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ON REPRESENTATIONS OF ERROR TERMS RELATED TO THE DERIVATIVES FOR SOME DIRICHLET SERIES

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Abstract. In previous papers, we examined several properties of an error term in a certain divisor problem related to the derivatives of the Riemann zeta-function. In this paper, we obtain representations of error terms related to the derivatives of some Dirichlet series, which can be regarded as generalized versions of a Dirichlet divisor problem and a Gauss circle problem. We also give the upper bounds of the error terms in terms of exponent pairs.

1. Introduction and statement of results

Let $\zeta(s)$ denote the Riemann zeta-function, and $\zeta^{(k)}(s)$ denote the kth derivative of $\zeta(s)$ with $\zeta^{(0)}(s)=\zeta(s)$. Further let $D_{(k,l)}(n)$ be the coefficient of Dirichlet series $(-1)^{k+l}\zeta^{(k)}(s)\zeta^{(l)}(s)$ in $\Re s>1$ for any non-negative integers k and l, namely

$$D_{(k,l)}(n)=\sum_{d|n}(\log d)^k\left(\log\frac{n}{d}\right)^l.$$

In the previous works, we investigated the upper bound estimates for the error term $\Delta_{(k,l)}(x)$, which is the error term on the summatory function $\sum_{n\le x}D_{(k,l)}(n)$. In particular, we treated the upper bound estimates for the case $k=l$ in [15], and general k and l in [7]. As other properties for this error term, the truncated Voronoï-type formula and a mean square formula for $\Delta_{(1,1)}(x)$ were derived in [15]. Furthermore, the Riesz means and differences between two kinds of mean values of the error term were treated in [2].

In [7], we derived the representation of $\Delta_{(k,l)}(x)$ called the "Chowla–Walum formula", that is, the sum of the periodic Bernoulli function. Actually, we

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proved that

$$\Delta_{(k,l)}(x) = -\{R_{k,l}(x) + R_{l,k}(x)\} + O\big((\log x)^{k+l}\big) \tag{1.1}$$

with

$$R_{\alpha,\beta}(x) = \sum_{j=0}^{\beta} \binom{\beta}{j} (-1)^j (\log x)^{\beta-j} \sum_{n\le\sqrt{x}} \psi\left(\frac{x}{n}\right)(\log n)^{\alpha+j},$$

where $\binom{k}{j}$ is the binomial coefficient and $\psi(x) = x - [x] - 1/2$ is the periodic Bernoulli function. Here $[x]$ denotes the greatest integer not exceeding x.

The formula (1.1) is derived by using the "Dirichlet hyperbola method", which is formulated as

$$\begin{aligned}\sum_{mn\le x} f(n)g(m) = \sum_{n\le x^c} f(n) \sum_{m\le x/n} g(m) + \sum_{n\le x^{1-c}} g(n) \sum_{m\le x/n} f(m) \\ - \Big(\sum_{n\le x^c} f(n)\Big)\Big(\sum_{n\le x^{1-c}} g(n)\Big)\end{aligned} \tag{1.2}$$

for any arithmetical functions f and g, where c is a real number with $0 \le c \le 1$ (see [1, Theorem 3.17]).

For an arithmetical function $f(n)$, we put $L(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ for $\Re s > \sigma_f$, where σ_f is an abscissa of absolute convergence of this series. Then we have

$$\zeta^{(k)}(s)L(s) = \sum_{n=1}^{\infty} \frac{d_{(k)}(n;f)}{n^s} \tag{1.3}$$

for $\Re s > \max(1, \sigma_f)$, where

$$d_{(k)}(n;f) = \sum_{d|n} f(d)\left(\log\frac{n}{d}\right)^k. \tag{1.4}$$

The aim of this paper is to study the summatory function

$$\sum_{n\le x} d_{(k)}(n;f), \tag{1.5}$$

and derive the Chowla–Walum type formula for the error term of (1.5). In particular, we treat two examples; one is $L(s) = (-1)^l\zeta^{(l)}(s-a)$ $(-1 < a < 0)$, and the other is $L(s) = (-1)^l L^{(l)}(s,\chi)$ (χ is the Dirichlet character mod 4).

1.1. The case $L(s) = (-1)^l\zeta^{(l)}(s-a)$. For a real number a $(-1 < a < 0)$, let $\sigma_a(n)$ denote the arithmetical function defined by

$$\sigma_a(n) = \sum_{d|n} d^a.$$

This is the coefficient of the Dirichlet series $\zeta(s)\zeta(s-a)$ for $\Re s > 1$. A generalized divisor problem is to study the behaviour of the error term $\Delta_a(x)$ defined by

$$\Delta_a(x) = \sum_{n \le x} \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a} x^{1+a}.$$

The Chowla–Walum type formula of $\Delta_a(x)$ is of the form

$$\Delta_a(x) = -\sum_{n \le \sqrt{x}} n^a \psi\left(\frac{x}{n}\right) - x^a \sum_{n \le \sqrt{x}} n^{-a} \psi\left(\frac{x}{n}\right) + O(1) \tag{1.6}$$

(cf. e.g. [6]). There are many researches on $\Delta_a(x)$, for example, the upper bound estimates, Voronoï-type representations and mean value formulas for $\Delta_a(x)$. We note that the studies of the function $\Delta_a(x)$ for $-1 < a < 0$ are deeply connected with the behaviour of the Riemann zeta function $\zeta(s)$ for $1/2 < \sigma < 1$. For the details of these topics, see [3], [4], [5], [6], [16].

Now, we shall consider a divisor problem for $(-1)^{k+l}\zeta^{(k)}(s)\zeta^{(l)}(s-a)$ with $-1 < a < 0$ (the case of $a = 0$ was already studied in [7]). In this case, $f(n) = n^a (\log n)^l$ in (1.4), and we put

$$\sigma_{(k,l,a)}(n) := d_{(k)}(n; f) = \sum_{d \mid n} d^a (\log d)^l \left(\log \frac{n}{d}\right)^k. \tag{1.7}$$

We consider the error term of $\sum_{n \le x} \sigma_{(k,l,a)}(n)$ and obtain the following theorem.

THEOREM 1.1. *Let $\sigma_{(k,l,a)}(n)$ be the arithmetical function defined by (1.7). Then we have*

$$\sum_{n \le x} \sigma_{(k,l,a)}(n) = x P_k(\log x) + x^{1+a} Q_l(\log x) + \Delta_{(k,l,a)}(x),$$

where $P_k(x)$ and $Q_l(x)$ are certain polynomials in x of degree k and l respectively, whose coefficients depend on k, l and a, and $\Delta_{(k,l,a)}(x)$ is the error term defined by

$$\Delta_{(k,l,a)}(x) = R_{(k,l)}(x; a) + x^a R_{(l,k)}(x; -a) + O\big((\log x)^{k+l}\big)$$

with

$$R_{(p,q)}(x; r) = -\sum_{j=0}^{p} \binom{p}{j} (-1)^j (\log x)^{p-j} \sum_{n \le \sqrt{x}} n^r \psi\left(\frac{x}{n}\right) (\log n)^{q+j}.$$

We can see that the formula in Theorem 1.1 implies the previous formula (1.6).

As a direct application of Theorem 1.1, using the theory of exponent pairs, we obtain the following corollary, which is the non-trivial estimate of $\Delta_{(k,l,a)}(x)$. For the theory of exponent pairs, refer to [10], [13] and [14].

