let us consider the map $\delta_{\mathscr{E}} : \mathscr{E} \to \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$ given by $\delta_{\mathscr{E}}(\xi) := \mathscr{V} T_\xi$ for all $\xi \in \mathscr{E}$. Then the pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ satisfies the conditions* 1, 2, *and* 3 *of Definition* 6.1.1.

(c) *Let us assume that the above statements hold. Then the pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is an action of $\mathcal{G}$ on $\mathscr{E}$ if, and only if, $\mathscr{V}$ is admissible.*

In the proof, we will use the following notation.

Notation 6.1.6. Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert C*-modules. Let $q \in \mathcal{L}(\mathcal{E})$ be a self-adjoint projection and $T \in \mathcal{L}(q\mathcal{E}, \mathcal{F})$. Let $\widetilde{T} : \mathcal{E} \to \mathcal{F}$ be the map defined by $\widetilde{T}\xi := Tq\xi$, for all $\xi \in \mathcal{E}$. Therefore, $\widetilde{T} \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ and $\widetilde{T}^* = qT^*$. By abuse of notation, we will still denote by T the adjointable operator $\widetilde{T}$.

Proof of Proposition 6.1.5. (a) By definition of the internal tensor product and Definition 6.1.1(1), there exists a unique isometric $(A \otimes S)$-linear map

$\mathscr{V} : \mathscr{E} \otimes_{\delta_A} (A \otimes S) \to \mathscr{E} \otimes S$ such that

$$\mathscr{V}(\xi \otimes_{\delta_A} x) = \delta_{\mathscr{E}}(\xi)x, \quad \text{for all } \xi \in \mathscr{E} \text{ and } x \in A \otimes S.$$

In other words, we have $\mathscr{V}T_\xi = \delta_{\mathscr{E}}(\xi)$ for all $\xi \in \mathscr{E}$. Now, it follows from Definition 6.1.1(2) that the ranges of $\mathscr{V}$ and $q_{\beta_{\mathscr{E}}\alpha}$ are equal. Then denote by v the range restriction of $\mathscr{V}$. Hence, the map $v^{-1}q_{\beta_{\mathscr{E}}\alpha}$ is an adjoint for $\mathscr{V}$. Indeed, for all $x \in \mathscr{E} \otimes S$ and $y \in \mathscr{E} \otimes_{\delta_A} (A \otimes S)$ we have

$$\begin{aligned}
\big\langle v^{-1}q_{\beta_{\mathscr{E}}\alpha}x, y\big\rangle &= \big\langle \mathscr{V}v^{-1}(q_{\beta_{\mathscr{E}}\alpha}x), \mathscr{V}y\big\rangle \quad (\mathscr{V} \text{ is isometric}) \\
&= \langle q_{\beta_{\mathscr{E}}\alpha}x, \mathscr{V}y\rangle \\
&= \langle x, \mathscr{V}y\rangle - \big\langle (1 - q_{\beta_{\mathscr{E}}\alpha})(x), \mathscr{V}y\big\rangle \\
&= \langle x, \mathscr{V}y\rangle \quad \big(\mathscr{V}y \in \operatorname{im}(q_{\beta_{\mathscr{E}}\alpha})\big).
\end{aligned}$$

Hence, $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ and then $\mathscr{V}^*\mathscr{V} = 1$ and $\mathscr{V}\mathscr{V}^* = \mathscr{V}v^{-1}q_{\beta_{\mathscr{E}}\alpha} = q_{\beta_{\mathscr{E}}\alpha}$.

The conditions 1 and 3 of Definition 6.1.4 are then fulfilled. Now, we have

$$\begin{aligned}
\mathscr{V}\big(\beta_{\mathscr{E}}\big(n^{\mathrm{o}}\big) \otimes_{\delta_A} 1\big)(\xi \otimes_{\delta_A} x) &= \delta_{\mathscr{E}}\big(\beta_{\mathscr{E}}\big(n^{\mathrm{o}}\big)\xi\big)x \\
&= \big(1_{\mathscr{E}} \otimes \beta\big(n^{\mathrm{o}}\big)\big)\delta_{\mathscr{E}}(\xi)x \\
&= \big(1_{\mathscr{E}} \otimes \beta\big(n^{\mathrm{o}}\big)\big)\mathscr{V}(\xi \otimes_{\delta_A} x),
\end{aligned}$$

for all $\xi \in \mathscr{E}$, $x \in A \otimes S$ and $n \in N$. Hence, the condition 2 of Definition 6.1.4 holds.

(b) is straightforward.

(c) Let $T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$. By using Notation 6.1.6 and the identifications (6.3), (6.7), we have $T \otimes_{\mathrm{id}_A \otimes \delta} 1 \in \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$. Now, we can define the extension

$$\mathrm{id}_{\mathscr{E}} \otimes \delta : \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$$

by setting

$$(\mathrm{id}_{\mathscr{E}} \otimes \delta)(T) := T \otimes_{\mathrm{id}_A \otimes \delta} 1, \quad \text{for all } T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S).$$

We also have $T \otimes_{\delta_A \otimes \mathrm{id}_S} 1 \in \mathcal{L}(A \otimes S \otimes S, (\mathscr{E} \otimes_{\delta_A} (A \otimes S)) \otimes S)$ by using again Notation 6.1.6 and the identifications (6.2), (6.6). Let us define the extension

$$\delta_{\mathscr{E}} \otimes \mathrm{id}_S : \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$$

by setting

$$(\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(T) := (\mathscr{V} \otimes_{\mathbb{C}} 1_S)(T \otimes_{\delta_A \otimes \mathrm{id}_S} 1), \quad \text{for all } T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S).$$

Therefore, for all $\xi \in \mathscr{E}$ we have:

$$\begin{aligned}
(\delta_{\mathscr{E}} \otimes \mathrm{id}_S)\delta_{\mathscr{E}}(\xi) &= (\mathscr{V} \otimes_{\mathbb{C}} 1_S)(\mathscr{V} \otimes_{\delta_A \otimes \mathrm{id}_S} 1)(T_\xi \otimes_{\delta_A \otimes \mathrm{id}_S} 1) \\
&\in \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S); \\
(\mathrm{id}_{\mathscr{E}} \otimes \delta)\delta_{\mathscr{E}}(\xi) &= (\mathscr{V} \otimes_{\mathrm{id}_A \otimes \delta} 1)(T_\xi \otimes_{\mathrm{id}_A \otimes \delta} 1) \\
&\in \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S);
\end{aligned}$$

where:

$$T_\xi \otimes_{\delta_A\otimes \mathrm{id}_S} 1 \in \mathcal{L}\big(A\otimes S\otimes S, \mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S)\big);$$
$$T_\xi \otimes_{\mathrm{id}_A\otimes\delta} 1 \in \mathcal{L}\big(A\otimes S\otimes S, \mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S)\big);$$

by using the identifications (6.2), (6.4) and (6.3), (6.5), respectively, and Notation 6.1.6. In particular, if $\mathscr{V}$ is admissible, then the condition 4 of Definition 6.1.1 holds.

Conversely, let us assume that the above condition is satisfied. In order to show that $\mathscr{V}$ is admissible, we only have to prove that the restrictions of the operators $T_\xi \otimes_{\delta_A\otimes\mathrm{id}_S} 1$ and $T_\xi \otimes_{\mathrm{id}_A\otimes\delta} 1$ to the Hilbert submodule $q_{\beta_A\alpha,12}q_{\beta\alpha,23}(A\otimes S\otimes S)$ are surjective.

Let $a\in A$, $x\in A\otimes S$ and $y\in A\otimes S\otimes S$. Let us set $z=(\delta_A\otimes\mathrm{id}_S)(\delta_A(a)x)y$. It is clear that $z\in q_{\beta_A\alpha,12}q_{\beta\alpha,23}(A\otimes S\otimes S)$. By a straightforward computation, we have

$$(T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1)(z)=\xi a\otimes_{\delta_A^2}(\delta_A\otimes\mathrm{id}_S)(x)y.$$

Hence, the restriction of $T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1$ to $q_{\beta_A\alpha,12}q_{\beta\alpha,23}(A\otimes S\otimes S)$ is surjective in virtue of (6.4) and the fact that $\mathscr{E}A=\mathscr{E}$. The same statement is also true for $T_\xi\otimes_{\mathrm{id}_A\otimes\delta}1$. □

Remarks 6.1.7. In the proof of Proposition 6.1.5, we have proved the statements below.

- By applying Notation 6.1.6 and the identifications (6.3), (6.7), we have obtained a linear map $\mathrm{id}_{\mathscr{E}}\otimes\delta : \mathcal{L}(A\otimes S,\mathscr{E}\otimes S)\to\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S)$ given by
$$(\mathrm{id}_{\mathscr{E}}\otimes\delta)(T):=T\otimes_{\mathrm{id}_A\otimes\delta}1,\quad \text{for all } T\in\mathcal{L}(A\otimes S,\mathscr{E}\otimes S);$$
- If $\delta_{\mathscr{E}}$ satisfies the conditions 1 and 2 of Definition 6.1.1, let $\mathscr{V}$ be the isometry associated with $\delta_{\mathscr{E}}$ (cf. Proposition 6.1.5(a)). By applying Notation 6.1.6 and the identifications (6.2), (6.6), the linear map $\delta_{\mathscr{E}}\otimes\mathrm{id}_S:\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)\to\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S)$ is defined by
$$(\delta_{\mathscr{E}}\otimes\mathrm{id}_S)(T):=(\mathscr{V}\otimes_{\mathbb{C}}1_S)(T\otimes_{\delta_A\otimes\mathrm{id}_S}1),\quad \text{for all } T\in\mathcal{L}(A\otimes S,\mathscr{E}\otimes S).$$

Note that the extensions $\mathrm{id}_{\mathscr{E}}\otimes\delta$ and $\delta_{\mathscr{E}}\otimes\mathrm{id}_S$ satisfy the following formulas:

$$\begin{aligned}(\mathrm{id}_{\mathscr{E}}\otimes\delta)(T)(\mathrm{id}_A\otimes\delta)(x)&=(\mathrm{id}_{\mathscr{E}}\otimes\delta)(Tx);\\ (\delta_{\mathscr{E}}\otimes\mathrm{id}_S)(T)(\delta_A\otimes\mathrm{id}_S)(x)&=(\delta_{\mathscr{E}}\otimes\mathrm{id}_S)(Tx);\end{aligned}\tag{6.8}$$

for all $x\in A\otimes S$ and $T\in\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)$.

Let us denote by $J:=\mathcal{K}(\mathscr{E}\oplus A)$ the linking C*-algebra associated with the Hilbert A-module $\mathscr{E}$. In the following, we apply the usual identifications $\mathcal{M}(J)=\mathcal{L}(\mathscr{E}\oplus A)$ and $\mathcal{M}(J\otimes S)=\mathcal{L}((\mathscr{E}\otimes S)\oplus(A\otimes S))$.

Definition 6.1.8. An action (β_J,δ_J) of $\mathcal{G}$ on J is said to be compatible with the action (β_A,δ_A) if:

1. $\delta_J : J \to \mathcal{M}(J \otimes S)$ is compatible with δ_A, *that is*, $\iota_{A\otimes S} \circ \delta_A = \delta_J \circ \iota_A$;
2. $\beta_J : N^{\mathrm{o}} \to \mathcal{M}(J)$ is compatible with β_A, *that is*, $\iota_A(\beta_A(n^{\mathrm{o}})a) = \beta_J(n^{\mathrm{o}}) \times \iota_A(a)$, for all $n \in N$ and $a \in A$.

Proposition 6.1.9. *Let (β_J, δ_J) be a compatible action of $\mathcal{G}$ on J. There exists a unique non-degenerate $*$-homomorphism $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ such that*

$$\beta_J\big(n^{\mathrm{o}}\big) = \begin{pmatrix} \beta_{\mathscr{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix}, \quad \textit{for all } n \in N.$$

Moreover, we have

$$q_{\beta_J \alpha} = \begin{pmatrix} q_{\beta_{\mathscr{E}}\alpha} & 0 \\ 0 & q_{\beta_A \alpha} \end{pmatrix}.$$

Proof. Note that since ι_A, β_A, and β_J are $*$-homomorphisms, the condition 2 of Definition 6.1.8 is equivalent to

$$\iota_A\big(a\beta_A\big(n^{\mathrm{o}}\big)\big) = \iota_A(a)\beta_J\big(n^{\mathrm{o}}\big), \quad \text{for all } a \in A, n \in N.$$

Therefore, there exists a map $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ necessarily unique such that

$$\beta_J\big(n^{\mathrm{o}}\big) = \begin{pmatrix} \beta_{\mathscr{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix},$$

for all $n \in N$. Then it is clear that $\beta_{\mathscr{E}}$ is a nondegenerate $*$-homomorphism and the last statement is then an immediate consequence. □

Remarks 6.1.10. Note that if β_A is injective, then so is β_J. For all $n \in N$, $\xi \in \mathscr{E}$ and $k \in \mathcal{K}(\mathscr{E})$, we have $\iota_{\mathcal{K}(\mathscr{E})}(\beta_{\mathscr{E}}(n^{\mathrm{o}})k) = \beta_J(n^{\mathrm{o}})\iota_{\mathcal{K}(\mathscr{E})}(k)$ and $\iota_{\mathscr{E}}(\beta_{\mathscr{E}}(n^{\mathrm{o}})\xi) = \beta_J(n^{\mathrm{o}})\iota_{\mathscr{E}}(\xi)$. In particular, we have $\beta_{\mathscr{E}}(n^{\mathrm{o}})\theta_{\xi,\eta} = \theta_{\beta_{\mathscr{E}}(n^{\mathrm{o}})\xi,\eta}$ for all $n \in N$ and $\xi, \eta \in \mathscr{E}$ (cf. Proposition 2.3.2(2)).

Proposition 6.1.11.

(a) *Let us assume that the C^*-algebra J is endowed with a compatible action (β_J, δ_J) of $\mathcal{G}$ such that $\delta_J(J) \subset \widetilde{\mathcal{M}}(J \otimes S)$. Then we have the following statements*:
- *there exists a unique linear map $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ such that $\iota_{\mathscr{E}\otimes S} \circ \delta_{\mathscr{E}} = \delta_J \circ \iota_{\mathscr{E}}$; furthermore, the pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is an action of $\mathcal{G}$ on $\mathscr{E}$, where $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ is the $*$-homomorphism defined in Proposition 6.1.9*;
- *there exists a unique faithful $*$-homomorphism $\delta_{\mathcal{K}(\mathscr{E})} : \mathcal{K}(\mathscr{E}) \to \widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}) \otimes S)$ such that $\iota_{\mathcal{K}(\mathscr{E}\otimes S)} \circ \delta_{\mathcal{K}(\mathscr{E})} = \delta_J \circ \iota_{\mathcal{K}(\mathscr{E})}$; moreover, the pair $(\beta_{\mathscr{E}}, \delta_{\mathcal{K}(\mathscr{E})})$ is an action of $\mathcal{G}$ on $\mathcal{K}(\mathscr{E})$.*

(b) *Conversely, let $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ be an action of $\mathcal{G}$ on the Hilbert A-module $\mathscr{E}$. Then there exists a faithful $*$-homomorphism $\delta_J : J \to \widetilde{\mathcal{M}}(J \otimes S)$ such that $\iota_{\mathscr{E}\otimes S} \circ \delta_{\mathscr{E}} = \delta_J \circ \iota_{\mathscr{E}}$. Moreover, we define a unique action (β_J, δ_J) of $\mathcal{G}$ on J compatible with (β_A, δ_A) by setting*

$$\beta_J\big(n^{\mathrm{o}}\big) = \begin{pmatrix} \beta_{\mathscr{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix}, \quad \textit{for all } n \in N.$$

Proof. (a) Let us assume that the C*-algebra J is endowed with a compatible action (β_J, δ_J) of $\mathcal{G}$. Let $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ be the $*$-homomorphism defined in Proposition 6.1.9. By strict continuity and Definition 6.1.8(1), we have $\delta_J(\iota_A(m)) = \iota_{A\otimes S}(\delta_A(m))$ for all $m \in \mathcal{M}(A)$. It then follows from Proposition 6.1.9 that

$$\begin{aligned}\delta_J\big(\iota_{\mathcal{K}(\mathscr{E})}(1_{\mathscr{E}})\big) &= \delta_J(1_J) - \delta_J\big(\iota_A(1_A)\big) = q_{\beta_J\alpha} - \iota_{A\otimes S}(q_{\beta_A\alpha}) \\ &= \iota_{\mathcal{K}(\mathscr{E}\otimes S)}(q_{\beta_{\mathscr{E}}\alpha}).\end{aligned}$$

Let $\xi \in \mathscr{E}$. We have $\iota_{\mathcal{K}(\mathscr{E})}(1_{\mathscr{E}})\iota_{\mathscr{E}}(\xi) = \iota_{\mathscr{E}}(\xi)$ and $\iota_{\mathscr{E}}(\xi)\iota_{\mathcal{K}(\mathscr{E})}(1_{\mathscr{E}}) = 0$. Hence,

$$\begin{aligned}&\iota_{\mathcal{K}(\mathscr{E}\otimes S)}(q_{\beta_{\mathscr{E}}\alpha})\delta_J\big(\iota_{\mathscr{E}}(\xi)\big) = \delta_J\big(\iota_{\mathscr{E}}(\xi)\big) \quad \text{and} \\ &\delta_J\big(\iota_{\mathscr{E}}(\xi)\big)\iota_{\mathcal{K}(\mathscr{E}\otimes S)}(q_{\beta_{\mathscr{E}}\alpha}) = 0.\end{aligned}$$

We have $\iota_{A\otimes S}(x)\delta_J(\iota_{\mathscr{E}}(\xi)) = \iota_{A\otimes S}(x)\iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q_{\beta_{\mathscr{E}}\alpha})\delta_J(\iota_{\mathscr{E}}(\xi)) = 0$, for all $x \in A \otimes S$. Now, let $(u_\lambda)_\lambda$ be an approximate unit of A. We have

$$\delta_J\big(\iota_{\mathscr{E}}(\xi)\big) = \lim_\lambda \delta_J\big(\iota_{\mathscr{E}}(\xi u_\lambda)\big) = \lim_\lambda \delta_J\big(\iota_{\mathscr{E}}(\xi)\big)\iota_{A\otimes S}\big(\delta_A(u_\lambda)\big).$$

Hence, $\delta_J(\iota_{\mathscr{E}}(\xi))\iota_{\mathscr{E}\otimes S}(\eta) = 0$ for all $\eta \in \mathscr{E} \otimes S$. Hence, there exists a unique linear map $\delta_{\mathscr{E}} : \mathscr{E} \to \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$ such that $\iota_{\mathscr{E}\otimes S} \circ \delta_{\mathscr{E}} = \delta_J \circ \iota_{\mathscr{E}}$ (cf. Lemma 2.3.4(1)). Moreover, $\delta_{\mathscr{E}}$ actually takes its values in the subspace $\widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ of $\mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$. Indeed, let us fix $\xi \in \mathscr{E}$ and $s \in S$. By assumption, we have that

$$\begin{aligned}&\iota_{\mathscr{E}\otimes S}\big((1_{\mathscr{E}} \otimes s)\delta_{\mathscr{E}}(\xi)\big) = (1_J \otimes s)\delta_J\big(\iota_{\mathscr{E}}(\xi)\big) \quad \text{and} \\ &\iota_{\mathscr{E}\otimes S}\big(\delta_{\mathscr{E}}(\xi)(1_A \otimes s)\big) = \delta_J\big(\iota_{\mathscr{E}}(\xi)\big)(1_J \otimes s)\end{aligned}$$

belong to $J \otimes S = \mathcal{K}((\mathscr{E} \otimes S) \oplus (A \otimes S))$. It then follows that $(1_{\mathscr{E}} \otimes s)\delta_{\mathscr{E}}(\xi) \in \mathscr{E} \otimes S$ and $\delta_{\mathscr{E}}(\xi)(1_A \otimes s) \in \mathscr{E} \otimes S$. The first condition of Definition 6.1.1 is easily derived from the compatibility of δ_J. The vector subspace of $\delta_J(1_J)((\mathscr{E} \oplus A) \otimes S)$ spanned by

$$\big\{\delta_J(\theta_{\xi\oplus a,\eta\oplus b})(\zeta);\, \xi,\eta \in \mathscr{E}, a,b \in A, \zeta \in (\mathscr{E} \oplus A) \otimes S\big\}$$

is dense. However, we have

$$\delta_J(\theta_{\xi\oplus a,\eta\oplus b})(\zeta) = \big(\delta_{\mathscr{E}}(\xi) \oplus \delta_A(a)\big)\big(\delta_{\mathscr{E}}(\eta) \oplus \delta_A(b)\big)^*(\zeta),$$

where $\delta_{\mathscr{E}}(\xi) \oplus \delta_A(a), \delta_{\mathscr{E}}(\eta) \oplus \delta_A(b) \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \oplus \mathcal{L}(A \otimes S) \subset \mathcal{L}(A \otimes S, (\mathscr{E} \oplus A) \otimes S)$. In particular, the vector subspace of $\delta_J(1_J)((\mathscr{E} \oplus A) \otimes S)$ spanned by

$$\big\{\delta_{\mathscr{E}}(\xi)x \oplus \delta_A(a)x;\, \xi \in \mathscr{E}, a \in A, x \in A \otimes S\big\}$$

is dense. Therefore, the relation $[\delta_{\mathscr{E}}(\mathscr{E})(A \otimes S)] = q_{\beta_{\mathscr{E}}\alpha}(\mathscr{E} \otimes S)$ follows since we also have $\delta_J(1_J)((\mathscr{E} \oplus A) \otimes S) = q_{\beta_{\mathscr{E}}\alpha}(\mathscr{E} \otimes S) \oplus q_{\beta_A\alpha}(A \otimes S)$.

Let us fix $\xi \in \mathscr{E}$ and $n \in N$. We have

$$\begin{aligned}\iota_{\mathscr{E}\otimes S}\big(\delta_{\mathscr{E}}\big(\beta_{\mathscr{E}}\big(n^{\mathrm{o}}\big)\xi\big)\big) &= \delta_J\big(\iota_{\mathscr{E}}\big(\beta_{\mathscr{E}}\big(n^{\mathrm{o}}\big)\xi\big)\big) \\ &= \delta_J\big(\beta_J\big(n^{\mathrm{o}}\big)\iota_{\mathscr{E}}(\xi)\big)\end{aligned}$$

$$= \big(1_J \otimes \beta\big(n^{\mathrm{o}}\big)\big)\delta_J\big(\iota_{\mathscr{E}}(\xi)\big)$$
$$= \iota_{\mathscr{E}\otimes S}\big(\big(1_{\mathscr{E}} \otimes \beta\big(n^{\mathrm{o}}\big)\delta_{\mathscr{E}}(\xi)\big)\big).$$

Hence, $\delta_{\mathscr{E}}(\beta_{\mathscr{E}}(n^{\mathrm{o}})\xi) = (1_{\mathscr{E}} \otimes \beta(n^{\mathrm{o}}))\delta_{\mathscr{E}}(\xi)$.

Let us consider the linear maps $\mathrm{id}_{\mathscr{E}} \otimes \delta : \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$ and $\delta_{\mathscr{E}} \otimes \mathrm{id}_S : \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$ (cf. Remarks 6.1.7). By using (6.8) and the compatibility of δ_J with δ_A and $\delta_{\mathscr{E}}$, it follows from a straightforward computation that

$$\iota_{\mathscr{E}\otimes S\otimes S}\big((\mathrm{id}_{\mathscr{E}} \otimes \delta)(T)\big) = (\mathrm{id}_J \otimes \delta)\big(\iota_{\mathscr{E}\otimes S}(T)\big);$$
$$\iota_{\mathscr{E}\otimes S\otimes S}\big((\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(T)\big) = (\delta_J \otimes \mathrm{id}_S)\big(\iota_{\mathscr{E}\otimes S}(T)\big);$$

for all $T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$. In particular, $\iota_{\mathscr{E}\otimes S\otimes S}((\mathrm{id}_{\mathscr{E}} \otimes \delta)\delta_{\mathscr{E}}(\xi)) = (\mathrm{id}_J \otimes \delta)\delta_J(\iota_{\mathscr{E}}(\xi))$ and $\iota_{\mathscr{E}\otimes S\otimes S}((\delta_{\mathscr{E}} \otimes \mathrm{id}_S)\delta_{\mathscr{E}}(\xi)) = (\delta_J \otimes \mathrm{id}_S)\delta_J(\iota_{\mathscr{E}}(\xi))$ for all $\xi \in \mathscr{E}$. Hence, for all $\xi \in \mathscr{E}$ we have $(\delta_{\mathscr{E}} \otimes \mathrm{id}_S)\delta_{\mathscr{E}}(\xi) = (\mathrm{id}_{\mathscr{E}} \otimes \delta)\delta_{\mathscr{E}}(\xi)$. Therefore, the pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is an action of $\mathcal{G}$ on $\mathscr{E}$.

We claim that there exists a unique $*$-homomorphism $\delta_{\mathcal{K}(\mathscr{E})} : \mathcal{K}(\mathscr{E}) \to \mathcal{M}(\mathcal{K}(\mathscr{E}) \otimes S)$ such that $\iota_{\mathcal{K}(\mathscr{E}\otimes S)} \circ \delta_{\mathcal{K}(\mathscr{E})} = \delta_J \circ \iota_{\mathcal{K}(\mathscr{E})}$. We recall that $\delta_J(\iota_{\mathcal{K}(\mathscr{E})}(1_{\mathscr{E}})) = \iota_{\mathcal{K}(\mathscr{E}\otimes S)}(q_{\beta_{\mathscr{E}}\alpha})$. We also have $\iota_{A\otimes S}(x)\iota_{\mathcal{K}(\mathscr{E}\otimes S)}(q_{\beta_{\mathscr{E}}\alpha}) = 0$ and $\iota_{\mathcal{K}(\mathscr{E}\otimes S)}(q_{\beta_{\mathscr{E}}\alpha})\iota_{A\otimes S}(x) = 0$ for all $x \in A \otimes S$. It follows that $\iota_{A\otimes S}(x) \times \delta_J(\iota_{\mathcal{K}(\mathscr{E})}(k)) = 0$ and $\delta_J(\iota_{\mathcal{K}(\mathscr{E})}(k))\iota_{A\otimes S}(x) = 0$ for all $k \in \mathcal{K}(\mathscr{E})$ and $x \in A \otimes S$. Hence, $\delta_J(\iota_{\mathcal{K}(\mathscr{E})}(k)) \in \iota_{\mathcal{K}(\mathscr{E}\otimes S)}(\mathcal{L}(\mathscr{E} \otimes S))$ (cf. Lemma 2.3.4) and the claim is proved since $\iota_{\mathcal{K}(\mathscr{E}\otimes S)}$ is faithful. Since $\iota_{\mathcal{K}(\mathscr{E}\otimes S)}$ is isometric and $\delta_J \circ \iota_{\mathcal{K}(\mathscr{E})}$ is strictly continuous, the $*$-homomorphism $\delta_{\mathcal{K}(\mathscr{E})}$ is strictly continuous and extend uniquely to a strictly continuous $*$-homomorphism $\delta_{\mathcal{K}(\mathscr{E})} : \mathcal{L}(\mathscr{E}) \to \mathcal{M}(\mathcal{K}(\mathscr{E}) \otimes S)$ such that $\delta_{\mathcal{K}(\mathscr{E})}(1_{\mathscr{E}}) = q_{\beta_{\mathscr{E}}\alpha}$. Moreover, for all $\xi, \eta \in \mathscr{E}$ we have (cf. Proposition 2.3.2(2))

$$\delta_{\mathcal{K}(\mathscr{E})}(\theta_{\xi,\eta}) = \delta_{\mathscr{E}}(\xi) \circ \delta_{\mathscr{E}}(\eta)^* = \theta_{\delta_{\mathscr{E}}(\xi),\delta_{\mathscr{E}}(\eta)} \in \mathcal{K}\big(\widetilde{\mathcal{M}}(\mathscr{E} \otimes S)\big) \subset \widetilde{\mathcal{M}}\big(\mathcal{K}(\mathscr{E}) \otimes S\big). \tag{6.9}$$

Hence, $\delta_{\mathcal{K}(\mathscr{E})}(\mathcal{K}(\mathscr{E})) \subset \widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}) \otimes S)$. We have $\delta_{\mathcal{K}(\mathscr{E})}(\beta_{\mathscr{E}}(n^{\mathrm{o}})) = (1_{\mathscr{E}} \otimes \beta(n^{\mathrm{o}})) \times q_{\beta_{\mathscr{E}}\alpha}$, for all $n \in N$ (cf. (6.9), Remarks 6.1.10). By strict continuity, we have the formulas:

$$\iota_{\mathcal{K}(\mathscr{E}\otimes S\otimes S)}(\mathrm{id}_{\mathcal{K}(\mathscr{E})} \otimes \delta)(T) = (\mathrm{id}_J \otimes \delta)\big(\iota_{\mathcal{K}(\mathscr{E}\otimes S)}(T)\big);$$
$$\iota_{\mathcal{K}(\mathscr{E}\otimes S\otimes S)}(\delta_{\mathcal{K}(\mathscr{E})} \otimes \mathrm{id}_S)(m)) = (\delta_J \otimes \mathrm{id}_S)\big(\iota_{\mathcal{K}(\mathscr{E}\otimes S)}(T)\big);$$

for all $T \in \mathcal{M}(\mathcal{K}(\mathscr{E}) \otimes S) = \mathcal{L}(\mathscr{E} \otimes S)$. By applying the above formulas to $T := \delta_{\mathcal{K}(\mathscr{E})}(k)$ for $k \in \mathcal{K}(\mathscr{E})$, we show that $(\delta_{\mathcal{K}(\mathscr{E})} \otimes \mathrm{id}_S)\delta_{\mathcal{K}(\mathscr{E})}(k) = (\mathrm{id}_{\mathcal{K}(\mathscr{E})} \otimes \delta)\delta_{\mathcal{K}(\mathscr{E})}(k)$.

(b) First, it is clear that β_J is a nondegenerate $*$-homomorphism. It is also clear that β_J is compatible with the fibration map β_A, *that is*, $\beta_J(n^{\mathrm{o}})\iota_A(a) = \iota_A(\beta_A(n^{\mathrm{o}})a)$, for all $a \in A$ and $n \in N$. Let $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ be the isometry associated with the action $\delta_{\mathscr{E}}$. Let $i : q_{\beta_A\alpha}(A \otimes S) \to A \otimes S$ be the

inclusion map. We verify that i is an $(A\otimes S)$-linear adjointable map and $i^*=q_{\beta_A\alpha}$. In particular, the map i is an isometry since we have $i^*i(x)=q_{\beta_A\alpha}x=x$ for all $x\in q_{\beta_A\alpha}(A\otimes S)$. Let

$$\mathscr{W}:=\mathscr{V}\oplus i\in\mathcal{L}\big(\big(\mathscr{E}\otimes_{\delta_A}(A\otimes S)\big)\oplus q_{\beta_A\alpha}(A\otimes S),(\mathscr{E}\otimes S)\oplus(A\otimes S)\big).$$

We have $\mathscr{W}^*\mathscr{W}=1$, then $\mathscr{W}$ is an isometry. Henceforth, we will use the following identification (see (6.1)):

$$\begin{aligned}\big(\mathscr{E}\otimes_{\delta_A}(A\otimes S)\big)\oplus q_{\beta_A\alpha}(A\otimes S)&=\big(\mathscr{E}\otimes_{\delta_A}(A\otimes S)\big)\oplus\big(A\otimes_{\delta_A}(A\otimes S)\big)\\&=(\mathscr{E}\oplus A)\otimes_{\delta_A}(A\otimes S).\end{aligned}$$

Hence, $\mathscr{W}\in\mathcal{L}((\mathscr{E}\oplus A)\otimes_{\delta_A}(A\otimes S),(\mathscr{E}\oplus A)\otimes S)$. Let us define

$$\delta_J(x):=\mathscr{W}(x\otimes_{\delta_A}1)\mathscr{W}^*\in\mathcal{M}(J\otimes S),\quad\text{for all } x\in J.$$

In that way, we define a strictly continuous $*$-homomorphism $\delta_J:J\to\mathcal{M}(J\otimes S)$ satisfying $\delta_J(1_J)=\mathscr{W}\mathscr{W}^*=q_{\beta_{\mathscr{E}}\alpha}\oplus q_{\beta_A\alpha}=q_{\beta_J\alpha}$. Let $a\in A$. Let us prove that $\iota_{A\otimes S}(\delta_A(a))=\delta_J(\iota_A(a))$. Since $\iota_{A\otimes S}(\delta_A(a))\mathscr{W}\mathscr{W}^*=\iota_{A\otimes S}(\delta_A(a))$ and $\delta_J(\iota_A(a))\mathscr{W}\mathscr{W}^*=\delta_J(\iota_A(a))$, it amounts to proving that $\iota_{A\otimes S}(\delta_A(a))\mathscr{W}=\delta_J(\iota_A(a))\mathscr{W}$, for all $a\in A$. Therefore, it is enough to prove that $\iota_{A\otimes S}(\delta_A(a))\mathscr{W}=\mathscr{W}(\iota_A(a)\otimes_{\delta_A}1)$ since $\mathscr{W}^*\mathscr{W}=1$. However, for all $\eta\in\mathscr{E}$, $b\in A$ and $x\in A\otimes S$ we have

$$\mathscr{W}\big((\eta\oplus b)\otimes_{\delta_A}x\big)=\mathscr{V}(\eta\otimes_{\delta_A}x)\oplus\delta_A(b)x=\delta_{\mathscr{E}}(\eta)x\oplus\delta_A(b)x.$$

We finally obtain

$$\begin{aligned}\mathscr{W}\big(\iota_A(a)\otimes_{\delta_A}1\big)\big((\eta\oplus b)\otimes_{\delta_A}x\big)&=\mathscr{W}\big((0\oplus ab)\otimes_{\delta_A}x\big)\\&=(\mathscr{V}\oplus i)\big(0\oplus\delta_A(ab)x\big)\\&=0\oplus\delta_A(a)\delta_A(b)x\\&=\iota_{A\otimes S}\big(\delta_A(a)\big)\big(\delta_{\mathscr{E}}(\eta)x\oplus\delta_A(b)x\big)\\&=\iota_{A\otimes S}\big(\delta_A(a)\big)\mathscr{W}\big((\eta\oplus b)\otimes_{\delta_A}x\big),\end{aligned}$$

for all $\eta\in\mathscr{E}$, $b\in A$, and $x\in A\otimes S$. By using similar arguments, we also prove that $\iota_{\mathscr{E}\otimes S}(\delta_{\mathscr{E}}(\xi))=\delta_J(\iota_{\mathscr{E}}(\xi))$ for all $\xi\in\mathscr{E}$. By strict continuity, we obtain the formulas:

$$\begin{aligned}(\delta_J\otimes\mathrm{id}_S)\iota_{A\otimes S}(m)&=\iota_{A\otimes S\otimes S}(\delta_A\otimes\mathrm{id}_S)(m);\\(\mathrm{id}_J\otimes\delta)\iota_{A\otimes S}(m)&=\iota_{A\otimes S\otimes S}(\mathrm{id}_A\otimes\delta)(m);\end{aligned}$$

for all $m\in\mathcal{M}(A\otimes S)$. By applying the above formulas to $m:=\delta_A(a)$ for $a\in A$ and by using again the compatibility of δ_J with δ_A, we obtain the formulas:

$$\begin{aligned}(\delta_J\otimes\mathrm{id}_S)\delta_J\big(\iota_A(a)\big)&=\iota_{A\otimes S\otimes S}(\delta_A\otimes\mathrm{id}_S)\delta_A(a);\\(\mathrm{id}_J\otimes\delta)\delta_J\big(\iota_A(a)\big)&=\iota_{A\otimes S\otimes S}(\mathrm{id}_A\otimes\delta)\delta_A(a).\end{aligned}$$

Hence, $(\delta_J\otimes\mathrm{id}_S)\delta_J(\iota_A(a))=(\mathrm{id}_J\otimes\delta)\delta_J(\iota_A(a))$ for all $a\in A$. In a similar way, we have $(\delta_J\otimes\mathrm{id}_S)\delta_J(\iota_{\mathscr{E}}(\xi))=(\mathrm{id}_J\otimes\delta)\delta_J(\iota_{\mathscr{E}}(\xi))$ for all $\xi\in\mathscr{E}$. However, since

J is generated by $\iota_{\mathscr{E}}(\mathscr{E}) \cup \iota_A(A)$ as a C*-algebra, the coassociativity of δ_J is then proved.

For all $\eta \in \mathscr{E}$, $b \in A$, $x \in A \otimes S$, and $n \in N$, we have

$$\begin{aligned}
\delta_J\big(\beta_J(n^{\mathrm{o}})\big)\mathscr{W}\big((\eta \oplus b) \otimes_{\delta_A} x\big) &= \mathscr{W}\big(\beta_J(n^{\mathrm{o}})(\eta \oplus b) \otimes_{\delta_A} x\big) \\
&= \mathscr{W}\big(\big(\beta_{\mathscr{E}}(n^{\mathrm{o}})\eta \oplus \beta_A(n^{\mathrm{o}})b\big) \otimes_{\delta_A} x\big) \\
&= \delta_{\mathscr{E}}\big(\beta_{\mathscr{E}}(n^{\mathrm{o}})\eta\big)x \oplus \delta_A\big(\beta_A(n^{\mathrm{o}})b\big)x \\
&= \big(1_J \otimes \beta(n^{\mathrm{o}})\big)\big(\delta_{\mathscr{E}}(\eta)x \oplus \delta_A(b)x\big) \\
&= \big(1_J \otimes \beta(n^{\mathrm{o}})\big)\mathscr{W}\big((\eta \oplus b) \otimes_{\delta_A} x\big).
\end{aligned}$$

Hence, $\delta_J(\beta_J(n^{\mathrm{o}})) = \delta_J(\beta_J(n^{\mathrm{o}}))\mathscr{W}\mathscr{W}^* = (1_J \otimes \beta(n^{\mathrm{o}}))\mathscr{W}\mathscr{W}^* = (1_J \otimes \beta(n^{\mathrm{o}})) \times \delta_J(1_J)$, for all $n \in N$. Therefore, (β_J, δ_J) is an action of $\mathcal{G}$ on J, compatible with (β_A, δ_A). □

Equivariant unitary equivalence. In this paragraph, we define a notion of equivariance for unitary equivalences of Hilbert modules acted upon by $\mathcal{G}$. We refer the reader to §A.3 for the definitions and notation used below.