Corollary 1.2. *Let (κ,λ) be any exponent pair. Then we have*

$$\Delta_{(k,l,a)}(x) \ll \begin{cases} x^{\frac{a}{2}+\frac{\lambda+\kappa}{2(\kappa+1)}}(\log x)^{k+l+1} & \text{for } a+\frac{\lambda-\kappa}{\kappa+1}>0,\\ x^{\frac{(1+a)\kappa}{1-\lambda+2\kappa}}(\log x)^{k+l+2} & \text{for } a+\frac{\lambda-\kappa}{\kappa+1}\le 0.\end{cases}$$

Specific exponent pairs give several upper bound estimates of $\Delta_{(k,l,a)}(x)$ under some condition on a. For example, if we take $(\kappa,\lambda)=(11/82,57/82)$, we have

$$\Delta_{(k,l,-1/2)}(x)=O\big(x^{11/94}(\log x)^{k+l+2}\big)$$

$(11/94=0.11702\ldots)$. We shall discuss this topic in Section 5.

1.2. The case $L(s)=(-1)^lL^{(l)}(s,\chi)$. Let $\chi(n)$ be the primitive Dirichlet character modulo 4, and $L(s,\chi)$ be the Dirichlet L-function associated to χ defined by

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s} \quad (\Re s>1).$$

The Gauss circle problem is to study the error term related to the Dirichlet series $4\zeta(s)L(s,\chi)$. Indeed, let $P(x)$ denote the error term in the circle problem defined by

$$P(x)=\sum_{1\le n\le x} r(n)-\pi x,$$

where $r(n)$ denotes the number of integer solutions of Diophantine equation $x^2+y^2=n$. The main object of the Gauss circle problem is to find the best possible estimation of $P(x)$. There are many results concerning several estimations of $P(x)$ and related topics on the circle problem. See e.g. [14] and [13] in details.

The Chowla–Walum type formula for $P(x)$ is of the form

$$P(x)=-4\sum_{n\le\sqrt{x}}\chi(n)\psi\Big(\frac{x}{n}\Big)-4\sum_{n\le\sqrt{x}}\psi\Big(\frac{x-n}{4n}\Big)+4\sum_{n\le\sqrt{x}}\psi\Big(\frac{x-3n}{4n}\Big)+O(1)$$

(cf. [10, Theorem 4.8]).

Now we treat the arithmetical function $r_{(k,l)}(n)$ defined by

$$r_{(k,l)}(n)=\sum_{d|n}\chi(d)(\log d)^l\Big(\log\frac{n}{d}\Big)^k, \tag{1.8}$$

which is the coefficient of the Dirichlet series $(-1)^{k+l}\zeta^{(k)}(s)L^{(l)}(s,\chi)$. In [8], we studied several arithmetical properties of the error term of the sum $\sum_{n\le x} r_{(1,1)}(n)$. Actually we derived the truncated Voronoï formula, the mean square formula and the non-trivial estimate of the error term.

Here, we shall derive the Chowla–Walum type formula of the error term related to the summatory function $\sum_{n\le x} r_{(k,l)}(n)$ for general k and l. We obtain the following theorem.

THEOREM 1.3. *Let $r_{(k,l)}(n)$ be the arithmetical function defined by (1.8). Then we have*

$$\sum_{n\le x} r_{(k,l)}(n) = -xQ_k(\log x;k,l) + P_{(k,l)}(x) + O\big((\log x)^{k+l}\big),$$

where $Q_d(x;k,l)$ is the polynomial in x of degree d whose coefficients depend on k and l,[1] *and $P_{(k,l)}(x)$ is the error term defined by*

$$P_{(k,l)}(x) = -\sum_{j=0}^{k}\binom{k}{j}(-1)^j(\log x)^{k-j}\sum_{n\le\sqrt{x}}\chi(n)\psi\left(\frac{x}{n}\right)(\log n)^{l+j}$$
$$+ R^{(1)}_{(k,l)}(x;\chi) + R^{(3)}_{(k,l)}(x;\chi) + O\big((\log x)^{k+l}\big)$$

with

$$R^{(\alpha)}_{(k,l)}(x;\chi) = \chi(-\alpha)\sum_{j=0}^{l}\binom{l}{j}(-1)^j(\log x)^{l-j}\sum_{n\le\sqrt{x}}\psi\left(\frac{y-\alpha n}{4n}\right)(\log n)^{k+j}$$

for $\alpha = 1$ and 3.

As an application of Theorem 1.3, we obtain the following corollary.

COROLLARY 1.4. *Let (κ,λ) be any exponent pair. Under the notations of Theorem 1.3, we have*

$$P_{(k,l)}(x) \ll \begin{cases} x^{1/3}(\log x)^{k+l} & \text{if } \kappa=\lambda=1/2,\\ x^{\frac{\kappa+\lambda}{2(\kappa+1)}}(\log x)^{k+l} + |P(x)|(\log x)^{k+l} & \text{if } \kappa\ne\lambda. \end{cases}$$

In particular, the exponent pair $(\kappa,\lambda) = (97/251, 132/251)$ gives the estimate

$$P_{(k,l)}(x) = O\big(x^{\frac{229}{696}}(\log x)^{k+l}\big).$$

Note that $229/696 = 0.329022\ldots$. In the case $k=l=0$, the best estimate (Huxley [12]) at present is

$$P_{(0,0)}(x) = O\big(x^{\frac{131}{416}}(\log x)^{\frac{18627}{8320}}\big)$$

$(131/416 = 0.314903\ldots)$. In this case, our Corollary 1.4 is weaker than the above estimate.

---

[1] Note that the leading term of $-xQ_k(\log x;k,l)$ is positive, since $c_f(0) < 0$ (cf. e.g. [9]).

2. Preliminaries

As a preparation for the purpose, we consider the summation formulas involving the log-function. We present these formulas as the following lemmas.

Lemma 2.1 ([7, Lemma 1]). *For a non-negative integer q, we have*

$$\sum_{n\le y}(\log n)^q = y\sum_{j=0}^{q} a_q(j)(\log y)^j - \psi(y)(\log y)^q + q\int_1^y \frac{\psi(t)(\log t)^{q-1}}{t}\,dt + c_q,$$

where $a_q(j) = (-1)^{q+j}q!/j!$ and c_q is given by

$$c_q = \begin{cases} -1/2 & \text{if } q=0,\\ (-1)^{q+1}q! & \text{if } q\ge 1.\end{cases}$$

Lemma 2.2. *For a non-negative integer q and a real number a with $-1 < a < 0$, we have*

$$\sum_{n\le y} n^a(\log n)^q = y^{1+a}\sum_{j=0}^{q}\frac{a_q(j)}{(1+a)^{q-j+1}}(\log y)^j - \psi(y)y^a(\log y)^q + \int_1^y t^{a-1}(a\log t + q)\psi(t)(\log t)^{q-1}\,dt + \frac{\tilde{c}_q}{(1+a)^{q+1}}$$

with

$$\tilde{c}_q = \begin{cases} (1-a)c_0 & \text{if } q=0,\\ c_q & \text{if } q\ge 1,\end{cases}$$

where $a_q(j)$ and c_q are the constants defined in Lemma 2.1.

Proof. This lemma can be proved by the Euler–Maclaurin summation formula and an integral formula

$$\int_1^y t^a(\log t)^q\,dt = y^{1+a}\sum_{j=0}^{q}\frac{a_q(j)}{(1+a)^{q-j+1}}(\log y)^j + \frac{(-1)^{q+1}q!}{(1+a)^{q+1}}. \tag{2.1}$$

□

Note that the formula (2.1) is valid for the cases $a=0$ and $q=0$.

Now we shall transform $\sum_{n\le x} d_{(k)}(n;f)$ by the hyperbola method (1.2). By the definition of $d_{(k)}(n;f)$ and (1.2) with $c=1/2$, we can see that

$$\sum_{n\le x} d_{(k)}(n;f) = \sum_{n\le\sqrt{x}} f(n)\sum_{m\le x/n}(\log m)^k + \sum_{n\le\sqrt{x}}(\log n)^k\sum_{m\le x/n} f(m)$$

$$- \Bigg(\sum_{n \le \sqrt{x}} f(n) \Bigg) \Bigg(\sum_{n \le \sqrt{x}} (\log n)^k \Bigg)$$
$$= S_1 + \sum_{n \le \sqrt{x}} (\log n)^k \sum_{m \le x/n} f(m) - S_2,$$

say.

In S_1, we have by Lemma 2.1 that

$$\begin{aligned} S_1 \; = \; & \sum_{n \le \sqrt{x}} f(n) \Bigg\{ \frac{x}{n} \sum_{j=0}^{k} a_k(j) \Big(\log \frac{x}{n} \Big)^j - \psi\Big(\frac{x}{n}\Big) \Big(\log \frac{x}{n} \Big)^k \\ & + k \int_1^{x/n} \frac{\psi(t)(\log t)^{k-1}}{t} \, dt + c_k \Bigg\} \\ =: \; & S_{11} + S_{12} + S_{13} + c_k \sum_{n \le \sqrt{x}} f(n). \end{aligned}$$

It is easy to see from the binomial expansion and partial summation that

$$\begin{aligned} S_{11} &= x \sum_{j=0}^{k} a_k(j) \sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu (\log x)^{j-\nu} \sum_{n \le \sqrt{x}} \frac{1}{n} f(n) (\log n)^\nu \\ &= \Bigg(\sqrt{x} \sum_{n \le \sqrt{x}} f(n) \Bigg) \sum_{j=0}^{k} \frac{a_k(j)}{2^j} (\log x)^j \\ &\quad - x \sum_{j=0}^{k} a_k(j) \sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu (\log x)^{j-\nu} \int_1^{\sqrt{x}} \Big\{ \frac{1}{t} (\log t)^\nu \Big\}' \sum_{n \le t} f(n) \, dt, \end{aligned}$$

where we have used the identity

$$\sum_{n=0}^{N} \binom{N}{n} \frac{(-1)^n}{2^n} = \frac{1}{2^N}. \tag{2.2}$$

Similarly, we have

$$S_{12} = - \sum_{j=0}^{k} \binom{k}{j} (-1)^j (\log x)^{k-j} \sum_{n \le \sqrt{x}} f(n) \psi\Big(\frac{x}{n}\Big) (\log n)^j.$$

In S_{13}, interchanging summation and integration we have

$$\begin{aligned} S_{13} &= k \Bigg(\sum_{n \le \sqrt{x}} f(n) \Bigg) \int_1^{\sqrt{x}} \frac{\psi(t)(\log t)^{k-1}}{t} \, dt \\ &\quad + k \int_{\sqrt{x}}^{x} \frac{\psi(t)(\log t)^{k-1}}{t} \sum_{n \le x/t} f(n) \, dt. \end{aligned}$$

Furthermore, in S_2, we again apply the formula in Lemma 2.1 to obtain

$$
\begin{aligned}
S_2 = &\Bigl(\sqrt{x}\sum_{n\le\sqrt{x}} f(n)\Bigr)\sum_{j=0}^{k}\frac{a_k(j)}{2^j}(\log x)^j \\
&- 2^{-k}\psi(\sqrt{x})(\log x)^k\sum_{n\le\sqrt{x}} f(n) \\
&+ k\Bigl(\sum_{n\le\sqrt{x}} f(n)\Bigr)\int_1^{\sqrt{x}}\frac{\psi(t)(\log t)^{k-1}}{t}\,dt + c_k\sum_{n\le\sqrt{x}} f(n).
\end{aligned}
$$

Therefore, we obtain the following lemma.

Lemma 2.3. *We have*

$$
\begin{aligned}
\sum_{n\le x} & d_{(k)}(n;f) \\
&= -x\sum_{j=0}^{k}a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu} \\
&\quad\times\int_1^{\sqrt{x}}\Bigl\{\frac{1}{t}(\log t)^\nu\Bigr\}'\sum_{n\le t} f(n)\,dt \\
&\quad-\sum_{j=0}^{k}\binom{k}{j}(-1)^j(\log x)^{k-j}\sum_{n\le\sqrt{x}} f(n)\psi\Bigl(\frac{x}{n}\Bigr)(\log n)^j \\
&\quad+k\int_{\sqrt{x}}^{x}\frac{\psi(t)(\log t)^{k-1}}{t}\sum_{n\le x/t} f(n)\,dt+\sum_{n\le\sqrt{x}}(\log n)^k\sum_{m\le x/n} f(m) \\
&\quad+2^{-k}\psi(\sqrt{x})(\log x)^k\sum_{n\le\sqrt{x}} f(n).
\end{aligned}
\tag{2.3}
$$

From now on, we assume some conditions of $\sum_{n\le x} f(n)$. Actually, we put

$$
\sum_{n\le x} f(n) = g(x) + E(x), \tag{2.4}
$$

where $g(x)$ is the “main term” and $E(x)$ is the “error term”. We assume that the function $g(x)$ is continuously differentiable and $E(x) = O(x^{\theta_1}(\log x)^{\theta_2})$, where θ_1 is a constant with $-1 < \theta_1 \le 0$ and θ_2 is a non-negative integer. Further we assume that the mean value of $E(x)$ is of the form

$$
\int_1^x E(t)\,dt = A_f x + B_f + O\bigl(x^{\theta_3}(\log x)^{\theta_4}\bigr)
$$

with some constants A_f, B_f, θ_3 and a non-negative integer θ_4. Note that $A_f = 0$ if $\theta_1 < 0$, and B_f is included in the O-term if $\theta_3 \ge 0$.

Now we shall transform the formula (2.3) under the assumption (2.4). Indeed, we have

$$
\begin{aligned}
(2.5)\qquad & \sum_{n\le x} d_{(k)}(n;f)\\
&= -x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^{\nu}(\log x)^{j-\nu}\int_1^{\sqrt{x}}\left\{\frac{1}{t}(\log t)^{\nu}\right\}' g(t)\,dt\\
&\quad -\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}(\log x)^{k-j}\sum_{n\le\sqrt{x}} f(n)\psi\left(\frac{x}{n}\right)(\log n)^{j}\\
&\quad +k\int_{\sqrt{x}}^{x}\frac{\psi(t)(\log t)^{k-1}}{t} g\left(\frac{x}{t}\right)dt+\sum_{n\le\sqrt{x}}(\log n)^{k} g\left(\frac{x}{n}\right)\\
&\quad +2^{-k}\psi(\sqrt{x})(\log x)^{k} g(\sqrt{x})+T(x)
\end{aligned}
$$

with

$$
\begin{aligned}
T(x) &= -x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^{\nu}(\log x)^{j-\nu}\\
&\qquad\times\int_1^{\sqrt{x}}\left\{\frac{1}{t}(\log t)^{\nu}\right\}' E(t)\,dt\\
&\qquad +k\int_{\sqrt{x}}^{x}\frac{\psi(t)(\log t)^{k-1}}{t} E\left(\frac{x}{t}\right)dt+\sum_{n\le\sqrt{x}}(\log n)^{k} E\left(\frac{x}{n}\right)\\
&\qquad +2^{-k}\psi(\sqrt{x})(\log x)^{k} E(\sqrt{x})\\
&= T_1+T_2+\sum_{n\le\sqrt{x}}(\log n)^{k} E\left(\frac{x}{n}\right)+T_3,
\end{aligned}
$$

say. By the assumption $E(x)=O(x^{\theta_1}(\log x)^{\theta_2})$ with the constants θ_j defined in (2.4), it is trivially seen that

$$
\begin{aligned}
T_2 &\ll x^{\theta_1}(\log x)^{\theta_2+k-1}\int_{\sqrt{x}}^{x} t^{-1-\theta_1}\,dt\\
&\ll (\log x)^{\theta_2+k-1}\times\begin{cases}\log x & \text{if } \theta_1=0,\\ x^{\theta_1/2} & \text{if } -1<\theta_1<0\end{cases}
\end{aligned}
$$

and $T_3=O(x^{\theta_1/2}(\log x)^{\theta_2+k})$. Hence, we have $T_2\ll T_3$.