Definition 6.1.12. Let A and B be two $\mathcal{G}$-C*-algebras and $\phi : A \to B$ a $\mathcal{G}$-equivariant $*$-isomorphism. Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert modules over respectively A and B acted upon by $\mathcal{G}$. A ϕ-compatible unitary operator $\Phi : \mathscr{E} \to \mathscr{F}$ is said to be $\mathcal{G}$-equivariant if we have

$$\delta_{\mathscr{F}}(\Phi\xi) = (\Phi \otimes \mathrm{id}_S)\delta_{\mathscr{E}}(\xi), \quad \text{for all } \xi \in \mathscr{E}.$$

We recall that the linear map $\Phi \otimes \mathrm{id}_S : \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \to \mathcal{L}(B \otimes S, \mathscr{F} \otimes S)$ (Notation A.3.6) is the extension of the $\phi \otimes \mathrm{id}_S$-compatible unitary operator $\Phi \otimes \mathrm{id}_S : \mathscr{E} \otimes S \to \mathscr{F} \otimes S$ (Proposition-Definition A.3.4).

Proposition 6.1.13. *With the notation and hypotheses of Definition* 6.1.12, *for all* $n \in N$ *we have* $\beta_{\mathscr{F}}(n^{\mathrm{o}}) \circ \Phi = \Phi \circ \beta_{\mathscr{E}}(n^{\mathrm{o}})$.

Proof. It is clear that $(\Phi \otimes \mathrm{id}_S)((1_{\mathscr{E}} \otimes s)T) = (1_{\mathscr{F}} \otimes s)(\Phi \otimes \mathrm{id}_S)(T)$ for all $s \in S$ and $T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$. Let $n \in N$ and $\xi \in \mathscr{E}$. We have $\delta_{\mathscr{F}}(\Phi(\beta_{\mathscr{E}}(n^{\mathrm{o}})\xi)) = \delta_{\mathscr{F}}(\beta_{\mathscr{F}}(n^{\mathrm{o}})\Phi\xi)$ by Definition 6.1.1(3). Hence, $\Phi(\beta_{\mathscr{E}}(n^{\mathrm{o}})\xi) = \beta_{\mathscr{F}}(n^{\mathrm{o}})\Phi\xi$ by Remarks 6.1.2(1). □

Definition 6.1.14. Two Hilbert C*-modules $\mathscr{E}$ and $\mathscr{F}$ acted upon by $\mathcal{G}$ are said to be $\mathcal{G}$-equivariantly unitarily equivalent if there exists a $\mathcal{G}$-equivariant unitary operator from $\mathscr{E}$ onto $\mathscr{F}$.

It is clear that the $\mathcal{G}$-equivariant unitary equivalence defines an equivalence relation on the class consisting of the Hilbert C*-modules acted upon by $\mathcal{G}$. In the following, we provide equivalent definitions of the $\mathcal{G}$-equivariant unitary equivalence in the two other pictures.

Let A and B be two $\mathcal{G}$-C*-algebras and $\phi : A \to B$ a $\mathcal{G}$-equivariant $*$-isomomorphism. Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert C*-modules over A and B respectively and $\Phi : \mathscr{E} \to \mathscr{F}$ a ϕ-compatible unitary operator.

LEMMA 6.1.15. *The linear map*

$$\mathscr{E} \otimes_{\delta_A} (A \otimes S) \to \mathscr{F} \otimes_{\delta_B} (B \otimes S);$$
$$\xi \otimes_{\delta_A} x \mapsto \Phi\xi \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)(x)$$

is a $\phi \otimes \mathrm{id}_S$-compatible unitary operator.

Proof. For all $\xi \in \mathscr{E}$, $a \in A$ and $x \in A \otimes S$, we have

$$\begin{aligned}\Phi(\xi a) \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)(x) &= \Phi(\xi)\phi(a) \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)(x)\\ &= \Phi\xi \otimes_{\delta_B} \delta_B\big(\phi(a)\big)(\phi \otimes \mathrm{id}_S)(x)\\ &= \Phi\xi \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)\big(\delta_A(a)x\big).\end{aligned}$$

Therefore, we have a well-defined linear map

$$\Psi : \mathscr{E} \odot_A (A \otimes S) \to \mathscr{F} \otimes_{\delta_B} (B \otimes S);$$
$$\xi \odot_A x \mapsto \Phi\xi \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)(x).$$

For all $\xi, \eta \in \mathscr{E}$, we have $\delta_B(\langle \Phi\xi, \Phi\eta\rangle) = \delta_B(\phi(\langle \xi, \eta\rangle)) = (\phi \otimes \mathrm{id}_S)\delta_A(\langle \xi, \eta\rangle)$. Therefore, for all $\xi, \eta \in \mathscr{E}$ and $x, y \in A \otimes S$, we have

$$\begin{aligned}&\big\langle \Phi\xi \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)(x), \Phi\eta \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)(y)\big\rangle\\ &= (\phi \otimes \mathrm{id}_S)\big(\langle \xi \otimes_{\delta_A} x, \eta \otimes_{\delta_A} y\rangle\big).\end{aligned}$$

Hence, $\langle \Psi(t), \Psi(t')\rangle = (\phi \otimes \mathrm{id}_S)(\langle t, t'\rangle)$ for all $t, t' \in \mathscr{E} \odot_A (A \otimes S)$. In particular, we have $\|\Psi(t)\| = \|t\|$ for all $t \in \mathscr{E} \odot_A (A \otimes S)$ ($\phi \otimes \mathrm{id}_S$ is isometric). Therefore, Ψ extends uniquely to a bounded operator from $\mathscr{E} \otimes_{\delta_A} (A \otimes S)$ to $\mathscr{F} \otimes_{\delta_B} (B \otimes S)$ still denoted by Ψ. We have $\langle \Psi(t), \Psi(t')\rangle = (\phi \otimes \mathrm{id}_S)(\langle t, t'\rangle)$ for all $t, t' \in \mathscr{E} \otimes_{\delta_A} (A \otimes S)$. Since Ψ is isometric and has a dense range, we conclude that Ψ is surjective. A staightforward computation shows that $\Psi(tx) = \Psi(t)(\phi \otimes \mathrm{id}_S)(x)$ for all $t \in \mathscr{E} \otimes_{\delta_A} (A \otimes S)$ and $x \in A \otimes S$. □

PROPOSITION 6.1.16. *Let $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ (resp., $(\beta_{\mathscr{F}}, \delta_{\mathscr{F}})$) be an action of $\mathcal{G}$ on $\mathscr{E}$ (resp., $\mathscr{F}$). Denote by $\mathscr{V}_{\mathscr{E}} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ (resp., $\mathscr{V}_{\mathscr{F}} \in \mathcal{L}(\mathscr{F} \otimes_{\delta_B} (B \otimes S), \mathscr{F} \otimes S)$) the isometry associated with $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ (resp., $(\beta_{\mathscr{F}}, \delta_{\mathscr{F}})$). Assume that $\Phi \circ \beta_{\mathscr{E}}(n^{\mathrm{o}}) = \beta_{\mathscr{F}}(n^{\mathrm{o}}) \circ \Phi$ for all $n \in N$. Then Φ is $\mathcal{G}$-equivariant if, and only if, we have*

$$\mathscr{V}_{\mathscr{F}}^*(\Phi \otimes \mathrm{id}_S)\mathscr{V}_{\mathscr{E}}(\xi \otimes_{\delta_A} x) = \Phi\xi \otimes_{\delta_B} (\phi \otimes \mathrm{id}_S)(x),$$

for all $\xi \in \mathscr{E}$ and $x \in A \otimes S$.

Proof. Let $\Psi : \mathscr{E} \otimes_{\delta_A} (A \otimes S) \to \mathscr{F} \otimes_{\delta_B} (B \otimes S)$ be the $\phi \otimes \mathrm{id}_S$-compatible unitary operator defined in the above lemma. For all $\xi \in \mathscr{E}$ and $x \in A \otimes S$, we have

$$\delta_{\mathscr{F}}(\Phi\xi)(\phi \otimes \mathrm{id}_S)(x) = \mathscr{V}_{\mathscr{F}}(\Phi\xi \otimes_{\delta_B} x) = \mathscr{V}_{\mathscr{F}} \circ \Psi(\xi \otimes_{\delta_A} x)$$

and $(\Phi\otimes \mathrm{id}_S)(\delta_{\mathscr{E}}(\xi))(\phi\otimes \mathrm{id}_S)(x) = (\Phi\otimes \mathrm{id}_S)(\delta_{\mathscr{E}}(\xi)x) = (\Phi\otimes \mathrm{id}_S)\mathscr{V}_{\mathscr{E}}(\xi\otimes_{\delta_A} x)$. Therefore, $\delta_{\mathscr{F}}\circ\Phi = (\Phi\otimes \mathrm{id}_S)\circ\delta_{\mathscr{E}}$ if, and only if, $\mathscr{V}_{\mathscr{F}}\circ\Psi = (\Phi\otimes \mathrm{id}_S)\mathscr{V}_{\mathscr{E}}$. In order for the formula $\mathscr{V}_{\mathscr{F}}\circ\Psi = (\Phi\otimes \mathrm{id}_S)\mathscr{V}_{\mathscr{E}}$ to hold true, it is necessary and sufficient that $\Psi = \mathscr{V}_{\mathscr{F}}^*(\Phi\otimes \mathrm{id}_S)\mathscr{V}_{\mathscr{E}}$. Indeed, it is necessary since $\mathscr{V}_{\mathscr{F}}^*\mathscr{V}_{\mathscr{F}} = 1$. It is also sufficient since we have $\mathscr{V}_{\mathscr{F}}\mathscr{V}_{\mathscr{F}}^* = q_{\beta_{\mathscr{F}}\alpha}$, $q_{\beta_{\mathscr{F}}\alpha}(\Phi\otimes \mathrm{id}_S) = (\Phi\otimes \mathrm{id}_S)q_{\beta_{\mathscr{E}}\alpha}$ (by assumption) and $q_{\beta_{\mathscr{E}}\alpha}\mathscr{V}_{\mathscr{E}} = \mathscr{V}_{\mathscr{E}}$. □

Remark 6.1.17. Let A be a $\mathcal{G}$-C*-algebra. Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert A-modules acted upon by $\mathcal{G}$. Let $\Phi\in\mathcal{L}(\mathscr{E},\mathscr{F})$ be a unitary. The following statements are equivalent:

(i) Φ is $\mathcal{G}$-equivariant;
(ii) $\Phi\circ\beta_{\mathscr{E}}(n^{\mathrm{o}}) = \beta_{\mathscr{F}}(n^{\mathrm{o}})\circ\Phi$ for all $n\in N$ and $\mathscr{V}_{\mathscr{F}}^*(\Phi\otimes 1_S)\mathscr{V}_{\mathscr{E}} = \Phi\otimes_{\delta_A} 1_{A\otimes S}$;
(iii) $\Phi\circ\beta_{\mathscr{E}}(n^{\mathrm{o}}) = \beta_{\mathscr{F}}(n^{\mathrm{o}})\circ\Phi$ for all $n\in N$ and $\mathscr{V}_{\mathscr{F}}(\Phi\otimes_{\delta_A} 1_{A\otimes S})\mathscr{V}_{\mathscr{E}}^* = q_{\beta_{\mathscr{E}}\alpha}(\Phi\otimes 1_S)$.

Proposition 6.1.18. *Let A and B be two $\mathcal{G}$-C*-algebras and $\phi: A\to B$ a $\mathcal{G}$-equivariant $*$-isomorphism. Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert modules over respectively A and B acted upon by $\mathcal{G}$. Let $\Phi:\mathscr{E}\to\mathscr{F}$ be a ϕ-compatible unitary operator. Denote by $f:\mathcal{K}(\mathscr{E}\oplus A)\to\mathcal{K}(\mathscr{F}\oplus B)$ the unique $*$-homomorphism such that $f\circ\iota_{\mathscr{E}} = \iota_{\mathscr{F}}\circ T$ and $f\circ\iota_A = \iota_B\circ\phi$ (cf. Proposition A.3.5). Then Φ is $\mathcal{G}$-equivariant if, and only if, f is $\mathcal{G}$-equivariant.*

Proof. Let $J := \mathcal{K}(\mathscr{E}\oplus A)$ and $L := \mathcal{K}(\mathscr{F}\oplus B)$. Assume that Φ is equivariant. The following formulas are immediate consequences of the definitions:

$$\iota_{B\otimes S}\circ(\phi\otimes \mathrm{id}_S)(m) = (f\otimes \mathrm{id}_S)\circ\iota_{A\otimes S}(m),\quad m\in\mathcal{M}(A\otimes S);$$
$$\iota_{\mathscr{F}\otimes S}\circ(\Phi\otimes \mathrm{id}_S)(T) = (f\otimes \mathrm{id}_S)\circ\iota_{\mathscr{E}\otimes S}(T),\quad T\in\mathcal{L}(A\otimes S,\mathscr{E}\otimes S).$$

By combining the first (resp., second) formula with the $\mathcal{G}$-equivariance of ϕ (resp., Φ) and the fact that $f\circ\iota_A = \iota_B\circ\phi$ (resp., $f\circ\iota_{\mathscr{E}} = \iota_{\mathscr{F}}\circ\Phi$), we obtain

$$\delta_L\circ f\big(\iota_A(a)\big) = (f\otimes \mathrm{id}_S)\delta_J\big(\iota_A(a)\big),\quad \text{for all } a\in A$$
$$\big(\text{resp., } \delta_L\circ f\big(\iota_{\mathscr{E}}(\xi)\big) = (f\otimes \mathrm{id}_S)\delta_J\big(\iota_{\mathscr{E}}(\xi)\big),\quad \text{for all } \xi\in\mathscr{E}\big).$$

Therefore, we have $\delta_L(f(x)) = (f\otimes \mathrm{id}_S)\delta_J(x)$ for all $x\in J$. Moreover, by definition of the fibration map on a linking C*-algebra (cf. Proposition 6.1.11) and the $\mathcal{G}$-equivariance of Φ, we have

$$f\big(\beta_J(n^{\mathrm{o}})\big) = \begin{pmatrix}\beta_{\mathscr{E}}(n^{\mathrm{o}}) & 0\\ 0 & \beta_A(n^{\mathrm{o}})\end{pmatrix}$$
$$= \begin{pmatrix}\Phi\circ\beta_{\mathscr{E}}(n^{\mathrm{o}})\circ\Phi^{-1} & 0\\ 0 & \phi(\beta_A(n^{\mathrm{o}}))\end{pmatrix} = \beta_L\big(n^{\mathrm{o}}\big),$$

for all $n\in N$. The converse is proved in a similar way. □

Continuous actions. In this paragraph, we introduce the notion of continuity for actions of the quantum groupoid $\mathcal{G}$ on Hilbert A-modules. If $\mathcal{G}$ is regular, we prove that any action of $\mathcal{G}$ is necessarily continuous.

DEFINITION 6.1.19. An action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ of $\mathcal{G}$ on a Hilbert A-module $\mathscr{E}$ is said to be continuous if we have $[(1_{\mathscr{E}} \otimes S)\delta_{\mathscr{E}}(\mathscr{E})] = (\mathscr{E} \otimes S)q_{\beta_A\alpha}$. A $\mathcal{G}$-equivariant Hilbert A-module is a Hilbert A-module $\mathscr{E}$ endowed with a continuous action of $\mathcal{G}$.

PROPOSITION 6.1.20. *Let $\mathscr{E}$ be a $\mathcal{G}$-equivariant Hilbert A-module. Let $B := \mathcal{K}(\mathscr{E})$. We have the following statements*:

1. *the action (β_B, δ_B) of $\mathcal{G}$ on B defined in Proposition 6.1.11 is strongly continuous*;
2. *we define a continuous action of $\mathcal{G}$ on the Hilbert B-module $\mathscr{E}^*$ by setting*:
   - $\beta_{\mathscr{E}^*}(n^{\mathrm{o}})T := \beta_A(n^{\mathrm{o}}) \circ T$, *for all $n \in N$ and $T \in \mathscr{E}^*$*,
   - $\delta_{\mathscr{E}^*}(T)x := \delta_{\mathscr{E}}(T^*)^* \circ x$, *for all $T \in \mathscr{E}^*$ and $x \in B \otimes S$*;

   where we have applied the usual identifications $B \otimes S = \mathcal{K}(\mathscr{E} \otimes S)$ and $\mathscr{E} = \mathcal{K}(A, \mathscr{E})$.

Proof. 1. We have $[\delta_B(B)(1_B \otimes S)] = [\delta_B(\theta_{\xi,\eta})(1_B \otimes y); \xi, \eta \in \mathscr{E}, y \in S]$. However, we have $\delta_B(\theta_{\xi,\eta})(1_B \otimes y) = \delta_{\mathscr{E}}(\xi)\delta_{\mathscr{E}}(\eta)^*(1_B \otimes y) = \delta_{\mathscr{E}}(\xi)((1_B \otimes y^*)\delta_{\mathscr{E}}(\eta))^*$ for all $y \in S$ and $\xi, \eta \in \mathscr{E}$. It then follows from the continuity of the action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ and Remarks 6.1.2 that

$$\big[\delta_B(B)(1_B \otimes S)\big] = \big[\delta_{\mathscr{E}}(\mathscr{E})q_{\beta_A\alpha}\big(\mathscr{E}^* \otimes S\big)\big] = \big[\delta_{\mathscr{E}}(\mathscr{E})\big(\mathscr{E}^* \otimes S\big)\big].$$

Now, by combining the formulas $[\delta_{\mathscr{E}}(\mathscr{E})(1_{\mathscr{E}} \otimes S)] = q_{\beta_{\mathscr{E}}\alpha}(\mathscr{E} \otimes S)$ and $B = [\mathscr{E}\mathscr{E}^*]$ with the fact that any element of S can be written as a product of two elements of S, we obtain $[\delta_B(B)(1_B \otimes S)] = [\delta_{\mathscr{E}}(\mathscr{E})(1_{\mathscr{E}} \otimes S)(\mathscr{E}^* \otimes S)] = q_{\beta_{\mathscr{E}}\alpha}(B \otimes S)$.

2. Straightforward. □

PROPOSITION 6.1.21. *Let $\mathscr{E}$ be a Hilbert A-module endowed with an action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ of $\mathcal{G}$ on $\mathscr{E}$. Let $J := \mathcal{K}(\mathscr{E} \oplus A)$ be the associated linking C*-algebra. Let (β_J, δ_J) be the action defined in Proposition 6.1.11. Then the action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is continuous if, and only if, the action (β_J, δ_J) is strongly continuous.*

Proof. Assume that the action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is continuous. Let $B := \mathcal{K}(\mathscr{E})$. Let us prove that (β_J, δ_J) is strongly continuous. Let $x \in J$ and $s \in S$. Let us write

$$x = \begin{pmatrix} b & \xi \\ \eta^* & a \end{pmatrix}, \quad \text{where } a \in A, b \in B \text{ and } \xi, \eta \in \mathscr{E}.$$

Then, we have

$$\begin{aligned}\delta_J(x)(1_J \otimes s) = {} & \iota_{B\otimes S}\big(\delta_B(b)(1_B \otimes s)\big) + \iota_{\mathscr{E}\otimes S}\big(\delta_{\mathscr{E}}(\xi)(1_A \otimes s)\big) \\ & + \iota_{\mathscr{E}^*\otimes S}\big(\delta_{\mathscr{E}^*}\big(\eta^*\big)(1_B \otimes s)\big) + \iota_{A\otimes S}\big(\delta_A(a)(1_A \otimes s)\big).\end{aligned}$$

Then the continuity of (β_J, δ_J) follows from the continuity of (β_A, δ_A), $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ and the continuity of (β_B, δ_B) and $(\beta_{\mathscr{E}^*}, \delta_{\mathscr{E}^*})$ (Proposition 6.1.20). Conversely, assume that (β_J, δ_J) is continuous. We have $\iota_{\mathscr{E}\otimes S}((\mathscr{E}\otimes S)q_{\beta_A\alpha}) = (\iota_{\mathscr{E}}(\mathscr{E})\otimes S)q_{\beta_J\alpha}$. Let $\zeta \in \mathscr{E}$ and $y \in S$. As in the above computation, we prove that $\iota_{\mathscr{E}\otimes S}((\zeta\otimes y)q_{\beta_A\alpha})$ is the norm limit of finite sums of elements of the following forms: $\iota_{B\otimes S}((1_B\otimes s)\delta_B(b))$, $\iota_{\mathscr{E}\otimes S}((1_{\mathscr{E}}\otimes s)\delta_{\mathscr{E}}(\xi))$, $\iota_{\mathscr{E}^*\otimes S}((1_{\mathscr{E}^*}\otimes s)\delta_{\mathscr{E}^*}(\eta^*))$, and $\iota_{A\otimes S}((1_A\otimes s)\delta_A(a))$. By multiplying by $\iota_{B\otimes S}(1_{B\otimes S})$ on the left and by $\iota_{A\otimes S}(1_{A\otimes S})$ on the right, we have that $\iota_{\mathscr{E}\otimes S}((\zeta\otimes y)q_{\beta_A\alpha})$ is the norm limit of finite sums of elements of the form $\iota_{\mathscr{E}\otimes S}((1_{\mathscr{E}}\otimes s)\delta_{\mathscr{E}}(\xi))$. The continuity of $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ follows from the fact that $\iota_{\mathscr{E}\otimes S}$ is isometric. □

Definition 6.1.22. A linking $\mathcal{G}$-C*-algebra is a quintuple $(J, \beta_J, \delta_J, e_1, e_2)$ consisting of a C*-algebra J endowed with a continuous action (β_J, δ_J) and nonzero self-adjoint projections $e_1, e_2 \in \mathcal{M}(J)$ satisfying the following conditions:

1. $e_1 + e_2 = 1_J$;
2. $[Je_jJ] = J$, for all $j = 1, 2$;
3. $\delta_J(e_j) = q_{\beta_J\alpha}(e_j \otimes 1_S)$, for all $j = 1, 2$.

Remarks 6.1.23.

- Let (A, β_A, δ_A) be a $\mathcal{G}$-C*-algebra and $m \in \mathcal{M}(A)$ such that $\delta_A(m) = q_{\beta_A\alpha}(m \otimes 1_S)$. Let $n \in N$, we have $[m, \beta_A(n^{\mathrm{o}})] = 0$. Indeed, since α and β commute pointwise we have $[q_{\beta_A\alpha}(1_A \otimes \beta(n^{\mathrm{o}})), q_{\beta_A\alpha}(m \otimes 1_S)] = 0$. It then follows that $\delta_A([m, \beta_A(n^{\mathrm{o}})]) = [\delta_A(m\beta_A(n^{\mathrm{o}})), \delta_A(\beta_A(n^{\mathrm{o}})m)] = 0$. Hence, $[m, \beta_A(n^{\mathrm{o}})] = 0$ by faithfulness of δ_A. In particular, we have $[q_{\beta_A\alpha}, m \otimes 1_S] = 0$.
- Let $(J, \beta_J, \delta_J, e_1, e_2)$ be a linking $\mathcal{G}$-C*-algebra. By restriction of the action (β_J, δ_J), the corner e_2Je_2 (resp., e_1Je_2) turns into a $\mathcal{G}$-C*-algebra (resp., $\mathcal{G}$-equivariant Hilbert C*-module over e_2Je_2). Furthermore, we also have the identification of $\mathcal{G}$-C*-algebras $\mathcal{K}(e_1Je_2) = e_1Je_1$.
- Conversely, if $(\mathscr{E}, \beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is a $\mathcal{G}$-equivariant Hilbert A-module, then the C*-algebra $J := \mathcal{K}(\mathscr{E} \oplus A)$ endowed with the continuous action (β_J, δ_J) (cf. Propositions 6.1.11, 6.1.21) and the projections $e_1 := \iota_{\mathscr{E}}(1_{\mathscr{E}})$ and $e_2 := \iota_A(1_A)$ is a linking $\mathcal{G}$-C*-algebra.

Lemma 6.1.24. *Let $\mathscr{E}$ be a Hilbert A-module endowed with an action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ of $\mathcal{G}$. We have $\mathscr{E} = [(\mathrm{id}_{\mathscr{E}} \otimes \omega)\delta_{\mathscr{E}}(\xi); \xi \in \mathscr{E}, \omega \in \mathcal{B}(\mathscr{H})_*]$ (cf. Proposition-Definition 2.3.6).*

Proof. We have $\mathscr{E} \supset [(\mathrm{id}_{\mathscr{E}} \otimes \omega)\delta_{\mathscr{E}}(\xi); \xi \in \mathscr{E}, \omega \in \mathcal{B}(\mathscr{H})_*]$ (cf. Proposition-Definition 2.3.6). To obtain the converse inclusion, we combine the fact that there exists $\omega \in \mathcal{B}(\mathscr{H})_*$ such that $(\mathrm{id}_{\mathscr{E}} \otimes \omega)(q_{\beta_{\mathscr{E}}\alpha}) = 1_{\mathscr{E}}$ with the formula $[\delta_{\mathscr{E}}(\mathscr{E})(1_B \otimes S)] = q_{\beta_{\mathscr{E}}\alpha}(\mathscr{E} \otimes S)$. □

Now, we can state the main results of this chapter.

THEOREM 6.1.25. *Let A be a $\mathcal{G}$-C*-algebra and $\mathscr{E}$ a Hilbert A-module acted upon by $\mathcal{G}$. Let $J := \mathcal{K}(\mathscr{E} \oplus A)$ be the associated linking C*-algebra endowed with the action (β_J, δ_J) defined in Proposition 6.1.11. If $\mathcal{G}$ is semi-regular (resp., regular), then the action (β_J, δ_J) is weakly (resp., strongly) continuous.*

Proof. Assume that $\mathcal{G}$ is semi-regular. Denote by $T := [(\mathrm{id}_J \otimes \omega)\delta_J(x);\, x \in J, \omega \in \mathcal{B}(\mathscr{H})_*]$ the C*-algebra of continuous elements (cf. Proposition 5.3.4). By combining the compatibility of δ_J with δ_A (resp., $\delta_{\mathscr{E}}$) with the fact that (β_A, δ_A) is (weakly) continuous (resp., Lemma 6.1.24), we obtain $\iota_A(A) \subset T$ (resp., $\iota_{\mathscr{E}}(\mathscr{E}) \subset T$). Hence, $J \subset T$. The converse inclusion also holds since $\delta_J(J) \subset \widetilde{\mathcal{M}}(J \otimes S)$. Hence, (β_J, δ_J) is weakly continuous. It follows from Theorem 5.3.6 that the action (β_J, δ_J) is strongly continuous if $\mathcal{G}$ is regular. □

COROLLARY 6.1.26. *Let $\mathscr{E}$ be a Hilbert A-module. If the quantum groupoid $\mathcal{G}$ is regular, then any action of $\mathcal{G}$ on $\mathscr{E}$ is continuous.*

Proof. This is an immediate consequence of Proposition 6.1.21 and Theorem 6.1.25. □

6.2. Case of a colinking measured quantum groupoid. Let us fix a colinking measured quantum groupoid $\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ between two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Let (A, β_A, δ_A) be a $\mathcal{G}$-C*-algebra. We follow all the notation of §3.3 (resp., Notation 5.2.1 and Proposition 5.2.2) concerning the objects associated with $\mathcal{G}$ (resp., (A, β_A, δ_A)).

In the following, we provide a description of Hilbert modules acted upon by $\mathcal{G}$ in terms of Hilbert modules acted upon by $\mathbb{G}_1$ and $\mathbb{G}_2$. Let us fix a Hilbert A-module $\mathscr{E}$ endowed with an action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ of $\mathcal{G}$.

NOTATION 6.2.1. We introduce some useful notation to describe the action $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$.

- Let $q_{\mathscr{E},j} := \beta_{\mathscr{E}}(\varepsilon_j)$ for $j = 1, 2$. Note that $q_{\mathscr{E},1}$ and $q_{\mathscr{E},2}$ are orthogonal self-adjoint projections of $\mathcal{L}(\mathscr{E})$ and $q_{\mathscr{E},1} + q_{\mathscr{E},2} = 1_{\mathscr{E}}$.
- Let $J := \mathcal{K}(\mathscr{E} \oplus A)$ be the linking C*-algebra associated with $\mathscr{E}$ endowed with the action (β_J, δ_J) of $\mathcal{G}$ (cf. Proposition 6.1.11(b)). Since $\beta_J(\mathbb{C}^2) \subset \mathcal{Z}(\mathcal{M}(J))$ (cf. 3.2.3 [2]), we have $\beta_{\mathscr{E}}(n)\xi = \xi\beta_A(n)$ in $\mathcal{L}(A, \mathscr{E})$ for all $n \in \mathbb{C}^2$ and $\xi \in \mathscr{E}$, *that is*, $(\beta_{\mathscr{E}}(n)\xi)a = \xi(\beta_A(n)a)$ for all $n \in \mathbb{C}^2$, $\xi \in \mathscr{E}$ and $a \in A$. Hence,

$$(q_{\mathscr{E},j}\xi)a = \xi(q_{A,j}a), \quad \text{for all } \xi \in \mathscr{E}, a \in A, j = 1, 2. \tag{6.10}$$

In particular, we have

$$\langle q_{\mathscr{E},j}\xi, q_{\mathscr{E},j}\eta\rangle = q_{A,j}\langle \xi, \eta\rangle, \quad \text{for all } \xi, \eta \in \mathscr{E}.$$

Indeed, fix $\xi, \eta \in \mathscr{E}$ and write $\xi = \xi' a$ and $\eta = \eta' b$ with $\xi', \eta' \in \mathscr{E}$ and $a, b \in A$. Since the projection $q_{A,j}$ is central in A, we have $\langle q_{\mathscr{E},j}\xi, q_{\mathscr{E},j}\eta\rangle = \langle (q_{\mathscr{E},j}\xi')a, (q_{\mathscr{E},j}\eta')b\rangle = \langle \xi'(q_{A,j}a), \eta'(q_{A,j}b)\rangle = q_{A,j}a^*\langle \xi', \eta'\rangle b = q_{A,j}\langle \xi, \eta\rangle$.

For $j=1,2$, we then define the following Hilbert A_j-module $\mathscr{E}_j := q_{\mathscr{E},j}\mathscr{E}$. Note that $\mathscr{E}=\mathscr{E}_1\oplus\mathscr{E}_2$.

- For $j,k=1,2$, let $\Pi_j^k : \mathscr{E}_k\otimes S_{kj}\to\mathscr{E}\otimes S$ be the inclusion map. It is clear that Π_j^k is a π_j^k-compatible operator (cf. Definition A.3.1). We can consider its canonical linear extension $\Pi_j^k : \mathcal{L}(A_k\otimes S_{kj},\mathscr{E}_k\otimes S_{kj})\to\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)$, up to the canonical injective maps $\mathscr{E}_k\otimes S_{kj}\to\mathcal{L}(A_k\otimes S_{kj},\mathscr{E}_k\otimes S_{kj})$ and $\mathscr{E}\otimes S\to\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)$, defined by $\Pi_j^k(T)(x) := \Pi_j^k\circ T((q_{A,k}\otimes p_{kj})x)$ for all $T\in\mathcal{L}(A_k\otimes S_{kj},\mathscr{E}_k\otimes S_{kj})$ and $x\in A\otimes S$.

LEMMA 6.2.2. *With the above notations and hypotheses, we have a canonical unitary equivalence of Hilbert $A\otimes S$-modules $\mathscr{E}\otimes_{\delta_A}(A\otimes S) = \bigoplus_{j,k=1,2}\mathscr{E}_j\otimes_{\delta_{A_j}^k}(A_k\otimes S_{kj})$.*

Proof. This is a straightforward verification to see that we define a unitary adjointable operator by the following formula:

$$\mathscr{E}\otimes_{\delta_A}(A\otimes S)\to\bigoplus_{j,k=1,2}\mathscr{E}_j\otimes_{\delta_{A_j}^k}(A_k\otimes S_{kj});$$
$$\xi\otimes_{\delta_A}x\mapsto\bigoplus_{j,k=1,2}q_{\mathscr{E},j}\xi\otimes_{\delta_{A_j}^k}(q_{A,k}\otimes p_{kj})x.$$

□

PROPOSITION-DEFINITION 6.2.3. *Let $\mathscr{V}\in\mathcal{L}(\mathscr{E}\otimes_{\delta_A}(A\otimes S),\mathscr{E}\otimes S)$ be the isometry associated with the action $(\beta_{\mathscr{E}},\delta_{\mathscr{E}})$ (cf. Proposition 6.1.5(a)). For all $j,k=1,2$, there exists a unique unitary*

$$\mathscr{V}_j^k\in\mathcal{L}\big(\mathscr{E}_j\otimes_{\delta_{A_j}^k}(A_k\otimes S_{kj}),\mathscr{E}_k\otimes S_{kj}\big)$$

such that

$$\mathscr{V}(\xi\otimes_{\delta_A}x) = \sum_{j,k=1,2}\mathscr{V}_j^k\big(q_{\mathscr{E},j}\xi\otimes_{\delta_{A_j}^k}(q_{A,k}\otimes p_{kj})x\big),\quad \text{for all } \xi\in\mathscr{E} \text{ and } x\in A\otimes S.$$

Proof. Let $j,k=1,2$. Fix $\xi\in\mathscr{E}$, $x\in A\otimes S$ and write $x=x'x''$ with $x',x''\in A\otimes S$. We have

$$\begin{aligned}
&\mathscr{V}\big(q_{\mathscr{E},j}\xi\otimes_{\delta_A}(q_{A,k}\otimes p_{kj})x\big)\\
&\quad=\big(1_{\mathscr{E}}\otimes\beta(\varepsilon_j)\big)\mathscr{V}\big(\xi\otimes_{\delta_A}(q_{A,k}\otimes p_{kj})x\big)\quad\text{(Definition 6.1.4(2))}\\
&\quad=\big(1_{\mathscr{E}}\otimes\beta(\varepsilon_j)\big)\mathscr{V}\big(\xi\otimes_{\delta_A}x'(q_{A,k}\otimes p_{kj})x''\big)\quad(q_{A,k},p_{kj}\text{ are central})\\
&\quad=\big(1_{\mathscr{E}}\otimes\beta(\varepsilon_j)\big)\mathscr{V}\big(\xi\otimes_{\delta_A}x'\big)(q_{A,k}\otimes p_{kj})x''\quad(\mathscr{V}\text{ is }A\otimes S\text{-linear}).
\end{aligned}$$

Now if $\eta\in\mathscr{E}$, $y,s\in S$ and $a\in A$, we have

$$\begin{aligned}
&\big(1_{\mathscr{E}}\otimes\beta(\varepsilon_j)\big)(\eta\otimes y)(q_{A,k}\otimes p_{kj})(a\otimes s)\\
&\quad=\eta(q_{A,k}a)\otimes\beta(\varepsilon_j)yp_{kj}s
\end{aligned}$$

$$= \eta(q_{A,k}a) \otimes p_{kj} ys$$
$$= q_{\mathscr{E},k}\eta a \otimes p_{kj} ys \in \mathscr{E}_k \otimes S_{kj}$$

by using (6.10) and the fact that p_{kj} is central in S. In particular, for all $\xi \in \mathscr{E}$ and $x \in A \otimes S$ we have $\mathscr{V}(q_{\mathscr{E},j}\xi \otimes_{\delta_A} (q_{A,k} \otimes p_{kj})x) \in \mathscr{E}_k \otimes S_{kj}$. By combining the fact that $\mathscr{V}$ is isometric with the fact that $x\delta^k_{A_j}(a) = x\delta_A(a)$ for all $a \in A_j$ and $x \in A_k \otimes S_{kj}$, we obtain a well-defined isometric $A_k \otimes S_{kj}$-linear map

$$\mathscr{V}^k_j : \mathscr{E}_j \otimes_{\delta^k_{A_j}} (A_k \otimes S_{kj}) \to \mathscr{E}_k \otimes S_{kj}; \qquad \xi \otimes_{\delta^k_{A_j}} x \mapsto \mathscr{V}(\xi \otimes_{\delta_A} x).$$

It follows from $\mathrm{im}(\mathscr{V}) = q_{\beta_{\mathscr{E}}\alpha}(\mathscr{E} \otimes S)$ (Definition 6.1.4(1)) that $\mathscr{V}^k_j$ is surjective. As a result, we have $\mathscr{V}^k_j \in \mathcal{L}(\mathscr{E}_j \otimes_{\delta^k_{A_j}} (A_k \otimes S_{kj}), \mathscr{E}_k \otimes S_{kj})$ and $\mathscr{V}^k_j$ is unitary. □

For $j,k,l = 1,2$ we have the following set of unitary equivalences of Hilbert modules:

$$A_j \otimes_{\delta^k_{A_j}} (A_k \otimes S_{kj}) \to A_k \otimes S_{kj},$$
$$a \otimes_{\delta^k_{A_j}} x \mapsto \delta^k_{A_j}(a)x; \tag{6.11}$$

$$(A_k \otimes S_{kj}) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}) \to A_l \otimes S_{lk} \otimes S_{kj},$$
$$x \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} y \mapsto \big(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}\big)(x)y; \tag{6.12}$$

$$(A_l \otimes S_{lj}) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} (A_l \otimes S_{lk} \otimes S_{kj}) \to A_l \otimes S_{lk} \otimes S_{kj},$$
$$x \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} y \mapsto \big(\mathrm{id}_{A_l} \otimes \delta^k_{lj}\big)(x)y; \tag{6.13}$$

$$\big(\mathscr{E}_j \otimes_{\delta^k_{A_j}} (A_k \otimes S_{kj})\big) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj})$$
$$\to \mathscr{E}_j \otimes_{(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj}), \tag{6.14}$$
$$(\xi \otimes_{\delta^k_{A_j}} x) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} y \mapsto \xi \otimes_{(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}} \big(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}\big)(x)y;$$

$$\big(\mathscr{E}_j \otimes_{\delta^l_{A_j}} (A_l \otimes S_{lj})\big) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} (A_l \otimes S_{lk} \otimes S_{kj})$$
$$\to \mathscr{E}_j \otimes_{(\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj}), \tag{6.15}$$
$$(\xi \otimes_{\delta^l_{A_j}} x) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} y \mapsto \xi \otimes_{(\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}} \big(\mathrm{id}_{A_l} \otimes \delta^k_{lj}\big)(x)y;$$

$$(\mathscr{E}_k \otimes S_{kj}) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}) \to \big(\mathscr{E}_k \otimes_{\delta^l_{A_k}} (A_l \otimes S_{lk})\big) \otimes S_{kj},$$
$$(\xi \otimes s) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (x \otimes t) \mapsto (\xi \otimes_{\delta^l_{A_k}} x) \otimes st; \tag{6.16}$$

$$(\mathscr{E}_l \otimes S_{lj}) \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} (A_l \otimes S_{lk} \otimes S_{kj}) \to \mathscr{E}_l \otimes S_{lk} \otimes S_{kj},$$
$$\xi \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} y \mapsto \big(\mathrm{id}_{\mathscr{E}_l} \otimes \delta^k_{lj}\big)(\xi)y. \tag{6.17}$$

Proposition 6.2.4. *For all $j,k,l = 1,2$, we have*

$$\big(\mathscr{V}^l_k \otimes_{\mathbb{C}} \mathrm{id}_{S_{kj}}\big)\big(\mathscr{V}^k_j \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} 1\big) = \mathscr{V}^l_j \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} 1.$$

For $j,k,l=1,2$, $\mathscr{V}_k^l \otimes_{\mathbb{C}} \mathrm{id}_{S_{kj}} \in \mathcal{L}((\mathscr{E}_k \otimes S_{kj}) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}), \mathscr{E}_l \otimes S_{lk} \otimes S_{kj})$ (6.16), $\mathscr{V}_j^k \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} 1 \in \mathcal{L}(\mathscr{E}_j \otimes_{(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj}), (\mathscr{E}_k \otimes S_{kj}) \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} (A_l \otimes S_{lk} \otimes S_{kj}))$ (6.14) and $\mathscr{V}_j^l \otimes_{\mathrm{id}_{A_l} \otimes \delta^k_{lj}} 1 \in \mathcal{L}(\mathscr{E}_j \otimes_{(\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}} (A_l \otimes S_{lk} \otimes S_{kj}), \mathscr{E}_l \otimes S_{lk} \otimes S_{kj})$ (6.17). Moreover, the composition $(\mathscr{V}_k^l \otimes_{\mathbb{C}} \mathrm{id}_{S_{kj}})(\mathscr{V}_j^k \otimes_{\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}}} 1)$ does make sense since $(\delta^l_{A_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j} = (\mathrm{id}_{A_l} \otimes \delta^k_{lj})\delta^l_{A_j}$.

Proof. Straightforward consequence of $(\mathscr{V} \otimes_{\mathbb{C}} \mathrm{id}_S)(\mathscr{V} \otimes_{\delta_A \otimes \mathrm{id}_S} 1) = \mathscr{V} \otimes_{\mathrm{id}_A \otimes \delta} 1$. □

Proposition-Definition 6.2.5. *For $j,k=1,2$, let $\delta^k_{\mathscr{E}_j} : \mathscr{E}_j \to \mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$ be the linear map defined by*

$$\delta^k_{\mathscr{E}_j}(\xi)x := \mathscr{V}_j^k(\xi \otimes_{\delta^k_{A_j}} x), \quad \text{for all } \xi \in \mathscr{E}_j \text{ and } x \in A_k \otimes S_{kj}.$$

For all $j,k,l=1,2$, we have:

(i) $\delta_{\mathscr{E}}(\xi) = \sum_{k,j=1,2} \Pi_j^k \circ \delta^k_{\mathscr{E}_j}(q_{\mathscr{E},j}\xi)$, *for all* $\xi \in \mathscr{E}$;

(ii) $\delta^k_{\mathscr{E}_j}(\mathscr{E}_j) \subset \widetilde{\mathcal{M}}(\mathscr{E}_k \otimes S_{kj})$;

(iii) $\delta^k_{\mathscr{E}_j}(\xi a) = \delta^k_{\mathscr{E}_j}(\xi)\delta^k_{A_j}(a)$ *and* $\langle \delta^k_{\mathscr{E}_j}(\xi), \delta^k_{\mathscr{E}_j}(\eta)\rangle = \delta^k_{A_j}(\langle \xi, \eta\rangle)$, *for all* $\xi,\eta \in \mathscr{E}_j$ *and* $a \in A_j$;

(iv) $[\delta^k_{\mathscr{E}_j}(\mathscr{E}_j)(1_{A_k} \otimes S_{kj})] = \mathscr{E}_k \otimes S_{kj}$; *in particular, we have*

$$\mathscr{E}_k = \big[(\mathrm{id}_{\mathscr{E}_k} \otimes \omega)\delta^k_{\mathscr{E}_j}(\xi); \omega \in \mathcal{B}(\mathscr{H}_{kj})_*, \xi \in \mathscr{E}_j\big]$$

(*cf. Proposition-Definition* 2.3.6).

(v) $\delta^l_{\mathscr{E}_k} \otimes \mathrm{id}_{S_{kj}}$ (*resp.*, $\mathrm{id}_{\mathscr{E}_l} \otimes \delta^k_{lj}$) *extends to a linear map from* $\mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$ (*resp.*, $\mathcal{L}(A_l \otimes S_{lj}, \mathscr{E}_l \otimes S_{lj})$) *to* $\mathcal{L}(A_l \otimes S_{lk} \otimes S_{kj}, \mathscr{E}_l \otimes S_{lk} \otimes S_{kj})$ *and for all* $\xi \in \mathscr{E}_j$ *we have*

$$\big(\delta^l_{\mathscr{E}_k} \otimes \mathrm{id}_{S_{kj}}\big)\delta^k_{\mathscr{E}_j}(\xi) = \big(\mathrm{id}_{\mathscr{E}_l} \otimes \delta^k_{lj}\big)\delta^l_{\mathscr{E}_j}(\xi) \in \mathcal{L}(A_l \otimes S_{lk} \otimes S_{kj}, \mathscr{E}_l \otimes S_{lk} \otimes S_{kj});$$

(vi) *if $\mathscr{E}$ is a $\mathcal{G}$-equivariant Hilbert A-module, then we have* $[(1_{\mathscr{E}_k} \otimes S_{kj}) \times \delta^k_{\mathscr{E}_j}(\mathscr{E}_j)] = \mathscr{E}_k \otimes S_{kj}$.

If $\mathscr{E}$ is a $\mathcal{G}$-equivariant Hilbert module, then $(\mathscr{E}_j, \delta^j_{\mathscr{E}_j})$ is a $\mathbb{G}_j$-equivariant Hilbert A_j-module and $\mathscr{V}_j^j$ is the associated unitary.

Proof. It is clear that $\delta^k_{\mathscr{E}_j} : \mathscr{E}_j \to \mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$ is a well-defined linear map. Moreover, statement (i) follows straightforwardly from Proposition-Definition 6.2.3 and the fact that $\delta_{\mathscr{E}}(\xi)x = \mathscr{V}(\xi \otimes_{\delta_A} x)$ for all $\xi \in \mathscr{E}$ and $x \in A \otimes S$. Let $\xi \in \mathscr{E}_j$ and $s \in S_{kj}$. We have

$$\Pi_j^k\big(\delta^k_{\mathscr{E}_j}(\xi)(1_{A_k} \otimes s)\big) = \delta_{\mathscr{E}}(\xi)(1_A \otimes s) \quad \text{and}$$
$$\Pi_j^k\big((1_{\mathscr{E}_k} \otimes s)\delta^k_{\mathscr{E}_j}(\xi)\big) = (1_{\mathscr{E}} \otimes s)\delta_{\mathscr{E}}(\xi)$$

It then follows from $\delta_{\mathscr{E}}(\mathscr{E}) \subset \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ that $\Pi_j^k(\delta_{\mathscr{E}_j}^k(\xi)(1_{A_k} \otimes s))$ and $\Pi_j^k((1_{\mathscr{E}_k} \otimes s)\delta_{\mathscr{E}_j}^k(\xi))$ belong to $\mathscr{E} \otimes S$. Moreover, $(q_{\mathscr{E},k} \otimes p_{kj})\Pi_j^k(T) = \Pi_j^k(T)$ for all $T \in \mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$; hence, $\Pi_j^k(\delta_{\mathscr{E}_j}^k(\xi)(1_{A_k} \otimes s)) \in \Pi_j^k(\mathscr{E}_k \otimes S_{kj})$ and $\Pi_j^k((1_{\mathscr{E}_k} \otimes s)\delta_{\mathscr{E}_j}^k(\xi)) \in \Pi_j^k(\mathscr{E}_k \otimes S_{kj})$. It then follows that $\delta_{\mathscr{E}_j}^k(\xi)(1_{A_k} \otimes s)$ and $(1_{\mathscr{E}_k} \otimes s)\delta_{\mathscr{E}_j}^k(\xi)$ belong to $\mathscr{E}_k \otimes S_{kj}$ by injectivity of $\Pi_j^k : \mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj}) \to \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$. Hence, statement (ii) is proved.

Let $\xi, \eta \in \mathscr{E}_j$. We have

$$\begin{aligned}\pi_j^k\big(\langle \delta_{\mathscr{E}_j}^k(\xi), \delta_{\mathscr{E}_j}^k(\eta)\rangle\big) &= \big\langle \Pi_j^k\big(\delta_{\mathscr{E}_j}^k(\xi)\big), \Pi_j^k\big(\delta_{\mathscr{E}_j}^k(\eta)\big)\big\rangle \\ &= \big\langle (q_{\mathscr{E},k} \otimes p_{kj})\delta_{\mathscr{E}}(\xi), (q_{\mathscr{E},k} \otimes p_{kj})\delta_{\mathscr{E}}(\eta)\big\rangle \\ &= (q_{A,k} \otimes p_{kj})\delta_A\big(\langle \xi, \eta\rangle\big) \\ &= \pi_j^k\big(\delta_{A_j}^k\big(\langle \xi, \eta\rangle\big)\big).\end{aligned}$$

Hence, $\langle \delta_{\mathscr{E}_j}^k(\xi), \delta_{\mathscr{E}_j}^k(\eta)\rangle = \delta_{A_j}^k(\langle \xi, \eta\rangle)$ by injectivity of π_j^k. The first formula of statement (iii) is derived immediately from the definition of $\delta_{\mathscr{E}_j}^k$.

The surjectivity of $\mathscr{V}_j^k$ is just a restatement of $[\delta_{\mathscr{E}_j}^k(\mathscr{E}_j)(A_k \otimes S_{kj})] = \mathscr{E}_k \otimes S_{kj}$. The identity $[\delta_{\mathscr{E}_j}^k(\mathscr{E}_j)(1_{A_k} \otimes S_{kj})] = \mathscr{E}_k \otimes S_{kj}$ follows by combining the previous formula with the first relation of (iii) and the relation $[\delta_{A_j}^k(A_j)(1_{A_k} \otimes S_{kj})] = A_k \otimes S_{kj}$. Let us prove the formula $\mathscr{E}_k = [(\mathrm{id}_{\mathscr{E}_k} \otimes \omega)\delta_{\mathscr{E}_j}^k(\xi); \omega \in \mathcal{B}(\mathscr{H}_{kj})_*, \xi \in \mathscr{E}_j]$. By statement (ii) and Proposition-Definition 2.3.6, we already have the relation $\mathscr{E}_k \supset [(\mathrm{id}_{\mathscr{E}_k} \otimes \omega)\delta_{\mathscr{E}_j}^k(\xi); \omega \in \mathcal{B}(\mathscr{H}_{kj})_*, \xi \in \mathscr{E}_j]$. Conversely, let us fix $\eta \in \mathscr{E}_k$. Let $\omega \in \mathcal{B}(\mathscr{H}_{kj})_*$ and $s \in S_{kj}$ such that $\omega(s) = 1$. It then follows from the formula $[\delta_{\mathscr{E}_j}^k(\mathscr{E}_j)(1_{A_k} \otimes S_{kj})] = \mathscr{E}_k \otimes S_{kj}$ that $\eta = (\mathrm{id}_{\mathscr{E}_k} \otimes \omega)(\eta \otimes s)$ is the norm limit of finite sums of elements of the form $(\mathrm{id}_{\mathscr{E}_k} \otimes \omega)(\delta_{\mathscr{E}_j}^k(\xi)(1_{A_k} \otimes y)) = (\mathrm{id}_{\mathscr{E}_k} \otimes y\omega)\delta_{\mathscr{E}_j}^k(\xi)$, where $\xi \in \mathscr{E}_j$ and $y \in S$. Therefore, statement (iv) is proved.

By using the identifications (6.12) and (6.16) (resp., (6.13) and (6.17)), the linear map

$$\begin{aligned}&\delta_{\mathscr{E}_k}^l \otimes \mathrm{id}_{S_{kj}} : \mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj}) \\ &\qquad \to \mathcal{L}(A_l \otimes S_{lk} \otimes S_{kj}, \mathscr{E}_l \otimes S_{lk} \otimes S_{kj}) \\ &\big(\text{resp., } \mathrm{id}_{\mathscr{E}_l} \otimes \delta_{lj}^k : \mathcal{L}(A_l \otimes S_{lj}, \mathscr{E}_l \otimes S_{lj}) \\ &\qquad \to \mathcal{L}(A_l \otimes S_{lk} \otimes S_{kj}, \mathscr{E}_l \otimes S_{lk} \otimes S_{kj})\big)\end{aligned}$$

is defined for all $T \in \mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$ (resp., $T \in \mathcal{L}(A_l \otimes S_{lj}, \mathscr{E}_l \otimes S_{lj})$) by

$$\begin{aligned}\big(\delta_{\mathscr{E}_k}^l \otimes \mathrm{id}_{S_{kj}}\big)(T) &:= \big(\mathscr{V}_k^j \otimes 1_{S_{kj}}\big)(T \otimes_{\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}}} 1) \\ \big(\text{resp., } \big(\mathrm{id}_{\mathscr{E}_l} \otimes \delta_{lj}^k\big)(T) &:= T \otimes_{\mathrm{id}_{A_l} \otimes \delta_{lj}^k} 1\big).\end{aligned}$$

The relation $(\delta^l_{\mathscr{E}_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{\mathscr{E}_j}(\xi) = (\mathrm{id}_{\mathscr{E}_l} \otimes \delta^k_{lj})\delta^l_{\mathscr{E}_j}(\xi)$ for $\xi \in \mathscr{E}_j$ is then derived from Proposition 6.2.4 as in the proof of Proposition 6.1.5. Assume that $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is continuous. Since p_{kj} is central in S, we have

$$\begin{aligned}\Pi^k_j(\mathscr{E}_k \otimes S_{kj}) &= (q_{\mathscr{E},j} \otimes p_{kj})(\mathscr{E} \otimes S)q_{\beta_A \alpha} \\ &= (q_{\mathscr{E},j} \otimes p_{kj})\big[(1_{\mathscr{E}} \otimes S)\delta_{\mathscr{E}}(\mathscr{E})\big] \\ &= \Pi^k_j\big[(1_{\mathscr{E}_k} \otimes S_{kj})\delta^k_{\mathscr{E}_j}(\mathscr{E}_j)\big]\end{aligned}$$

and statement (vi) is proved. □

From this concrete description of $\mathcal{G}$-equivariant Hilbert C*-modules, we can also provide a corresponding description of the $\mathcal{G}$-equivariant unitary equivalences between them.

LEMMA 6.2.6. *Let A and B be $\mathcal{G}$-C*-algebras. Let $\mathscr{E}$ and $\mathscr{F}$ be Hilbert C*-modules over A and B respectively acted upon by $\mathcal{G}$.*

1. *Let $\Phi : \mathscr{E} \to \mathscr{F}$ be a $\mathcal{G}$-equivariant unitary equivalence over a $\mathcal{G}$-equivariant $*$-isomorphism $\phi : A \to B$. For $j = 1, 2$, there exists a unique map $\Phi_j : \mathscr{E}_j \to \mathscr{F}_j$ satisfying the formula $\Phi(\xi) = \Phi_1(q_{\mathscr{E},1}\xi) + \Phi_2(q_{\mathscr{E},2}\xi)$ for all $\xi \in \mathscr{E}$. Moreover, we have:*
   (i) *for $j = 1, 2$, the map Φ_j is a unitary equivalence over the $*$-isomorphism $\phi_j : A_j \to B_j$ (cf. Proposition 5.2.3(1));*
   (ii) *for all $j, k = 1, 2$, we have*

$$(\Phi_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{\mathscr{E}_j} = \delta^k_{\mathscr{F}_j} \circ \Phi_j. \tag{6.18}$$

   In particular, Φ_j is a $\mathbb{G}_j$-equivariant ϕ_j-compatible unitary operator.
2. *Conversely, for $j = 1, 2$ let $\Phi_j : \mathscr{E}_j \to \mathscr{F}_j$ be a $\mathbb{G}_j$-equivariant unitary equivalence over a $\mathbb{G}_j$-equivariant $*$-isomorphism $\phi_j : A_j \to B_j$ such that (5.1) and (6.18) hold for all $j, k = 1, 2$. Then the map $\Phi : \mathscr{E} \to \mathscr{F}$, defined by $\Phi(\xi) := \Phi_1(q_{\mathscr{E},1}\xi) + \Phi_2(q_{\mathscr{E},2}\xi)$ for all $\xi \in \mathscr{E}$, is a $\mathcal{G}$-equivariant unitary equivalence over the $\mathcal{G}$-equivariant $*$-isomorphism $\phi : A \to B$ (cf. Proposition 5.2.3(2)).*

Proof. 1. Let $j = 1, 2$. Since Φ is $\mathcal{G}$-equivariant, we have $\Phi \circ q_{\mathscr{E},j} = q_{\mathscr{F},j} \circ \Phi$. It then follows that $\Phi(\mathscr{E}_j) \subset \mathscr{F}_j$. Let us denote $\Phi_j := \Phi|_{\mathscr{E}_j} : \mathscr{E}_j \to \mathscr{F}_j$. For $\xi \in \mathscr{E}$, we have $\xi = q_{\mathscr{E},1}\xi + q_{\mathscr{E},2}\xi$; hence, $\Phi(\xi) = \Phi_1(q_{\mathscr{E},1}\xi) + \Phi_2(q_{\mathscr{E},2}\xi)$. Moreover, such a decomposition of Φ is unique since $\mathscr{F}_1$ and $\mathscr{F}_2$ are orthogonal in $\mathscr{F}$. Statement (i) is straightforward. Let $j, k = 1, 2$ and $x \in A_k \otimes S_{kj}$. For all $T \in \mathcal{L}(A_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$ we have

$$(\Phi \otimes \mathrm{id}_S)\big(\Pi^k_j(T)\big)(\phi \otimes \mathrm{id}_S)(x) = (\Phi_k \otimes \mathrm{id}_{S_{kj}})(T)(\phi_k \otimes \mathrm{id}_{S_{kj}})(x).$$

In particular, $(\Phi \otimes \mathrm{id}_S)(\Pi^k_j(\delta^k_{\mathscr{E}_j}(\xi)))(\phi \otimes \mathrm{id}_S)(x) = (\Phi_k \otimes \mathrm{id}_{S_{kj}})(\delta^k_{\mathscr{E}_j}(\xi))(\phi_k \otimes \mathrm{id}_{S_{kj}})(x)$ for all $\xi \in \mathscr{E}_j$; hence, $(\Phi \otimes \mathrm{id}_S)(\delta_{\mathscr{E}}(\xi))(\phi \otimes \mathrm{id}_S)(x) = (\Phi_k \otimes \mathrm{id}_{S_{kj}}) \times (\delta^k_{\mathscr{E}_j}(\xi))(\phi_k \otimes \mathrm{id}_{S_{kj}})(x)$ (Proposition-Definition 6.2.5(i)) for all $\xi \in \mathscr{E}_j$. We

also have $\delta_{\mathscr{F}}(\Phi(\xi))(q_{A,k}\otimes p_{kj}) = \Pi_j^k(\delta^k_{\mathscr{F}_j}(\Phi_j(\xi)))$ for all $\xi\in\mathscr{E}_j$. Hence, $\delta_{\mathscr{F}}(\Phi(\xi))(\phi\otimes\mathrm{id}_S)(x) = \delta^k_{\mathscr{F}_j}(\Phi_j(\xi))(\phi_k\otimes\mathrm{id}_{S_{kj}})(x)$ for all $\xi\in\mathscr{E}_j$ and statement (ii) is proved.

2. Straightforward. □

EXAMPLE 6.2.7. Let (β_N,δ_N) be the trivial action (cf. Examples 5.1.4). Let $i=1,2$. Consider the Hilbert N-module $\mathscr{E} := \mathscr{H}_{i1}\oplus\mathscr{H}_{i2}$. Let $\mathscr{V}\in\mathcal{L}(\mathscr{E}\otimes_{\delta_N}(N\otimes S),\mathscr{E}\otimes S)$ and $\beta_{\mathscr{E}}: N\to\mathcal{L}(\mathscr{E})$ be the maps defined by the formulas:

$$\mathscr{V}(\xi\otimes 1) = \sum_{k=1,2} V^i_{kj}(\xi\otimes 1),\quad \xi\in\mathscr{H}_{ij};\qquad \beta_{\mathscr{E}}(\varepsilon_j) = p_{ij},\quad j=1,2.$$

Then the pair $(\mathscr{V},\beta_{\mathscr{E}})$ is an action of $\mathcal{G}$ on $\mathscr{E}$.

6.3. Induction of equivariant Hilbert C*-modules. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent regular locally compact quantum groups.

Fix a $\mathbb{G}_1$-C*-algebra (A_1,δ_{A_1}) and a $\mathbb{G}_1$-equivariant Hilbert A_1-module $(\mathscr{E}_1,\delta_{\mathscr{E}_1})$. We denote by $J_1 := \mathcal{K}(\mathscr{E}_1\oplus A_1)$ the associated linking C*-algebra endowed with the continuous action δ_{J_1} of $\mathbb{G}_1$.

NOTATION 6.3.1. Let us fix some notation.

- Let $\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{11}: \mathcal{L}(A_1\otimes S_{11},\mathscr{E}_1\otimes S_{11})\to\mathcal{L}(A_1\otimes S_{12}\otimes S_{21},\mathscr{E}_1\otimes S_{12}\otimes S_{21})$ be the unique linear extension of $\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{11}:\mathscr{E}_1\otimes S_{11}\to\mathcal{L}(A_1\otimes S_{12}\otimes S_{21},\mathscr{E}_1\otimes S_{12}\otimes S_{21})$ such that $(\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{11})(T)(\mathrm{id}_{A_1}\otimes\delta^2_{11})(x) = (\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{11})(Tx)$ for all $x\in\mathcal{M}(A_1\otimes S_{11})$ and $T\in\mathcal{L}(A_1\otimes S_{11},\mathscr{E}_1\otimes S_{11})$.
- Let $\delta^{(2)}_{\mathscr{E}_1}:\mathscr{E}_1\to\mathcal{L}(A_1\otimes S_{12}\otimes S_{21},\mathscr{E}_1\otimes S_{12}\otimes S_{21})$ be the linear map defined by $\delta^{(2)}_{\mathscr{E}_1}(\xi) := (\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{11})\delta_{\mathscr{E}_1}(\xi)$ for all $\xi\in\mathscr{E}_1$.
- Consider the Banach subspace of $\mathcal{L}(A_1\otimes S_{12},\mathscr{E}_1\otimes S_{12})$ defined by (cf. Proposition-Definition 2.3.6):

$$\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1) := \big[(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{\mathscr{E}_1}(\xi);\,\xi\in\mathscr{E}_1,\omega\in\mathcal{B}(\mathscr{H}_{21})_*\big].$$

PROPOSITION 6.3.2. *We have* $[\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)(1_{A_1}\otimes S_{12})] = \mathscr{E}_1\otimes S_{12} = [(1_{\mathscr{E}_1}\otimes S_{11})\,\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)]$. *In particular*, $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)\subset\widetilde{\mathcal{M}}(\mathscr{E}_1\otimes S_{12})$.

Proof. Let us prove the formula $[\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)(1_{A_1}\otimes S_{12})] = \mathscr{E}_1\otimes S_{12}$. Fix $\xi\in\mathscr{E}_1$, $s\in S_{12}$ and $\omega\in\mathcal{B}(\mathscr{H}_{21})_*$. Write $\omega = s'\omega'$ with $s'\in S_{21}$ and $\omega'\in\mathcal{B}(\mathscr{H}_{21})_*$. It follows from $S_{12}\otimes S_{21} = [\delta^2_{11}(S_{11})(1_{S_{12}}\otimes S_{21})]$ that

$$\begin{aligned}&(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega)\big(\delta^{(2)}_{\mathscr{E}_1}(\xi)\big)(1_{A_1}\otimes s)\\ &\quad= \big(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega'\big)\big(\delta^{(2)}_{\mathscr{E}_1}(\xi)\big(1_{\mathscr{E}_1}\otimes s\otimes s'\big)\big)\end{aligned}$$

is the norm limit of finite sums of elements of the form

$$\eta = \big(\mathrm{id}_{\mathscr{E}_1}\otimes\mathrm{id}_{S_{12}}\otimes\omega'\big)\big(\delta^{(2)}_{\mathscr{E}_1}(\xi)\big(1_{\mathscr{E}_1}\otimes\delta^2_{11}\big(t'\big)(1_{S_{12}}\otimes t)\big)\big),$$
$$\text{with } t'\in S_{11} \text{ and } t\in S_{21}.$$

It follows from $[\delta_{\mathscr{E}_1}(\mathscr{E}_1)(1_{A_1}\otimes S_{11})]=\mathscr{E}_1\otimes S_{11}$ that

$$\eta=\big(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega'\big)\big(\big(\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{11}\big)\big(\delta_{\mathscr{E}_1}(\xi)\big(1_{\mathscr{E}_1}\otimes t'\big)\big)\big(1_{\mathscr{E}_1\otimes S_{12}}\otimes t\big)\big)$$

is the norm limit of finite sums of elements of the form

$$\eta'=\big(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega'\big)\big(\zeta\otimes\delta^2_{11}\big(t''\big)(1_{S_{12}}\otimes t)\big),\quad\text{with }\zeta\in\mathscr{E}_1\text{ and }t''\in S_{21}.$$

By using $S_{12}=[(\mathrm{id}_{S_{12}}\otimes\omega)(\delta^2_{11}(y));\omega\in\mathcal{B}(\mathscr{H}_{21})_*,y\in S_{11}]$, we obtain

$$\eta'=\zeta\otimes\big(\mathrm{id}_{S_{12}}\otimes t\omega'\big)\big(\delta^2_{11}\big(t''\big)\big)\in\mathscr{E}_1\otimes S_{12}.$$

Hence, $(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega)(\delta^{(2)}_{\mathscr{E}_1}(\xi))(1_{A_1}\otimes s)\in\mathscr{E}_1\otimes S_{12}$ for all $\xi\in\mathscr{E}_1$, $\omega\in\mathcal{B}(\mathscr{H}_{21})_*$ and $s\in S_{12}$. Therefore, the inclusion

$$\big[\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)(1_{A_1}\otimes S_{12})\big]\subset\mathscr{E}_1\otimes S_{12}$$

is proved. The converse inclusion is obtained by following backwards the above argument.

By a similar argument, we prove by using the relation $(1_{\mathscr{E}_1}\otimes S_{12})\delta_{\mathscr{E}_1}(\mathscr{E}_1)\subset\mathscr{E}_1\otimes S_{12}$ that $(1_{\mathscr{E}_1}\otimes S_{12})\,\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)\subset\mathscr{E}_1\otimes S_{12}$. Hence, $[(1_{\mathscr{E}_1}\otimes S_{12})\,\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)]\subset\mathscr{E}_1\otimes S_{12}$. For the converse inclusion, it suffices to follow backwards the proof as above and to use the continuity of the action $\delta_{\mathscr{E}_1}$. □

Lemma 6.3.3. *For all $a\in A_1$, $\xi\in\mathscr{E}_1$, $k\in\mathcal{K}(\mathscr{E}_1)$ and $\omega\in\mathcal{B}(\mathscr{H}_{21})_*$, we have*:

1. $\iota_{A_1\otimes S_{11}}(\mathrm{id}_{A_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{A_1}(a)=(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{J_1}(\iota_{A_1}(a))$;
2. $\iota_{\mathscr{E}_1\otimes S_{12}}(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{\mathscr{E}_1}(\xi)=(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{J_1}(\iota_{\mathscr{E}_1}(\xi))$;
3. $\iota_{\mathcal{K}(\mathscr{E}_1\otimes S_{12})}(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)\otimes S_{12}}\otimes\omega)\delta^{(2)}_{\mathcal{K}(\mathscr{E}_1)}(k)=(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{J_1}(\iota_{\mathcal{K}(\mathscr{E}_1)}(k))$.

Proof. These formulas are straightforward consequences of definitions and the compatibility of δ_{J_1} with δ_{A_1} and $\delta_{\mathscr{E}_1}$ and $\delta_{\mathcal{K}(\mathscr{E}_1)}$ (2.7 (b), 2.8 (a) [3], Proposition-Definition 2.3.6). □

Proposition 6.3.4. *Let $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1)$ be the induced C*-algebra. Then $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)$ is a Hilbert $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1)$-module for the right action by composition and the $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1)$-valued inner product given by $\langle\xi,\eta\rangle:=\xi^*\circ\eta$ for $\xi,\eta\in\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)$.*

Proof. Let $\omega,\omega'\in\mathcal{B}(\mathscr{H}_{21})_*$, $a\in A_1$ and $\xi\in\mathscr{E}_1$. Let $\eta:=(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{\mathscr{E}_1}(\xi)$ and $x:=(\mathrm{id}_{A_1\otimes S_{12}}\otimes\omega')\delta^{(2)}_{A_1}(a)$. We have

$$\begin{aligned}
&\iota_{\mathscr{E}_1\otimes S_{12}}(\eta x)\\
&\quad=\iota_{\mathscr{E}_1\otimes S_{12}}(\eta)\iota_{A_1\otimes S_{12}}(a)\quad\text{(Proposition 2.3.2(1))}\\
&\quad=(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega)\delta^{(2)}_{J_1}\big(\iota_{\mathscr{E}_1}(\xi)\big)\big(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega'\big)\delta^{(2)}_{J_1}\big(\iota_A(a)\big)\quad\text{(Lemma 6.3.3)}\\
&\quad=\big(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega\otimes\omega'\big)\big(\delta^{(2)}_{J_1}\big(\iota_{\mathscr{E}_1}(\xi)\big)_{123}\delta^{(2)}_{J_1}\big(\iota_A(a)\big)_{124}\big).
\end{aligned}$$

In virtue of 4.2 a) [2], we have $\iota_{\mathscr{E}_1\otimes S_{12}}(\eta x)\in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1)$. Therefore, $\iota_{\mathscr{E}_1\otimes S_{12}}(\eta x)$ is the norm limit of finite sums of elements of the form

$$y=(\mathrm{id}_{J_1\otimes S_{12}}\otimes\phi)\delta_{J_1}^{(2)}\begin{pmatrix}k & \zeta\\ \chi^* & b\end{pmatrix},$$
$$\text{with } k\in\mathcal{K}(\mathscr{E}_1),\zeta\in\mathscr{E}_1,\chi^*\in\mathscr{E}_1^*,b\in A_1,\phi\in\mathcal{B}(\mathscr{H}_{21})_*.$$

We have (Lemma 6.3.3)

$$\begin{aligned}y=\ &\iota_{\mathcal{K}(\mathscr{E}_1\otimes S_{12})}(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)\otimes S_{12}}\otimes\phi)\delta_{\mathcal{K}(\mathscr{E}_1)}^{(2)}(k)\\ &+\iota_{\mathscr{E}_1\otimes S_{12}}(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\phi)\delta_{\mathscr{E}_1}^{(2)}(\zeta)\\ &+\iota_{\mathscr{E}_1\otimes S_{12}}(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\phi)\delta_{\mathscr{E}_1}^{(2)}(\chi)^*\\ &+\iota_{A_1\otimes S_{12}}(\mathrm{id}_{A_1\otimes S_{12}}\otimes\phi)\delta_{A_1}^{(2)}(b).\end{aligned}$$

By multiplying on the left (resp., right) by $\iota_{\mathcal{K}(\mathscr{E}_1\otimes S_{12})}(1_{\mathscr{E}_1\otimes S_{12}})$ (resp., $\iota_{A_1\otimes S_{12}}(1_{A_1\otimes S_{12}})$), we obtain (Lemma 2.3.4) that $\iota_{\mathscr{E}_1\otimes S_{12}}(\eta x)$ is the norm limit of finite sums of elements of the form

$$\iota_{\mathscr{E}_1\otimes S_{12}}(\mathrm{id}_{\mathscr{E}_1}\otimes\mathrm{id}_{S_{12}}\otimes\phi)\delta_{\mathscr{E}_1}^{(2)}(\zeta),\quad \text{with } \zeta\in\mathscr{E}_1 \text{ and } \phi\in\mathcal{B}(\mathscr{H}_{21})_*.$$

Since $\iota_{\mathscr{E}_1\otimes S_{12}}$ is isometric, we have proved that $\eta x\in\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1)$.

Let us prove that $\zeta^*\circ\chi\in\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ for all $\zeta,\chi\in\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1)\subset\mathcal{L}(A_1\otimes S_{12},\mathscr{E}_1\otimes S_{12})$. Let us fix $\xi,\eta\in\mathscr{E}_1$ and $\omega,\psi\in\mathcal{B}(\mathscr{H}_{21})_*$. Let us denote $\zeta:=(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\omega)\delta_{\mathscr{E}_1}^{(2)}(\xi)$ and $\chi:=(\mathrm{id}_{\mathscr{E}_1\otimes S_{12}}\otimes\psi)\delta_{\mathscr{E}_1}^{(2)}(\eta)$. We have

$$\begin{aligned}&\iota_{A_1\otimes S_{12}}\big(\zeta^*\circ\chi\big)\\ &\quad=\iota_{\mathscr{E}_1\otimes S_{12}}(\zeta)^*\iota_{\mathscr{E}_1\otimes S_{12}}(\chi)\quad\text{(Proposition 2.3.2(3))}\\ &\quad=(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega)\delta_{J_1}^{(2)}\big(\iota_{\mathscr{E}_1}(\xi)^*\big)(\mathrm{id}_{J_1\otimes S_{12}}\otimes\psi)\delta_{J_1}^{(2)}\big(\iota_{\mathscr{E}_1}(\eta)\big)\quad\text{(Lemma 6.3.3)}\\ &\quad=(\mathrm{id}_{J_1\otimes S_{12}}\otimes\omega\otimes\psi)\big(\delta_{J_1}^{(2)}\big(\iota_{\mathscr{E}_1}(\xi)^*\big)_{123}\delta_{J_1}^{(2)}\big(\iota_{\mathscr{E}_1}(\eta)\big)_{124}\big).\end{aligned}$$

Hence, $\iota_{A_1\otimes S_{12}}(\zeta^*\circ\chi)\in\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1)$ (4.2 a) [2]). As above, we prove that $\iota_{A_1\otimes S_{12}}(\zeta^*\circ\chi)$ is the norm limit of finite sums of elements of the form $\iota_{A_1\otimes S_{12}}(\mathrm{id}_{A_1\otimes S_{12}}\otimes\phi)\delta_{A_1}^{(2)}(a)$ with $a\in A_1$ and $\phi\in\mathcal{B}(\mathscr{H}_{21})_*$. We have proved that $\zeta^*\circ\chi\in\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ since $\iota_{A_1\otimes S_{12}}$ is isometric. □

Let us denote $(A_2,\delta_{A_2}):=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1,\delta_{A_1})$ and $(J_2,\delta_{J_2}):=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1,\delta_{J_1})$ the induced $\mathbb{G}_2$-C*-algebra of (A_1,δ_{A_1}) and (J_1,δ_{J_1}), respectively. We also denote $\mathscr{E}_2:=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1)$ the induced Hilbert A_2-module as defined above.

In the technical lemma below, we make the identification $\mathcal{M}(A)=\mathcal{L}(A)$. We first recall a well-known corollary of the Cohen–Hewitt factorization theorem.

LEMMA 6.3.5. *Let A be a C*-algebra and $\mathcal{E}$ a Hilbert A-module. If $T:A\to\mathcal{E}$ is a map such that $T(ab)=T(a)b$ for all $a,b\in A$, then T is linear and continuous.*

LEMMA 6.3.6. *Let A be a C*-algebra, $B \subset \mathcal{M}(A)$ a nondegenerate C*-subalgebra and $\mathcal{E}$ a Hilbert A-module. Let $\mathcal{F} \subset \mathcal{L}(A,\mathcal{E})$ be a Hilbert B-module (where B is acting on the right by composition and the B-valued inner product is given by $\langle \eta_1, \eta_2 \rangle := \eta_1^* \circ \eta_2$, for all $\eta_1, \eta_2 \in \mathcal{F}$) such that $[\mathcal{F}A] = \mathcal{E}$.*

(i) *There exists a unique map $i : \mathcal{L}(B,\mathcal{F}) \to \mathcal{L}(A,\mathcal{E})$ such that $i(T)(ba) = (Tb)a$ for all $T \in \mathcal{L}(B,\mathcal{F})$, $b \in B$ and $a \in A$. Moreover, i is an injective linear map whose image is $\mathrm{im}(i) = \{S \in \mathcal{L}(A,\mathcal{E});\, SB \subset \mathcal{F}, S^*\mathcal{F} \subset B\}$.*

(ii) *There exists a unique map $j : \mathcal{L}(\mathcal{F} \oplus B) \to \mathcal{L}(\mathcal{E} \oplus A)$ such that $j(x)(\eta a) = (x\eta)a$ for all $x \in \mathcal{L}(\mathcal{F} \oplus B)$, $\eta \in \mathcal{F} \oplus B$ and $a \in A$. Moreover, j is a unital faithful $*$-homomorphism.*

Proof. (i) We have $A = BA$. Let $T \in \mathcal{L}(B,\mathcal{F})$. Let (u_λ) be an approximate unit of B, we have $(Tb)a = \lim_\lambda [T(u_\lambda b)]a = \lim_\lambda [T(u_\lambda)b]a = \lim_\lambda T(u_\lambda)ba$, for all $b \in B$ and $a \in A$. In particular, we have $(Tb)a = (Tb')a'$ for all $b, b' \in B$ and $a, a' \in A$ such that $ba = b'a'$. Therefore, $i(T)$ is well-defined. Moreover, we have $i(T)(aa') = (i(T)a)a'$ for all $a, a' \in A$. Indeed, let us fix $a, a' \in A$. Let us write $a = ba''$ with $b \in B$ and $a'' \in A$. We have $i(T)(aa') = i(T)(b(a''a')) = (Tb)a''a' = i(T)(ba'')a' = (i(T)a)a'$. By Lemma 6.3.5, it then follows that $i(T)$ is a bounded linear map. By a straightforward computation, we have $\langle i(T)(ba'), \eta a \rangle = \langle ba', T^*(\eta)a \rangle$, for all $b \in B$, $a, a' \in A$ and $\eta \in \mathcal{F}$. Hence, $\langle i(T)x, \eta a \rangle = \langle x, T^*(\eta)a \rangle$ for all $x, a \in A$ and $\eta \in \mathcal{F}$. Let $S \in \mathcal{L}(\mathcal{F}, B)$. We have

$$\left\langle x, \sum_{l=1}^{n} S(\eta_l)a_l \right\rangle = \left\langle i\big(S^*\big)x, \sum_{l=1}^{n} \eta_l a_l \right\rangle,$$
$$\text{for all } a_1, \dots a_n \in A, \eta_1, \dots, \eta_n \in \mathcal{F} \text{ and } x \in A.$$

As a consequence, the following map

$$i'(S) : \langle \mathcal{F}A \rangle \to A; \qquad \sum_{l=1}^{n} \eta_l a_l \mapsto \sum_{l=1}^{n} S(\eta_l)a_l$$

is well-defined and we have $\langle x, i'(S)(\xi) \rangle = \langle i(S^*)x, \xi \rangle$ for all $\xi \in \langle \mathcal{F}A \rangle$ and $x \in A$. It follows from the boundedness of the linear operator $i(S^*)$ and the Cauchy–Schwarz inequality that $\|i'(S)\xi\|^2 = \|\langle i'(S)\xi, i'(S)\xi \rangle\| = \|\langle \xi, i(S^*)(i'(S)\xi) \rangle\| \leqslant \|\xi\| \|i(S^*)(i'(S)\xi)\| \leqslant \|\xi\| \|i(S^*)\| \|i'(S)\xi\|$ for all $\xi \in \mathcal{E}$. Hence, $\|i'(S)\xi\| \leqslant \|i(S^*)\| \|\xi\|$ for all $\xi \in \mathcal{E}$, which proves the continuity of $i'(S)$ since $i'(S)$ is linear by definition. In particular, $i'(S)$ extends uniquely to a bounded linear map $i'(S) : \mathcal{E} \to A$. By continuity of the inner product, we have proved that $i'(S) \in \mathcal{L}(\mathcal{E}, A)$ and $i'(S)^* = i(S^*)$. As a result, we have well-defined maps $i : \mathcal{L}(A,\mathcal{E}) \to \mathcal{L}(B,\mathcal{F})$ and $i' : \mathcal{L}(\mathcal{F}, B) \to \mathcal{L}(\mathcal{E}, A)$ such that $i(T)^* = i'(T^*)$ for all $T \in \mathcal{L}(A,\mathcal{E})$. It is clear that i is linear and injective.