On T_1, we see that $\int_{\sqrt{x}}^{\infty}\{\frac{1}{t}(\log t)^{\nu}\}' E(t)\,dt=O(x^{-(1-\theta_1)/2}(\log x)^{\theta_2+\nu})$ by the assumption of $E(x)$, and therefore $\int_1^{\infty}\{\frac{1}{t}(\log t)^{\nu}\}' E(t)\,dt$ is convergent. More precisely, by the assumption $\int_1^{x} E(t)\,dt=A_f x+B_f+O(x^{\theta_3}(\log x)^{\theta_4})$

and integration by parts, we can see that

$$\begin{aligned}&\int_{\sqrt{x}}^{\infty}\left\{\frac{1}{t}(\log t)^{\nu}\right\}' E(t)\,dt\\&\quad=-\left\{\frac{1}{t}(\log t)^{\nu}\right\}'\Big|_{t=\sqrt{x}}\int_1^{\sqrt{x}}E(t)\,dt-\int_{\sqrt{x}}^{\infty}\left\{\frac{1}{t}(\log t)^{\nu}\right\}''\int_1^t E(u)\,du\,dt\\&\quad=-A_f x^{-1/2}(\log\sqrt{x})^{\nu}+O\big(x^{-1+\theta_3/2}(\log x)^{\theta_4+\nu}\big).\end{aligned}$$

Hence by putting $c_f(\nu)=\int_1^{\infty}\{\frac{1}{t}(\log t)^{\nu}\}'E(t)\,dt$ we have

$$\begin{aligned}T_1=&-x\sum_{j=0}^{k}a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^{\nu}(\log x)^{j-\nu}c_f(\nu)\\&-A_f x^{1/2}\sum_{j=0}^{k}\frac{a_k(j)}{2^j}(\log x)^j+O\big(x^{\theta_3/2}(\log x)^{\theta_4+k}\big)\end{aligned}$$

by (2.2).

Combining all the above results, we obtain

$$\begin{aligned}T(x)=&-x\sum_{j=0}^{k}a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^{\nu}(\log x)^{j-\nu}c_f(\nu)\\&-A_f x^{1/2}\sum_{j=0}^{k}\frac{a_k(j)}{2^j}(\log x)^j+\sum_{n\le\sqrt{x}}(\log n)^k E\left(\frac{x}{n}\right)\\&+O\big(x^{\theta_1/2}(\log x)^{\theta_2+k}\big)+O\big(x^{\theta_3/2}(\log x)^{\theta_4+k}\big).\end{aligned}\tag{2.6}$$

As for the second preliminary, we transform the fourth term on the right-hand side in (2.5). We put this part as U. By partial summation, we have

$$\begin{aligned}U&=g(\sqrt{x})\sum_{n\le\sqrt{x}}(\log n)^k+x\int_1^{\sqrt{x}}t^{-2}g'\left(\frac{x}{t}\right)\sum_{n\le t}(\log n)^k\,dt\\&=U_1+U_2,\end{aligned}$$

say. As for the function U_1, we have by Lemma 2.1 that

$$\begin{aligned}U_1=g(\sqrt{x})\Bigg\{&\sqrt{x}\sum_{j=0}^{k}\frac{a_k(j)}{2^j}(\log x)^j-\psi(\sqrt{x})(\log\sqrt{x})^k\\&+k\int_1^{\sqrt{x}}\frac{\psi(t)(\log t)^{k-1}}{t}\,dt+c_k\Bigg\}\\=U_{11}&-g(\sqrt{x})\psi(\sqrt{x})(\log\sqrt{x})^k+U_{12}+c_k g(\sqrt{x}).\end{aligned}$$

Note that the second term on the right-hand side in the above is canceled by the fifth term on the right-hand side in (2.5).

On U_2, we have

$$U_2 = \int_{\sqrt{x}}^{x} g'(t)\left\{\frac{x}{t}\sum_{j=0}^{k} a_k(j)\left(\log\frac{x}{t}\right)^j - \psi\left(\frac{x}{t}\right)\left(\log\frac{x}{t}\right)^k + k\int_1^{x/t}\frac{\psi(u)(\log u)^{k-1}}{u}\,du + c_k\right\}dt \tag{2.7}$$

$$= U_{21} - \int_{\sqrt{x}}^{x} g'(t)\psi\left(\frac{x}{t}\right)\left(\log\frac{x}{t}\right)^k dt + U_{22} + c_k\{g(x) - g(\sqrt{x})\},$$

say. It is easy to see that

$$U_{22} = k\int_1^{\sqrt{x}}\frac{\psi(u)(\log u)^{k-1}}{u} g\left(\frac{x}{u}\right)du - U_{12}.$$

On U_{21}, we have

$$U_{21} = x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu}\int_{\sqrt{x}}^{x} g'(t)\frac{(\log t)^\nu}{t}\,dt$$

$$= -U_{11} - x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu} \times \int_{\sqrt{x}}^{x} g(t)\left\{\frac{1}{t}(\log t)^\nu\right\}' dt + a_k(0)g(x)$$

by (2.2) and the formula

$$\sum_{n=0}^{N}\binom{N}{n}(-1)^n = \begin{cases} 1 & \text{if } N = 0,\\ 0 & \text{if } N \geq 1.\end{cases}$$

Collecting these estimates we have

$$U = -x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu}\int_{\sqrt{x}}^{x} g(t)\left\{\frac{1}{t}(\log t)^\nu\right\}' dt$$
$$+ k\int_1^{\sqrt{x}}\frac{\psi(u)(\log u)^{k-1}}{u} g\left(\frac{x}{u}\right)du - \int_{\sqrt{x}}^{x} g'(t)\psi\left(\frac{x}{t}\right)\left(\log\frac{x}{t}\right)^k dt$$
$$- 2^{-k} g(\sqrt{x})\psi(\sqrt{x})(\log x)^k + \big(a_k(0) + c_k\big)g(x).$$

Therefore, as the preparation of the proofs, we obtain the following lemma.