It remains to prove that $\mathrm{im}(i) = \{S \in \mathcal{L}(A,\mathcal{E});\, SB \subset \mathcal{F}, S^*\mathcal{F} \subset B\}$. Let $T \in \mathcal{L}(B,\mathcal{F})$ and $b \in B \subset \mathcal{L}(A)$. For all $a \in A$, we have $[i(T) \circ b]a = i(T)(ba) = (Tb)a$. Hence $i(T) \circ b = T(b) \in \mathcal{F}$. Fix $\eta \in \mathcal{F}$. Write $\eta = \zeta b$ with $\zeta \in \mathcal{F}$

and $b \in B$. For all $a \in A$, we have $[i(T)^* \circ \eta]a = [i'(T^*)\eta]a = i'(T^*)(\eta a) = i'(T^*)(\zeta(ba)) = (T^*\zeta)ba = T^*(\zeta b)a = T^*(\eta)a$. Hence, $i(T)^* \circ \eta = T^*(\eta) \in B$. Conversely, let us fix $S \in \mathcal{L}(A, \mathcal{E})$ such that $SB \subset \mathcal{F}$ and $S^*\mathcal{F} \subset B$. Let $T : A \to \mathcal{E}$ and $T' : \mathcal{F} \to B$ be the maps defined by:

$$T(b) := S \circ b, \quad b \in B; \qquad T'(\eta) := S^* \circ \eta, \quad \eta \in \mathcal{F}.$$

For all $b \in B$ and $\eta \in \mathcal{F}$, we have $\langle T(b), \eta\rangle = (S \circ b)^* \circ \eta = b^*(S^* \circ \eta) = \langle b, T'(\eta)\rangle$. Hence $T \in \mathcal{L}(A, \mathcal{E})$ and $T^* = T'$. Moreover, we have $i(T)(ba) = T(b)a = S(ba)$ for all $a \in A$ and $b \in B$. Thus, we have $S = i(T)$.

(ii) Since $[\mathcal{F}A] = \mathcal{E}$, we have $[(\mathcal{F} \oplus B)A] = \mathcal{E} \oplus A$, which proves the uniqueness of j. Let $i_{12} := i$ and $i_{21} := i'$. By a similar argument as in statement (i), we prove that there exists a unique map $i_{11} : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{E})$ such that $i_{11}(T)(\eta a) = (T\eta)a$ for all $\eta \in \mathcal{F}$ and $a \in A$. The nondegenerate inclusion of C*-algebras $B \subset \mathcal{M}(A)$ extends to a unital $*$-homomorphism $i_{22} : \mathcal{M}(B) \to \mathcal{M}(A)$. Then we consider the map $j : \mathcal{L}(\mathcal{F} \oplus B) \to \mathcal{L}(\mathcal{E} \oplus A)$ defined by $j(x) := (i_{kl}(x_{kl}))_{k,l=1,2}$ for all $x = (x_{kl})_{k,l=1,2} \in \mathcal{L}(\mathcal{F} \oplus B)$. It is clear that $j(x)(\eta a) = (x\eta)a$ for all $x \in \mathcal{L}(\mathcal{F} \oplus B)$, $\eta \in \mathcal{F} \oplus B$ and $a \in A$. The fact that j is a unital faithful $*$-homomorphism is then straightforward. □

Remarks 6.3.7. With the notation and hypotheses of the previous proposition, we have:

(i) for all $T \in \mathcal{L}(B, \mathcal{F})$, $j(\iota_{\mathcal{F}}(T)) = \iota_{\mathcal{E}}(i(T))$;
(ii) for all $m \in \mathcal{M}(B)$, $j(\iota_B(m)) = \iota_A(m)$, where we identify $\mathcal{M}(B) \subset \mathcal{M}(A)$.

Lemma 6.3.8. *Let $j = 1, 2$. We have a canonical embedding*

$$\mathcal{L}(A_2 \otimes S_{2j}, \mathscr{E}_2 \otimes S_{2j}) \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{2j}, \mathscr{E}_1 \otimes S_{12} \otimes S_{2j});$$
$$T \mapsto \widetilde{T},$$

where for $T \in \mathcal{L}(A_2 \otimes S_{2j}, \mathscr{E}_2 \otimes S_{2j})$ the operator $\widetilde{T} \in \mathcal{L}(A_1 \otimes S_{12} \otimes S_{2j}, \mathscr{E}_1 \otimes S_{12} \otimes S_{2j})$ is defined by $\widetilde{T}(xa) = T(x)a$ for all $x \in A_2 \otimes S_{2j}$ and $a \in A_1 \otimes S_{12} \otimes S_{2j}$. Moreover, the image of $\mathcal{L}(A_2 \otimes S_{2j}, \mathscr{E}_2 \otimes S_{2j}) \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{2j}, \mathscr{E}_1 \otimes S_{12} \otimes S_{2j})$ is

$$\big\{X \in \mathcal{L}(A_1 \otimes S_{12} \otimes S_{2j}, \mathscr{E}_1 \otimes S_{12} \otimes S_{2j});$$
$$X(A_2 \otimes S_{2j}) \subset \mathscr{E}_2 \otimes S_{2j} \text{ and } X^*(\mathscr{E}_2 \otimes S_{2j}) \subset A_2 \otimes S_{2j}\big\}.$$

Proof. This follows from Lemma 6.3.6 with $A := A_1 \otimes S_{12} \otimes S_{2j}$, $B := A_2 \otimes S_{2j}$, $\mathcal{E} := \mathscr{E}_1 \otimes S_{12} \otimes S_{2j}$ and $\mathcal{F} := \mathscr{E}_2 \otimes S_{2j} \subset \mathcal{L}(A_1 \otimes S_{12} \otimes S_{2j}, \mathscr{E}_1 \otimes S_{12} \otimes S_{2j})$. The assumptions of Lemma 6.3.6 are satisfied in this case in virtue of Propositions 5.2.5(1) and 6.3.2. □

Notation 6.3.9. Let

$$\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2 : \mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12})$$
$$\to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{22}, \mathscr{E}_1 \otimes S_{12} \otimes S_{22})$$

be the unique linear extension of $\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2 : \mathscr{E}_1 \otimes S_{12} \to \mathcal{L}(A_1 \otimes S_{12} \otimes S_{22}, \mathscr{E}_1 \otimes S_{12} \otimes S_{22})$ such that $(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(T)(\mathrm{id}_{A_1} \otimes \delta_{12}^2)(x) = (\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(Tx)$ for all $x \in \mathcal{M}(A_1 \otimes S_{12})$ and $T \in \mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12})$.

Proposition-Definition 6.3.10. *There exists a unique linear map*

$$\delta_{\mathscr{E}_2} : \mathscr{E}_2 \to \mathcal{L}(A_2 \otimes S_{22}, \mathscr{E}_2 \otimes S_{22})$$

satisfying the relation $[\delta_{\mathscr{E}_2}(\xi)a]b = (\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\xi)(ab)$ *for all* $\xi \in \mathscr{E}_2$, $a \in A_2 \otimes S_{22}$ *and* $b \in A_1 \otimes S_{12} \otimes S_{22}$.

Proof. Let us prove the inclusion $(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\mathscr{E}_2)(A_2 \otimes S_{22}) \subset \mathscr{E}_2 \otimes S_{22}$. It follows from Lemma 6.3.3(2) that $\iota_{\mathscr{E}_1 \otimes S_{12}}(\mathscr{E}_2) \subset J_2$. Fix $\xi \in \mathscr{E}_2$ and $x \in A_2 \otimes S_{22}$. We have

$$\begin{aligned}
&\iota_{\mathscr{E}_1 \otimes S_{12} \otimes S_{22}}\big((\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\xi)x\big) \\
&\quad = \big(\mathrm{id}_{J_1} \otimes \delta_{12}^2\big)\big(\iota_{\mathscr{E}_1 \otimes S_{12}}(\xi)\big)\iota_{A_1 \otimes S_{12} \otimes S_{22}}(x) \\
&\quad = \delta_{J_2}\big(\iota_{\mathscr{E}_1 \otimes S_{12}}(\xi)\big)\iota_{A_2 \otimes S_{22}}(x) \in J_2 \otimes S_{22}.
\end{aligned}$$

As in the proof of Proposition 6.3.4, $\iota_{\mathscr{E}_1 \otimes S_{12} \otimes S_{22}}((\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\xi)x)$ is the norm limit of finite sums of elements of the form $\iota_{\mathscr{E}_1 \otimes S_{12} \otimes S_{22}}((\mathrm{id}_{\mathscr{E}_1 \otimes S_{12}} \otimes \omega)\delta_{\mathscr{E}_1}^{(2)}(\eta) \otimes s)$ with $\eta \in \mathscr{E}_1$, $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$ and $s \in S_{22}$. Hence, $(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\xi)x \in \mathscr{E}_2 \otimes S_{22}$ since $\iota_{\mathscr{E}_1 \otimes S_{12} \otimes S_{22}}$ is isometric. Therefore, we have $(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\mathscr{E}_2)(A_2 \otimes S_{22}) \subset \mathscr{E}_2 \otimes S_{22}$. The inclusion $(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\mathscr{E}_2)^*(\mathscr{E}_2 \otimes S_{22}) \subset A_2 \otimes S_{22}$ is obtained by a similar argument. Then the existence and uniqueness of the operator $\delta_{\mathscr{E}_2}(\xi) \in \mathcal{L}(A_2 \otimes S_{22}, \mathscr{E}_2 \otimes S_{22})$ follows as an application of Lemma 6.3.8 with $j = 2$. It is clear that the map $\delta_{\mathscr{E}_2} : \mathscr{E}_2 \to \mathcal{L}(A_2 \otimes S_{22}, \mathscr{E}_2 \otimes S_{22})$ is linear. □

In the following, we prove that $\delta_{\mathscr{E}_2}$ is a continuous action of $\mathbb{G}_2$ on $\mathscr{E}_2$. We also show that the induction procedure for equivariant Hilbert modules is equivalent to that of §4.3 [2].

Notation 6.3.11. Let $e_{1,1} := \iota_{\mathcal{K}(\mathscr{E}_1)}(1_{\mathscr{E}_1}) \in \mathcal{M}(J_1)$ and $e_{2,1} := \iota_{A_1}(1_{A_1}) \in \mathcal{M}(J_1)$, where we identify $\mathcal{M}(J_1) = \mathcal{L}(\mathscr{E}_1 \oplus A_1)$. Let $(J_2, \delta_{J_2}, e_{1,2}, e_{2,2})$ be the induced linking $\mathbb{G}_2$-C*-algebra, with $e_{l,2} := e_{l,1} \otimes 1_{S_{12}} \in \mathcal{M}(J_2)$ for $l = 1, 2$ (cf. 4.14 [2]). Consider $e_{2,2}J_2e_{2,2}$ and $e_{1,2}J_2e_{2,2}$ endowed with their structure of $\mathbb{G}_2$-C*-algebra and $\mathbb{G}_2$-equivariant Hilbert $e_{2,2}J_2e_{2,2}$-module [3]. Recall that the morphism $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{A_1} : A_2 \to J_2$; $x \mapsto (\iota_{A_1} \otimes \mathrm{id}_{S_{12}})(x)$ induces a $\mathbb{G}_2$-equivariant $*$-isomormorphism $A_2 \to e_{2,2}J_2e_{2,2}$ (cf. 4.17, 4.18 [2]).

Proposition 6.3.12. *We use the above notation*:

(i) *The map* $\delta_{\mathscr{E}_2} : \mathscr{E}_2 \to \mathcal{L}(A_2 \otimes S_{22}, \mathscr{E}_2 \otimes S_{22})$ *is a continuous action of* $\mathbb{G}_2$ *on* $\mathscr{E}_2$.

(ii) *There exists a unique bounded linear map* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{\mathscr{E}_1} : \mathscr{E}_2 \to J_2$ *such that*

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{\mathscr{E}_1}\big((\mathrm{id}_{\mathscr{E}_1 \otimes S_{12}} \otimes \omega)\delta_{\mathscr{E}_1}^{(2)}(\xi)\big) = (\mathrm{id}_{J_1 \otimes S_{12}} \otimes \omega)\delta_{J_1}^{(2)}\big(\iota_{\mathscr{E}_1}(\xi)\big),$$

for all $\xi \in \mathscr{E}_1$ *and* $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. *Moreover, we have* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\mathscr{E}_2) = e_{1,2}J_2e_{2,2}$ *and* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}$ *induces a* $\mathbb{G}_2$*-equivariant unitary equivalence* $\mathscr{E}_2 \to e_{1,2}J_2e_{2,2}$; $\xi \mapsto \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\xi)$ *over the* $\mathbb{G}_2$*-equivariant* $*$*-isomorphism* $A_2 \to e_{2,2}J_2e_{2,2}$; $a \mapsto \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1}(a)$.

(iii) *There exists a unique* $*$*-homomorphism* $\tau : \mathcal{K}(\mathscr{E}_2 \oplus A_2) \to J_2$ *such that* $\tau \circ \iota_{\mathscr{E}_2} = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}$ *and* $\tau \circ \iota_{A_2} = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1}$. *Moreover,* τ *is an isomorphism of linking* $\mathbb{G}_2$*-*C^**-algebras.*

(iv) *If* $T \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{K}(\mathscr{E}_1)) \subset \mathcal{L}(\mathscr{E}_1 \otimes S_{12})$ *and* $\eta \in \mathscr{E}_2 \subset \mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12})$, *then we have* $T \circ \eta \in \mathscr{E}_2$. *Moreover, for all* $T \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{K}(\mathscr{E}_1))$, *we have* $[\eta \mapsto T \circ \eta] \in \mathcal{K}(\mathscr{E}_2)$. *More precisely, the map* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{K}(\mathscr{E}_1)) \to \mathcal{K}(\mathscr{E}_2)$; $T \mapsto [\eta \mapsto T \circ \eta]$ *is a* $\mathbb{G}_2$*-equivariant* $*$*-isomorphism.*

Proof. Let us denote $B := e_{2,2}J_2e_{2,2}$ and $\mathscr{F} := e_{1,2}J_2e_{2,2}$ for short.

(i)–(ii) We have $\iota_{\mathscr{E}_1 \otimes S_{12}}(\mathscr{E}_2) \subset J_2$ (cf. Lemma 6.3.3). Let $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1} := \iota_{\mathscr{E}_1 \otimes S_{12}}{\restriction_{\mathscr{E}_2}} : \mathscr{E}_2 \to J_2$. It also follows from the formulas $\delta_{J_1}^{(2)}(e_{l,1}) = e_{l,2} \otimes 1_{S_{21}}$ for $l = 1, 2$ (4.14 [2]) that

$$\begin{aligned} e_{1,2}J_2e_{2,2} &= \big[(\mathrm{id}_{J_1 \otimes S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(e_{1,1}xe_{2,1});\, x \in J_1, \omega \in \mathcal{B}(\mathscr{H}_{21})_*\big] \\ &= \big[(\mathrm{id}_{J_1 \otimes S_{12}} \otimes \omega)\delta_{J_1}^{(2)}\big(\iota_{\mathscr{E}_1}(\xi)\big);\, \xi \in \mathscr{E}_1, \omega \in \mathcal{B}(\mathscr{H}_{21})_*\big] \\ &= \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\mathscr{E}_2). \end{aligned}$$

Let $\xi, \eta \in \mathscr{E}_2$. Since $\langle \xi, \eta \rangle \in A_2$, we have (cf. Proposition 2.3.2(3))

$$\begin{aligned} \big\langle \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\xi), \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\eta) \big\rangle &= \iota_{\mathscr{E}_1 \otimes S_{12}}(\xi)^* \iota_{\mathscr{E}_1 \otimes S_{12}}(\eta) = \iota_{A_1 \otimes S_{12}}\big(\langle \xi, \eta \rangle\big) \\ &= \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1}\big(\langle \xi, \eta \rangle\big). \end{aligned}$$

We also have $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\xi) = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\xi)\, \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1}(a)$ for all $a \in A_2$ and $\xi \in \mathscr{E}_2$ (cf. Proposition 2.3.2(1)). The map $\Phi : \mathscr{E}_2 \to \mathscr{F}$; $\xi \mapsto \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{\mathscr{E}_1}(\xi)$ is a unitary equivalence of Hilbert modules over the $*$-isomorphism $\phi : A_2 \to B$; $a \mapsto \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1}(a)$.

Let us prove that $(\Phi \otimes \mathrm{id}_{S_{22}}) \circ \delta_{\mathscr{E}_2} = \delta_{\mathscr{F}} \circ \Phi$. It is immediately verified that for $\xi \in \mathscr{E}_1 \otimes S_{12}$, the formula $(\iota_{\mathscr{E}_1 \otimes S_{12}} \otimes \mathrm{id}_{S_{22}})(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2)(\xi) = (\mathrm{id}_{J_1} \otimes \delta_{12}^2)(\iota_{\mathscr{E}_1 \otimes S_{12}}(\xi))$ holds true. Let us fix $\xi \in \mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12})$. For all $a \in A_1 \otimes S_{12}$ and $x \in A_1 \otimes S_{12} \otimes S_{22}$, we have

$$\begin{aligned} &\iota_{\mathscr{E}_1 \otimes S_{12} \otimes S_{22}}\big(\big(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2\big)(\xi)\big)\iota_{A_1 \otimes S_{12} \otimes S_{22}}\big(\big(\mathrm{id}_{A_1} \otimes \delta_{12}^2\big)(a)x\big) \\ &\quad = \big(\mathrm{id}_{J_1} \otimes \delta_{12}^2\big)\big(\iota_{\mathscr{E}_1 \otimes S_{12}}(\xi a)\big)\iota_{A_1 \otimes S_{12} \otimes S_{22}}(x) \end{aligned}$$

and $(\mathrm{id}_{J_1} \otimes \delta_{12}^2)(\iota_{\mathscr{E}_1 \otimes S_{12}}(\xi a)) = (\mathrm{id}_{J_1} \otimes \delta_{12}^2)(\iota_{\mathscr{E}_1 \otimes S_{12}}(\xi))\iota_{A_1 \otimes S_{12} \otimes S_{22}}((\mathrm{id}_{A_1} \otimes \delta_{12}^2)(a))$. Hence,

$$\begin{aligned} &\iota_{\mathscr{E}_1 \otimes S_{12} \otimes S_{22}}\big(\big(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2\big)(\xi)\big)\iota_{A_1 \otimes S_{12} \otimes S_{22}}\big(\big(\mathrm{id}_{A_1} \otimes \delta_{12}^2\big)(a)x\big) \\ &\quad = \big(\mathrm{id}_{J_1} \otimes \delta_{12}^2\big)\big(\iota_{\mathscr{E}_1 \otimes S_{12}}(\xi)\big)\iota_{A_1 \otimes S_{12} \otimes S_{22}}\big(\big(\mathrm{id}_{A_1} \otimes \delta_{12}^2\big)(a)x\big). \end{aligned}$$

Thus, $\iota_{\mathscr{E}_1\otimes S_{12}\otimes S_{22}}((\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{12})(\xi))\iota_{A_1\otimes S_{12}\otimes S_{22}}(x) = (\mathrm{id}_{J_1}\otimes\delta^2_{12})(\iota_{\mathscr{E}_1\otimes S_{12}}(\xi))\times \iota_{A_1\otimes S_{12}\otimes S_{22}}(x)$ for all $\xi\in\mathcal{L}(A_1\otimes S_{12},\mathscr{E}_1\otimes S_{12})$ and $a\in A_1\otimes S_{12}\otimes S_{22}$ in virtue of the non-degeneracy of $\mathrm{id}_{A_1}\otimes\delta^2_{12}$. Let us fix $\xi\in\mathscr{E}_2$. For all $x\in A_2\otimes S_{22}$ and $y\in A_1\otimes S_{12}\otimes S_{21}$ we have

$$\begin{aligned}
&\big[(\Phi\otimes\mathrm{id}_{S_{22}})\delta_{\mathscr{E}_2}(\xi)(\phi\otimes\mathrm{id}_{S_{22}})(x)\big]\iota_{A_1\otimes S_{12}\otimes S_{22}}(y)\\
&\quad=\iota_{\mathscr{E}_1\otimes S_{12}\otimes S_{22}}\big(\big(\delta_{\mathscr{E}_2}(\xi)x\big)y\big)\\
&\quad=\iota_{\mathscr{E}_1\otimes S_{12}\otimes S_{22}}\big(\big(\mathrm{id}_{\mathscr{E}_1}\otimes\delta^2_{12}\big)(\xi)(xy)\big)\\
&\quad=\big(\mathrm{id}_{J_1}\otimes\delta^2_{12}\big)\big(\iota_{\mathscr{E}_1\otimes S_{12}}(\xi)\big)\iota_{A_1\otimes S_{12}\otimes S_{22}}(xy)\\
&\quad=\big[\delta_{\mathscr{F}}\big(\Phi(\xi)\big)(\phi\otimes\mathrm{id}_{S_{22}})(x)\big]\iota_{A_1\otimes S_{12}\otimes S_{22}}(y),
\end{aligned}$$

which also holds for all $y\in\mathcal{M}(A_1\otimes S_{12}\otimes S_{22})$ by strict continuity. In particular, by applying this formula for $y\in A_2\otimes S_{22}$, we have then proved that

$$(\Phi\otimes\mathrm{id}_{S_{22}})\big(\delta_{\mathscr{E}_2}(\xi)\big)(\phi\otimes\mathrm{id}_{S_{22}})(x)=\delta_{\mathscr{F}}\big(\Phi(\xi)\big)(\phi\otimes\mathrm{id}_{S_{22}})(x)$$

for all $x\in A_2\otimes S_{22}$. Hence, $(\Phi\otimes\mathrm{id}_{S_{22}})(\delta_{\mathscr{E}_2}(\xi))=\delta_{\mathscr{F}}(\Phi(\xi))$ for all $\xi\in\mathscr{E}_2$. This proves that $\delta_{\mathscr{E}_2}$ is a continuous action of $\mathbb{G}_2$ on $\mathscr{E}_2$ and Φ is $\mathbb{G}_2$-equivariant.

(iii) There exists a unique unital faithful $*$-homomorphism

$$j:\mathcal{L}(\mathscr{E}_2\oplus A_2)\to\mathcal{L}\big((\mathscr{E}_1\otimes S_{12})\oplus(A_1\otimes S_{12})\big)$$

such that $j(x)(\eta a)=(x\eta)a$ for all $x\in\mathcal{L}(\mathscr{E}_2\oplus A_2)$, $\eta\in\mathscr{E}_2\oplus A_2$ and $a\in A_1\otimes S_{12}$ (Lemma 6.3.8(ii), with $A:=A_1\otimes S_{12}$, $B:=A_2$, $\mathcal{E}:=\mathscr{E}_1\otimes S_{12}$ and $\mathcal{F}:=\mathscr{E}_2$). Now, it should be noted that we have the following canonical identifications:

$$\begin{aligned}
J_2\subset\mathcal{M}(J_1\otimes S_{12})&=\mathcal{L}\big((\mathscr{E}_1\oplus A_1)\otimes S_{12}\big)\\
&=\mathcal{L}\big((\mathscr{E}_1\otimes S_{12})\oplus(A_1\otimes S_{12})\big).
\end{aligned}$$

We have $j(\iota_{\mathscr{E}_2}(\xi))=\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}\iota_{\mathscr{E}_1}(\xi)$ for all $\xi\in\mathscr{E}_2$ and $j(\iota_{A_2}(b))=\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}\iota_{A_1}(b)$ for all $b\in A_2$ (cf. Remarks 6.3.7). In particular, we have $j(\mathcal{K}(\mathscr{E}_2\oplus A_2))\subset J_2$. Let $\tau:=j{\restriction}_{\mathcal{K}(\mathscr{E}_2\oplus A_2)}\colon\mathcal{K}(\mathscr{E}_2\oplus A_2)\to J_2$. Since J_2 is generated by $e_{1,2}J_2e_{2,2}$ and $e_{2,2}J_2e_{2,2}$ as a C*-algebra, τ has dense range (cf. (ii)); moreover, τ is also isometric (faithful), therefore τ is surjective. Thus, we have proved that τ is a $*$-isomorphism. The $\mathbb{G}_2$-equivariance of τ is derived from straightforward computations.

(iv) Consider the $\mathbb{G}_2$-equivariant $*$-isomorphism

$$\varphi:\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}\big(\mathcal{K}(\mathscr{E}_1)\big)\to e_{1,2}J_2e_{1,2};\ k\mapsto\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}\iota_{\mathcal{K}(\mathscr{E}_1)}(k)$$

(cf. 4.18 [2], note that $\mathcal{K}(\mathscr{F})=e_{1,2}J_2e_{1,2}$). By statement (ii), τ induces by restriction a $\mathbb{G}_2$-equivariant $*$-isomorphism $\tau:f_{1,2}\mathcal{K}(\mathscr{E}_2\oplus A_2)f_{1,2}\to e_{1,2}J_2e_{1,2}$, where $f_{1,2}:=\iota_{\mathscr{E}_2}(1_{\mathscr{E}_2})$ and $f_{2,2}:=\iota_{A_2}(1_{A_2})$. We have an isomorphism $\psi:\mathcal{K}(\mathscr{E}_2)\to f_{1,2}\mathcal{K}(\mathscr{E}_2\oplus A_2)f_{1,2};\ k\mapsto\iota_{\mathcal{K}(\mathscr{E}_2)}(k)$ of $\mathbb{G}_2$-C*-algebras. Hence, $\chi:=\psi^{-1}\circ\tau^{-1}\circ\varphi:\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathcal{K}(\mathscr{E}_1))\to\mathcal{K}(\mathscr{E}_2)$ is an isomorphism of $\mathbb{G}_2$-C*-algebras.

It is clear that $\chi(T)\xi = T \circ \xi$ for all $T \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathcal{K}(\mathscr{E}_1)) \subset \mathcal{L}(\mathscr{E}_1 \otimes S_{12})$ and $\xi \in \mathscr{E}_2 \subset \mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12})$. □

Proposition-Definition 6.3.13. *Let us fix some notation. Consider*:

- *two* $\mathbb{G}_1$*-*C^**-algebras* A_1 *and* B_1;
- *two* $\mathbb{G}_1$*-equivariant Hilbert modules* $\mathscr{E}_1$ *and* $\mathscr{F}_1$ *over* A_1 *and* B_1, *respectively*;
- *a* $\mathbb{G}_1$*-equivariant unitary equivalence* $\Phi_1 : \mathscr{E}_1 \to \mathscr{F}_1$ *over a* $\mathbb{G}_1$*-equivariant* $*$*-isomorphism* $\phi_1 : A_1 \to B_1$.

Denote by:

- $A_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ *and* $B_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1)$ *the induced* $\mathbb{G}_2$*-*C^**-algebras*;
- $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\phi_1) : A_2 \to B_2$ *the induced* $\mathbb{G}_2$*-equivariant* $*$*-isomorphism*;
- $\mathscr{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1)$ *and* $\mathscr{F}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{F}_1)$ *the induced* $\mathbb{G}_2$*-equivariant Hilbert modules over* A_2 *and* B_2, *respectively*;
- $\Phi_1 \otimes \mathrm{id}_{S_{12}} : \mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12}) \to \mathcal{L}(B_1 \otimes S_{12}, \mathscr{F}_1 \otimes S_{12})$ *the unique linear map such that* $(\Phi_1 \otimes \mathrm{id}_{S_{12}})(T)(\phi_1 \otimes \mathrm{id}_{S_{12}})(x) = (\Phi_1 \otimes \mathrm{id}_{S_{12}})(Tx)$ *for all* $\mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12})$ *and* $x \in A_1 \otimes S_{12}$ (*cf. Notation* A.3.6).

Then, $(\Phi_1 \otimes \mathrm{id}_{S_{12}})(\mathscr{E}_2) \subset \mathscr{F}_2$ *and the map* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\Phi_1) := (\Phi_1 \otimes \mathrm{id}_{S_{12}})\restriction_{\mathscr{E}_2} : \mathscr{E}_2 \to \mathscr{F}_2$ *is a* $\mathbb{G}_2$*-equivariant unitary equivalence over* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\phi_1) : A_2 \to B_2$. *Moreover, for all* $\xi \in \mathscr{E}_1$ *and* $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$ *we have* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\Phi_1)((\mathrm{id}_{\mathscr{E}_1 \otimes S_{12}} \otimes \omega)\delta_{\mathscr{E}_1}^{(2)}(\xi)) = (\mathrm{id}_{\mathscr{F}_1 \otimes S_{12}} \otimes \omega)\delta_{\mathscr{F}_1}^{(2)}(\Phi_1\xi)$.

Proof. Denote by $J_1 := \mathcal{K}(\mathscr{E}_1 \oplus A_1)$ and $K_1 := \mathcal{K}(\mathscr{F}_1 \oplus B_1)$ the linking $\mathbb{G}_1$-C*-algebras, whose linking structures are respectively defined by: $e_{1,1} := \iota_{\mathscr{E}_1}(1_{\mathscr{E}_1})$, $e_{2,1} := \iota_{A_1}(1_{A_1})$; $f_{1,1} := \iota_{\mathscr{F}_1}(1_{\mathscr{F}_1})$, $f_{2,1} := \iota_{B_1}(1_{B_1})$. We also denote by $(J_2, \delta_{J_2}, e_{1,2}, e_{2,2})$ and $(K_2, \delta_{K_2}, f_{1,2}, f_{2,2})$ the induced linking $\mathbb{G}_2$-C*-algebras, where $e_{l,2} := e_{l,1} \otimes 1_{S_{12}}$ and $f_{l,2} := f_{l,1} \otimes 1_{S_{12}}$ for $l = 1, 2$ (cf. 4.14 [2]). There exists a unique $*$-isomorphism $\tau_1 : J_1 \to K_1$ such that $\tau_1 \circ \iota_{\mathscr{E}_1} = \iota_{\mathscr{F}_1} \circ \Phi_1$ and $\tau_1 \circ \iota_{A_1} = \iota_{B_1} \circ \phi_1$ (cf. Propositions A.3.5 and 6.1.18). We then denote by

$$\tau_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \tau_1 : J_2 \to K_2$$

the induced morphism. Since τ_2 is an isomorphism of linking $\mathbb{G}_2$-C*-algebras, it induces a $\mathbb{G}_2$-equivariant unitary equivalence $\Psi : e_{1,2}J_2e_{2,2} \to f_{1,2}K_2f_{2,2}$ over the isomorphism of $\mathbb{G}_2$-C*-algebras $\psi : e_{2,2}J_2e_{2,2} \to f_{2,2}K_2f_{2,2}$. Since $\tau_1 \circ \iota_{A_1} = \iota_{B_1} \circ \phi_1$, we have

$$\tau_2 \circ \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{A_1} = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} \iota_{B_1} \circ \phi_2.$$

Therefore, by composition of $\mathbb{G}_2$-equivariant unitary equivalences (cf. Remarks A.3.2(2)) and by applying Proposition 6.3.12, we obtain a $\mathbb{G}_2$-equivariant ϕ_2-compatible unitary operator $\Phi_2 : \mathscr{E}_2 \to \mathscr{F}_2$. By a straightforward computation, we show that $\Phi_2 = (\Phi_1 \otimes \mathrm{id}_{S_{12}})\restriction_{\mathscr{E}_2}$. □

By exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$, we define as above an induction procedure for $\mathbb{G}_2$-equivariant Hilbert modules.

In the following, we investigate the composition of $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}$ and $\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}$. Let A_1 be a $\mathbb{G}_1$-C*-algebra and $\mathscr{E}_1$ a $\mathbb{G}_1$-equivariant Hilbert A_1-module. Denote by:

- $A_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ and $\mathscr{E}_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1) \subset \mathcal{L}(A_1 \otimes S_{12}, \mathscr{E}_1 \otimes S_{12})$ the induced $\mathbb{G}_2$-C*-algebra and the induced $\mathbb{G}_2$-equivariant Hilbert A_2-module;
- $C = \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2)$ and $\mathscr{F} := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathscr{E}_2) \subset \mathcal{L}(A_2 \otimes S_{21}, \mathscr{E}_2 \otimes S_{21})$ the induced $\mathbb{G}_1$-C*-algebra and the induced $\mathbb{G}_1$-equivariant Hilbert C-module.

PROPOSITION 6.3.14. *With the above notation and hypotheses, we have the following statements*:

1. *there exists a unique map* $\Pi_1 : \mathscr{E}_1 \to \mathscr{F}$ *such that*

$$\big(\Pi_1(\xi)x\big)a = \delta_{\mathscr{E}_1}^{(2)}(\xi)(xa),$$
$$\text{for all } \xi \in \mathscr{E}_1, x \in A_2 \otimes S_{21} \text{ and } a \in A_1 \otimes S_{12} \otimes S_{21};$$

   moreover, Π_1 *is a* $\mathbb{G}_1$*-equivariant unitary equivalence over the* $\mathbb{G}_1$*-equivariant* $*$*-isomorphism* $\pi_1 : A_1 \to C$; $a \mapsto \delta_{A_1}^{(2)}(a)$;

2. $\delta_{\mathscr{E}_1}^2 : \mathscr{E}_1 \to \widetilde{\mathcal{M}}(\mathscr{E}_2 \otimes S_{21})$; $\xi \mapsto \Pi_1(\xi)$ *is a well-defined linear map such that*:
   (i) $\delta_{\mathscr{E}_1}^2(\xi a) = \delta_{\mathscr{E}_1}^2(\xi)\delta_{A_1}^2(a)$ *and* $\langle \delta_{\mathscr{E}_1}^2(\xi), \delta_{\mathscr{E}_1}^2(\eta)\rangle = \delta_{A_1}^2(\langle \xi, \eta\rangle)$ *for all* $\xi, \eta \in \mathscr{E}_1$ *and* $a \in A_1$,
   (ii) $[\delta_{\mathscr{E}_1}^2(\mathscr{E}_1)(1_{A_2} \otimes S_{21})] = \mathscr{E}_2 \otimes S_{21} = [(1_{\mathscr{E}_2} \otimes S_{21})\delta_{\mathscr{E}_1}^2(\mathscr{E}_1)]$.

Proof. 1. The existence and uniqueness of Π_1 is an immediate application of Lemma 6.3.8 with $j = 1$ and the proof is very similar to that of Proposition 6.3.10. The fact that Π_1 is a $\mathbb{G}_1$-equivariant unitary equivalence over π_1 is a straightforward consequence of Proposition 6.3.12(ii), (iii) and Proposition 5.2.6(2).

2. Statement (ii) and the fact that $\delta_{\mathscr{E}_1}^2$ takes its values in $\widetilde{\mathcal{M}}(\mathscr{E}_2 \otimes S_{21})$ are proved by combining the formulas $[\mathscr{F}(1_{A_2} \otimes S_{21})] = \mathscr{E}_2 \otimes S_{21} = [(1_{\mathscr{E}_2} \otimes S_{21})\mathscr{F}]$ (cf. Proposition 6.3.2) with the fact that Π_1 is bijective. Statement (i) follows from the compatibility of Π_1 with π_1. □

We have proved the following result:

THEOREM 6.3.15. *Let* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *be two monoidally equivalent regular locally compact quantum groups. The map*

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} : (\mathscr{E}_1, \delta_{\mathscr{E}_1})$$
$$\mapsto \big(\mathscr{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1), \delta_{\mathscr{E}_2} : \xi \in \mathscr{E}_2 \mapsto \big[x \in A_2 \otimes S_{22} \mapsto \big(\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{12}^2\big)(\xi)x\big]\big),$$

where $\mathscr{E}_1$ *is a Hilbert module over the* $\mathbb{G}_1$*-C*-algebra* A_1 *and* $A_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ *denotes the induced* $\mathbb{G}_2$*-C*-algebra, is a one-to-one correspondence up to unitary equivalence. The inverse map, up to unitary equivalence, is*

$$\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1} : (\mathscr{F}_2, \delta_{\mathscr{F}_2})$$
$$\mapsto \big(\mathscr{F}_1 := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathscr{F}_2), \delta_{\mathscr{F}_1} : \xi \in \mathscr{F}_1 \mapsto \big[x \in B_1 \otimes S_{11} \mapsto \big(\mathrm{id}_{\mathscr{F}_2} \otimes \delta_{21}^1\big)(\xi)x\big]\big),$$

where $\mathscr{F}_2$ is a Hilbert module over the $\mathbb{G}_2$-C^*-algebra B_2 and $B_1 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_2)$ denotes the induced $\mathbb{G}_1$-C^*-algebra.

Proof. This is a consequence of Propositions 6.3.14, 6.3.13, and the corresponding results obtained by exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$. □

Let B_1 be a $\mathbb{G}_1$-C*-algebra. Let us denote by $B_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1)$ the induced $\mathbb{G}_2$-C*-algebra. Let $\delta_{B_j}^k : B_j \to \mathcal{M}(B_k \otimes S_{kj})$ for $j,k = 1,2$ be the $*$-homomorphisms defined in Notation 5.2.7.

NOTATION 6.3.16. Let $\mathscr{E}_1$ be a $\mathbb{G}_1$-equivariant Hilbert B_1-module. Let us denote by $\mathscr{F}_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{F}_1)$ the induced $\mathbb{G}_2$-equivariant Hilbert B_2-module. We have four linear maps

$$\delta_{\mathscr{F}_j}^k : \mathscr{F}_j \to \mathcal{L}(B_k \otimes S_{kj}, \mathscr{F}_k \otimes S_{kj}), \quad \text{for } j,k = 1,2,$$

defined as follows:

- $\delta_{\mathscr{F}_1}^1 := \delta_{\mathscr{F}_1}$ and $\delta_{\mathscr{F}_2}^2 := \delta_{\mathscr{F}_2}$;
- $\delta_{\mathscr{F}_1}^2 : \mathscr{F}_1 \to \mathcal{L}(B_2 \otimes S_{21}, \mathscr{F}_2 \otimes S_{21})$ is the unique linear map such that
$$\big(\delta_{\mathscr{F}_1}^2(\xi)x\big)b = \delta_{\mathscr{F}_1}^{(2)}(\xi)(xb)$$
for all $\xi \in \mathscr{F}_1$, $x \in B_2 \otimes S_{21}$ and $b \in B_1 \otimes S_{12} \otimes S_{22}$, where $\delta_{\mathscr{F}_1}^{(2)}(\xi) := (\mathrm{id}_{\mathscr{E}_1} \otimes \delta_{11}^2)\delta_{\mathscr{F}_1}(\xi)$ (cf. Proposition 6.3.14);
- $\delta_{\mathscr{F}_2}^1 : \mathscr{F}_2 \to \mathcal{L}(B_1 \otimes S_{12}, \mathscr{F}_1 \otimes S_{12})$ is the unique linear map such that for all $\xi \in \mathscr{F}_2$, $x \in \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(B_2) \otimes S_{12}$ and $y \in B_2 \otimes S_{21} \otimes S_{12}$, we have
$$\big[(\Pi_1 \otimes \mathrm{id}_{S_{12}})\big(\delta_{\mathscr{F}_2}^1(\xi)\big)x\big]y = \delta_{\mathscr{F}_2}^{(1)}(\xi)(xy),$$
where $\delta_{\mathscr{F}_2}^{(1)}(\xi) := (\mathrm{id}_{\mathscr{F}_1} \otimes \delta_{22}^1)\delta_{\mathscr{F}_2}(\xi)$ and $\Pi_1 : \mathscr{F}_1 \to \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathscr{F}_2)$ (cf. Proposition 6.3.14(1)).

LEMMA 6.3.17. *For all $j,k,l = 1,2$, we have the following statements*:

1. $\delta_{\mathscr{F}_j}^k(\mathscr{F}_j) \subset \widetilde{\mathcal{M}}(\mathscr{F}_k \otimes S_{kj})$;
2. $\delta_{\mathscr{F}_j}^k(\xi b) = \delta_{\mathscr{F}_j}^k(\xi)\delta_{B_j}^k(b)$ *and* $\langle \delta_{\mathscr{F}_j}^k(\xi), \delta_{\mathscr{F}_j}^k(\eta)\rangle = \delta_{B_j}^k(\langle \xi, \eta\rangle)$ *for all* $\xi, \eta \in \mathscr{F}_j$ *and* $b \in B_j$;
3. $[\delta_{\mathscr{F}_j}^k(\mathscr{F}_j)(1_{B_k} \otimes S_{kj})] = \mathscr{F}_k \otimes S_{kj} = [(1_{\mathscr{F}_k} \otimes S_{kj})\delta_{\mathscr{F}_j}^k(\mathscr{F}_j)]$;
4. $\delta_{\mathscr{F}_k}^l \otimes \mathrm{id}_{S_{kj}}$ (*resp.*, $\mathrm{id}_{\mathscr{F}_l} \otimes \delta_{lj}^k$) *extends uniquely to a linear map from* $\mathcal{L}(B_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$ *to* $\mathcal{L}(B_l \otimes S_{lk} \otimes S_{kj}, \mathscr{E}_l \otimes S_{lk} \otimes S_{kj})$ *such that*
$$\big(\delta_{\mathscr{F}_k}^l \otimes \mathrm{id}_{S_{kj}}\big)(T)\big(\delta_{B_k}^l \otimes \mathrm{id}_{S_{kj}}\big)(x) = \big(\delta_{\mathscr{F}_k}^l \otimes \mathrm{id}_{S_{kj}}\big)(Tx)$$
$$\big(\textit{resp.}, \ \big(\mathrm{id}_{\mathscr{F}_l} \otimes \delta_{lj}^k\big)(T)\big(\mathrm{id}_{B_l} \otimes \delta_{lj}^k\big)(x) = \mathrm{id}_{\mathscr{F}_l} \otimes \delta_{lj}^k\big)(Tx)\big)$$
for all $T \in \mathcal{L}(B_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$ *and* $x \in B_k \otimes S_{kj}$;
5. $(\delta_{\mathscr{F}_k}^l \otimes \mathrm{id}_{S_{kj}})\delta_{\mathscr{F}_j}^k = (\mathrm{id}_{\mathscr{F}_l} \otimes \delta_{lj}^k)\delta_{\mathscr{F}_j}^l$.

Proof. Let $C := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(B_2)$, $D := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(C)$, $\mathscr{K} = \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathscr{F}_2)$ and $\mathscr{L} = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{K})$. There exists a unique $\mathbb{G}_2$-equivariant unitary equivalence $\Pi_2 : \mathscr{F}_2 \to \mathscr{L}$ (Proposition 6.3.14(1), after exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$) over the $\mathbb{G}_2$-equivariant $*$-isomorphism $\pi_2 : B_2 \to D$.

1. This statement will follow straightforwardly from the third one.

2. This statement has already been proved for $(j,k) = (1,1)$ (by definition), $(j,k) = (2,2)$ (cf. Proposition 6.3.12(i)) and for $(j,k) = (1,2)$ (cf. Proposition 6.3.14). Moreover, the case $(j,k) = (2,1)$ follows from the formulas $\delta^1_{\mathscr{F}_2} = (\Pi_1^{-1} \otimes \mathrm{id}_{S_{12}})\Pi_2$ and $\delta^1_{B_2} = (\pi_1^{-1} \otimes \mathrm{id}_{S_{12}})\pi_2$.

3. This statement is true by assumption for $(j,k) = (1,1)$, for $(j,k) = (2,2)$ (cf. Proposition 6.3.12(i)) and for $(j,k) = (1,2)$ (cf. Proposition 6.3.14(2)(ii)). By Propositions 6.3.2 and 6.3.14, we have $[\mathscr{L}(C \otimes S_{12})] = \mathscr{K} \otimes S_{12}$, $\mathscr{L} = \Pi_2(\mathscr{F}_2)$ and $\mathscr{K} = \Pi_1(\mathscr{F}_1)$. Therefore, we have

$$\begin{aligned}\big[\delta^1_{\mathscr{F}_2}(\mathscr{F}_2)(B_1 \otimes S_{12})\big] &= \big[\big(\Pi_1^{-1} \otimes \mathrm{id}_{S_{12}}\big)(\mathscr{L})\big(\pi_1^{-1} \otimes \mathrm{id}_{S_{12}}\big)(D \otimes S_{12})\big] \\ &= \big[\big(\Pi_1^{-1} \otimes \mathrm{id}_{S_{12}}\big)\big(\mathscr{L}(D \otimes S_{12})\big)\big] \\ &= \mathscr{F}_1 \otimes S_{12}.\end{aligned}$$

It then follows from the second statement and the fact that $[\delta^1_{B_2}(B_2)(1_{B_1} \otimes S_{12})] = B_1 \otimes S_{12}$ that $[\delta^1_{\mathscr{F}_2}(\mathscr{F}_2)(1_{B_1} \otimes S_{12})] = \mathscr{F}_1 \otimes S_{12}$, which is statement 3 for $(j,k) = (2,1)$.

4. Let $j,k,l = 1,2$. The uniqueness of the extensions is obvious by the nondegeneracy of δ^k_{lj} and $\delta^l_{B_k}$. The linear map $\mathrm{id}_{\mathscr{F}_l} \otimes \delta^k_{lj} : \mathcal{L}(B_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj}) \to \mathcal{L}(B_l \otimes S_{lk} \otimes S_{kj}, \mathscr{E}_l \otimes S_{lk} \otimes S_{kj})$ is defined by

$$\big(\mathrm{id}_{\mathscr{F}_l} \otimes \delta^k_{lj}\big)(T) := T \otimes_{\mathrm{id}_{B_l} \otimes \delta^k_{lj}} 1, \quad \text{for all } T \in \mathcal{L}(B_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj}),$$

where we use the identifications (6.13) and (6.17). As in 2.4 (a) [3], there exists a unique unitary $\mathscr{V}^l_k \in \mathcal{L}(\mathscr{F}_k \otimes_{\delta^l_{B_k}} (B_l \otimes S_{lk}), \mathscr{F}_l \otimes S_{lk})$ such that

$$\mathscr{V}^l_k(\xi \otimes_{\delta^l_{B_k}} x) = \delta^l_{\mathscr{F}_k}(\xi)x, \quad \text{for all } \xi \in \mathscr{F}_k \text{ and } x \in B_l \otimes S_{lk}.$$

The linear extension $\delta^l_{\mathscr{F}_k} \otimes \mathrm{id}_{S_{kj}} : \mathcal{L}(B_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj}) \to \mathcal{L}(B_l \otimes S_{lk} \otimes S_{kj}, \mathscr{E}_l \otimes S_{lk} \otimes S_{kj})$ is defined by $(\delta^l_{\mathscr{F}_k} \otimes \mathrm{id}_{S_{kj}})(T) := (\mathscr{V}^l_k \otimes_{\mathbb{C}} 1)(T \otimes_{\delta^l_{B_k} \otimes \mathrm{id}_{S_{kj}}} 1)$ for all $T \in \mathcal{L}(B_k \otimes S_{kj}, \mathscr{E}_k \otimes S_{kj})$, up to the identifications (6.12) and (6.16).

5. The formula $(\delta^l_{\mathscr{F}_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{\mathscr{F}_j} = (\mathrm{id}_{\mathscr{F}_l} \otimes \delta^k_{lj})\delta^l_{\mathscr{F}_j}$ is derived from Proposition 6.3.12 after long but straightforward computations. □

Let us consider the C*-algebra $B := B_1 \oplus B_2$ endowed with the continuous action (β_B, δ_B) (cf. Proposition 5.2.9).

Proposition 6.3.18. *Let $\mathscr{F}_1$ be a $\mathbb{G}_1$-equivariant Hilbert B_1-module. Let $\mathscr{F}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{F}_1)$ be the induced $\mathbb{G}_2$-equivariant Hilbert B_2-module. Consider the Hilbert B-module $\mathscr{F} := \mathscr{F}_1 \oplus \mathscr{F}_2$. Denote by $\Pi^k_j : \mathcal{L}(B_k \otimes S_{kj}, \mathscr{F}_k \otimes S_{kj}) \to \mathcal{L}(B \otimes S, \mathscr{F} \otimes S)$ the linear extension of the canonical injection $\mathscr{F}_k \otimes S_{kj} \to$*

$\mathscr{F} \otimes S$. Let us consider the linear maps $\delta_{\mathscr{F}} : \mathscr{F} \to \mathcal{L}(B \otimes S, \mathscr{F} \otimes S)$ and $\beta_{\mathscr{F}} : \mathbb{C}^2 \to \mathcal{L}(\mathscr{F})$ defined by:

$$\delta_{\mathscr{F}}(\xi) := \sum_{k,j=1,2} \Pi_j^k \circ \delta_{\mathscr{F}_j}^k(\xi_j), \quad \xi = (\xi_1, \xi_2) \in \mathscr{F};$$

$$\beta_{\mathscr{F}}(\lambda, \mu) := \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad (\lambda, \mu) \in \mathbb{C}^2.$$

Then, the triple $(\mathscr{F}, \beta_{\mathscr{F}}, \delta_{\mathscr{F}})$ is a $\mathcal{G}$-equivariant Hilbert B-module.

Proof. Let us consider $J_1 := \mathcal{K}(\mathscr{F}_1 \oplus B_1)$ (resp., $\mathcal{K}(\mathscr{F}_2 \oplus B_2)$) the linking $\mathbb{G}_1$-C*-algebra (resp., linking $\mathbb{G}_2$-C*-algebra) associated with $\mathscr{F}_1$ (resp., $\mathscr{F}_2$). Let $J_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1)$ be the induced $\mathbb{G}_2$-C*-algebra. Let us consider $J := J_1 \oplus J_2$ endowed with the continuous action (β_J, δ_J) of $\mathcal{G}$ (see above). We denote $L := \mathcal{K}(\mathscr{F} \oplus B)$ the linking C*-algebra associated with $\mathscr{F}$ and we identify $L = J_1 \oplus \mathcal{K}(\mathscr{F}_2 \oplus B_2)$. We have an isomorphism of linking C*-algebras $f := \mathrm{id}_{J_1} \oplus \tau : L \to J$ (Proposition 6.3.12(ii)). Let (β_L, δ_L) be the continuous action of $\mathcal{G}$ on L obtained by transport of structure, *i.e.*:

$$\delta_L(x) := \big(f^{-1} \otimes \mathrm{id}_S\big)\delta_J\big(f(x)\big), \quad x \in L;$$

$$\beta_L(n) := f^{-1}\big(\beta_J(n)\big), \quad n \in \mathbb{C}^2.$$

By straightforward computations, we show that (β_L, δ_L) is compatible with (β_B, δ_B) (cf. Definition 6.1.8) and we prove that $\delta_L(\iota_{\mathscr{F}}(\xi)) = \iota_{\mathscr{F} \otimes S}(\delta_{\mathscr{F}}(\xi))$, for all $\xi \in \mathscr{F}$. Therefore, the result follows from Propositions 6.1.11(a) and 6.1.21. □

Proposition 6.3.19. *Let $(\mathscr{E}, \beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ be a $\mathcal{G}$-equivariant Hilbert A-module. In the following, we use the notation of Proposition-Definition 6.2.5. Let $j, k = 1, 2$ with $j \neq k$. Let*

$$\widetilde{A}_j := \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}\big(A_k, \delta_{A_k}^k\big) \quad \text{and} \quad \widetilde{\mathscr{E}}_j := \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}\big(\mathscr{E}_k, \delta_{\mathscr{E}_k}^k\big).$$

If $\xi \in \mathscr{E}_j$, then we have $\delta_{\mathscr{E}_j}^k(\xi) \in \widetilde{\mathscr{E}}_j \subset \widetilde{\mathcal{M}}(\mathscr{E}_k \otimes S_{kj})$ and the map $\widetilde{\Pi}_j : \mathscr{E}_j \to \widetilde{\mathscr{E}}_j$; $\xi \mapsto \delta_{\mathscr{E}_j}^k(\xi)$ is a $\mathbb{G}_j$-equivariant unitary equivalence over $\widetilde{\pi}_j : A_j \to \widetilde{A}_j$ (cf. Proposition 5.2.8).

Proof. We have $\mathscr{E}_j = [(\mathrm{id}_{\mathscr{E}_j} \otimes \omega)\delta_{\mathscr{E}_k}^j(\xi); \omega \in \mathcal{B}(\mathscr{H}_{jk})_*, \xi \in \mathscr{E}_k]$ (cf. Proposition-Definition 6.2.5(iv)) and for all $\xi \in \mathscr{E}_j$ and $\omega \in \mathcal{B}(\mathscr{H}_{jk})_*$ we have

$$\begin{aligned} \delta_{\mathscr{E}_j}^k(\mathrm{id}_{\mathscr{E}_j} \otimes \omega)\delta_{\mathscr{E}_k}^j(\xi) &= (\mathrm{id}_{\mathscr{E}_k \otimes S_{kj}} \otimes \omega)\big(\delta_{\mathscr{E}_j}^k \otimes \mathrm{id}_{S_{jk}}\big)\delta_{\mathscr{E}_k}^j(\xi) \\ &= (\mathrm{id}_{\mathscr{E}_k \otimes S_{kj}} \otimes \omega)\delta_{\mathscr{E}_j}^{(k)}(\xi) \end{aligned}$$

(cf. Proposition-Definition 6.2.5(v)), where $\delta_{\mathscr{E}_j}^{(k)}(\xi) := (\mathrm{id}_{\mathscr{E}_j} \otimes \delta_{jj}^k)\delta_{\mathscr{E}_j}^j(\xi)$. As a consequence, statement 1 is proved as well as the surjectivity of $\widetilde{\Pi}_j$. The fact

that $\widetilde{\Pi}_j$ is a $\mathbb{G}_j$-equivariant $\widetilde{\pi}_j$-compatible unitary operator is just a restatement of Proposition-Definition 6.2.5(iii) and $(\mathrm{id}_{\mathscr{E}_k} \otimes \delta^j_{kj})\delta^k_{\mathscr{E}_j} = (\delta^k_{\mathscr{E}_j} \otimes \mathrm{id}_{S_{jj}})\delta^j_{\mathscr{E}_j}$ (Proposition-Definition 6.2.5(v)). □

THEOREM 6.3.20. *Let $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ be a colinking measured quantum groupoid between two regular monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Let $j = 1, 2$. The map $(\mathscr{E}, \beta_{\mathscr{E}}, \delta_{\mathscr{E}}) \mapsto (\mathscr{E}_j, \delta^j_{\mathscr{E}_j})$ is a one-to-one correspondence up to unitary equivalence (cf. Proposition-Definition 6.2.5 and Lemma 6.2.6(1)). The inverse map, up to unitary equivalence, is $(\mathscr{F}_j, \delta_{\mathscr{F}_j}) \mapsto (\mathscr{F}, \beta_{\mathscr{F}}, \delta_{\mathscr{F}})$ (cf. Propositions 6.3.18, 6.3.13 and Lemma 6.2.6(2)).*

Proof. Let A be a $\mathcal{G}$-C*-algebra and $\mathscr{E}$ a $\mathcal{G}$-equivariant Hilbert A-module. Let us use all the notation introduced in §6.2. Let us denote:

$$(B_1, \delta_{B_1}) := \big(A_1, \delta^1_{A_1}\big), \qquad (B_2, \delta_{B_2}) := \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(B_1, \delta_{B_1});$$
$$(\mathscr{F}_1, \delta_{\mathscr{F}_1}) := \big(\mathscr{E}_1, \delta^1_{\mathscr{E}_1}\big), \qquad (\mathscr{F}_2, \delta_{\mathscr{F}_2}) := \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{F}_1, \delta_{\mathscr{F}_1}).$$

Let us endow the C*-algebra $B := B_1 \oplus B_2$ with the continuous action (β_B, δ_B) of $\mathcal{G}$ and $\mathscr{F} := \mathscr{F}_1 \oplus \mathscr{F}_2$ with the structure of $\mathcal{G}$-equivariant Hilbert B-module $(\beta_{\mathscr{F}}, \delta_{\mathscr{F}})$ (cf. Propositions 5.2.9, 6.3.18). Let $\psi_A : A \to B$ the canonical $\mathcal{G}$-equivariant $*$-isomorphism defined for all $a \in A$ by $\psi_A(a) := (q_{A,1}a, \widetilde{\pi}_2(q_{A,2}a))$ (cf. 4.10 [2]). Then we consider the map $\Psi : \mathscr{E} \to \mathscr{F}$ given by

$$\Psi(\xi) := \big(q_{\mathscr{E},1}\xi, \widetilde{\Pi}_2(q_{\mathscr{E},2}\xi)\big), \quad \text{for all } \xi \in \mathscr{E}.$$

It is clear from Proposition 6.3.19 that Ψ is a ψ_A-compatible unitary operator. Let us consider the $\mathcal{G}$-C*-algebras $K := \mathcal{K}(\mathscr{E} \oplus A)$ and $L := \mathcal{K}(\mathscr{F} \oplus B)$. Let $f : K \to L$ be the associated isomorphism of linking C*-algebras (cf. Proposition A.3.5). In virtue of Proposition 6.1.18, it only remains to prove that f is $\mathcal{G}$-equivariant. We also consider the $\mathbb{G}_1$-C*-algebra $J_1 := \mathcal{K}(\mathscr{F}_1 \oplus B_1)$ and the induced $\mathbb{G}_2$-C*-algebra $J_2 := \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(J_1)$. We recall that we have a canonical isomorphism $\tau : \mathcal{K}(\mathscr{F}_2 \oplus B_2) \to J_2$ (cf. Proposition 6.3.12(ii)). Let us endow the C*-algebra $J := J_1 \oplus J_2$ with the continuous action (β_J, δ_J) of $\mathcal{G}$. Therefore, it amounts to proving that the $*$-isomorphism $(\mathrm{id}_{J_1} \oplus \tau)f : K \to J$ is $\mathcal{G}$-equivariant (we identify $L = J_1 \oplus \mathcal{K}(\mathscr{F}_2 \oplus B_2)$). We apply the notation of §6.2 to the $\mathcal{G}$-C*-algebra K and identify $K_j := q_{K,j}K = \mathcal{K}(\mathscr{E}_j \oplus A_j)$ for $j = 1, 2$. Let us consider as above (by exchanging the roles of A and K) the $\mathcal{G}$-equivariant $*$-isomorphism $\psi_K : K \to J$. By evaluating on elements of the form $\iota_{\mathscr{E}}(\xi)$ for $\xi \in \mathscr{E}$ and $\iota_A(a)$ for $a \in A$, it is staightforward to see that $(\mathrm{id}_{J_1} \oplus \tau)f = \psi_K$. □

7. Takesaki–Takai duality and equivariant Morita equivalence

In this section, we fix a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$ on the finite-dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$ and we use all the

notation introduced in §3.1 and §3.2. We will use the notation and results of §5.1, §5.3, and §6.1.

Equivariant Hilbert bimodules and Morita equivalence. In this paragraph, we introduce the notion of equivariant representation of a $\mathcal{G}$-C*-algebra on a Hilbert module acted upon by $\mathcal{G}$. We then introduce the notion of equivariant Morita equivalence.

NOTATION 7.1. Let A and B be C*-algebras. Let $\mathscr{E}$ be a Hilbert B-module. If $\gamma : A \to \mathcal{L}(\mathscr{E})$ is a $*$-homomorphism then, up to the identification $\mathcal{M}(\mathcal{K}(\mathscr{E}) \otimes S) = \mathcal{L}(\mathscr{E} \otimes S)$, we can extend $\gamma \otimes \mathrm{id}_S$ to a $*$-homomorphism $\gamma \otimes \mathrm{id}_S : \widetilde{\mathcal{M}}(A \otimes S) \to \mathcal{L}(\mathscr{E} \otimes S)$ (cf. §1).

As in 2.9 [3], we have:

DEFINITION 7.2. Let A and B be two $\mathcal{G}$-C*-algebras, $\mathscr{E}$ a Hilbert B-module, $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ an action of $\mathcal{G}$ on $\mathscr{E}$ and $\gamma : A \to \mathcal{L}(\mathscr{E})$ a $*$-representation. We say that γ is $\mathcal{G}$-equivariant if we have the following.

1. $\delta_{\mathscr{E}}(\gamma(a)\xi) = (\gamma \otimes \mathrm{id}_S)(\delta_A(a)) \circ \delta_{\mathscr{E}}(\xi)$, for all $a \in A$ and $\xi \in \mathscr{E}$;
2. $\beta_{\mathscr{E}}(n^{\mathrm{o}}) \circ \gamma(a) = \gamma(\beta_A(n^{\mathrm{o}})a)$, for all $n \in N$ and $a \in A$.

REMARKS 7.3.

1. Provided that the second condition in the above definition is verified, the first condition is equivalent to:

$$\mathscr{V}\big(\gamma(a) \otimes_{\delta_B} 1\big)\mathscr{V}^* = (\gamma \otimes \mathrm{id}_S)\delta_A(a), \quad \text{for all } a \in A, \tag{7.1}$$

   where $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_B} (B \otimes S), \mathscr{E} \otimes S)$ denotes the isometry defined in Proposition 6.1.5(a). Indeed, we can interpret it as follows: $\mathscr{V}(\gamma(a) \otimes_{\delta_B} 1) = (\gamma \otimes \mathrm{id}_S)(\delta_A(a))\mathscr{V}$, for all $a \in A$. Moreover, for all $a \in A$ we have

$$\begin{aligned}(\gamma \otimes \mathrm{id}_S)\big(\delta_A(a)\big)\mathscr{V}\mathscr{V}^* &= (\gamma \otimes \mathrm{id}_S)\big(\delta_A(a)\big)q_{\beta_{\mathscr{E}}\alpha} \\ &= (\gamma \otimes \mathrm{id}_S)\big(\delta_A(a)q_{\beta_A\alpha}\big) \\ &= (\gamma \otimes \mathrm{id}_S)\delta_A(a).\end{aligned}$$

   Hence, $(\mathscr{V}(\gamma(a) \otimes_{\delta_B} 1) = (\gamma \otimes \mathrm{id}_S)(\delta_A(a))\mathscr{V} \Leftrightarrow \mathscr{V}(\gamma(a) \otimes_{\delta_B} 1)\mathscr{V}^* = (\gamma \otimes \mathrm{id}_S)\delta_A(a))$, for all $a \in A$.
2. We recall that the action $\delta_{\mathcal{K}(\mathscr{E})}$ of $\mathcal{G}$ on $\mathcal{K}(\mathscr{E})$ is defined by $\delta_{\mathcal{K}(\mathscr{E})}(k) := \mathscr{V}(k \otimes_{\delta_B} 1)\mathscr{V}^*$ for all $k \in \mathcal{K}(\mathscr{E})$. Hence, (7.1) can be restated as follows: $\delta_{\mathcal{K}(\mathscr{E})}(\gamma(a)) = (\gamma \otimes \mathrm{id}_S)\delta_A(a)$ for all $a \in A$. In particular, if γ is nondegenerate, then Definition 7.2 simply means that the $*$-homomorphism $\gamma : A \to \mathcal{M}(\mathcal{K}(\mathscr{E}))$ is $\mathcal{G}$-equivariant (cf. Definition 5.1.10).
3. If $\gamma : A \to \mathcal{L}(\mathscr{E})$ is a nondegenerate $*$-representation such that

$$\delta_{\mathscr{E}}\big(\gamma(a)\xi\big) = (\gamma \otimes \mathrm{id}_S)\big(\delta_A(a)\big) \circ \delta_{\mathscr{E}}(\xi), \quad \text{for all } a \in A \text{ and } \xi \in \mathscr{E},$$

   then we have $\beta_{\mathscr{E}}(n^{\mathrm{o}}) \circ \gamma(a) = \gamma(\beta_A(n^{\mathrm{o}})a)$ for all $n \in N$ and $a \in A$. Indeed, this will be inferred from Remark 5.1.11 and the previous remark.

DEFINITION 7.4 (*cf.* §6 [22]). Let A and B be two C*-algebras. An imprimitivity A-B-bimodule is an A-B-bimodule $\mathscr{E}$, which is a full left Hilbert A-module for an A-valued inner product ${}_A\langle\cdot,\cdot\rangle$ and a full right Hilbert B-module for a B-valued inner product $\langle\cdot,\cdot\rangle_B$ such that ${}_A\langle\xi,\eta\rangle\zeta = \xi\langle\eta,\zeta\rangle_B$ for all $\xi,\eta,\zeta\in\mathscr{E}$.

REMARKS 7.5. Let A and B be two C*-algebras and $\mathscr{E}$ an imprimitivity A-B-bimodule. We recall that the norms defined by the inner products ${}_A\langle\cdot,\cdot\rangle$ on ${}_A\mathscr{E}$ and $\langle\cdot,\cdot\rangle_B$ on $\mathscr{E}_B$ coincide. We also recall that the left (resp., right) action of A (resp., B) on $\mathscr{E}$ defines a non-degenerate $*$-homomorphism $\gamma : A\to\mathcal{L}(\mathscr{E}_B)$ (resp., $\rho : B\to\mathcal{L}({}_A\mathscr{E})$).

DEFINITION 7.6. Let A and B be two $\mathcal{G}$-C*-algebras. A $\mathcal{G}$-equivariant imprimitivity A-B-bimodule is an imprimitivity A-B-bimodule $\mathscr{E}$ endowed with a continuous action of $\mathcal{G}$ on $\mathscr{E}_B$ such that the left action $\gamma : A\to\mathcal{L}(\mathscr{E}_B)$ is $\mathcal{G}$-equivariant.

EXAMPLES 7.7. Let A and B be two $\mathcal{G}$-C*-algebras.

(i) B is a $\mathcal{G}$-equivariant imprimitivity B-B-bimodule for the inner products given by ${}_B\langle x,y\rangle := xy^*$ and $\langle x,y\rangle_B := x^*y$ for all $x,y\in B$.

(ii) Let $\mathscr{E}$ be a $\mathcal{G}$-equivariant Hilbert B-module. If $\mathscr{E}$ is full, then $\mathscr{E}$ is a $\mathcal{G}$-equivariant imprimitivity $\mathcal{K}(\mathscr{E})$-B-bimodule for the natural left action and the inner product given by ${}_{\mathcal{K}(\mathscr{E})}\langle\xi,\eta\rangle := \theta_{\xi,\eta}$ for all $\xi,\eta\in\mathscr{E}$. Conversely, if $\mathscr{E}$ is a $\mathcal{G}$-equivariant imprimitivity A-B-bimodule, then the the left action $\gamma : A\to\mathcal{L}(\mathscr{E}_B)$ induces an isomorphism of $\mathcal{G}$-C*-algebras $A\simeq\mathcal{K}(\mathscr{E}_B)$.

(iii) Let $(J,\beta_J,\delta_J,e_1,e_2)$ be a linking $\mathcal{G}$-C*-algebra (cf. Definition 6.1.22). Let $A := e_1Je_1$ and $B := e_2Je_2$ be the corner C*-algebras endowed with the continuous actions of $\mathcal{G}$ induced by (β_J,δ_J). Let us endow $\mathscr{E} := e_1Je_2$ with its structure of $\mathcal{G}$-equivariant Hilbert B-module (cf. Remarks 6.1.23). Then, $\mathscr{E}$ is a $\mathcal{G}$-equivariant imprimitivity A-B-module whose actions and inner products are defined as in (i).

(iv) If $\mathscr{E}$ is a $\mathcal{G}$-equivariant imprimitivity A-B-bimodule, then $\mathscr{E}^*$ turns into a $\mathcal{G}$-equivariant imprimitivity B-A-bimodule for the actions and inner products given by the following formulas: $b\xi^*a := (a^*\xi b^*)^*$, for $\xi^*\in\mathscr{E}^*$, $a\in A$ and $b\in B$; ${}_B\langle\xi^*,\eta^*\rangle := \langle\xi,\eta\rangle_B$ and $\langle\xi^*,\eta^*\rangle_A := {}_A\langle\xi,\eta\rangle$, for $\xi^*,\eta^*\in\mathscr{E}^*$.

PROPOSITION 7.8. *Let A and B be $\mathcal{G}$-C*-algebras. The following statements are equivalent*:

(i) *there exists a $\mathcal{G}$-equivariant imprimitivity A-B-bimodule*;

(ii) *there exists a full $\mathcal{G}$-equivariant Hilbert B-module $\mathscr{E}$ such that we have an isomorphism $A\simeq\mathcal{K}(\mathscr{E})$ of $\mathcal{G}$-C*-algebras*;

(iii) *there exists a linking $\mathcal{G}$-C*-algebra $(J,\beta_J,\delta_J,e_1,e_2)$ such that we have $\mathcal{G}$-equivariant $*$-isomorphisms $A\simeq e_1Je_1$ and $B\simeq e_2Je_2$.*

Proof. This is a straightforward consequence of Examples 7.7(ii), (iii), Proposition 6.1.11(b), and Remarks 6.1.23. □

Now, we investigate the tensor product construction (cf. 2.10 [3] for the quantum group case).

PROPOSITION 7.9. *Let C (resp., B) be a $\mathcal{G}$-C^*-algebra. Let $\mathscr{E}_1$ (resp., $\mathscr{E}_2$) be a Hilbert module over C (resp., B) endowed with an action $(\beta_{\mathscr{E}_1}, \delta_{\mathscr{E}_1})$ (resp., $(\beta_{\mathscr{E}_2}, \delta_{\mathscr{E}_2})$) of $\mathcal{G}$. Let $\gamma_2 : C \to \mathcal{L}(\mathscr{E}_2)$ be a $\mathcal{G}$-equivariant $*$-representation. Consider the Hilbert B-module $\mathscr{E} := \mathscr{E}_1 \otimes_{\gamma_2} \mathscr{E}_2$. Denote by*

$$\Delta(\xi_1, \xi_2) := \big(\delta_{\mathscr{E}_1}(\xi_1) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} 1\big) \circ \delta_{\mathscr{E}_2}(\xi_2), \quad \text{for } \xi_1 \in \mathscr{E}_1 \text{ and } \xi_2 \in \mathscr{E}_2.$$

We have $\Delta(\xi_1, \xi_2) \in \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ for all $\xi_1 \in \mathscr{E}_1$ and $\xi_2 \in \mathscr{E}_2$. Let $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ be the $*$-homomorphism defined by

$$\beta_{\mathscr{E}}\big(n^{\mathrm{o}}\big) := \beta_{\mathscr{E}_1}\big(n^{\mathrm{o}}\big) \otimes_{\gamma_2} 1, \quad \text{for all } n \in N.$$

There exists a unique map $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ defined by the formula $\delta_{\mathscr{E}}(\xi_1 \otimes_{\gamma_2} \xi_2) := \Delta(\xi_1, \xi_2)$ for $\xi_1 \in \mathscr{E}_1$ and $\xi_2 \in \mathscr{E}_2$ such that the pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$ is an action of $\mathcal{G}$ on $\mathscr{E}$.

The operator $\delta_{\mathscr{E}_1}(\xi_1)$ is considered here as an element of $\mathcal{L}(\widetilde{C} \otimes S, \mathscr{E}_1 \otimes S) \supset \widetilde{\mathcal{M}}(\mathscr{E}_1 \otimes S)$. In particular, we have $\delta_{\mathscr{E}_1}(\xi_1) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} 1 \in \mathcal{L}(\mathscr{E}_2 \otimes S, \mathscr{E} \otimes S)$ since we use the identifications:

$$\begin{aligned} (\widetilde{C} \otimes S) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} (\mathscr{E}_2 \otimes S) &= \mathscr{E}_2 \otimes S, \\ x \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} \eta &\mapsto (\widetilde{\gamma}_2 \otimes \mathrm{id}_S)(x)\eta; \end{aligned} \tag{7.2}$$

$$\begin{aligned} (\mathscr{E}_1 \otimes S) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} (\mathscr{E}_2 \otimes S) &= \mathscr{E} \otimes S, \\ (\xi_1 \otimes s) \otimes_{\widetilde{\gamma}_2 \otimes \mathrm{id}_S} (\xi_2 \otimes t) &\mapsto (\xi_1 \otimes_{\gamma_2} \xi_2) \otimes st. \end{aligned} \tag{7.3}$$

Proof. The proof is basically the same as that of 2.10 [3]. For example, we refer the reader to it for the proof of the fact that $\Delta(\xi_1, \xi_2) \in \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ for all $\xi_1 \in \mathscr{E}_1$ and $\xi_2 \in \mathscr{E}$. Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be the isometries associated with $\delta_{\mathscr{E}_1}$ and $\delta_{\mathscr{E}_2}$. Since $\mathscr{V}_2$ intertwines the left actions $c \mapsto \gamma_2(c) \otimes_{\delta_B} 1$ and $(\gamma_2 \otimes \mathrm{id}_S)\delta_C$ of C, there exists a unique isometry $\widetilde{\mathscr{V}_2} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_B} (B \otimes S), \mathscr{E}_1 \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} (\mathscr{E}_2 \otimes S))$ such that

$$\begin{aligned} &\widetilde{\mathscr{V}_2}\big((\xi_1 \otimes_{\gamma_2} \xi_2) \otimes_{\delta_B} x\big) = \xi_1 \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} \mathscr{V}_2(\xi_2 \otimes_{\delta_B} x), \\ &\quad \text{for all } \xi_1 \in \mathscr{E}_1, \xi_2 \in \mathscr{E}_2 \text{ and } x \in B \otimes S. \end{aligned}$$

Let us prove that $\widetilde{\mathscr{V}_2}$ is a unitary. It amounts to proving that $\widetilde{\mathscr{V}_2}$ is surjective. Since $\mathrm{im}(\mathscr{V}_2) = \mathrm{im}(q_{\beta_{\mathscr{E}_2}\alpha})$, we have $\mathrm{im}(\widetilde{\mathscr{V}_2}) = [\xi \otimes_{(\gamma_2 \otimes \mathrm{id}_S)\delta_C} q_{\beta_{\mathscr{E}_2}\alpha}\eta; \xi \in \mathscr{E}_1, \eta \in \mathscr{E}_2 \otimes S]$. Let $\xi \in \mathscr{E}_1$ and $\eta \in \mathscr{E}_2 \otimes S$. Write $\xi = \xi' c$ with $\xi' \in \mathscr{E}_1$ and

$c \in C$. Since $\mathscr{V}_2\mathscr{V}_2^* = q_{\mathscr{E}_2\alpha}$, we have $(\gamma_2 \otimes \mathrm{id}_S)\delta_C(c)q_{\beta_{\mathscr{E}_2}\alpha} = (\gamma_2 \otimes \mathrm{id}_S)\delta_C(c)$ (cf. Remarks 7.3). Hence,

$$\begin{aligned}\xi \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} q_{\beta_{\mathscr{E}_2}\alpha}\eta &= \xi' \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} (\gamma_2 \otimes \mathrm{id}_S)\delta_C(c)q_{\beta_{\mathscr{E}_2}\alpha}\eta \\ &= \xi' \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} (\gamma_2 \otimes \mathrm{id}_S)\delta_C(c)\eta \\ &= \xi \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} \eta.\end{aligned}$$

Therefore, we have shown that $\mathrm{im}(\widetilde{\mathscr{V}_2}) = \mathscr{E}_1 \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} (\mathscr{E}_2 \otimes S)$, which proves that $\widetilde{\mathscr{V}_2}$ is unitary. Let us identify

$$\begin{aligned}\big(\mathscr{E}_1 \otimes_{\delta_C} (C \otimes S)\big) \otimes_{\gamma_2\otimes\mathrm{id}_S} (\mathscr{E}_2 \otimes S) &\to \mathscr{E}_1 \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} (\mathscr{E}_2 \otimes S), \\ (\xi_1 \otimes_{\delta_C} x) \otimes_{\gamma_2\otimes\mathrm{id}_S} \eta &\mapsto \xi_1 \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} (\gamma_2 \otimes \mathrm{id}_S)(x)\eta\end{aligned}$$

and $(\mathscr{E}_1 \otimes S) \otimes_{\gamma_2\otimes\mathrm{id}_S} (\mathscr{E}_2 \otimes S) = \mathscr{E} \otimes S$ (cf. (7.3)). Let

$$\mathscr{V} := (\mathscr{V}_1 \otimes_{\gamma_2\otimes\mathrm{id}_S} 1)\widetilde{\mathscr{V}_2} \in \mathcal{L}\big(\mathscr{E} \otimes_{\delta_B} (B \otimes S), \mathscr{E} \otimes S\big).$$

It follows from the formulas $\widetilde{\mathscr{V}_2}^*\widetilde{\mathscr{V}_2} = 1$, $\widetilde{\mathscr{V}_2}\widetilde{\mathscr{V}_2}^* = 1$, $\mathscr{V}_1^*\mathscr{V}_1 = 1$ and $\mathscr{V}_1\mathscr{V}_1^* = q_{\beta_{\mathscr{E}_1}\alpha}$ that $\mathscr{V}^*\mathscr{V} = 1$ and $\mathscr{V}\mathscr{V}^* = q_{\beta_{\mathscr{E}_1}\alpha} \otimes_{\gamma_2\otimes\mathrm{id}_S} 1 = q_{\beta_{\mathscr{E}}\alpha}$ (by definition of $\beta_{\mathscr{E}}$).