Lemma 2.4. *Let $d_{(k)}(n;f)$ be the function defined by* (1.4). *Under the assumption* (2.4), *we have*

$$\sum_{n\le x} d_{(k)}(n;f) \tag{2.8}$$
$$= -x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^{\nu}(\log x)^{j-\nu}\int_1^x\left\{\frac{1}{t}(\log t)^{\nu}\right\}' g(t)\,dt$$
$$+ k\int_1^x \frac{\psi(t)(\log t)^{k-1}}{t}g\left(\frac{x}{t}\right)dt$$
$$- \int_{\sqrt{x}}^x g'(t)\psi\left(\frac{x}{t}\right)\left(\log\frac{x}{t}\right)^k dt$$
$$- \sum_{j=0}^{k}\binom{k}{j}(-1)^j(\log x)^{k-j}\sum_{n\le\sqrt{x}} f(n)\psi\left(\frac{x}{n}\right)(\log n)^j$$
$$+ \big(a_k(0)+c_k\big)g(x)+T(x)$$

with

$$T(x) = -x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^{\nu}(\log x)^{j-\nu}c_f(\nu) \tag{2.9}$$
$$- A_f x^{1/2}\sum_{j=0}^{k}\frac{a_k(j)}{2^j}(\log x)^j + \sum_{n\le\sqrt{x}}(\log n)^k E\left(\frac{x}{n}\right)$$
$$+ O\big(x^{\theta_1/2}(\log x)^{\theta_2+k}\big) + O\big(x^{\theta_3/2}(\log x)^{\theta_4+k}\big).$$

3. Proof of Theorem 1.1

Let $f(n)=n^a(\log n)^l$ and $\sigma_{(k,l,a)}(n)=d_{(k)}(n;f)$. By Lemma 2.2 we find that the main term $g(y)$ and the error term $E(y)$ in $\sum_{n\le y} f(n)$ are of the forms

$$g(y) = y^{1+a}\sum_{j=0}^{l}\frac{a_l(j)}{(1+a)^{l-j+1}}(\log y)^j \tag{3.1}$$

and

$$E(y) = -y^a\psi(y)(\log y)^l + \int_1^y t^{a-1}(a\log t + l)\psi(t)(\log t)^{l-1}\,dt$$
$$+ \frac{\tilde{c}_l}{(1+a)^{l+1}},$$

respectively. From the definition of $a_l(j)$, we find that

$$g'(y) = y^a(\log y)^l. \tag{3.2}$$

Since the integral on the right-hand side of $E(y)$ converges when $y\to\infty$ we have the following expressions:

$$\begin{aligned} E(y) &= A_f - y^a\psi(y)(\log y)^l - \int_y^\infty t^{a-1}(a\log t + l)\psi(t)(\log t)^{l-1}\,dt \\ &= A_f + O\big(y^a(\log y)^l\big), \end{aligned}$$

and

$$\int_1^y E(t)\,dt = A_f y + B_f + O\big(y^a(\log y)^l\big), \tag{3.3}$$

where

$$A_f = \frac{\tilde{c}_l}{(1+a)^{l+1}} + \int_1^\infty t^{a-1}(a\log t + l)\psi(t)(\log t)^{l-1}\,dt,$$

and

$$B_f = -A_f - \int_1^\infty t^a\psi(t)(\log t)^l\,dt - \int_1^\infty t^a(a\log t + l)\psi(t)(\log t)^{l-1}\,dt.$$

Hence, we can take $\theta_1=\theta_2=0$, $\theta_3=a$ and $\theta_4=l$. We remark that $\theta_3=a<0$ in this case (compare with the choice in Section 6).

Now we shall consider the formula (2.8).

The fourth term on the right-hand side in (2.8) coincides with $R_{(k,l)}(x;a)$.

By (3.1), (3.2) and integration by parts we find that

$$\begin{aligned} &\int_1^x \left\{\frac{1}{t}(\log t)^\nu\right\}' g(t)\,dt \\ &\quad= \left[\frac{(\log t)^\nu}{t}g(t)\right]_1^x - \int_1^x t^{a-1}(\log t)^{\nu+l}\,dt \\ &\quad= (\log x)^\nu x^{-1}g(x) - \delta_\nu - \int_1^\infty t^{a-1}(\log t)^{\nu+l}\,dt \\ &\qquad+ \int_x^\infty t^{a-1}(\log t)^{\nu+l}\,dt, \end{aligned} \tag{3.4}$$

where $\delta_\nu=0$ if $\nu\ge 1$ and $\delta_0=g(1)$. We should note that the integral in the second and third lines are convergent by the condition $a<0$. Hence, the first term on the right-hand side in (2.8) becomes

$$\begin{aligned} &= -g(x)\sum_{j=0}^k a_k(j)(\log x)^j \sum_{\nu=0}^j \binom{j}{\nu}(-1)^\nu \\ &\quad+ x\sum_{j=0}^k a_k(j)\sum_{\nu=0}^j\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu}\left(\delta_\nu + \int_1^\infty t^{a-1}(\log t)^{\nu+l}\,dt\right) \\ &\quad- x\sum_{j=0}^k a_k(j)\sum_{\nu=0}^j\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu}\int_x^\infty t^{a-1}(\log t)^{\nu+l}\,dt. \end{aligned} \tag{3.5}$$

The first line on the right-hand side of (3.5) equals to $-g(x)a_k(0)$ since the sum over ν vanishes for $j \geq 1$. We consider the third line of (3.5), which we denote by $J(x)$. By putting back the binomial expansion we have

$$J(x) = -x \sum_{j=0}^{k} a_k(j) \int_x^\infty t^{a-1} \left(\log \frac{x}{t}\right)^j (\log t)^l \, dt.$$

If we change the variable by $\frac{x}{t} = u$, we have

$$\begin{aligned} J(x) &= -x^{1+a} \sum_{j=0}^{k} a_k(j) \int_0^1 u^{-a-1} (\log u)^j \left(\log \frac{x}{u}\right)^l du \\ &= -x^{1+a} \sum_{j=0}^{k} a_k(j) \sum_{\nu=0}^{l} \binom{l}{\nu} (-1)^\nu (\log x)^{l-\nu} \\ &\quad \times \int_0^1 u^{-a-1} (\log u)^{j+\nu} \, du. \end{aligned} \tag{3.6}$$

Hence by (3.5) and (3.6), we can see that the first term on the right-hand side of (2.8) has the form

$$x \sum_{j=0}^{k} A_{1,j}(a,k,l)(\log x)^j + x^{1+a} \sum_{j=0}^{l} A_{2,j}(a,k,l)(\log x)^j.$$

Next, we treat the second term on the right-hand side in (2.8). Let $\psi_1(y) = \int_1^y \psi(t)\, dt$. Noting that $\psi_1(y) = O(1)$ uniformly in y, we have by integration by parts that

$$\begin{aligned} k \int_1^x \frac{\psi(t)(\log t)^{k-1}}{t} g\left(\frac{x}{t}\right) dt &= -k \int_1^x \psi_1(t) \left\{ \frac{(\log t)^{k-1}}{t} g\left(\frac{x}{t}\right) \right\}' dt \\ &\quad + O\big(x^{-1}(\log x)^{k-1}\big). \end{aligned}$$

By (3.1), we get

$$\begin{aligned} &\frac{(\log t)^{k-1}}{t} g\left(\frac{x}{t}\right) \\ &= x^{1+a} \sum_{j=0}^{l} \frac{a_l(j)}{(1+a)^{l-j+1}} \sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu (\log x)^{j-\nu} \frac{(\log t)^{\nu+k-1}}{t^{2+a}}, \end{aligned}$$

and thus

$$\begin{aligned} &k \int_1^x \frac{\psi(t)(\log t)^{k-1}}{t} g\left(\frac{x}{t}\right) dt \\ &= -kx^{1+a} \sum_{j=0}^{l} \frac{a_l(j)}{(1+a)^{l-j+1}} \sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu (\log x)^{j-\nu} \end{aligned}$$

$$\times \int_1^x \psi_1(t) \left\{ \frac{(\log t)^{\nu+k-1}}{t^{2+a}} \right\}' dt$$
$$+ O\big(x^{-1}(\log x)^{k-1}\big).$$

Since $-1 < a < 0$, we see that the integral in the above is convergent absolutely. We obtain

$$k \int_1^x \frac{\psi(t)(\log t)^{k-1}}{t} g\left(\frac{x}{t}\right) dt = x^{1+a} \sum_{j=0}^{l} A_{3,j}(a,k,l)(\log x)^j$$
$$+ O\big(x^{-1}(\log x)^{k+l-1}\big).$$

As for the third one, we have by noting the formula (3.2) that

$$\int_{\sqrt{x}}^x g'(t)\psi\left(\frac{x}{t}\right)\left(\log\frac{x}{t}\right)^k dt$$
$$= x \int_1^{\sqrt{x}} u^{-2}\psi(u) g'\left(\frac{x}{u}\right)(\log u)^k\, du$$
$$= x^{1+a} \int_1^{\sqrt{x}} u^{-2-a}\psi(u)\left(\log\frac{x}{u}\right)^l (\log u)^k\, du$$
$$= x^{1+a} \sum_{\nu=0}^{l} \binom{l}{\nu} (-1)^\nu (\log x)^{l-\nu}$$
$$\times \int_1^{\sqrt{x}} u^{-2-a}\psi(u)(\log u)^{\nu+k}\, du.$$

Since

$$\int_1^{\sqrt{x}} u^{-2-a}\psi(u)(\log u)^{\nu+k}\, du = \int_1^\infty u^{-2-a}\psi(u)(\log u)^{\nu+k}\, du$$
$$+ O\big(x^{-1-a/2}(\log x)^{\nu+k}\big),$$

we get

$$\int_{\sqrt{x}}^x g'(t)\psi\left(\frac{x}{t}\right)\left(\log\frac{x}{t}\right)^k dt$$
$$= x^{1+a}\sum_{j=0}^{l} A_{4,j}(a,k,l)(\log x)^j + O\big(x^{a/2}(\log x)^{k+l}\big).$$

Clearly $a_k(0)g(x)$ in the fifth term of (2.8) cancels with the first line of (3.5), hence it becomes $c_k g(x)$, which we write

$$c_k g(x) = x^{1+a}\sum_{j=0}^{l} A_{5,j}(a,k,l)(\log x)^j.$$

It remains to consider the formula of (2.9). By the definition of $E(x)$, we have

$$\begin{aligned}\sum_{n\le\sqrt{x}}(\log n)^k E\left(\frac{x}{n}\right) &= \sum_{n\le\sqrt{x}}(\log n)^k\biggl\{-\left(\frac{x}{n}\right)^a\psi\left(\frac{x}{n}\right)\left(\log\frac{x}{n}\right)^l \\ &\qquad + \int_{x/n}^{\infty} t^{a-1}(a\log t+l)\psi(t)(\log t)^{l-1}\,dt + A_f\biggr\} \\ &= V_1+V_2+V_3,\end{aligned}$$

say. For V_1, it is easy to see that

$$V_1 = -x^a\sum_{\nu=0}^{l}\binom{l}{\nu}(-1)^\nu(\log x)^{l-\nu}\sum_{n\le\sqrt{x}} n^{-a}\psi\left(\frac{x}{n}\right)(\log n)^{k+\nu},$$

which coincides with $x^a R_{(l,k)}(x;-a)$. Also it is easily seen from Lemma 2.1 that

$$V_3 = A_f\sqrt{x}\sum_{j=0}^{k} a_k(j)(\log\sqrt{x})^k + O\bigl((\log x)^k\bigr).$$

For V_2, changing the summation and integration, we have

$$\begin{aligned}V_2 &= \int_{\sqrt{x}}^{x} t^{a-1}(a\log t+l)\psi(t)(\log t)^{l-1}\sum_{x/t<n\le\sqrt{x}}(\log n)^k\,dt \\ &\qquad + \left(\sum_{n\le\sqrt{x}}(\log n)^k\right)\int_{x}^{\infty} t^{a-1}(a\log t+l)\psi(t)(\log t)^{l-1}\,dt \\ &= -\int_{\sqrt{x}}^{x} t^{a-1}(a\log t+l)\psi(t)(\log t)^{l-1}\sum_{n\le x/t}(\log n)^k\,dt \\ &\qquad + \left(\sum_{n\le\sqrt{x}}(\log n)^k\right)\int_{\sqrt{x}}^{\infty} t^{a-1}(a\log t+l)\psi(t)(\log t)^{l-1}\,dt \\ &= V_{21}+V_{22},\end{aligned}$$

say. It is easy to see that $V_{22}\ll x^{a/2}(\log x)^{k+l}$. For V_{21}, we apply Lemma 2.2 and get

$$\begin{aligned}V_{21} &= -x\sum_{j=0}^{k} a_k(j)\int_{\sqrt{x}}^{x} t^{a-2}(a\log t+l)\psi(t)(\log t)^{l-1}\left(\log\frac{x}{t}\right)^j dt \\ &\qquad + O\bigl(x^{a/2}(\log x)^{k+l}\bigr) \\ &= -x\sum_{j=0}^{k} a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu}\end{aligned}$$

$$\times \int_{\sqrt{x}}^{x} t^{a-2}(a\log t + l)\psi(t)(\log t)^{l-1+\nu}\,dt$$
$$+ O\big(x^{a/2}(\log x)^{k+l}\big)$$
$$\ll x^{a/2}(\log x)^{k+l}.$$

Collecting these estimates we obtain that

$$\sum_{n\le\sqrt{x}} (\log n)^k E\left(\frac{x}{n}\right) = x^a R_{(l,k)}(x;-a) + A_f\sqrt{x}\sum_{j=0}^{k} a_k(j)(\log\sqrt{x})^k + O\big((\log x)^k\big) + O\big(x^{a/2}(\log x)^{k+l}\big).$$

The second term on the right-hand side above is canceled by the second term on the right-hand side in (2.9). Clearly the first term of $T(x)$ has the form $x\sum_{j=0}^{k} A_{6,j}(a,k,l)(\log x)^j$.

Collecting all formulas, we obtain the assertion of Theorem 1.1.

Remark. We can see that the coefficient of $x(\log x)^k$, which is the greatest term for the main term of $\sum_{n\le x}\sigma_{(k,l,a)}(n)$, does not vanish. Actually, we can see this fact as follows: The explicit value of such coefficient is equal to the sum of the terms from $A_{1,k}(a,k,l)$ and $A_{6,k}(a,k,l)$ in the case $j=k$ and $\nu=0$. By the definitions of $A_{1,k}(a,k,l)$ and $A_{6,k}(1,k,l)$, we have that the corresponding term coming from $A_{1,k}(a,k,l)$ is

$$= \frac{(-1)^l l!}{(1+a)^{l+1}} + \int_1^\infty t^{a-1}(\log t)^l\,dt$$

and the term corresponding to $A_{6,k}(a,k,l)$ is

$$= -c_f(0) = (-1)^{l-1}\zeta^{(l)}(1-a) - \frac{(-1)^l l!}{(1+a)^l} - \int_1^\infty t^{a-1}(\log t)^l\,dt.$$

Then we have that the coefficient of $x(\log x)^k$ is $(-1)^{l-1}\zeta^{(l)}(1-a)$. We can trivially see that $\zeta^{(l)}(1-a)\neq 0$ for all non-negative integer l and the real number a with $-1<a<0$, hence we see that the coefficient of $x(\log x)^k$ is not equal to zero.

4. Proof of Corollary 1.2

In order to prove Corollary 1.2, we apply the formula of $\psi(x)$ in the following lemma.