Let $n \in N$. On one hand, we have

$$\widetilde{\mathscr{V}_2}\big(\beta_{\mathscr{E}}(n^{\mathrm{o}}) \otimes_{\delta_B} 1\big) = \big(\beta_{\mathscr{E}_1}(n^{\mathrm{o}}) \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} 1\big)\widetilde{\mathscr{V}_2}$$

(by definition of $\beta_{\mathscr{E}}$ and $\widetilde{\mathscr{V}_2}$). On the other, we have

$$\begin{aligned}&(\mathscr{V}_1 \otimes_{\gamma_2\otimes\mathrm{id}_S} 1)\big(\beta_{\mathscr{E}_1}(n^{\mathrm{o}}) \otimes_{(\gamma_2\otimes\mathrm{id}_S)\delta_C} 1\big) \\ &\quad = \big(\big(1_{\mathscr{E}_1} \otimes \beta(n^{\mathrm{o}})\big) \otimes_{\gamma_2\otimes\mathrm{id}_S} 1\big)(\mathscr{V}_1 \otimes_{\gamma_2\otimes\mathrm{id}_S} 1).\end{aligned}$$

Hence, we have proved that $\mathscr{V}(\beta_{\mathscr{E}}(n^{\mathrm{o}}) \otimes_{\delta_B} 1) = (1 \otimes \beta(n^{\mathrm{o}}))\mathscr{V}$ for all $n \in N$. Exactly as in the proof of 2.10 [3], we have $\mathscr{V}T_{\xi_1\otimes_{\gamma_2}\xi_2} = \Delta(\xi_1, \xi_2)$ for all $\xi_1 \in \mathscr{E}_1$ and $\xi_2 \in \mathscr{E}_2$ (cf. Notation 6.1.3 for the definition of T_ξ). In particular, $\mathscr{V}T_\xi \in \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ for all $\xi \in \mathscr{E}$. It then follows from Proposition 6.1.5(b) that the pair $(\beta_{\mathscr{E}}, \delta_{\mathscr{E}})$, where $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ is defined for all $\xi \in \mathscr{E}$ by $\delta_{\mathscr{E}}(\xi) := \mathscr{V}T_\xi$, satisfies the conditions 1, 2, and 3 of Definition 6.1.4. The coassociativity condition of $\delta_{\mathscr{E}}$ is derived from those of $\delta_{\mathscr{E}_1}$ and $\delta_{\mathscr{E}_2}$ exactly as in the proof of 2.10 [3]. □

PROPOSITION 7.10. *We use all the notation and hypotheses of Proposition 7.9. If A is a $\mathcal{G}$-C^*-algebra and $\gamma_1 : A \to \mathcal{L}(\mathscr{E}_1)$ is a $\mathcal{G}$-equivariant $*$-representation, then $\gamma : A \to \mathcal{L}(\mathscr{E}_1 \otimes_{\gamma_2} \mathscr{E}_2)$ the $*$-representation defined by $\gamma(a) := \gamma_1(a) \otimes_{\gamma_2} 1$ for all $a \in A$ is $\mathcal{G}$-equivariant.*

Proof. Through the identification (7.3), for all $x \in A \otimes S$ the operator $(\gamma_1 \otimes \mathrm{id}_S)(x) \otimes_{\widetilde{\gamma}_2\otimes\mathrm{id}_S} 1$ is identified to $(\gamma \otimes \mathrm{id}_S)(x)$. This identification also holds for $x \in \widetilde{\mathcal{M}}(A \otimes S)$ (by using the fact that any element of S can be written as a product of two elements of S). In particular, for all $a \in A$ the operator $(\gamma_1 \otimes \mathrm{id}_S)\delta_A(a) \otimes_{\widetilde{\gamma}_2\otimes\mathrm{id}_S} 1$ is identified to $(\gamma \otimes \mathrm{id}_S)\delta_A(a)$. Hence, $\delta_{\mathscr{E}}(\gamma(a)\xi) =$

$(\gamma \otimes \mathrm{id}_S)\delta_A(a) \circ \delta_{\mathscr{E}}(\xi)$ for all $\xi \in \mathscr{E}$ and $a \in A$ by definition of $\delta_{\mathscr{E}}$. The relation $\beta_{\mathscr{E}}(n^{\mathrm{o}}) \circ \gamma(a) = \gamma(\beta_A(n^{\mathrm{o}})a)$ for $n \in N$ and $a \in A$ is straightforward. □

From now on, we assume the quantum groupoid $\mathcal{G}$ to be regular. We recall that any action of the quantum groupoid $\mathcal{G}$ on a Hilbert module is necessarily continuous (cf. Corollary 6.1.26).

Proposition-Definition 7.11. *Let A, C and B be $\mathcal{G}$-C*-algebras. Let $\mathscr{E}_1$ (resp., $\mathscr{E}_2$) be a $\mathcal{G}$-equivariant imprimitivity A-C-bimodule (resp., C-B-bimodule). Denote by $\mathscr{E}_1 \otimes_C \mathscr{E}_2$ the internal tensor product $\mathscr{E}_1 \otimes_{\gamma_2} \mathscr{E}_2$, where $\gamma_2 : C \to \mathcal{L}(\mathscr{E}_2)$ is the $\mathcal{G}$-equivariant $*$-representation defined by the left action of C on $\mathscr{E}_2$. The Hilbert B-module $\mathscr{E}_1 \otimes_C \mathscr{E}_2$ endowed with the action of $\mathcal{G}$ defined in Proposition 7.9 is a $\mathcal{G}$-equivariant imprimitivity A-B-bimodule for the left action of A and the A-valued inner product defined by the formulas*:

- $a(\xi_1 \otimes_C \xi_2) := a\xi_1 \otimes_C \xi_2$, *for all* $a \in A$, $\xi_1 \in \mathscr{E}_1$ *and* $\xi_2 \in \mathscr{E}_2$;
- ${}_A\langle \xi_1 \otimes_C \xi_2, \eta_1 \otimes_C \eta_2 \rangle := {}_A\langle \xi_1, \eta_1 {}_C\langle \xi_2, \eta_2 \rangle \rangle$, *for all* $\xi_1, \eta_1 \in \mathscr{E}_1$ *and* $\xi_2, \eta_2 \in \mathscr{E}_2$.

Proof. It is known that $\mathscr{E}_1 \otimes_C \mathscr{E}_2$ is an imprimitivity A-B-bimodule. The rest of the proof is contained in Propositions 7.9 and 7.10. □

Proposition 7.12. *Let A and B be $\mathcal{G}$-C*-algebras. Let $\mathscr{E}$ be a $\mathcal{G}$-equivariant imprimitivity A-B-bimodule. Then the map $\mathscr{E}^* \otimes_A \mathscr{E} \to B$; $\xi^* \otimes_A \eta \mapsto \langle \xi, \eta \rangle_B$ defines an isomorphism of $\mathcal{G}$-equivariant imprimitivity B-B-bimodules.*

Proof. It is known that the map $\Phi : \mathscr{E}^* \otimes_A \mathscr{E} \to B$; $\xi^* \otimes_A \eta \mapsto \langle \xi, \eta \rangle_B$ is an isomorphism of imprimitivity B-B-bimodules. The $\mathcal{G}$-equivariance of Φ is a restatement of the formula $\delta_B(\langle \xi, \eta \rangle_B) = \delta_{\mathscr{E}}(\xi)^* \circ \delta_{\mathscr{E}}(\eta)$ for $\xi, \eta \in \mathscr{E}$. □

Definition 7.13. Let A and B be $\mathcal{G}$-C*-algebras. We say that A and B are $\mathcal{G}$-equivariantly Morita equivalent if there exists a $\mathcal{G}$-equivariant imprimitivity A-B-bimodule. The $\mathcal{G}$-equivariant Morita equivalence is a reflexive (Examples 7.7(i)), symmetric (Examples 7.7(iv)) and transitive (Proposition-Definition 7.11) relation on the class of $\mathcal{G}$-C*-algebras.

Biduality and equivariant Morita equivalence. In this paragraph, the measured quantum groupoid $\mathcal{G}$ is assumed to be regular. Let us fix a $\mathcal{G}$-C*-algebras A. We show that there is a canonical $\mathcal{G}$-equivariant Morita equivalence between A and the double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$.

Notation 7.14. Denote by $\mathcal{K} := \mathcal{K}(\mathscr{H})$ for short. Consider the Hilbert A-modules $\mathcal{E}_0 := A \otimes \mathscr{H}$ and $\mathcal{E}_{A,R} := q_{\beta_A \widehat{\alpha}}(A \otimes \mathscr{H})$. Let $\mathcal{V} \in \mathcal{L}(\mathscr{H} \otimes S)$ be the unique partial isometry such that $(\mathrm{id}_{\mathcal{K}} \otimes L)(\mathcal{V}) = V$.

Proposition 7.15. *There exists a unique bounded linear map $\delta_{\mathcal{E}_0} : \mathcal{E}_0 \to \mathcal{L}(A \otimes S, \mathcal{E}_0 \otimes S)$ such that $\delta_{\mathcal{E}_0}(a \otimes \zeta) = \mathcal{V}_{23}\delta_A(a)_{13}(1_A \otimes \zeta \otimes 1_S)$, for all $a \in A$ and $\zeta \in \mathscr{H}$.*

Proof. If B is a C*-algebra and $\mathscr{K}$ a Hilbert space, we identify $\mathcal{M}(B)\otimes\mathscr{K}$ with a closed vector subspace of $\mathcal{L}(B, B\otimes\mathscr{K})$. We have $(\delta_A\otimes \mathrm{id}_{\mathscr{H}})(\xi)\in \mathcal{L}(A\otimes S, A\otimes S\otimes\mathscr{H})$ and $(\delta_A\otimes \mathrm{id}_{\mathscr{H}})(\xi)^* = (\delta_A\otimes \mathrm{id}_{\mathscr{H}^*})(\xi^*)$ for $\xi\in\mathcal{E}_0$. Let $\sigma\in\mathcal{L}(S\otimes\mathscr{H}, \mathscr{H}\otimes S)$ be the flip map. Denote by $\delta_{\mathcal{E}_0}:\mathcal{E}_0\to\mathcal{L}(A\otimes S, \mathcal{E}_0\otimes S)$ the map defined by $\delta_{\mathcal{E}_0}(\xi) := \mathcal{V}_{23}\sigma_{23}(\delta_A\otimes \mathrm{id}_{\mathscr{H}})(\xi)$ for $\xi\in\mathcal{E}_0$. It is clear that $\delta_{\mathcal{E}_0}:\mathcal{E}_0\to\mathcal{L}(A\otimes S, \mathcal{E}_0\otimes S)$ is linear map satsifying the formula $\delta_{\mathcal{E}_0}(a\otimes\xi) = \mathcal{V}_{23}\delta_A(a)_{13}(1_A\otimes\xi\otimes 1_S)$ for all $a\in A$ and $\zeta\in\mathscr{H}$. □

Proposition 7.16. *We have the following statements*:

1. $\delta_{\mathcal{E}_0}(\xi)^*\delta_{\mathcal{E}_0}(\eta) = \delta_A(\langle q_{\beta_A\widehat{\alpha}}\xi, q_{\beta_A\widehat{\alpha}}\eta\rangle)$, *for all* $\xi,\eta\in\mathcal{E}_0$;
2. $\delta_{\mathcal{E}_0}(\xi a) = \delta_{\mathcal{E}_0}(\xi)\delta_A(a)$, *for all* $\xi\in\mathcal{E}_0$ *and* $a\in A$;
3. $\delta_{\mathcal{E}_0}(q_{\beta_A\widehat{\alpha}}\xi) = \delta_{\mathcal{E}_0}(\xi)$, *for all* $\xi\in\mathcal{E}_0$;
4. $\delta_{\mathcal{E}_0}(\mathcal{E}_0)(A\otimes S)\subset\mathcal{E}_{A,R}\otimes S$.

Proof. 1. Let $\xi,\eta\in\mathcal{E}_0$, we have $\delta_{\mathcal{E}_0}(\xi)^*\delta_{\mathcal{E}_0}(\eta) = (\delta_A\otimes\mathrm{id})(\xi^*)\sigma_{23}^*\mathcal{V}_{23}^*\mathcal{V}_{23}\times\sigma_{23}(\delta_A\otimes\mathrm{id})(\eta)$. We have $\sigma^*\mathcal{V}^*\mathcal{V}\sigma = q_{\beta\widehat{\alpha}}$. Let $n,n'\in N$. For all $a\in A$ and $\zeta\in\mathscr{H}$, we have

$$\begin{aligned}
&\big(1_A\otimes\beta\big(n^{\mathrm{o}}\big)\otimes\widehat{\alpha}\big(n'\big)\big)(\delta_A\otimes\mathrm{id}_{\mathscr{H}})(a\otimes\zeta)\\
&\quad= \big(1_A\otimes\beta\big(n^{\mathrm{o}}\big)\big)\delta_A(a)\otimes\widehat{\alpha}\big(n'\big)\zeta\\
&\quad= \delta_A\big(\beta_A\big(n^{\mathrm{o}}\big)a\big)\otimes\widehat{\alpha}\big(n'\big)\zeta\\
&\quad= (\delta_A\otimes\mathrm{id}_{\mathscr{H}})\big(\big(\beta_A\big(n^{\mathrm{o}}\big)\otimes\widehat{\alpha}\big(n'\big)\big)(a\otimes\zeta)\big).
\end{aligned}$$

Hence, $(1_A\otimes\beta(n^{\mathrm{o}})\otimes\widehat{\alpha}(n'))(\delta_A\otimes\mathrm{id}_{\mathscr{H}})(\eta) = (\delta_A\otimes\mathrm{id}_{\mathscr{H}})((\beta_A(n^{\mathrm{o}})\otimes\widehat{\alpha}(n'))\eta)$. It then follows that $\sigma_{23}^*\mathcal{V}_{23}^*\mathcal{V}_{23}\sigma_{23}(\delta_A\otimes\mathrm{id}_{\mathscr{H}})(\eta) = (\delta_A\otimes\mathrm{id}_{\mathscr{H}})(q_{\beta_A\widehat{\alpha}}\eta)$. We finally have

$$\begin{aligned}
\delta_{\mathcal{E}_0}(\xi)^*\delta_{\mathcal{E}_0}(\eta) &= (\delta_A\otimes\mathrm{id}_{\mathscr{H}^*})\big(\xi^*\big)(\delta_A\otimes\mathrm{id}_{\mathscr{H}})(q_{\beta_A\widehat{\alpha}}\eta)\\
&= \delta_A\big(\langle\xi, q_{\beta_A\widehat{\alpha}}\eta\rangle\big)\\
&= \delta_A\big(\langle q_{\beta_A\widehat{\alpha}}\xi, q_{\beta_A\widehat{\alpha}}\eta\rangle\big),
\end{aligned}$$

where the last equality follows from the fact that $q_{\beta_A\widehat{\alpha}}\in\mathcal{L}(\mathcal{E}_0)$ is a self-adjoint projection.

2. Let us fix $a,b\in A$, and $\zeta\in\mathscr{H}$. We have $\delta_A(a)_{13}(1_A\otimes\zeta\otimes 1_S) = (1_A\otimes\zeta\otimes 1_S)\delta_A(a)$ in $\mathcal{L}(A\otimes S, \mathcal{E}_0\otimes S)$. Hence, $\delta_{\mathcal{E}_0}((b\otimes\zeta)a) = \delta_{\mathcal{E}_0}(b\otimes\zeta)\delta_A(a)$.

3. Let $a\in A$ and $\zeta\in\mathscr{H}$. For all $n,n'\in N$, we have

$$\begin{aligned}
&\delta_{\mathcal{E}_0}\big(\beta_A\big(n^{\mathrm{o}}\big)a\otimes\widehat{\alpha}\big(n'\big)\zeta\big)\\
&\quad= \mathcal{V}_{23}\big(1_A\otimes\widehat{\alpha}\big(n'\big)\otimes 1_S\big)\delta_A\big(\beta_A\big(n^{\mathrm{o}}\big)a\big)_{13}(1_A\otimes\zeta\otimes 1_S)\\
&\quad= \mathcal{V}_{23}\big(1_A\otimes\widehat{\alpha}\big(n'\big)\otimes\beta\big(n^{\mathrm{o}}\big)\big)\delta_A(a)_{13}(1_A\otimes\zeta\otimes 1_S).
\end{aligned}$$

Hence, $\delta_{\mathcal{E}_0}(q_{\beta_A\widehat{\alpha}}(a\otimes\zeta)) = \mathcal{V}_{23}q_{\widehat{\alpha}\beta,23}\delta_A(a)_{13}(1_A\otimes\zeta\otimes 1_S) = \delta_{\mathcal{E}_0}(a\otimes\zeta)$.

4. It suffices to show that $q_{\beta_A\widehat{\alpha},12}\delta_{\mathcal{E}_0}(\xi) = \delta_{\mathcal{E}_0}(\xi)$ for all $\xi\in\mathcal{E}_0$. We recall (cf. Proposition 3.1.5) that $(\widehat{\alpha}(n)\otimes 1_S)\mathcal{V} = \mathcal{V}(1_{\mathcal{K}}\otimes\alpha(n))$ for all $n\in N$. Hence, $q_{\beta_A\widehat{\alpha},12}\mathcal{V}_{23} = \mathcal{V}_{23}q_{\beta_A\alpha,13}$. It then follows from $q_{\beta_A\alpha} = \delta_A(1_A)$ that

$q_{\beta_A\widehat{\alpha},12}\mathcal{V}_{23}\delta_A(a)_{13} = \mathcal{V}_{23}\delta_A(a)_{13}$ for all $a \in A$. Hence, $q_{\beta_A\widehat{\alpha},12}\delta_{\mathcal{E}_0}(a\otimes\zeta) = \delta_{\mathcal{E}_0}(a\otimes\zeta)$ for all $a\in A$ and $\zeta\in\mathscr{H}$. □

NOTATION 7.17. According to the previous proposition, $\delta_{\mathcal{E}_0}$ restricts to a linear map

$$\delta_{\mathcal{E}_{A,R}} : \mathcal{E}_{A,R} \to \mathcal{L}(A\otimes S, \mathcal{E}_{A,R}\otimes S),$$

which satisfies the following statements:

- $\delta_{\mathcal{E}_{A,R}}(\xi)^*\delta_{\mathcal{E}_{A,R}}(\eta) = \delta_A(\langle\xi,\eta\rangle)$, for all $\xi,\eta\in\mathcal{E}_{A,R}$;
- $\delta_{\mathcal{E}_{A,R}}(\xi a) = \delta_{\mathcal{E}_{A,R}}(\xi)\delta_A(a)$, for all $\xi\in\mathcal{E}_{A,R}$ and $a\in A$.

Since $[\widehat{\alpha}(n'), \beta(n^{\mathrm{o}})] = 0$ for all $n,n'\in N$, we have $[1_A\otimes\beta(n^{\mathrm{o}}), q_{\beta_A\widehat{\alpha}}] = 0$ for all $n\in N$. We then have a nondegenerate $*$-homomorphism

$$\beta_{\mathcal{E}_{A,R}} : N^{\mathrm{o}} \to \mathcal{L}(\mathcal{E}_{A,R}); \qquad n\mapsto \big(1_A\otimes\beta\big(n^{\mathrm{o}}\big)\big)\restriction_{\mathcal{E}_{A,R}}.$$

Since β and $\widehat{\alpha}$ commute pointwise and $\mathcal{V}\mathcal{V}^* = q_{\beta\alpha}$, we have $[\mathcal{V}_{23}\mathcal{V}_{23}^*, q_{\beta_A\widehat{\alpha},12}] = 0$. Hence, $q_{\beta_{\mathcal{E}_{A,R}}\alpha} = \mathcal{V}_{23}\mathcal{V}_{23}^*\restriction_{\mathcal{E}_{A,R}\otimes S}\in\mathcal{L}(\mathcal{E}_{A,R}\otimes S)$.

PROPOSITION 7.18. *We have the following statements*:

1. $\delta_{\mathcal{E}_{A,R}}(\mathcal{E}_{A,R}) \subset \widetilde{\mathcal{M}}(\mathcal{E}_{A,R}\otimes S)$;
2. $[\delta_{\mathcal{E}_{A,R}}(\mathcal{E}_{A,R})(A\otimes S)] = q_{\beta_{\mathcal{E}_{A,R}}\alpha}(\mathcal{E}_{A,R}\otimes S)$;
3. $\delta_{\mathcal{E}_{A,R}}(\beta_{\mathcal{E}_{A,R}}(n^{\mathrm{o}})\xi) = (1_{\mathcal{E}_{A,R}}\otimes\beta(n^{\mathrm{o}}))\delta_{\mathcal{E}_{A,R}}(\xi)$, *for all* $\xi\in\mathcal{E}_{A,R}$ *and* $n\in N$.

Proof. 1. Let us prove that $\delta_{\mathcal{E}_{A,R}}(\xi)(1_A\otimes s)\in\mathcal{E}_{A,R}\otimes S$ for all $\xi\in\mathcal{E}_{A,R}$ and $s\in S$. It amounts to proving that $\delta_{\mathcal{E}_0}(\xi)(1_A\otimes s)\in\mathcal{E}_0\otimes S$ for all $\xi\in\mathcal{E}_0$ and $s\in S$ (cf. Proposition 7.16(3), (4)). Let $a\in A$ and $\zeta\in\mathscr{H}$. It follows from the relation $\delta_A(A)(1_A\otimes S)\subset A\otimes S$ that $\delta_{\mathcal{E}_0}(a\otimes\zeta)(1_A\otimes s) = (1_A\otimes\mathcal{V}(\zeta\otimes 1_S))\delta_A(a)(1_A\otimes s)$ is the norm limit of finite sums of the form $\sum_i a_i\otimes\mathcal{V}(\zeta\otimes s_i)$, where $a_i\in A$ and $s_i\in S$. Hence, $\delta_{\mathcal{E}_0}(a\otimes\zeta)(1_A\otimes s)\in\mathcal{E}_0\otimes S$.

Now, let us prove that $(1_{\mathcal{E}_{A,R}}\otimes y)\delta_{\mathcal{E}_{A,R}}(\xi)\in\mathcal{E}_{A,R}\otimes S$ for all $\xi\in\mathcal{E}_{A,R}$ and $y\in S$. This also amounts to proving that $(1_{\mathcal{E}_0}\otimes y)\delta_{\mathcal{E}_0}(\xi)\in\mathcal{E}_0\otimes S$ for all $\xi\in\mathcal{E}_0$ and $y\in S$. Let $a\in A$, $\zeta\in\mathscr{H}$ and $y\in S$, we have $(1_{\mathcal{E}_0}\otimes y)\delta_{\mathcal{E}_0}(a\otimes\zeta) = (1_A\otimes(1_{\mathscr{H}}\otimes y)\mathcal{V}(\zeta\otimes 1_S))\delta_A(a)$. Write $\zeta = \rho(x)\eta$ with $x\in\widehat{S}$ and $\eta\in\mathscr{H}$. We have $(1_{\mathcal{K}}\otimes y)\mathcal{V}(\zeta\otimes 1_S) = (\rho\otimes\mathrm{id}_S)((1_{\widehat{S}}\otimes y)V(x\otimes 1_S))(\eta\otimes 1_S)$. Since $\mathcal{G}$ is regular, we have $(1_{\widehat{S}}\otimes y)V(x\otimes 1_S)\in\widehat{S}\otimes S$ (cf. Corollary 4.10(2)). Hence, $(1_{\mathcal{E}_0}\otimes y)\delta_{\mathcal{E}_0}(a\otimes\zeta)$ is the norm limit of finite sums of elements of the form $(1_A\otimes\rho(x')\eta\otimes y')\delta_A(a)$ with $x'\in\widehat{S}$ and $y'\in S$. Hence, $(1_{\mathcal{E}_0}\otimes y)\delta_{\mathcal{E}_0}(a\otimes\zeta)\in\mathcal{E}_0\otimes S$ since $(1_A\otimes S)\delta_A(A)\subset A\otimes S$.

2. Since $\mathcal{V}\mathcal{V}^*\mathcal{V} = \mathcal{V}$, we have $\mathcal{V}_{23}\mathcal{V}_{23}^*\delta_{\mathcal{E}_0}(\xi) = \delta_{\mathcal{E}_0}(\xi)$ for all $\xi\in\mathcal{E}_0$. It then follows that $q_{\beta_{\mathcal{E}_{A,R}}\alpha}\delta_{\mathcal{E}_{A,R}}(\xi) = \delta_{\mathcal{E}_{A,R}}(\xi)$, for all $\xi\in\mathcal{E}_{A,R}$. By the first statement, we then obtain

$$\delta_{\mathcal{E}_{A,R}}(\mathcal{E}_{A,R})(A\otimes S)\subset q_{\beta_{\mathcal{E}_{A,R}}\alpha}(\mathcal{E}_{A,R}\otimes S).$$

Conversely, let $a \in A$, $\zeta \in \mathscr{H}$ and $s \in S$. Since $\mathcal{V}_{23} q_{\beta_A \alpha,13} = q_{\beta_A \widehat{\alpha},12} \mathcal{V}_{23}$, we have

$$\begin{aligned} q_{\beta_{\mathcal{E}_{A,R}} \alpha}\big(q_{\beta_A \widehat{\alpha}}(a \otimes \zeta) \otimes s\big) &= \mathcal{V}_{23} \mathcal{V}_{23}^* q_{\beta_A \widehat{\alpha},12}(a \otimes \zeta \otimes s) \\ &= \mathcal{V}_{23} q_{\beta_A \alpha,13}\big(a \otimes \mathcal{V}^*(\zeta \otimes s)\big). \end{aligned}$$

Hence, $q_{\beta_{\mathcal{E}_{A,R}} \alpha}(q_{\beta_A \widehat{\alpha}}(a \otimes \zeta) \otimes s)$ is the norm limit of finite sums of elements of the form:

$$\begin{aligned} &\mathcal{V}_{23} q_{\beta_A \alpha,13}\big(a \otimes \zeta' \otimes s'\big) \\ &\quad = \mathcal{V}_{23}\big(q_{\beta_A \alpha}\big(a \otimes s'\big)\big)_{23}\big(1_A \otimes \zeta' \otimes 1_S\big), \quad \text{where } \zeta' \in \mathscr{H}, s' \in S. \end{aligned}$$

By continuity of the action (δ_A, β_A), $\mathcal{V}_{23} q_{\beta_A \alpha,13}(a \otimes \zeta' \otimes s')$ is the norm limit of finite sums of the form $\sum_i \mathcal{V}_{23} \delta_A(a_i)_{13}(1_A \otimes \zeta' \otimes s_i) = \sum_i \delta_{\mathcal{E}_{A,R}}(q_{\beta_A \widehat{\alpha}}(a_i \otimes \zeta'))(1_A \otimes s_i)$, where $a_i \in A$ and $s_i \in S$. As a result, we have

$$q_{\beta_{\mathcal{E}_{A,R}} \alpha}\big(q_{\beta_A \widehat{\alpha}}(a \otimes \zeta) \otimes s\big) \in \big[\delta_{\mathcal{E}_{A,R}}(\mathcal{E}_{A,R})(A \otimes S)\big]$$

for all $a \in A$, $\zeta \in \mathscr{H}$ and $s \in S$. Hence, $q_{\beta_{\mathcal{E}_{A,R}} \alpha}(\mathcal{E}_{A,R} \otimes S) \subset [\delta_{\mathcal{E}_{A,R}}(\mathcal{E}_{A,R})(A \otimes S)]$.

3. Let $\xi = q_{\beta_A \widehat{\alpha}}(a \otimes \zeta)$, with $a \in A$ and $\zeta \in \mathscr{H}$. We have

$$\beta_{\mathcal{E}_{A,R}}\big(n^{\mathrm{o}}\big)\xi = \big(1_A \otimes \beta\big(n^{\mathrm{o}}\big)\big) q_{\beta_A \widehat{\alpha}}(a \otimes \zeta) = q_{\beta_A \widehat{\alpha}}\big(a \otimes \beta\big(n^{\mathrm{o}}\big)\zeta\big).$$

Moreover, we have $\mathcal{V}(\beta(n^{\mathrm{o}}) \otimes 1_S) = (1_{\mathcal{K}} \otimes \beta(n^{\mathrm{o}}))\mathcal{V}$ for all $n \in N$ (cf. Proposition 3.1.5). It then follows that

$$\begin{aligned} \delta_{\mathcal{E}_{A,R}}\big(\beta_{\mathcal{E}_{A,R}}\big(n^{\mathrm{o}}\big)\xi\big) &= \delta_{\mathcal{E}_0}\big(a \otimes \beta\big(n^{\mathrm{o}}\big)\zeta\big) = \big(1_{\mathcal{E}_0} \otimes \beta\big(n^{\mathrm{o}}\big)\big)\delta_{\mathcal{E}_0}(a \otimes \zeta) \\ &= \big(1_{\mathcal{E}_{A,R}} \otimes \beta\big(n^{\mathrm{o}}\big)\big)\delta_{\mathcal{E}_{A,R}}(\xi). \end{aligned}$$ □

Consequently, $\delta_{\mathcal{E}_{A,R}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathcal{E}_{A,R}} \otimes \delta$ extend to linear maps from $\mathcal{L}(A \otimes S, \mathcal{E}_{A,R} \otimes S)$ to $\mathcal{L}(A \otimes S \otimes S, \mathcal{E}_{A,R} \otimes S \otimes S)$ (cf. Remarks 6.1.7) and we have:

$$\begin{aligned} (\delta_{\mathcal{E}_{A,R}} \otimes \mathrm{id}_S)(T)(\delta_A \otimes \mathrm{id}_S)(x) &= (\delta_{\mathcal{E}_{A,R}} \otimes \mathrm{id}_S)(Tx); \\ (\mathrm{id}_{\mathcal{E}_{A,R}} \otimes \delta)(T)(\mathrm{id}_A \otimes \delta)(x) &= (\mathrm{id}_{\mathcal{E}_{A,R}} \otimes \delta)(Tx); \end{aligned}$$

for all $x \in A \otimes S$ and $T \in \mathcal{L}(A \otimes S, \mathcal{E}_{A,R} \otimes S)$.

Proposition 7.19. *For all* $\xi \in \mathcal{E}_{A,R}$, $(\delta_{\mathcal{E}_{A,R}} \otimes \mathrm{id}_S)\delta_{\mathcal{E}_{A,R}}(\xi) = (\mathrm{id}_{\mathcal{E}_{A,R}} \otimes \delta)\delta_{\mathcal{E}_{A,R}}(\xi)$.

Proof. Let $a \in A$, $\zeta \in \mathscr{H}$ and $x \in A \otimes S$. Let $\xi := q_{\beta_A \widehat{\alpha}}(a \otimes \zeta)$. We have

$$\begin{aligned} &(\delta_{\mathcal{E}_{A,R}} \otimes \mathrm{id}_S)\delta_{\mathcal{E}_{A,R}}(\xi)(\delta_A \otimes \mathrm{id}_S)(x) \\ &\quad = (\delta_{\mathcal{E}_0} \otimes \mathrm{id}_S)\big(\mathcal{V}_{23}\big(\delta_A(a)x\big)_{13}(1_A \otimes \zeta \otimes 1_S)\big). \end{aligned}$$

For all $b \in A$, $\zeta' \in \mathscr{H}$ and $s' \in S$, we have

$$(\delta_{\mathcal{E}_0} \otimes \mathrm{id}_S)\big(b \otimes \zeta' \otimes s'\big) = \mathcal{V}_{23}\delta_A(b)_{13}\big(1_A \otimes \zeta' \otimes 1_S \otimes s'\big).$$

Hence, $(\delta_{\mathcal{E}_0}\otimes \mathrm{id}_S)(b\otimes X) = \mathcal{V}_{23}\delta_A(b)_{13}X_{24} \in \mathcal{L}(A\otimes S\otimes S, A\otimes \mathscr{H}\otimes S\otimes S)$ for all $b\in A$ and $X\in \mathscr{H}\otimes S$. In particular, we have

$$\begin{aligned}(\delta_{\mathcal{E}_0}\otimes \mathrm{id}_S)\big(\mathcal{V}_{23}(b\otimes\zeta\otimes s)\big) &= (\delta_{\mathcal{E}_0}\otimes \mathrm{id}_S)\big(b\otimes \mathcal{V}(\zeta\otimes s)\big)\\ &= \mathcal{V}_{23}\delta_A(b)_{13}\mathcal{V}_{24}(1_A\otimes\zeta\otimes 1_S\otimes s)\\ &= \mathcal{V}_{23}\mathcal{V}_{24}\delta_A(b)_{13}(1_A\otimes\zeta\otimes 1_S\otimes s).\end{aligned}$$

However, we have $(\mathrm{id}_{\mathcal{K}}\otimes\delta)(\mathcal{V}) = \mathcal{V}_{12}\mathcal{V}_{13}$. Hence, $\mathcal{V}_{23}\mathcal{V}_{24} = (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)(\mathcal{V}_{23})$. Moreover, we have $\delta_A(b)_{13}(1_A\otimes\zeta\otimes 1_S\otimes s) = (\delta_{A,13}\otimes \mathrm{id}_S)(b\otimes s)(1_A\otimes\zeta\otimes 1_S\otimes 1_S)$, for all $b\in A$ and $s\in S$, where $\delta_{A,13}: A\to \mathcal{M}(A\otimes\mathcal{K}\otimes S)$ is the strictly continuous $*$-homomorphism defined by $\delta_{A,13}(a) := \delta_A(a)_{13}$ for all $a\in A$. As a result, for all $Y\in A\otimes S$ we have

$$\begin{aligned}&(\delta_{\mathcal{E}_0}\otimes \mathrm{id}_S)\big(\mathcal{V}_{23}Y_{13}(1_A\otimes\zeta\otimes 1_S)\big)\\ &\quad= (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)(\mathcal{V}_{23})(\delta_{A,13}\otimes \mathrm{id}_S)(Y)(1_A\otimes\zeta\otimes 1_S\otimes 1_S).\end{aligned}$$

In particular, we have

$$\begin{aligned}&(\delta_{\mathcal{E}_0}\otimes \mathrm{id}_S)\big(\delta_{\mathcal{E}_0}(a\otimes\zeta)x\big)\\ &\quad= (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)(\mathcal{V}_{23})(\delta_{A,13}\otimes \mathrm{id}_S)\big(\delta_A(a)x\big)(1_A\otimes\zeta\otimes 1_S\otimes 1_S)\\ &\quad= (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)(\mathcal{V}_{23})(\delta_{A,13}\otimes \mathrm{id}_S)\delta_A(a)(1_A\otimes\zeta\otimes 1_S\otimes 1_S)(\delta_A\otimes \mathrm{id}_S)(x).\end{aligned}$$

Moreover, we have $(\delta_{A,13}\otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)\delta_{A,13}$. Hence,

$$\begin{aligned}&(\delta_{\mathcal{E}_{A,R}}\otimes \mathrm{id}_S)\delta_{\mathcal{E}_{A,R}}(\xi)x\\ &\quad= (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)\big(\mathcal{V}_{23}\delta_A(a)_{13}\big)(1_A\otimes\zeta\otimes 1_S\otimes 1_S)x,\end{aligned}$$

for all $x\in q_{\beta_A\alpha,12}(A\otimes S\otimes S)$. In particular, if $x\in q_{\beta_A\alpha,12}q_{\beta\alpha,23}(A\otimes S\otimes S)$ we have

$$\begin{aligned}&(\delta_{\mathcal{E}_{A,R}}\otimes \mathrm{id}_S)\delta_{\mathcal{E}_{A,R}}(\xi)x\\ &\quad= (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)\big(\mathcal{V}_{23}\delta_A(a)_{13}\big)q_{\beta\alpha,34}(1_A\otimes\zeta\otimes 1_S\otimes 1_S)x\\ &\quad= (\mathrm{id}_{A\otimes\mathcal{K}}\otimes\delta)\big(\mathcal{V}_{23}\delta_A(a)_{13}\big)(\mathrm{id}_{\mathcal{E}_0}\otimes\delta)(1_A\otimes\zeta\otimes 1_S)x\\ &\quad= (\mathrm{id}_{\mathcal{E}_0}\otimes\delta)\delta_{\mathcal{E}_0}(a\otimes\zeta)x\\ &\quad= (\mathrm{id}_{\mathcal{E}_{A,R}}\otimes\delta)\delta_{\mathcal{E}_{A,R}}(\xi)x.\end{aligned}$$

Hence, $(\delta_{\mathcal{E}_{A,R}}\otimes \mathrm{id}_S)\delta_{\mathcal{E}_{A,R}}(\xi) = (\mathrm{id}_{\mathcal{E}_{A,R}}\otimes\delta)\delta_{\mathcal{E}_{A,R}}(\xi)$. □

Now, we can assemble the previous results (see also Corollary 6.1.26) in the statement below.

PROPOSITION 7.20. *The triple* $(\mathcal{E}_{A,R}, \beta_{\mathcal{E}_{A,R}}, \delta_{\mathcal{E}_{A,R}})$ *is a* $\mathcal{G}$*-equivariant Hilbert* A*-module.*

Let D be the bidual $\mathcal{G}$-C*-algebra of A. We have a canonical $\mathcal{G}$-equivariant $*$-isomorphism $\phi: (A\rtimes\mathcal{G})\rtimes\widehat{\mathcal{G}}\to D$ of $\mathcal{G}$-C*-algebras (cf. Proposition 5.1.22).

Let $j_D : \mathcal{M}(D) \to \mathcal{L}(\mathcal{E}_0)$ be the unique faithful continuous $*$-homomorphism for the strict/$*$-strong topologies such that $j_D(1_D) = q_{\beta_A\widehat{\alpha}}$.