Lemma 4.1 ([11, p. 245]). *Let h be any real number. We have*

$$\psi(x) = -\frac{1}{2\pi i}\sum_{1\le|h|\le H}\frac{e(hx)}{h} + O\big(E_H(x)\big) \tag{4.1}$$

with $e(y) = \exp(2\pi i y)$, *where* $E_H(x)$ *is the function estimated and represented as*

$$E_H(x) = \min\left(1, \frac{1}{H\|x\|}\right) = \sum_{h=-\infty}^{\infty} b(h)e(hx)$$

with

$$b(0) \ll \frac{\log H}{H} \quad \textit{and} \quad b(h) \ll \min\left(\frac{\log H}{H}, \frac{H}{h^2}\right) \quad (h \neq 0). \tag{4.2}$$

By using this lemma, we have the following lemma.

LEMMA 4.2. *Let* r *be a real number and let* $\mathfrak{g}_N(x)$ *be the function defined by*

$$\mathfrak{g}_N(x) = \sum_{N<n\leq 2N} n^r \psi\left(\frac{x}{n}\right) \tag{4.3}$$

with $1 \leq N \leq \sqrt{x}$. *Then we have*

$$\mathfrak{g}_N(x) \ll x^{\frac{\kappa}{\kappa+1}} N^{r+\frac{\lambda-\kappa}{\kappa+1}} \log x \tag{4.4}$$

for any exponent pair (κ, λ).

Proof. Substituting (4.1) in (4.3), we have

$$\mathfrak{g}_N(x) = -\frac{1}{2\pi i} \sum_{1\leq |h|\leq H} \frac{1}{h} \sum_{N<n\leq 2N} e\left(\frac{hx}{n}\right) + O\left(\sum_{N<n\leq 2N} n^r E_H\left(\frac{x}{n}\right)\right). \tag{4.5}$$

Let (κ, λ) be an exponent pair. By partial summation and applying this exponent pair, we get

$$\begin{aligned}\sum_{N<n\leq 2N} n^r e\left(\frac{hx}{n}\right) &\ll \max_{N<t\leq 2N} \left|\sum_{N<n\leq t} e\left(\frac{hx}{n}\right)\right| N^r \\ &\ll N^r \left\{\left(\frac{hx}{N^2}\right)^{\kappa} (t-N)^{\lambda} + \frac{N^2}{hx}\right\} \\ &\ll h^{\kappa} x^{\kappa} N^{a+\lambda-2\kappa}.\end{aligned}$$

Hence the first term on the right-hand side of (4.5) is evaluated as as

$$\ll H^{\kappa} x^{\kappa} N^{a+\lambda-2\kappa} \times \begin{cases} \log H & \kappa = 0, \\ 1 & \kappa > 0. \end{cases}$$

On the other hand by using (4.2), the second term on the right-hand side of (4.5) is bounded as

$$\sum_{N<n\leq 2N} n^r E_H\left(\frac{x}{n}\right) \ll \left(x^{\kappa} N^{a+\lambda-2\kappa} H^{\kappa} + \frac{N^{a+1}}{H}\right) \log H.$$

Hence, we get

$$\mathfrak{g}_N(x) \ll \left(x^{\kappa} N^{a+\lambda-2\kappa} H^{\kappa} + \frac{N^{a+1}}{H} \right) \log H \tag{4.6}$$

for $H \geq 1$. Note that $\mathfrak{g}_N(x) \ll N^{a+1}$ trivially, hence (4.6) holds for all $H > 0$. Taking $H = x^{-\kappa/(\kappa+1)} N^{(1-\lambda+2\kappa)/(\kappa+1)}$, we get (4.4). □

Proof of Corollary 1.2. Let r be a real number and let

$$G(x,r) = \sum_{n \leq \sqrt{x}} n^r \psi\left(\frac{x}{n}\right) (\log x)^A$$

for a non-negative integer A. In view of Theorem 1.1, we have to estimate the upper bounds of $G(x,a)$ and $x^a G(x,-a)$.

Let

$$g(t) = \sum_{n \leq t} n^r \psi\left(\frac{x}{n}\right).$$

By partial summation, we have

$$\begin{aligned} G(x,r) &= g(\sqrt{x})(\log \sqrt{x})^A - A \int_1^{\sqrt{x}} g(t)(\log t)^{A-1} \frac{1}{t}\, dt \\ &\ll \Big\{ \max_{t \leq \sqrt{x}} \big| g(t) \big| \Big\} (\log x)^A. \end{aligned}$$

Hence, it is enough to evaluate $|g(t)|$ for t in the range $1 \leq t \leq \sqrt{x}$. Let $N_j = t/2^j$. Then by the standard decomposition technique we have

$$g(t) = \sum_{j=1}^{j_0} \mathfrak{g}_{N_j}(x) + O(1),$$

where $j_0 = [\log t / \log 2]$.

For $G(x,a)$ we take $r = a$ for $-1 < a < 0$ in (4.4). We consider the three cases.

Case 1. Suppose that $a + \frac{\lambda-\kappa}{\kappa+1} > 0$. In this case, we have

$$\begin{aligned} g(t) &\ll x^{\frac{\kappa}{\kappa+1}} \sum_j \left(\frac{t}{2^j} \right)^{a + \frac{\lambda-\kappa}{\kappa+1}} \log x \\ &\ll x^{\frac{\kappa}{\kappa+1}} t^{a + \frac{\lambda-\kappa}{\kappa+1}} \log x. \end{aligned}$$

Case 2. Suppose that $a + \frac{\lambda-\kappa}{\kappa+1} < 0$. Let t be a real number such that $0 < t < 1$. By the theory of exponent pairs,

$$(\kappa_t, \lambda_t) = t(0,1) + (1-t)(\kappa,\lambda) = \big((1-t)\kappa, t/2 + (1-t)\lambda \big)$$

is also an exponent pair. If we take

$$t_0 = \frac{-a(\kappa+1) - (\lambda-\kappa)}{1 - a\kappa - (\lambda-\kappa)},$$

then we have $a(\kappa_{t_0}+1) + \lambda_{t_0} - \kappa_{t_0} = 0$. We also note that $0 < t_0 < 1$ by the assumption $a + \frac{\lambda-\kappa}{\kappa+1} < 0$. Applying this exponent pair in (4.4), we get

$$\mathfrak{g}_N(x) \ll x^{\frac{\kappa_0}{\kappa_0+1}} \log x = x^{\frac{(1+a)\kappa}{1-\lambda+2\kappa}} \log x,$$

and hence

$$g(t) \ll x^{\frac{(1+a)\kappa}{1-\lambda+2\kappa}} (\log x)^2.$$

Case 3. Suppose that $a + \frac{\lambda-\kappa}{\kappa+1} = 0$. As in the case 2, we get

$$g(t) \ll x^{\frac{\kappa}{\kappa+1}} (\log x)^2.$$

From these estimates, we have

$$G(x,a) \ll \begin{cases} x^{\frac{a}{2}+\frac{\lambda+\kappa}{2(\kappa+1)}} (\log x)^{A+1}, & a + \frac{\lambda-\kappa}{\kappa+1} > 0, \\ x^{\frac{(1+a)\kappa}{1-\lambda+2\kappa}} (\log x)^{A+2}, & a + \frac{\lambda-\kappa}{\kappa+1} \le 0. \end{cases} \tag{4.7}$$

For $x^a G(x,-a)$ we take $r = -a > 0$ in (4.4). Since $-a + \frac{\lambda-\kappa}{\kappa+1} > 1$ we have

$$x^a G(x,-a) \ll x^{\frac{a}{2}+\frac{\lambda+\kappa}{2(\kappa+1)}} (\log x)^{A+1}. \tag{4.8}$$

Therefore, by the definition of $\Delta_{(k,l,a)}(x)$ in Theorem 1.1, we obtain

$$\Delta_{(k,l,a)}(x) \ll \begin{cases} x^{\frac{a}{2}+\frac{\lambda+\kappa}{2(\kappa+1)}} (\log x)^{k+l+1}, & a + \frac{\lambda-\kappa}{\kappa+1} > 0, \\ x^{\frac{(1+a)\kappa}{1-\lambda+2\kappa}} (\log x)^{k+l+2} + x^{\frac{a}{2}+\frac{\lambda+\kappa}{2(\kappa+1)}} (\log x)^{k+l+1}, & a + \frac{\lambda-\kappa}{\kappa+1} \le 0. \end{cases}$$

Remarking that $\frac{a}{2} + \frac{\lambda+\kappa}{2(\kappa+1)} \le \frac{(1+a)\kappa}{1-\lambda+2\kappa}$ for $a + \frac{\lambda-\kappa}{\kappa+1} \le 0$, we obtain the assertion of Corollary 1.2. □

5. Some remarks on Corollary 1.2

We first recall that the constant a satisfies the condition $-1 < a < 0$.

If we take the trivial exponent pair $(\kappa, \lambda) = (0,1)$, we have $a + \frac{\lambda-\kappa}{\kappa+1} > 0$ and hence

$$\Delta_{(k,l,a)}(x) = O\big(x^{\frac{a+1}{2}} (\log x)^{k+l+1}\big)$$

for $-1 < a < 0$. Similarly, if we take $(\kappa, \lambda) = (1/2, 1/2)$, we have $a + \frac{\lambda-\kappa}{\kappa+1} \le 0$ and hence

$$\Delta_{(k,l,a)}(x) = O\big(x^{\frac{a+1}{3}} (\log x)^{k+l+1}\big)$$

for $-1 < a < 0$.

Here are some other examples. Each exponent pair is taken from [14].

(i) Put $(\kappa, \lambda) = (1/6, 4/6)$, then

$$\Delta_{(k,l,a)}(x) \ll \begin{cases} x^{\frac{a}{2}+\frac{5}{14}} (\log x)^{k+l+1} & \text{for } -\frac{3}{7} < a < 0, \\ x^{\frac{(a+1)}{4}} (\log x)^{k+l+2} & \text{for } -1 < a \le -\frac{3}{7}. \end{cases}$$

(ii) Put $(\kappa,\lambda)=(2/18,13/18)$, then

$$\Delta_{(k,l,a)}(x)\ll\begin{cases} x^{\frac{a}{2}+\frac{3}{8}}(\log x)^{k+l+1} & \text{for } -\frac{11}{20}<a<0,\\ x^{\frac{2}{9}(a+1)}(\log x)^{k+l+2} & \text{for } -1<a\le-\frac{11}{20}.\end{cases}$$

(iii) Put $(\kappa,\lambda)=(11/82,57/82)$, then

$$\Delta_{(k,l,a)}(x)\ll\begin{cases} x^{\frac{a}{2}+\frac{34}{93}}(\log x)^{k+l+1} & \text{for } -\frac{46}{93}<a<0,\\ x^{\frac{11(a+1)}{47}}(\log x)^{k+l+2} & \text{for } -1<a\le-\frac{46}{93}.\end{cases}$$

Thus, the third one gives the special estimate

$$\Delta_{(k,l,-1/2)}(x)=O\big(x^{\frac{11}{94}}(\log x)^{k+l+2}\big).$$

6. Proof of Theorem 1.3

We put $f(n)=\chi(n)(\log n)^l$, then $d_{(k)}(n;f)=r_{(k,l)}(n)$ in this setting. By the formula

$$\sum_{n\le y}\chi(n)=\frac{1}{2}-\psi\left(\frac{y-1}{4}\right)+\psi\left(\frac{y-3}{4}\right) \tag{6.1}$$

(cf. [6, Lemma 4.7]), we have by partial summation that

$$\sum_{n\le y}f(n)=\left\{-\psi\left(\frac{y-1}{4}\right)+\psi\left(\frac{y-3}{4}\right)\right\}(\log y)^l+A_f+O\big(y^{-1}(\log y)^{l-1}\big),$$

where A_f is a constant. Hence, we have

$$\int_1^y\sum_{n\le t}f(n)\,dt=A_fy+O\big((\log y)^l\big).$$

Thus if we put $g(x)=0$, $E(x)=\sum_{n\le x}f(n)$, $\theta_1=\theta_3=0$ and $\theta_2=\theta_4=l$, we can see that the function $f(n)$ of this setting satisfies all assumptions of the error term $E(x)$ in Section 2.

On this setting, the formula (2.5) is reduced to

$$\begin{aligned}\sum_{n\le x}r_{(k,l)}(n)=&-\sum_{j=0}^{k}\binom{k}{j}(-1)^j(\log x)^{k-j}\sum_{n\le\sqrt{x}}\chi(n)\psi\left(\frac{x}{n}\right)(\log n)^{l+j}\\&-x\sum_{j=0}^{k}a_k(j)\sum_{\nu=0}^{j}\binom{j}{\nu}(-1)^\nu(\log x)^{j-\nu}c_f(\nu)\\&-A_fx^{1/2}\sum_{j=0}^{k}\frac{a_k(j)}{2^j}(\log x)^j+\sum_{n\le\sqrt{x}}(\log n)^kE\left(\frac{x}{n}\right)\\&+O\big((\log x)^{k+l}\big).\end{aligned}$$

Here the second term on the right-hand side contributes the main term $-xQ_k(\log x; k, l)$ of the theorem; the first one above is the first term of $P_{(k,l)}(x)$.

On the fourth term in the right-hand side of the above formula, applying partial summation and substituting the formula (6.1) into it, we have

$$\begin{aligned}
&\sum_{n\le\sqrt{x}} (\log n)^k E\left(\frac{x}{n}\right) \\
&\quad= \sum_{n\le\sqrt{x}} (\log n)^k \sum_{m\le x/n} \chi(m)(\log m)^l \\
&\quad= \sum_{n\le\sqrt{x}} (\log n)^k \left\{ \left(-\psi\left(\frac{x/n-1}{4}\right) + \psi\left(\frac{x/n-3}{4}\right)\right)\left(\log\frac{x}{n}\right)^l + A_f \right\} \\
&\qquad + O\big((\log x)^{k+l-1}\big) \\
&\quad= \sum_{j=0}^{l} \binom{l}{j} (-1)^j (\log x)^{l-j} \sum_{n\le\sqrt{x}} \left\{-\psi\left(\frac{x-n}{4n}\right) + \psi\left(\frac{x-3n}{4n}\right)\right\}(\log n)^{k+j} \\
&\qquad + A_f\left(x^{1/2}\sum_{j=0}^{k} \frac{a_k(j)}{2^j}(\log x)^j + O\big((\log x)^k\big)\right) + O\big((\log x)^{k+l-1}\big).
\end{aligned}$$

Hence, the term containing A_f cancels and the remaining terms containing $\psi(x)$ contribute $R^{(1)}_{(k,l)}(x,\chi)$ and $R^{(3)}_{(k,l)}(x,\chi)$ of $P_{(k,l)}(x)$ of the theorem. The proof of Theorem 1.3 is complete.

The assertions of Corollary 1.4 can be proved by using Theorem 1.3 and the method used in [10] and [7, Section 5]. We omit the details of the proof of this corollary.

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