PROPOSITION 7.21. *The $*$-representation $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}} \to \mathcal{L}(\mathcal{E}_{A,R})$; $x \mapsto \phi(x)\restriction_{\mathcal{E}_{A,R}}$ is $\mathcal{G}$-equivariant.*

Proof. We have to prove that $\delta_{\mathcal{E}_0}(d\xi) = (j_D \otimes \mathrm{id}_S)(\delta_D(d)) \circ \delta_{\mathcal{E}_0}(\xi)$ for all $d \in D$ and $\xi \in \mathcal{E}_0$ (cf. Remarks 7.3(3) and Proposition 7.16(3)). Let us prove it in three steps:

• Let $b \in A$, $x \in \widehat{S}$ and $\zeta \in \mathscr{H}$. We have

$$\delta_{\mathcal{E}_0}\big(b \otimes \lambda(x)\zeta\big) = \big(1_A \otimes \mathcal{V}\big(\lambda(x) \otimes 1_S\big)\big)\delta_A(b)_{13}(1_A \otimes \zeta \otimes 1_S).$$

However, $[\mathcal{V}, \lambda(x) \otimes 1_S] = 0$ (as $\lambda(\widehat{S}) \subset \widehat{M}$ and $V \in \widehat{M}' \otimes M$). Hence,

$$\delta_{\mathcal{E}_0}\big(b \otimes \lambda(x)\zeta\big) = \big(1_A \otimes \lambda(x) \otimes 1_S\big)\delta_{\mathcal{E}_0}(b \otimes \zeta)$$

and then $\delta_{\mathcal{E}_0}((1_A \otimes \lambda(x))\xi) = (1_A \otimes \lambda(x) \otimes 1_S)\delta_{\mathcal{E}_0}(\xi)$, for all $x \in \widehat{S}$ and $\xi \in \mathcal{E}_0$.

• Let $y \in S$. Since $L(y) \in M \subset \widehat{\alpha}(N)'$, we have

$$\begin{aligned}\mathcal{V}\big(L(y) \otimes 1_S\big) &= \mathcal{V}q_{\widehat{\alpha}\beta}\big(L(y) \otimes 1_S\big) = \mathcal{V}\big(L(y) \otimes 1_S\big)q_{\widehat{\alpha}\beta} \\ &= \mathcal{V}\big(L(y) \otimes 1_S\big)\mathcal{V}^*\mathcal{V} = (L \otimes \mathrm{id}_S)\delta(y)\mathcal{V}.\end{aligned}$$

For all $b \in A$ and $\zeta \in \mathscr{H}$, we have

$$\begin{aligned}\delta_{\mathcal{E}_0}\big(\big(1_A \otimes L(y)\big)(b \otimes \zeta)\big) &= \big(1_A \otimes \mathcal{V}\big(L(y) \otimes 1_S\big)\big)\delta_A(b)_{13}(1_A \otimes \zeta \otimes 1_S) \\ &= \big(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)\big)\delta_{\mathcal{E}_0}(b \otimes \zeta).\end{aligned}$$

Hence, $\delta_{\mathcal{E}_0}((1_A \otimes L(y))\xi) = (1_A \otimes (L \otimes \mathrm{id}_S)\delta(y))\delta_{\mathcal{E}_0}(\xi)$ for all $y \in S$ and $\xi \in \mathcal{E}_0$.

In virtue of the first two steps, for all $\xi \in \mathcal{E}_{A,R}$ we have

$$\begin{aligned}&\delta_{\mathcal{E}_{A,R}}\big(\big(1_A \otimes \lambda(x)L(y)\big)\xi\big) \\ &\quad= \big(1_A \otimes \lambda(x) \otimes 1_S\big)\big(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)\big)\delta_{\mathcal{E}_{A,R}}(\xi).\end{aligned}$$

• Let $s \in S$. We have (cf. Proposition 3.1.3)

$$\begin{aligned}\big(R(s) \otimes 1\big)V &= (U \otimes 1)\Sigma\big(1 \otimes L(s)\big)\Sigma\big(U^* \otimes 1\big)V \\ &= (U \otimes 1)\Sigma\big(1 \otimes L(s)\big)W\Sigma\big(U^* \otimes 1\big).\end{aligned}$$

Besides, $(1 \otimes L(s))W = (1 \otimes L(s))WW^*W = WW^*(1 \otimes L(s))W = W\delta(s)$ since we have $WW^* = q_{\alpha\widehat{\beta}}$ and $L(s) \in M \subset \widehat{\beta}(N^{\mathrm{o}})'$. Therefore, since $(U \otimes 1)\Sigma W = V(U \otimes 1)\Sigma$ we have $(R(s) \otimes 1)V = V\Sigma(1 \otimes U)\delta(s)(1 \otimes U^*)\Sigma$. Hence, $(R(s) \otimes 1_S)\mathcal{V} = \mathcal{V}\sigma(\mathrm{id}_S \otimes R)(\delta(s))\sigma^*$ for all $s \in S$. We then have $((\mathrm{id}_A \otimes R)(x) \otimes 1_S)\mathcal{V}_{23} = \mathcal{V}_{23}\sigma_{23}(\mathrm{id}_{A\otimes S} \otimes R)((\mathrm{id}_A \otimes \delta)(x))\sigma_{23}^*$ for all $x \in A \otimes S$. But, since R and δ are strictly continuous this equality also holds for all $x \in \mathcal{M}(A \otimes S)$. In particular, we have $\pi_R(a)_{12}\mathcal{V}_{23} = \mathcal{V}_{23}\sigma_{23}(\mathrm{id}_{A\otimes S} \otimes R)(\delta_A^2(a))\sigma_{23}^*$ for all $a \in A$. By coassociativity of δ_A, we have

$$\pi_R(a)_{12}\mathcal{V}_{23} = \mathcal{V}_{23}\sigma_{23}(\delta_A \otimes \mathrm{id}_{\mathcal{K}})\big(\pi_R(a)\big)\sigma_{23}^*, \quad \text{for all } a \in A.$$

It then follows that

$$\begin{aligned}\pi_R(a)_{12}\delta_{\mathcal{E}_0}(\xi) &= \mathcal{V}_{23}\sigma_{23}(\delta_A\otimes \mathrm{id}_{\mathcal{K}})\big(\pi_R(a)\big)(\delta_A\otimes \mathrm{id}_{\mathscr{H}})(\xi)\\ &= \mathcal{V}_{23}\sigma_{23}(\delta_A\otimes \mathrm{id}_{\mathscr{H}})\big(\pi_R(a)\xi\big)\\ &= \delta_{\mathcal{E}_0}\big(\pi_R(a)\xi\big),\end{aligned}$$

for all $a\in A$ and $\xi\in\mathcal{E}_0$. In particular, $\pi_R(a)_{12}\delta_{\mathcal{E}_0}(\xi_0)=\delta_{\mathcal{E}_0}(\pi_R(a)\xi_0)$ for all $a\in A$ and $\xi\in\mathcal{E}_0$.

We have proved that for all $a\in A$, $x\in\widehat{S}$, $y\in S$ and $\xi\in\mathcal{E}_0$, we have

$$\begin{aligned}&\delta_{\mathcal{E}_0}\big(\pi_R(a)\big(1_A\otimes\lambda(x)L(y)\big)\xi\big)\\ &\quad=\pi_R(a)_{12}\big(1_A\otimes\lambda(x)\otimes 1_S\big)\big(1_A\otimes(L\otimes\mathrm{id}_S)\delta(y)\big)\delta_{\mathcal{E}_0}(\xi).\end{aligned}$$

However, for all $a\in A$, $x\in\widehat{S}$ and $y\in S$ we have (cf. 3.37 d) [2])

$$\begin{aligned}&(j_D\otimes\mathrm{id}_S)\delta_D\big(\pi_R(a)\big(1_A\otimes\lambda(x)L(y)\big)\big)\\ &\quad=\pi_R(a)_{12}\big(1_A\otimes\lambda(x)\otimes 1_S\big)\big(1_A\otimes(L\otimes\mathrm{id}_S)\delta(y)\big).\end{aligned}$$

If $d=\pi_R(a)(1_A\otimes\lambda(x)L(y))\in D$, where $a\in A$, $x\in\widehat{S}$ and $y\in S$, we have proved that $\delta_{\mathcal{E}_0}(d\xi)=(j_D\otimes\mathrm{id}_S)(\delta_D(d))\circ\delta_{\mathcal{E}_0}(\xi)$ for all $\xi\in\mathcal{E}_0$. Thus, the statement is proved since $D=[\pi_R(a)(1_A\otimes\lambda(x)L(y));\, a\in A, x\in\widehat{S}, y\in S]$. □

THEOREM 7.22. *The $\mathcal{G}$-C^*-algebras $(A\rtimes\mathcal{G})\rtimes\widehat{\mathcal{G}}$ and A are Morita equivalent via the $\mathcal{G}$-equivariant imprimitivity $(A\rtimes\mathcal{G})\rtimes\widehat{\mathcal{G}}$-$A$-bimodule $\mathcal{E}_{A,R}$.*

Proof. Let us prove that the Hilbert A-module $\mathcal{E}_{A,R}$ is full. Fix $x\in A$ and write $x=a^*b$ for $a,b\in A$. There exists $\omega\in\mathcal{B}(\mathscr{H})_*$ such that $(\mathrm{id}_A\otimes\omega)(q_{\beta_A\widehat{\alpha}})=1_A$. Hence, x is the norm limit of finite sums of elements of the form $a^*(\mathrm{id}_A\otimes\omega_{\xi,\eta})(q_{\beta_A\widehat{\alpha}})b$, where $\xi,\eta\in\mathscr{H}$. However, for all $\xi,\eta\in\mathscr{H}$ we have

$$\begin{aligned}a^*(\mathrm{id}_A\otimes\omega_{\xi,\eta})(q_{\beta_A\widehat{\alpha}})b &= (a\otimes\xi)^*q_{\beta_A\widehat{\alpha}}(b\otimes\eta)\\ &= \big\langle q_{\beta_A\widehat{\alpha}}(a\otimes\xi), q_{\beta_A\widehat{\alpha}}(b\otimes\eta)\big\rangle.\end{aligned}$$

Hence, $A=[\langle\xi,\eta\rangle;\,\xi,\eta\in\mathcal{E}_{A,R}]$. Now, we recall that $D=q_{\beta_A\widehat{\alpha}}(A\otimes\mathcal{K})q_{\beta_A\widehat{\alpha}}$ (cf. Theorem 5.1.24). It is easily seen that the left action of $(A\rtimes\mathcal{G})\rtimes\widehat{\mathcal{G}}$ (cf. Proposition 7.21) induces a $\mathcal{G}$-equivariant $*$-isomorphism $(A\rtimes\mathcal{G})\rtimes\widehat{\mathcal{G}}\simeq\mathcal{K}(\mathcal{E}_{A,R})$. □

Appendix

A.1. Normal linear forms, weights and operator-valued weights on von Neumann algebras [8]. Let M be a von Neumann algebra. Denote by M_* (resp., M_*^+) the Banach space (resp., positive cone) of the normal linear forms (resp., positive normal linear forms) on M. Let $\omega\in M_*$ and $a,b\in M$.

Denote by $a\omega \in M_*$ and $\omega b \in M_*$ the normal linear functionals on M given for all $x \in M$ by:

$$(a\omega)(x) := \omega(xa); \qquad (\omega b)(x) := \omega(bx).$$

We have $a'(a\omega) = (a'a)\omega$ and $(\omega b)b' = \omega(bb')$, for all $a, a, b, b' \in M$. We also denote

$$a\omega b := a(\omega b) = (a\omega)b; \qquad \omega_a := a^*\omega a.$$

If $\omega \in M_*^+$, then $\omega_a \in M_*^+$. Note that $(\omega_a)_b = \omega_{ab}$ for all $a, b \in M$. If $\omega \in M_*$ we define $\overline{\omega} \in M_*$ by setting

$$\overline{\omega}(x) := \overline{\omega\big(x^*\big)}, \quad \text{for all } x \in M.$$

Let $\mathscr{H}$ be a Hilbert space and let us fix $\xi, \eta \in \mathscr{H}$. Denote by $\omega_{\xi,\eta} \in \mathcal{B}(\mathscr{H})_*$ the normal linear form defined by

$$\omega_{\xi,\eta}(x) := \langle \xi, x\eta \rangle, \quad \text{for all } x \in \mathcal{B}(\mathscr{H}).$$

Note that we have $\overline{\omega}_{\xi,\eta} = \omega_{\eta,\xi}$, $a\omega_{\xi,\eta} = \omega_{\xi,a\eta}$ and $\omega_{\xi,\eta}a = \omega_{a^*\xi,\eta}$ for all $a \in \mathcal{B}(\mathscr{H})$.

A.1.1. *Tensor product of normal linear forms.* Let M and N be von Neumann algebras, $\phi \in M_*$ and $\psi \in N_*$. There exists a unique $\phi \otimes \psi \in (M \otimes N)_*$ such that $(\phi \otimes \psi)(x \otimes y) = \phi(x)\psi(y)$ for all $x \in M$ and $y \in N$. Moreover, $\|\phi \otimes \psi\| \leqslant \|\phi\| \cdot \|\psi\|$. Actually, it is known that we have an (completely) isometric identification $M_* \widehat{\otimes}_\pi N_* = (M \otimes N)_*$, where $\widehat{\otimes}_\pi$ denotes the projective tensor product of Banach spaces. In particular, any $\omega \in (M \otimes N)_*$ is the norm limit of finite sums of the form $\sum_i \phi_i \otimes \psi_i$, where $\phi_i \in M_*$ and $\psi_i \in N_*$.

A.1.2. *Slicing with normal linear forms.* We will also need to slice maps with normal linear forms. Let M_1 and M_2 be von Neumann algebras, $\omega_1 \in (M_1)_*$ and $\omega_2 \in (M_2)_*$. Therefore, the maps $\omega_1 \odot \mathrm{id} : M_1 \odot M_2 \to M_1$ and $\mathrm{id} \odot \omega_2 : M_1 \odot M_2 \to M_2$ extend uniquely to norm continuous normal linear maps $\omega_1 \otimes \mathrm{id} : M_1 \otimes M_2 \to M_2$ and $\mathrm{id} \otimes \omega_2 : M_1 \otimes M_2 \to M_1$. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, for $\xi \in \mathscr{H}$ and $\eta \in \mathscr{K}$ we define $\theta_\xi \in \mathcal{B}(\mathscr{K}, \mathscr{H} \otimes \mathscr{K})$ and $\theta'_\eta \in \mathcal{B}(\mathscr{H}, \mathscr{H} \otimes \mathscr{K})$ by setting:

$$\theta_\xi(\zeta) := \xi \otimes \zeta, \quad \text{for all } \zeta \in \mathscr{K}; \qquad \theta'_\eta(\zeta) := \zeta \otimes \eta, \quad \text{for all } \zeta \in \mathscr{H}.$$

If $T \in \mathcal{B}(\mathscr{H} \otimes \mathscr{K})$, $\phi \in \mathcal{B}(\mathscr{K})_*$ and $\omega \in \mathcal{B}(\mathscr{H})_*$, then the operators $(\mathrm{id} \otimes \phi)(T) \in \mathcal{B}(\mathscr{H})$ and $(\omega \otimes \mathrm{id})(T) \in \mathcal{B}(\mathscr{K})$ are determined by the formulas:

$$\big\langle \xi_1, (\mathrm{id} \otimes \phi)(T)\xi_2 \big\rangle = \phi\big(\theta_{\xi_1}^* T \theta_{\xi_2}\big), \quad \xi_1, \xi_2 \in \mathscr{H};$$
$$\big\langle \eta_1, (\omega \otimes \mathrm{id})(T)\eta_2 \big\rangle = \omega\big(\theta_{\eta_1}'^* T \theta'_{\eta_2}\big), \quad \eta_1, \eta_2 \in \mathscr{K}.$$

In particular, we have:

$$(\mathrm{id} \otimes \omega_{\eta_1,\eta_2})(T) = \theta_{\eta_1}'^* T \theta'_{\eta_2}, \quad \eta_1, \eta_2 \in \mathscr{K};$$
$$(\omega_{\xi_1,\xi_2} \otimes \mathrm{id})(T) = \theta_{\xi_1}^* T \theta_{\xi_2}, \quad \xi_1, \xi_2 \in \mathscr{H}.$$

Let us recall some formulas that will be used several times. For all $\phi \in \mathcal{B}(\mathscr{K})_*$, $\omega \in \mathcal{B}(\mathscr{H})_*$ and $T \in \mathcal{B}(\mathscr{H} \otimes \mathscr{K})$, we have:

$$x(\mathrm{id} \otimes \phi)(T)y = (\mathrm{id} \otimes \phi)\big((x \otimes 1)T(y \otimes 1)\big),$$
$$(y\omega x \otimes \mathrm{id})(T) = (\omega \otimes \mathrm{id})\big((x \otimes 1)T(y \otimes 1)\big)$$

for all $x, y \in \mathcal{B}(\mathscr{H})$;

$$a(\omega \otimes \mathrm{id})(T)b = (\omega \otimes \mathrm{id})\big((1 \otimes a)T(1 \otimes b)\big),$$
$$(\mathrm{id} \otimes b\phi a)(T) = (\mathrm{id} \otimes \phi)\big((1 \otimes a)T(1 \otimes b)\big)$$

for all $a, b \in \mathcal{B}(\mathscr{K})$. We also have

$$(\mathrm{id} \otimes \phi)(T)^* = (\mathrm{id} \otimes \overline{\phi})\big(T^*\big),$$
$$(\omega \otimes \mathrm{id})(T)^* = (\overline{\omega} \otimes \mathrm{id})\big(T^*\big),$$
$$(\phi \otimes \mathrm{id})(\Sigma_{\mathscr{H}\otimes\mathscr{K}} T \Sigma_{\mathscr{K}\otimes\mathscr{H}}) = (\mathrm{id} \otimes \phi)(T),$$
$$(\mathrm{id} \otimes \omega)(\Sigma_{\mathscr{H}\otimes\mathscr{K}} T \Sigma_{\mathscr{K}\otimes\mathscr{H}}) = (\omega \otimes \mathrm{id})(T),$$

for all $T \in \mathcal{B}(\mathscr{H} \otimes \mathscr{K})$, $\phi \in \mathcal{B}(\mathscr{K})_*$ and $\omega \in \mathcal{B}(\mathscr{H})_*$.

Definition A.1.3. A weight φ on M is a map $\varphi : M_+ \to [0, \infty]$ such that:

- for all $x, y \in M_+$, $\varphi(x + y) = \varphi(x) + \varphi(y)$;
- for all $x \in M_+$ and $\lambda \in \mathbb{R}_+$, $\varphi(\lambda x) = \lambda\varphi(x)$.

We denote by $\mathfrak{N}_\varphi := \{x \in M;\, \varphi(x^*x) < \infty\}$ the left ideal of square φ-integrable elements of M, $\mathfrak{M}_\varphi^+ := \{x \in M_+;\, \varphi(x) < \infty\}$ the cone of positive φ-integrable elements of M and $\mathfrak{M}_\varphi := \langle \mathfrak{M}_\varphi^+ \rangle$ the space of φ-integrable elements of M.

Definition A.1.4. Let φ be a weight on M. The opposite weight of φ is the weight φ^{o} on M^{o} given by $\varphi^{\mathrm{o}}(x^{\mathrm{o}}) := \varphi(x)$ for all $x \in M_+$. Then, we have $\mathfrak{N}_{\varphi^{\mathrm{o}}} = (\mathfrak{N}_\varphi^*)^{\mathrm{o}}$, $\mathfrak{M}_{\varphi^{\mathrm{o}}}^+ = (\mathfrak{M}_\varphi^+)^{\mathrm{o}}$ and $\mathfrak{M}_{\varphi^{\mathrm{o}}} = (\mathfrak{M}_\varphi)^{\mathrm{o}}$.

Definition A.1.5. A weight φ on M is called:

- semi-finite, if $\mathfrak{N}_\varphi$ is σ-weakly dense in M;
- faithful, if for $x \in M_+$ the condition $\varphi(x) = 0$ implies $x = 0$;
- normal, if $\varphi(\sup_{i\in\mathcal{I}} x_i) = \sup_{i\in\mathcal{I}} \varphi(x_i)$ for all increasing bounded net $(x_i)_{i\in\mathcal{I}}$ of M_+.

From now on, we will mainly use normal semi-finite faithful (n.s.f.) weights. Fix a n.s.f. weight φ on M.

Definition A.1.6. We define an inner product on $\mathfrak{N}_\varphi$ by setting

$$\langle x, y \rangle_\varphi := \varphi\big(x^* y\big), \quad \text{for all } x, y \in \mathfrak{N}_\varphi.$$

We denote by $(\mathscr{H}_\varphi, \Lambda_\varphi)$ the Hilbert space completion of $\mathfrak{N}_\varphi$ with respect to this inner product, where $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathscr{H}_\varphi$ is the canonical map. There exists a

unique unital normal $*$-representation $\pi_\varphi : M \to \mathcal{B}(\mathscr{H}_\varphi)$ such that

$$\pi_\varphi(x)\Lambda_\varphi(y) = \Lambda_\varphi(xy), \quad \text{for all } x \in M \text{ and } y \in \mathfrak{N}_\varphi.$$

The triple $(\mathscr{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$ is called the G.N.S. construction for (M, φ).

Remarks A.1.7. The linear map Λ_φ is called the G.N.S. map. We have that $\Lambda_\varphi(\mathfrak{N}_\varphi)$ is dense in $\mathscr{H}_\varphi$ and $\langle \Lambda_\varphi(x), \Lambda_\varphi(y)\rangle_\varphi = \varphi(x^*y)$ for all $x, y \in \mathfrak{N}_\varphi$. In particular, Λ_φ is injective. Moreover, we also call π_φ the G.N.S. representation.

We recall below the main objects of the Tomita–Takesaki modular theory.

Proposition-Definition A.1.8. *Let M be a von Neumann algebra and φ a n.s.f. weight on M. The anti-linear map $\Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi) \to \Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi)$; $\Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*)$ is closable and its closure is a possibly unbounded anti-linear map $\mathcal{T}_\varphi : D(\mathcal{T}_\varphi) \subset \mathscr{H}_\varphi \to \mathscr{H}_\varphi$ such that $D(\mathcal{T}_\varphi) = \operatorname{im} \mathcal{T}_\varphi$ and $\mathcal{T}_\varphi \circ \mathcal{T}_\varphi(x) = x$ for all $x \in D(\mathcal{T}_\varphi)$.*

Let $\mathcal{T}_\varphi = J_\varphi \nabla_\varphi^{1/2}$ be the polar decomposition of $\mathcal{T}_\varphi$. The anti-unitary $J_\varphi : \mathscr{H}_\varphi \to \mathscr{H}_\varphi$ is called the modular conjugation for φ and the injective positive self-adjoint operator ∇_φ is called the modular operator for φ.

Proposition-Definition A.1.9. *There exists a unique one-parameter group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ of automorphisms on M, called the modular automorphism group of φ, such that*

$$\pi_\varphi\big(\sigma_t^\varphi(x)\big) = \nabla_\varphi^{\mathrm{i}t} \pi_\varphi(x) \nabla_\varphi^{-\mathrm{i}t}, \quad \textit{for all } t \in \mathbb{R} \textit{ and } x \in M.$$

Then, for all $t \in \mathbb{R}$ and $x \in M$ we have $\sigma_t^\varphi(x) \in \mathfrak{N}_\varphi$ and $\Lambda_\varphi(\sigma_t^\varphi(x)) = \nabla_\varphi^{it} \Lambda_\varphi(x)$.

Proposition-Definition A.1.10. *The map $C_M : M \to M'$; $x \mapsto J_\varphi \times \pi_\varphi(x)^* J_\varphi$ is a normal unital $*$-antihomomorphism.*

Definition A.1.11. Let N be a von Neumann algebra. The extended positive cone of N is the set N_+^{ext} consisting of the maps $m : N_*^+ \to [0, \infty]$, which satisfy the following conditions:

- for all $\omega_1, \omega_2 \in N_*^+$, $m(\omega_1 + \omega_2) = m(\omega_1) + m(\omega_2)$;
- for all $\omega \in N_*^+$ and $\lambda \in \mathbb{R}_+$, $m(\lambda\omega) = \lambda m(\omega)$;
- m is lower semicontinuous with respect to the norm topology on N_*.

Notation A.1.12. Let N be a von Neumann algebra.

1. From now on, we will identify N_+ with its part inside N_+^{ext}. Accordingly, if $m \in N_+^{\mathrm{ext}}$ and $\omega \in N_*^+$ we will denote by $\omega(m)$ the evaluation of m at ω.
2. Let $a \in N$ and $m \in N_+^{\mathrm{ext}}$, we define $a^*ma \in N_+^{\mathrm{ext}}$ by setting $\omega(a^*ma) := a\omega a^*(m)$ for all $\omega \in N_*^+$. If $m, n \in N_+^{\mathrm{ext}}$ and $\lambda \in \mathbb{R}_+$, we also define $m + n \in N_+^{\mathrm{ext}}$ and $\lambda m \in N_+^{\mathrm{ext}}$ by setting $\omega(m + n) := \omega(m) + \omega(n)$ and $\omega(\lambda m) := \lambda\omega(m)$ for all $\omega \in N_*^+$.

DEFINITION A.1.13. Let $N \subset M$ be a unital normal inclusion of von Neumann algebras. An operator-valued weight from M to N is a map $T : M_+ \to N_+^{\mathrm{ext}}$ such that:

- for all $x, y \in M_+$, $T(x+y) = T(x) + T(y)$;
- for all $x \in M_+$, $\forall \lambda \in \mathbb{R}_+$, $T(\lambda x) = \lambda T(x)$;
- for all $x \in M_+$ and $a \in N$, $T(a^* x a) = a^* T(x) a$.

Let $\mathfrak{N}_T := \{x \in M;\, T(x^*x) \in N_+\}$, $\mathfrak{M}_T^+ := \{x \in M_+;\, T(x) \in N_+\}$ and $\mathfrak{M}_T := \langle \mathfrak{M}_T^+ \rangle$.

DEFINITION A.1.14. Let $N \subset M$ be a unital normal inclusion of von Neumann algebras. An operator-valued weight T from M to N is said to be:

- semi-finite, if $\mathfrak{N}_T$ is σ-weakly dense in M;
- faithful, if for $x \in M_+$ the condition $T(x) = 0$ implies $x = 0$;
- normal, if for every increasing bounded net $(x_i)_{i \in \mathcal{I}}$ of elements of M_+ and $\omega \in N_*^+$, we have $\omega(T(\sup_{i \in \mathcal{I}} x_i)) = \lim_{i \in \mathcal{I}} \omega(T(x_i))$.

Note that if $T : M_+ \to N_+^{\mathrm{ext}}$ is an operator-valued weight, it extends uniquely to a semi-linear map $\overline{T} : M_+^{\mathrm{ext}} \to N_+^{\mathrm{ext}}$. This will allow us to compose n.s.f. operator-valued weights. Indeed, let $P \subset N \subset M$ be unital normal inclusions of von Neumann algebras. Let S (resp., T) be an operator-valued weight from N (resp., M) to P (resp., N). We define an operator-valued weight from M to P by setting $(S \circ T)(x) := \overline{S}(T(x))$ for all $x \in N_+$.

A.2. Relative tensor product of Hilbert spaces and fiber product of von Neumann algebras. In this paragraph, we will recall the definitions, notations and important results concerning the relative tensor product and the fiber product which are the main technical tools of the theory of measured quantum groupoids. For more information, we refer the reader to [7].

In the whole section, N is a von Neumann algebra endowed with a n.s.f. weight φ. Let π (resp., γ) be a normal unital $*$-representation of N (resp., N^{o}) on a Hilbert space $\mathcal{H}$ (resp., $\mathcal{K}$).

Relative tensor product. The Hilbert space $\mathcal{H}$ (resp., $\mathcal{K}$) may be considered as a left (resp., right) N-module. Moreover, $\mathscr{H}_\varphi$ is an N-bimodule whose actions are given by

$$x\xi := \pi_\varphi(x)\xi \quad \text{and}$$
$$\xi y := J_\varphi \pi_\varphi\big(y^*\big) J_\varphi \xi, \quad \text{for all } \xi \in \mathscr{H}_\varphi \text{ and } x, y \in N.$$

DEFINITION A.2.1. We define the set of right (resp., left) bounded vectors with respect to φ and π (resp., γ) to be:

$$\mathcal{R}_\varphi(\pi, \mathcal{H}) := \big\{\xi \in \mathcal{H};\, \exists C \in \mathbb{R}_+, \forall x \in \mathfrak{N}_\varphi, \big\|\pi(x)\xi\big\| \leqslant C \big\|\Lambda_\varphi(x)\big\|\big\},$$
$$\big(\text{resp., } (\mathcal{K}, \gamma)_\varphi := \big\{\xi \in \mathcal{K};\, \exists C \in \mathbb{R}_+, \forall x \in \mathfrak{N}_\varphi^*, \big\|\gamma\big(x^{\mathrm{o}}\big)\xi\big\| \leqslant C \big\|\Lambda_{\varphi^{\mathrm{o}}}\big(x^{\mathrm{o}}\big)\big\|\big\}\big).$$

If $\xi \in {}_{\varphi}(\pi, \mathcal{H})$, we denote by $R_{\xi}^{\pi,\varphi} \in \mathcal{B}(\mathscr{H}_{\varphi}, \mathcal{H})$ (or simply R_{ξ}^{π} if φ is understood) the unique bounded operator such that

$$R_{\xi}^{\pi,\varphi}\Lambda_{\varphi}(x) = \pi(x)\xi, \quad \text{for all } x \in \mathfrak{N}_{\varphi}.$$

Similarly, if $\xi \in (\mathcal{K}, \gamma)_{\varphi}$ we denote $L_{\xi}^{\gamma,\varphi} \in \mathcal{B}(\mathscr{H}_{\varphi}, \mathcal{K})$ (or simply L_{ξ}^{γ} if φ is understood) the unique bounded operator such that

$$L_{\xi}^{\gamma,\varphi}J_{\varphi}\Lambda_{\varphi}\big(x^*\big) = \gamma\big(x^{\mathrm{o}}\big)\xi, \quad \text{for all } x \in \mathfrak{N}_{\varphi}^*,$$

where we have used the identification $\mathscr{H}_{\varphi^{\mathrm{o}}} \to \mathscr{H}_{\varphi}$; $\Lambda_{\varphi^{\mathrm{o}}}(x^{\mathrm{o}}) \mapsto J_{\varphi}\Lambda_{\varphi}(x^*)$.

Note that $\xi \in \mathcal{K}$ is left bounded with respect to φ and γ if, and only if, it is right bounded with respect to the n.s.f. weight $\varphi^{\mathrm{c}} := \varphi \circ C_N^{-1}$ on N' and the normal unital $*$-representation $\gamma^{\mathrm{c}} := \gamma \circ C_N^{-1}$ of N'. It is important to note that $(\mathcal{K}, \gamma)_{\varphi}$ (resp., ${}_{\varphi}(\pi, \mathcal{H})$) is dense in $\mathcal{K}$ (resp., $\mathcal{H}$) (cf. Lemma 2 of [7]).

If $\xi \in {}_{\varphi}(\pi, \mathcal{H})$ (resp., $\xi \in (\mathcal{K}, \gamma)_{\varphi}$), we have that $R_{\xi}^{\pi,\varphi}$ (resp., $L_{\xi}^{\gamma,\varphi}$) is left (resp., right) N-linear. Therefore, for all $\xi, \eta \in {}_{\varphi}(\pi, \mathcal{H})$ (resp., $(\mathcal{K}, \gamma)_{\varphi}$) we have

$$\big(R_{\xi}^{\pi,\varphi}\big)^* R_{\eta}^{\pi,\varphi} \in \pi_{\varphi}(N)' = C_N(N) \quad \text{and} \quad R_{\xi}^{\pi,\varphi}\big(R_{\eta}^{\pi,\varphi}\big)^* \in \pi(N)'$$
$$\big(\text{resp., } \big(L_{\xi}^{\gamma,\varphi}\big)^* L_{\eta}^{\gamma,\varphi} \in \pi_{\varphi}(N) \quad \text{and} \quad L_{\xi}^{\gamma,\varphi}\big(L_{\eta}^{\gamma,\varphi}\big)^* \in \gamma\big(N^{\mathrm{o}}\big)'\big).$$

Notation A.2.2 (*cf.* 2.1 [16]). Let

$$\mathcal{K}_{\pi,\varphi} := \big[R_{\xi}^{\pi,\varphi}\big(R_{\eta}^{\pi,\varphi}\big)^*; \xi, \eta \in {}_{\varphi}(\pi, \mathcal{H})\big]$$
$$\big(\text{resp., } \mathcal{K}_{\gamma,\varphi} := \big[L_{\xi}^{\gamma,\varphi}\big(L_{\eta}^{\gamma,\varphi}\big)^*; \xi, \eta \in (\mathcal{H}, \gamma)_{\varphi}\big]\big).$$

Note that $\mathcal{K}_{\pi,\varphi}$ (resp., $\mathcal{K}_{\gamma,\varphi}$) is a weakly dense ideal of $\pi(N)'$ (resp., $\gamma(N^{\mathrm{o}})'$) (cf. Proposition 3 of [7]). If φ is understood, we denote $\mathcal{K}_{\pi}$ (resp., $\mathcal{K}_{\gamma}$) instead of $\mathcal{K}_{\pi,\varphi}$ (resp., $\mathcal{K}_{\gamma,\varphi}$).

Notation A.2.3. Let $\xi, \eta \in {}_{\varphi}(\pi, \mathcal{H})$ (resp., $(\mathcal{K}, \gamma)_{\varphi}$), we denote

$$\langle \xi, \eta \rangle_{N^{\mathrm{o}}} := C_N^{-1}\big(\big(R_{\xi}^{\pi,\varphi}\big)^* R_{\eta}^{\pi,\varphi}\big)^{\mathrm{o}} \in N^{\mathrm{o}}$$
$$\big(\text{resp., } \langle \xi, \eta \rangle_N := \pi_{\varphi}^{-1}\big(\big(L_{\xi}^{\gamma,\varphi}\big)^* L_{\eta}^{\gamma,\varphi}\big) \in N\big).$$

Proposition A.2.4. *For all $\xi, \eta \in {}_{\varphi}(\pi, \mathcal{H})$ (resp., $\xi, \eta \in (\mathcal{K}, \gamma)_{\varphi}$) and $y \in N$ analytic for $(\sigma_t^{\varphi})_{t \in \mathbb{R}}$, we have*:

1. $\langle \xi, \eta \rangle_{N^{\mathrm{o}}}^* = \langle \eta, \xi \rangle_{N^{\mathrm{o}}}$ (*resp.*, $\langle \xi, \eta \rangle_N^* = \langle \eta, \xi \rangle_N$);
2. $\langle \xi, \eta y^{\mathrm{o}} \rangle_{N^{\mathrm{o}}} = \langle \xi, \eta \rangle_{N^{\mathrm{o}}} \sigma_{\mathrm{i}/2}^{\varphi}(y)^{\mathrm{o}}$ (*resp.*, $\langle \xi, \eta y \rangle_N = \langle \xi, \eta \rangle_N \sigma_{-\mathrm{i}/2}^{\varphi}(y)$).

Lemma A.2.5. *For all $\xi_1, \xi_2 \in {}_{\varphi}(\pi, \mathcal{H})$ and $\eta_1, \eta_2 \in (\mathcal{K}, \gamma)_{\varphi}$, we have*

$$\big\langle \eta_1, \gamma\big(\langle \xi_1, \xi_2 \rangle_{N^{\mathrm{o}}}\big)\eta_2 \big\rangle_{\mathcal{K}} = \big\langle \xi_1, \pi\big(\langle \eta_1, \eta_2 \rangle_N\big)\xi_2 \big\rangle_{\mathcal{H}}.$$

Definition A.2.6. The relative tensor product

$$\mathcal{K} \underset{\varphi}{{}_{\gamma}\otimes_{\pi}} \mathcal{H} \quad (\text{or simply denoted by } \mathcal{K} {}_{\gamma}\otimes_{\pi} \mathcal{H})$$

is the Hausdorff completion of the pre-Hilbert space $(\mathcal{K},\gamma)_\varphi \odot {}_\varphi(\pi,\mathcal{H})$, whose inner product is given by

$$\langle \eta_1\otimes\xi_1, \eta_2\otimes\xi_2\rangle := \big\langle \eta_1, \gamma\big(\langle\xi_1,\xi_2\rangle_{N^\circ}\big)\eta_2\big\rangle_{\mathcal{K}} = \big\langle \xi_1, \pi\big(\langle\eta_1,\eta_2\rangle_N\big)\xi_2\big\rangle_{\mathcal{H}},$$

for all $\eta_1,\eta_2\in(\mathcal{K},\gamma)_\varphi$ and $\xi_1,\xi_2\in{}_\varphi(\pi,\mathcal{H})$. If $\eta\in(\mathcal{K},\gamma)_\varphi$ and $\xi\in{}_\varphi(\pi,\mathcal{H})$, we will denote by

$$\eta\,{}_\gamma\underset{\varphi}{\otimes}_\pi\,\xi \quad \text{(or simply } \eta\,{}_\gamma\otimes_\pi\,\xi)$$

the image of $\eta\otimes\xi$ by the canonical map $(\mathcal{K},\gamma)_\varphi\odot{}_\varphi(\pi,\mathcal{H})\to\mathcal{K}\,{}_\gamma\otimes_\pi\,\mathcal{H}$ (isometric dense range).

Remarks A.2.7.

1. By applying this construction to (N°,φ°) instead of (N,φ) we obtain the relative tensor product $\mathcal{H}\,{}_\pi\underset{\varphi^\circ}{\otimes}_\gamma\,\mathcal{K}$.
2. The relative tensor product $\mathcal{K}\,{}_\gamma\otimes_\pi\,\mathcal{H}$ is also the Hausdorff completion of the pre-Hilbert space $(\mathcal{K},\gamma)_\varphi\odot\mathcal{H}$ (resp., $\mathcal{K}\odot{}_\varphi(\pi,\mathcal{H})$), whose inner product is given by:
$$\langle \eta_1\otimes\xi_1, \eta_2\otimes\xi_2\rangle := \big\langle \xi_1, \pi\big(\langle\eta_1,\eta_2\rangle_N\big)\xi_2\big\rangle_{\mathcal{H}}$$
$$\big(\text{resp., } \langle \eta_1\otimes\xi_1, \eta_2\otimes\xi_2\rangle := \big\langle \eta_1, \gamma\big(\langle\xi_1,\xi_2\rangle_{N^\circ}\big)\eta_2\big\rangle_{\mathcal{K}}\big).$$
3. Moreover, for all $\eta\in\mathcal{K},\xi\in{}_\varphi(\pi,\mathcal{H})$ and $y\in N$ analytic for $(\sigma^\varphi_t)_{t\in\mathbb{R}}$ we have
$$\gamma\big(y^\circ\big)\eta\,{}_\gamma\otimes_\pi\,\xi = \eta\,{}_\gamma\otimes_\pi\,\pi\big(\sigma^\varphi_{-i/2}(y)\big)\xi.$$

A.2.8. The relative flip map is the isomorphism $\sigma^{\gamma\pi}_\varphi$ from $\mathcal{K}\,{}_\gamma\underset{\varphi}{\otimes}_\pi\,\mathcal{H}$ onto $\mathcal{H}\,{}_\pi\underset{\varphi^\circ}{\otimes}_\gamma\,\mathcal{K}$ given by:

$$\sigma^{\gamma\pi}_\varphi(\eta\,{}_\gamma\underset{\varphi}{\otimes}_\pi\,\xi) := \xi\,{}_\pi\underset{\varphi^\circ}{\otimes}_\gamma\,\eta,$$
$$\text{for all } \xi\in(\mathcal{K},\gamma)_\varphi \text{ and } \eta\in{}_\varphi(\pi,\mathcal{H}) \text{ (or simply } \sigma_{\gamma\pi}).$$

Note that $\sigma^{\gamma\pi}_\varphi$ is unitary and $(\sigma^{\gamma\pi}_\varphi)^* = \sigma^{\pi\gamma}_{\varphi^\circ}$. Then we can define a relative flip $*$-homomorphism

$$\varsigma^{\gamma\pi}_\varphi : \mathcal{B}(\mathcal{K}\,{}_\gamma\underset{\varphi}{\otimes}_\pi\,\mathcal{H}) \to \mathcal{B}\big(\mathcal{H}\,{}_\pi\underset{\varphi^\circ}{\otimes}_\gamma\,\mathcal{K}\big) \quad \text{(or simply denoted by } \varsigma_{\gamma\pi})$$

by setting $\varsigma^{\gamma\pi}_\varphi(X) := \sigma^{\gamma\pi}_\varphi X(\sigma^{\gamma\pi}_\varphi)^*$ for all $X\in\mathcal{B}(\mathcal{K}\,{}_\gamma\underset{\varphi}{\otimes}_\pi\,\mathcal{H})$.

Fiber product of von Neumann algebras. We continue to use the notation of the previous paragraph.

Proposition-Definition A.2.9. *Let $\mathcal{K}_i$ and $\mathcal{H}_i$ be Hilbert spaces, and $\gamma_i : N^\circ\to\mathcal{B}(\mathcal{K}_i)$ and $\pi_i : N\to\mathcal{B}(\mathcal{H}_i)$ be unital normal $*$-homomorphisms for $i=1,2$. Let $T\in\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)$ and $S\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ such that $T\circ\gamma_1(n^\circ)=\gamma_2(n^\circ)\circ T$ and $S\circ\pi_1(n)=\pi_2(n)\circ S$ for all $n\in N$. Then the linear map*

$$(\mathcal{K}_1,\gamma_1)_\varphi\odot{}_\varphi(\pi_1,\mathcal{H}_1)\to\mathcal{K}_2\,{}_{\gamma_2}\otimes_{\pi_2}\,\mathcal{H}_2; \qquad \xi\odot\eta\mapsto T\xi\,{}_{\gamma_2}\otimes_{\pi_2}\,S\eta$$

extends uniquely to a bounded operator ${}_{\gamma_2}T_{\gamma_1}\otimes_{\pi_2} S_{\pi_1} \in \mathcal{B}(\mathcal{K}_1\,{}_{\gamma_1}\otimes_{\pi_1}\mathcal{H}_1, \mathcal{K}_2\,{}_{\gamma_2}\otimes_{\pi_2}\mathcal{H}_2)$ *(or simply denoted by* $T_{\gamma_1}\otimes_{\pi_2} S$*), whose adjoint operator is* ${}_{\gamma_1}T^*{}_{\gamma_2}\otimes_{\pi_1} S^*{}_{\pi_2}$ *(or simply* $T^*{}_{\gamma_2}\otimes_{\pi_1} S^*$*). In particular, if* $x \in \gamma(N^{\mathrm{o}})'$ *and* $y \in \pi(N)'$*, then the linear map*

$$(\mathcal{K},\gamma)_\varphi \odot {}_\varphi(\pi,\mathcal{H}) \to \mathcal{K}\,{}_\gamma\otimes_\pi \mathcal{H}; \qquad \xi\odot\eta \mapsto x\xi\,{}_\gamma\otimes_\pi\, y\eta$$

extends uniquely to a bounded operator on $\mathcal{K}\,{}_\gamma\otimes_\pi\mathcal{H}$ *denoted by* $x\,{}_\gamma\otimes_\pi\, y \in \mathcal{B}(\mathcal{K}\,{}_\gamma\otimes_\pi\mathcal{H})$.

Remark A.2.10. With the notations of Proposition-Definition A.2.9, let $T:\mathcal{K}_1\to\mathcal{H}_2$ and $S:\mathcal{H}_1\to\mathcal{K}_2$ be bounded antilinear maps such that $T\circ\gamma_1(n^{\mathrm{o}})^* = \pi_2(n)\circ T$ and $S\circ\pi_1(n) = \gamma_2(n^{\mathrm{o}})^*\circ S$ for all $n\in N$. In a similar way, we define ${}_{\pi_2}T_{\gamma_1}\otimes_{\gamma_2} S_{\pi_1} \in \mathcal{B}(\mathcal{K}_1\,{}_{\gamma_1}\otimes_{\pi_1}\mathcal{H}_1, \mathcal{H}_2\,{}_{\pi_2}\otimes_{\gamma_2}\mathcal{K}_2)$ (or simply $T_{\gamma_1}\otimes_{\gamma_2} S$). Note that these notation are different from those used in [17], [20].

Let $M\subset\mathcal{B}(\mathcal{K})$ and $P\subset\mathcal{B}(\mathcal{H})$ be two von Neumann algebras. Let us assume that $\pi(N)\subset P$ and $\gamma(N^{\mathrm{o}})\subset M$.

Definition A.2.11. The fiber product $M\,{}_\gamma\star_\pi P$ of M and P over N is the commutant of $\{x\,{}_\gamma\otimes_\pi\, y;\, x\in M', y\in P'\}\subset\mathcal{B}(\mathcal{K}\,{}_\gamma\otimes_\pi\mathcal{H})$. Then $M\,{}_\gamma\star_\pi P$ is a von Neumann algebra.

Note that we have $\varsigma_{\gamma\pi}(M\,{}_\gamma\star_\pi P) = P\,{}_\pi\star_\gamma M$. We still denote by $\varsigma_{\gamma\pi}: M\,{}_\gamma\star_\pi P \to P\,{}_\pi\star_\gamma M$ the restriction of $\varsigma_{\gamma\pi}$ to $M\,{}_\gamma\star_\pi P$.

A.2.12. *Slicing with normal linear forms.* Now, let us recall how to slice with normal linear forms. For $\xi\in(\mathcal{K},\gamma)_\varphi$ and $\eta\in{}_\varphi(\pi,\mathcal{H})$, we consider the following bounded linear maps:

$$\lambda^{\gamma\pi}_\xi : \mathcal{H}\to\mathcal{K}\,{}_\gamma\otimes_\pi\mathcal{H}, \qquad \zeta\mapsto\xi\,{}_\gamma\otimes_\pi\,\zeta;$$
$$\rho^{\gamma\pi}_\eta : \mathcal{K}\to\mathcal{K}\,{}_\gamma\otimes_\pi\mathcal{H}, \qquad \zeta\mapsto\zeta\,{}_\gamma\otimes_\pi\,\eta.$$

Let $T\in\mathcal{B}(\mathcal{K}\,{}_\gamma\otimes_\pi\mathcal{H})$ and $\omega\in\mathcal{B}(\mathcal{H})_*$ (resp., $\omega\in\mathcal{B}(\mathcal{K})_*$). By using the fact that $(\mathcal{K},\gamma)_\varphi$ (resp., ${}_\varphi(\pi,\mathcal{H})$) is dense in $\mathcal{H}$ (resp., $\mathcal{K}$), there exists a unique $(\mathrm{id}\,{}_\gamma\star_\pi\omega)(T)\in\mathcal{B}(\mathcal{K})$ (resp., $(\omega\,{}_\gamma\star_\pi\mathrm{id})(T)\in\mathcal{B}(\mathcal{H})$) such that

$$\big\langle\xi_1,(\mathrm{id}\,{}_\gamma\star_\pi\omega)(T)\xi_2\big\rangle = \omega\big(\big(\lambda^{\gamma\pi}_{\xi_1}\big)^* T\lambda^{\gamma\pi}_{\xi_2}\big), \quad \text{for all } \xi_1,\xi_2\in(\mathcal{K},\gamma)_\varphi$$
$$\big(\text{resp., }\big\langle\eta_1,(\omega\,{}_\gamma\star_\pi\mathrm{id})(T)\eta_2\big\rangle = \omega\big(\big(\rho^{\gamma\pi}_{\eta_1}\big)^* T\rho^{\gamma\pi}_{\eta_2}\big), \quad \text{for all } \eta_1,\eta_2\in{}_\varphi(\pi,\mathcal{H})\big).$$

In particular, we have:

$$(\mathrm{id}\,{}_\gamma\star_\pi\omega_{\eta_1,\eta_2})(T) = \big(\rho^{\gamma\pi}_{\eta_1}\big)^* T\rho^{\gamma\pi}_{\eta_2}\in\mathcal{B}(\mathcal{K}), \quad \text{for all } \eta_1,\eta_2\in{}_\varphi(\pi,\mathcal{H});$$
$$(\omega_{\xi_1,\xi_2}\,{}_\gamma\star_\pi\,\mathrm{id})(T) = \big(\lambda^{\gamma\pi}_{\xi_1}\big)^* T\lambda^{\gamma\pi}_{\xi_2}\in\mathcal{B}(\mathcal{H}), \quad \text{for all } \xi_1,\xi_2\in(\mathcal{K},\gamma)_\varphi.$$

If $x\in M\,{}_\gamma\star_\pi P$, then for all $\omega\in\mathcal{B}(\mathcal{H})_*$ (resp., $\omega\in\mathcal{B}(\mathcal{K})_*$) we have $(\mathrm{id}\,{}_\gamma\star_\pi\omega)(x)\in M$ (resp., $(\omega\,{}_\gamma\star_\pi\mathrm{id})(x)\in P$). We refrain from writing the details but we can easily define the slice maps if T takes its values in a different relative tensor product. Note that we can extend the notion of slice maps for normal linear forms to normal semi-finite weights.

Fiber product over a finite-dimensional von Neumann algebra. Now, let us assume that

$$N := \bigoplus_{1\leqslant l\leqslant k} \mathrm{M}_{n_l}(\mathbb{C}) \quad \text{and} \quad \varphi := \bigoplus_{1\leqslant l\leqslant k} \mathrm{Tr}_l(F_l -),$$

where F_l is a positive invertible matrix of $\mathrm{M}_{n_l}(\mathbb{C})$ and Tr_l is the nonnormalized trace on $\mathrm{M}_{n_l}(\mathbb{C})$. Denote by $(F_{l,i})_{1\leqslant i\leqslant n_l}$ the eigenvalues of F_l.

Proposition-Definition A.2.13 (§7 [13]). *The bounded linear map*

$$v_\varphi^{\gamma\pi} : \mathcal{K}\otimes\mathcal{H} \to \mathcal{K}\underset{\varphi}{{}_\gamma\otimes_\pi}\mathcal{H};$$
$$\xi\otimes\eta \mapsto \xi\underset{\varphi}{{}_\gamma\otimes_\pi}\eta \quad (\textit{or simply denoted by } v_{\gamma\pi})$$

is a coisometry if, and only if, we have $\sum_{1\leqslant i\leqslant n_l} F_{l,i}^{-1} = 1$ *for all* $1\leqslant l\leqslant k$.

In the following, we assume the above condition to be satisfied.

Proposition-Definition A.2.14 (§7 [13]). *Let us denote*

$$q_\varphi^{\gamma\pi} := \big(v_\varphi^{\gamma\pi}\big)^* v_\varphi^{\gamma\pi} \quad (\textit{or simply } q_{\gamma\pi}).$$

Then $q_\varphi^{\gamma\pi}$ *is a self-adjoint projection of* $\mathcal{B}(\mathcal{K}\otimes\mathcal{H})$ *such that*

$$q_\varphi^{\gamma\pi} = \sum_{1\leqslant l\leqslant k}\ \sum_{1\leqslant i,j\leqslant n_l} F_{l,i}^{-1/2} F_{l,j}^{-1/2} \gamma\big(e_{ij}^{(l)\mathrm{o}}\big)\otimes\pi\big(e_{ji}^{(l)}\big),$$

where, for all $1\leqslant l\leqslant k$, $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ *is a system of matrix units (s.m.u.) diagonalizing* F_l, *that is,* $F_l = \sum_{1\leqslant i\leqslant n_l} F_{l,i} e_{ii}^{(l)}$. *Moreover,* $M\,{}_\gamma\!\star_\pi P \to q_\varphi^{\gamma\pi}(M\otimes P)q_\varphi^{\gamma\pi}$; $x\mapsto (v_\varphi^{\gamma\pi})^* x v_\varphi^{\gamma\pi}$ *is a unital normal* $*$*-isomorphism.*

Since N is finite-dimensional, the inner product given by $\langle x,y\rangle := \varphi(x^*y)$ for all $x,y\in N$ defines a structure of finite-dimensional Hilbert space on N. We have a (bounded) linear map $\mu_\varphi : N\otimes N\to N$ defined for all $x,y\in N$ by $\mu_\varphi(x\otimes y) = xy$, where $N\otimes N$ is endowed with its canonical structure of finite-dimensional Hilbert space.

Proposition-Definition A.2.15. *For* $i=1,2$, *let* $\pi_i : N\to\mathcal{B}(\mathcal{H}_i)$ *be a unital normal* $*$*-representation of* N *on a Hilbert space* $\mathcal{H}_i$. *Let us denote*

$$q_\varphi^{\pi_1\pi_2} := (\pi_1\otimes\pi_2)\big(\mu_\varphi^*(1_N)\big) \in \mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2) \quad (\textit{or simply } q_{\pi_1\pi_2}).$$

We denote $q_\varphi^{\pi_1} := q_\varphi^{\pi_1\pi_1}$ *(or simply* q_{π_1}*) for short. Then we have*

$$q_\varphi^{\pi_1\pi_2} = \sum_{1\leqslant l\leqslant k}\ \sum_{1\leqslant i,j\leqslant n_l} F_{l,j}^{-1}\pi_1\big(e_{ij}^{(l)}\big)\otimes\pi_2\big(e_{ji}^{(l)}\big),$$

where, for all $1\leqslant l\leqslant k$, $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ *is a s.m.u. diagonalizing* F_l.

Proof. For $1 \leqslant l \leqslant k$, fix a s.u.m. $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ of $\mathrm{M}_{n_l}(\mathbb{C})$ diagonalizing F_l. It suffices to prove that

$$\mu_\varphi^*(1_N) = \sum_{1\leqslant l\leqslant k} \sum_{1\leqslant i,j\leqslant n_l} F_{l,j}^{-1} e_{ij}^{(l)} \otimes e_{ji}^{(l)}.$$

Since $1_N = \sum_{1\leqslant l\leqslant k} \sum_{1\leqslant i\leqslant n_l} e_{ii}^{(l)}$, it is enough to prove that

$$\mu_\varphi^*\big(e_{rs}^{(l)}\big) = \sum_{1\leqslant j\leqslant n_l} F_{l,j}^{-1} e_{rj}^{(l)} \otimes e_{js}^{(l)}, \quad \text{for all } 1 \leqslant r,s \leqslant n_l.$$

Let $(f_{ij}^{(l)})$ be the family of N given by $f_{ij}^{(l)} := F_{l,j}^{-1/2} e_{ij}^{(l)}$ for all $1 \leqslant l \leqslant k$ and $1 \leqslant i,j \leqslant n_l$. It is clear that $(f_{ij}^{(l)})$ is an orthonormal basis of N. We have

$$\varphi\big(e_{sq}^{(l)}\big) = \mathrm{Tr}_l\big(F_l e_{sq}^{(l)}\big) = \sum_{i=1}^{n_l} F_{l,i}\, \mathrm{Tr}_l\big(e_{ii}^{(l)} e_{sq}^{(l)}\big) = F_{l,s}\, \mathrm{Tr}_l\big(e_{sq}^{(l)}\big) = F_{l,s}\delta_q^s.$$

We have

$$\begin{aligned}
\mu_\varphi^*\big(e_{rs}^{(l)}\big) &= \sum_{l',l''=1}^{k} \sum_{i,j\ =1}^{n_{l'}} \sum_{p,q\ =1}^{n_{l''}} \big\langle \mu_\varphi^*\big(e_{rs}^{(l)}\big), f_{ij}^{(l')} \otimes f_{pq}^{(l'')} \big\rangle f_{ij}^{(l')} \otimes f_{pq}^{(l'')} \\
&= \sum_{l',l''=1}^{k} \sum_{i,j\ =1}^{n_{l'}} \sum_{p,q\ =1}^{n_{l''}} \delta_{l''}^{l'} \delta_p^j F_{l',j}^{-1} F_{l'',q}^{-1} \big\langle e_{rs}^{(l)}, e_{iq}^{(l')} \big\rangle e_{ij}^{(l')} \otimes e_{pq}^{(l'')} \\
&= \sum_{l'=1}^{k} \sum_{i,j,q\ =1}^{n_{l'}} \delta_{l'}^{l} \delta_i^r F_{l',j}^{-1} F_{l',q}^{-1} \varphi\big(e_{sq}^{(l)}\big) e_{ij}^{(l')} \otimes e_{jq}^{(l')} \\
&= \sum_{j,q=1}^{n_l} F_{l,j}^{-1} F_{l,q}^{-1} F_{l,s} \delta_q^s e_{rj}^{(l)} \otimes e_{jq}^{(l)} \\
&= \sum_{j=1}^{n_l} F_{l,j}^{-1} e_{rj}^{(l)} \otimes e_{js}^{(l)}.
\end{aligned}$$

□

Remarks A.2.16.

1. For $i = 1,2$, let $\gamma_i : N^{\mathrm{o}} \to \mathcal{B}(\mathcal{K}_i)$ be a unital normal $*$-representation of N^{o} on a Hilbert space $\mathcal{K}_i$. In a similar way, we define $q_{\varphi^{\mathrm{o}}}^{\gamma_1\gamma_2} \in \mathcal{B}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ (or simply $q_{\gamma_1\gamma_2}$) such that

$$q_{\varphi^{\mathrm{o}}}^{\gamma_1\gamma_2} = \sum_{1\leqslant l\leqslant k} \sum_{1\leqslant i,j\leqslant n_l} F_{l,j}^{-1} \gamma_1\big(e_{ij}^{(l)\mathrm{o}}\big) \otimes \gamma_2\big(e_{ji}^{(l)\mathrm{o}}\big),$$

   where, for all $1 \leqslant l \leqslant k$, $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ is a s.m.u. diagonalizing F_l.
2. It should be noted that $q_\varphi^{\pi_1\pi_2}$ and $q_{\varphi^{\mathrm{o}}}^{\gamma_1\gamma_2}$ are self-adjoint but not idempotent in general. If N is commutative (*i.e.*, $N = \mathbb{C}^k$), then $q_\varphi^{\pi_1\pi_2}$ and $q_{\varphi^{\mathrm{o}}}^{\gamma_1\gamma_2}$ are projections.

Case of the nonnormalized Markov trace. In this paragraph, we take for φ the nonnormalized Markov trace on $N = \bigoplus_{1\leqslant l\leqslant k} \mathrm{M}_{n_l}(\mathbb{C})$, *that is*, $\epsilon = \bigoplus_{1\leqslant l\leqslant k} n_l \cdot \mathrm{Tr}_l$. From now on, the operators $q_\epsilon^{\gamma\pi}$, $q_{\epsilon^{\mathrm{o}}}^{\pi\gamma}$, $q_\epsilon^{\pi_1\pi_2}$ and $q_{\epsilon^{\mathrm{o}}}^{\gamma_1\gamma_2}$ will be simply denoted by $q_{\gamma\pi}$, $q_{\pi\gamma}$, $q_{\pi_1\pi_2}$ and $q_{\gamma_1\gamma_2}$. As a corollary of Proposition-Definition A.2.14, we have the following.

PROPOSITION A.2.17. *For all s.u.m.* $(e_{ij}^{(l)})_{1\leqslant l\leqslant k, 1\leqslant i,j\leqslant n_l}$ *of* N, *we have*

$$q_{\gamma\pi} = \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \gamma\big(e_{ij}^{(l)\mathrm{o}}\big) \otimes \pi\big(e_{ji}^{(l)}\big) \quad \textit{and}$$

$$q_{\pi\gamma} = \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \pi\big(e_{ij}^{(l)}\big) \otimes \gamma\big(e_{ji}^{(l)\mathrm{o}}\big).$$

As a corollary of Proposition-Definition A.2.15, we have:

PROPOSITION A.2.18. *For all s.u.m.* $(e_{ij}^{(l)})_{1\leqslant l\leqslant k, 1\leqslant i,j\leqslant n_l}$ *of* N, *we have*

$$q_{\pi_1\pi_2} = \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \pi_1\big(e_{ij}^{(l)}\big) \otimes \pi_2\big(e_{ji}^{(l)}\big) \quad \textit{and}$$

$$q_{\gamma_1\gamma_2} = \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \gamma_1\big(e_{ij}^{(l)\mathrm{o}}\big) \otimes \gamma_2\big(e_{ji}^{(l)\mathrm{o}}\big).$$

The following result is a slight generalization of Proposition A.2.17 to the setting of C*-algebras.

PROPOSITION-DEFINITION A.2.19 (2.6 [2]). *Let* A, B *be two* C*-*algebras. We consider two non-degenerate* $*$*-homomorphisms* $\gamma_A : N^{\mathrm{o}} \to \mathcal{M}(A)$ *and* $\pi_B : N \to \mathcal{M}(B)$. *There exists a unique self-adjoint projection* $q_{\gamma_A\pi_B} \in \mathcal{M}(A \otimes B)$ *(resp.,* $q_{\pi_B\gamma_A} \in \mathcal{M}(B \otimes A)$*) such that*

$$q_{\gamma_A\pi_B} = \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \gamma_A\big(e_{ij}^{(l)\mathrm{o}}\big) \otimes \pi_B\big(e_{ji}^{(l)}\big)$$

$$\left(\textit{resp.,}\ q_{\pi_B\gamma_A} = \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \pi_B\big(e_{ij}^{(l)}\big) \otimes \gamma_A\big(e_{ji}^{(l)\mathrm{o}}\big)\right),$$

for all s.u.m. $(e_{ij}^{(l)})_{1\leqslant l\leqslant k, 1\leqslant i,j\leqslant n_l}$ *of* N.

Proof. The uniqueness of such a self-adjoint projection is straightforward. In virtue of the Gelfand–Naimark theorem, we can consider faithful nondegenerate $*$-homomorphisms $\theta_A : A \to \mathcal{B}(\mathcal{K})$ and $\theta_B : B \to \mathcal{B}(\mathcal{H})$. Let us denote $\gamma := \theta_A \circ \gamma_A$ and $\pi := \theta_B \circ \pi_B$. Then $\gamma : N^{\mathrm{o}} \to \mathcal{B}(\mathcal{K})$ and $\pi : N \to \mathcal{B}(\mathcal{H})$ are normal unital $*$-representations. Let us fix an arbitrary s.u.m. $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ for $\mathrm{M}_{n_l}(\mathbb{C})$ for each $1 \leqslant l \leqslant k$. We define a self-adjoint projection $q_{\gamma_A\pi_B} \in \mathcal{M}(A \otimes B)$ by setting:

$$q_{\gamma_A\pi_B} := \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \gamma_A\big(e_{ij}^{(l)\mathrm{o}}\big) \otimes \pi_B\big(e_{ji}^{(l)}\big).$$

By Proposition A.2.17, we have $q_{\gamma\pi} = (\theta_A \otimes \theta_B)(q_{\gamma_A\pi_B})$. By using again Proposition A.2.17 and the fact that $\theta_A \otimes \theta_B$ is faithful, we obtain that $q_{\gamma_A\pi_B}$ is independent of the chosen systems of matrix units. Moreover, the definition of $q_{\gamma_A\pi_B}$ shows that $q_{\gamma_A\pi_B}$ is also independent of the chosen faithful nondegenerate $*$-homomorphisms θ_A and θ_B. □

In a similar way, we have the following generalization of Proposition A.2.18 to the setting of C*-algebras.

PROPOSITION A.2.20. *For $i = 1, 2$, let B_i (resp., A_i) be a C*-algebra and $\pi_i : N \to \mathcal{M}(B_i)$ (resp., $\gamma_i : N^{\mathrm{o}} \to \mathcal{M}(A_i)$) a nondegenerate $*$-homomorphism. Then there exists a unique $q_{\pi_1\pi_2} \in \mathcal{M}(B_1 \otimes B_2)$ (resp., $q_{\gamma_1\gamma_2} \in \mathcal{M}(A_1 \otimes A_2)$) such that*

$$q_{\pi_1\pi_2} = \sum_{1 \leqslant l \leqslant k} n_l^{-1} \sum_{1 \leqslant i,j \leqslant n_l} \pi_1\big(e_{ij}^{(l)}\big) \otimes \pi_2\big(e_{ji}^{(l)}\big)$$

$$\Big(\textit{resp.}, \; q_{\gamma_1\gamma_2} = \sum_{1 \leqslant l \leqslant k} n_l^{-1} \sum_{1 \leqslant i,j \leqslant n_l} \gamma_1\big(e_{ij}^{(l)\mathrm{o}}\big) \otimes \gamma_2\big(e_{ji}^{(l)\mathrm{o}}\big)\Big),$$

for all s.u.m. $(e_{ij}^{(l)})_{1 \leqslant l \leqslant k, 1 \leqslant i,j \leqslant n_l}$ of N.

In the following, we adopt a multi-index notation to simplify formulas and computations.

NOTATION A.2.21.

1. Consider the index sets $\mathscr{I} := \{(l,i,j);\, 1 \leqslant l \leqslant k, 1 \leqslant i,j \leqslant n_l\}$ and $\mathscr{I}_0 := \mathscr{I} \sqcup \{\varnothing\}$.
2. For $I = (l,i,j) \in \mathscr{I}$, we denote $\overline{I} := (l,j,i) \in \mathscr{I}$. Denote also $\overline{\varnothing} := \varnothing$. The map $\mathscr{I}_0 \to \mathscr{I}_0$; $I \mapsto \overline{I}$ is involutive.
3. A pair of indices $(I,J) \in \mathscr{I} \times \mathscr{I}$ is said to be composable if we have $I = (l,i,m)$ and $J = (l,m,j)$ for some indices $1 \leqslant l \leqslant k$ and $1 \leqslant i,m,j \leqslant n_l$. In this case, we denote $IJ := (l,i,j) \in \mathscr{I}$. We also denote $IJ = \varnothing$ if I and J are not composable, $I = \varnothing$ or $J = \varnothing$. This defines a map $\mathscr{I}_0 \times \mathscr{I}_0 \to \mathscr{I}_0$; $(I,J) \mapsto IJ$. It is clear that $\overline{IJ} = \overline{JI}$ for all $I, J \in \mathscr{I}_0$.

Let us fix a s.u.m. $(e_{ij}^{(l)})_{1 \leqslant l \leqslant k, 1 \leqslant i,j \leqslant n_l}$ of N.

NOTATION A.2.22.

1. Denote by $\varepsilon_I := e_{ij}^{(l)}$ for $I = (l,i,j) \in \mathscr{I}$ and $\varepsilon_\varnothing := 0$. Denote by $e_I := \pi(\varepsilon_I)$ and $f_I := \gamma(\varepsilon_I^{\mathrm{o}})$ for $I \in \mathscr{I}_0$. Denote by $n_I := n_l$ for $I = (l,i,j) \in \mathscr{I}$ and $n_\varnothing := 1$. Notice that we have $n_{\overline{I}} = n_I$ for all $I \in \mathscr{I}_0$.
2. Since $(\varepsilon_I)_{I \in \mathscr{I}}$ is a basis of N, for $x \in N$ we denote $x = \sum_{I \in \mathscr{I}} x_I \cdot \varepsilon_I$, with $x_I \in \mathbb{C}$ for $I \in \mathscr{I}$. Note that $x^* = \sum_{I \in \mathscr{I}} \overline{x_{\overline{I}}} \cdot \varepsilon_I$.

REMARKS A.2.23. 1. For all $I, J \in \mathscr{I}_0$, we have $\varepsilon_I^* = \varepsilon_{\overline{I}}$ and $\varepsilon_I \varepsilon_J = \varepsilon_{IJ}$. For all $I, J \in \mathscr{I}_0$, we have:

$$e_I^* = e_{\overline{I}}, \qquad e_I e_J = e_{IJ}; \qquad f_I^* = f_{\overline{I}}, \qquad f_I f_J = f_{JI}.$$

2. We have $q_{\gamma\pi} = \sum_{I \in \mathscr{I}} n_I^{-1} f_I \otimes e_{\overline{I}}$, $q_{\pi\gamma} = \sum_{I \in \mathscr{I}} n_I^{-1} e_I \otimes f_{\overline{I}}$, $q_\pi = \sum_{I \in \mathscr{I}} e_I \otimes e_{\overline{I}}$ and $q_\gamma = \sum_{I \in \mathscr{I}} f_I \otimes f_{\overline{I}}$.

A.3. Unitary equivalence of Hilbert C*-modules. In the following, we recall the notion of morphism between Hilbert modules over possibly different C*-algebras.

DEFINITION A.3.1. Let A and B be two C*-algebras and $\phi : A \to B$ a $*$-homomorphism. Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert C*-modules over A and B respectively. A ϕ-compatible operator from $\mathscr{E}$ to $\mathscr{F}$ is a linear map $\Phi : \mathscr{E} \to \mathscr{F}$ such that:

(i) for all $\xi \in \mathscr{E}$ and $a \in A$, $\Phi(\xi a) = \Phi(\xi)\phi(a)$;
(ii) for all $\xi, \eta \in \mathscr{E}$, $\langle \Phi\xi, \Phi\eta \rangle = \phi(\langle \xi, \eta \rangle)$.

Furthermore, if ϕ is a $*$-isomorphism and Φ is surjective, we say that Φ is ϕ-compatible unitary operator (or a unitary equivalence over ϕ) from $\mathscr{E}$ onto $\mathscr{F}$.

REMARKS A.3.2.

1. It follows from (ii) that $\Phi : \mathscr{E} \to \mathscr{F}$ is bounded and even isometric if ϕ is faithful. Indeed, we have $\|\langle \Phi\xi, \Phi\eta \rangle\| = \|\phi(\langle \xi, \eta \rangle)\| = \|\langle \xi, \eta \rangle\|$ for all $\xi, \eta \in \mathscr{E}$. Then, for all $\xi \in \mathscr{E}$ we have $\|\Phi\xi\|^2 = \|\langle \Phi\xi,\ \Phi\xi \rangle\| = \|\langle \xi,\ \xi \rangle\| = \|\xi\|^2$. In particular, if ϕ is a $*$-isomorphism and Φ is a ϕ-compatible unitary operator, then Φ is bijective and the inverse map $\Phi^{-1} : \mathscr{F} \to \mathscr{E}$ is a ϕ^{-1}-compatible unitary operator.
2. It is clear that $\mathrm{id}_{\mathscr{E}}$ is a id_A-compatible unitary operator. Let A, B, and C be C*-algebras and $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ be Hilbert modules over A, B, and C, respectively. Let $\phi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms (resp., $*$-isomorphisms). If $\Phi : \mathscr{E} \to \mathscr{F}$ is a ϕ-compatible operator (resp., unitary operator) and $\Psi : \mathscr{F} \to \mathscr{G}$ a ψ-compatible operator (resp., unitary operator), then $\Psi \circ \Phi : \mathscr{E} \to \mathscr{G}$ is a $\psi \circ \phi$-compatible operator (resp., unitary operator).
3. Let $\Phi : \mathscr{E} \to \mathscr{F}$ be a unitary equivalence over a given $*$-isomorphism ϕ. If $T \in \mathcal{L}(\mathscr{E})$, then the map $\Phi \circ T \circ \Phi^{-1} : \mathscr{F} \to \mathscr{F}$ is an adjointable operator whose adjoint operator is $\Phi^{-1} \circ T^* \circ \Phi$. We define a $*$-isomorphism $\mathcal{L}(\mathscr{E}) \to \mathcal{L}(\mathscr{F})$; $T \mapsto \Phi \circ T \circ \Phi^{-1}$. Note that $\Phi \circ \theta_{\xi,\eta} \circ \Phi^{-1} = \theta_{\Phi\xi,\Phi\eta}$ for all $\xi, \eta \in \mathscr{E}$. In particular, for all $k \in \mathcal{K}(\mathscr{E})$ we have $\Phi \circ k \circ \Phi^{-1} \in \mathcal{K}(\mathscr{F})$. More precisely, the map $\mathcal{K}(\mathscr{E}) \to \mathcal{K}(\mathscr{F})$; $k \mapsto \Phi \circ k \circ \Phi^{-1}$ is a $*$-isomorphism.

The notion of unitary equivalence defines an equivalence relation on the class consisting of all Hilbert C*-modules (cf. Remarks A.3.2(1), (2)). Actually,

this notion of morphism between Hilbert modules over possibly different C*-algebra can be understood in terms of unitary adjointable operator between two Hilbert modules over the same C*-algebra.

PROPOSITION A.3.3. *Let A and B be two C*-algebras and $\phi : A \to B$ a $*$-isomorphism. Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert C*-modules over A and B, respectively.*

1. *If $\Phi : \mathscr{E} \to \mathscr{F}$ is a surjective ϕ-compatible unitary operator, then there exists a unique unitary adjointable operator $U \in \mathcal{L}(\mathscr{E} \otimes_\phi B, \mathscr{F})$ such that $U(\xi \otimes_\phi b) = \Phi(\xi)b$, for all $\xi \in \mathscr{E}$ and $b \in B$.*
2. *Conversely, if $U \in \mathcal{L}(\mathscr{E} \otimes_\phi B, \mathscr{F})$ is a unitary, then there exists a unique ϕ-compatible unitary operator $\Phi : \mathscr{E} \to \mathscr{F}$ such that $\Phi(\xi)b = U(\xi \otimes_\phi b)$ for all $\xi \in \mathscr{E}$ and $b \in B$.*

As an application of the above proposition, we can the state the following result.

PROPOSITION-DEFINITION A.3.4. *Let A_1, B_1, A_2, and B_2 be C*-algebras, $\phi_1 : A_1 \to B_1$ and $\phi_2 : A_2 \to B_2$ $*$-isomorphisms. Let $\mathscr{E}_1$, $\mathscr{F}_1$, $\mathscr{E}_2$, and $\mathscr{F}_2$ be Hilbert C*-modules over A_1, B_1, A_2, and B_2, respectively. Let $\Phi_1 : \mathscr{E}_1 \to \mathscr{F}_1$ and $\Phi_2 : \mathscr{E}_2 \to \mathscr{F}_2$ be unitary equivalences over ϕ_1 and ϕ_2 respectively. Then the linear map $\mathscr{E}_1 \odot \mathscr{E}_2 \to \mathscr{F}_1 \otimes \mathscr{F}_2$; $\xi_1 \otimes \xi_2 \mapsto \Phi_1(\xi_1) \otimes \Phi_2(\xi_2)$ extends to a bounded linear map $\Phi_1 \otimes \Phi_2 : \mathscr{E}_1 \otimes \mathscr{E}_2 \to \mathscr{F}_1 \otimes \mathscr{F}_2$. Moreover, $\Phi_1 \otimes \Phi_2$ is a $\phi_1 \otimes \phi_2$-compatible unitary operator.*

The notion of unitary equivalence can also be understood in terms of isomorphism between the associated linking C*-algebras.

PROPOSITION A.3.5. *Let A and B be two C*-algebras and $\phi : A \to B$ a $*$-isomorphism. Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert C*-modules over A and B, respectively.*

1. *If $\Phi : \mathscr{E} \to \mathscr{F}$ is a ϕ-compatible unitary operator, then there exists a unique $*$-homomorphism $f : \mathcal{K}(\mathscr{E} \oplus A) \to \mathcal{K}(\mathscr{F} \oplus B)$ such that $f \circ \iota_{\mathscr{E}} = \iota_{\mathscr{F}} \circ \Phi$ and $f \circ \iota_A = \iota_B \circ \phi$. Moreover, f is a $*$-isomorphism.*
2. *Conversely, let $f : \mathcal{K}(\mathscr{E} \oplus A) \to \mathcal{K}(\mathscr{F} \oplus B)$ be a $*$-isomorphism such that $f \circ \iota_A = \iota_B \circ \phi$. Then there exists a unique map $\Phi : \mathscr{E} \to \mathscr{F}$ such that $f \circ \iota_{\mathscr{E}} = \iota_{\mathscr{F}} \circ \Phi$. Moreover, Φ is a ϕ-compatible unitary operator.*

Proof. 1. The $*$-homomorphism $f : \mathcal{K}(\mathscr{E} \oplus A) \to \mathcal{K}(\mathscr{F} \oplus B)$ is defined by (cf. Remarks A.3.2(3)):

$$f\begin{pmatrix} k & \xi \\ \eta^* & a \end{pmatrix} := \begin{pmatrix} \Phi \circ k \circ \Phi^{-1} & \Phi\xi \\ (\Phi\eta)^* & \phi(a) \end{pmatrix},$$
$$\text{for all } k \in \mathcal{K}(\mathscr{E}), \xi, \eta \in \mathscr{E} \text{ and } a \in A.$$

2. This is a straightforward consequence of Lemma 2.3.4(1). □

NOTATION A.3.6. Let A, B be C*-algebras and $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert C*-modules over A and B, respectively. Let $\phi : A \to B$ be a $*$-isomorphism and $\Phi : \mathscr{E} \to \mathscr{F}$ a ϕ-compatible unitary operator. If $T \in \mathcal{L}(A, \mathscr{E})$, we define the map $\widetilde{\Phi}(T) := \Phi \circ T \circ \phi^{-1} : B \to \mathscr{F}$. By a straightforward computation, we show that $\widetilde{\Phi}(T) \in \mathcal{L}(B, \mathscr{F})$ whose adjoint operator is $\widetilde{\Phi}(T)^* = \phi \circ T^* \circ \Phi^{-1}$. We have a bounded linear map $\widetilde{\Phi} : \mathcal{L}(A, \mathscr{E}) \to \mathcal{L}(B, \mathscr{F})$, which is an extension of Φ up to the canonical injections $\mathscr{E} \to \mathcal{L}(A, \mathscr{E})$ and $\mathscr{F} \to \mathcal{L}(B, \mathscr{F})$.

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Jonathan Crespo, Vrije Universiteit Brussel, Vakgroep Wiskunde, Pleinlaan 2, B-1050 Brussels, Belgium

E-mail address: jonathan.crespo@wanadoo.fr; jonathan.crespo@vub.ac.be

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