Illinois Journal of Mathematics
Volume 60, Numbers 3–4, Fall and Winter 2016, Pages 833–844
S 0019-2082

KOSZUL FACTORIZATION AND THE COHEN–GABBER THEOREM

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Abstract. We present a sharpened version of the Cohen–Gabber theorem for equicharacteristic, complete local domains $(A, \mathfrak{m}, k)$ with algebraically closed residue field and dimension $d > 0$. Namely, we show that for any prime number p, $\operatorname{Spec} A$ admits a dominant, finite map to $\operatorname{Spec} k[[X_1, \ldots, X_d]]$ with generic degree relatively prime to p. Our result follows from Gabber's original theorem, elementary Hilbert–Samuel multiplicity theory, and a "factorization" of the map induced on the Grothendieck group $\mathbf{G}_0(A)$ by the Koszul complex.

0. Introduction

When $(A, \mathfrak{m}, k)$ is a complete, equicharacteristic, d-dimensional local domain, the familiar Cohen structure theorem says that there exists a power series subring $R = k[[X_1, \ldots, X_d]]$ over which A is finite. When k has characteristic zero, the map on fraction fields is automatically separable, but arranging for this condition in characteristic $p > 0$ requires a theorem of Gabber [ILO14, IV.2.1.1] (see also [KS] for an elementary proof):

Cohen–Gabber Theorem. *Let $(A, \mathfrak{m}, k)$ be a complete, local ring of characteristic $p > 0$. Suppose that A is reduced and of equidimension d. Then there exists a subring $R \cong k[[X_1, \ldots, X_d]]$ such that the map $R \to A$ is finite and generically étale.*

The theorem is a crucial ingredient in the proof of the existence of so-called ℓ'-alterations: given a variety X/k and an $\ell \neq \operatorname{char} k$, there exists a regular, connected scheme Y, equipped with a proper map $Y \to X$ whose generic

Received October 20, 2016; received in final form April 7, 2017.
This work was partially supported by an NSF RTG Grant (DMS-1246844).
2010 *Mathematics Subject Classification.* 13H15, 13D40, 19A49.

degree is finite and prime to ℓ. We refer the reader to [ILO14] for a more precise statement and some generalizations to mixed-characteristic. From the perspective of commutative algebra, the existence of such a generically étale Noether normalization simplifies arguments involving tight closures and test elements (see, for example, [HH90, (6.3)] and [HH00, §4]).

When the residue field is algebraically closed, we show that generic separability may be achieved in an even stronger sense:

Theorem. *Let $(A, \mathfrak{m}, k)$ be a complete, equicharacteristic local domain of positive dimension whose residue field is algebraically closed. Then for any prime $p > 0$, there exists a regular subring $R = k[[X_1, \dots X_d]]$ such that $R \to A$ is finite and p is relatively prime to the generic degree $[K(A) : K(R)]$.*

Note that when A has characteristic $p > 0$, the condition that p does not divide $[K(A) : K(R)]$ will force the map to be generically étale. Since the generic degree of the map is just the Hilbert–Samuel multiplicity of A with respect to the parameter ideal $(X_1, \dots, X_d)$, the technical heart of the proof is the following statement, which appears below as Theorem 2.6:

Theorem. *Let $(A, \mathfrak{m})$ be a complete, reduced local ring of equidimension $d > 0$ with algebraically closed residue field. Suppose that the residue fields $k(\mathfrak{p})$ for each minimal prime $\mathfrak{p} \subseteq A$ all have the same characteristic. Then for each prime $p \in \mathbb{Z}$, there exists a parameter ideal $I = (f_1, f_2, \dots, f_d) \subseteq A$ such that $e_I(A)$ is relatively prime to p.*

When $d = 1$, we prove the result via direct calculation (Lemma 2.1). To reduce to the one-dimensional case, we carefully select $d - 1$ parameters to cut down the dimension of A and exploit the relationship between the Koszul complex and multiplicity with respect to parameter ideals. In particular, taking Koszul homology gives well-defined endomorphisms on the Grothendieck group of A which can be decomposed into simple factors. We develop this notion of "Koszul factorizations" systematically in Section 1. The first two parts of Section 2 are dedicated to proving the main theorems. We close, in Section 2.3, by indicating some analogous results in the mixed-characteristic setting.

1. Koszul factorization

Notations 1.1. Fix a Noetherian ring A. For a closed subset $Y \neq \emptyset$ of $\operatorname{Spec} A$, we define $\mathcal{M}(Y)$ to be the Serre subcategory of finitely-generated A-modules M whose support lies inside of Y. By $\mathbf{G}_0(Y)$ we shall mean the Grothendieck group $K_0(\mathcal{M}(Y))$. Given a module $M \in \mathcal{M}(Y)$, we shall write $[M]$ for the corresponding class in $\mathbf{G}_0(Y)$.

DEFINITION 1.2. Let $\mathbf{x}=(x_1,\dots,x_n)$ be a collection of elements of A whose vanishing set $V(\mathbf{x})$ has nonempty intersection with Y. Define $\Phi_{\mathbf{x}}:\mathbf{G}_0(Y)\to\mathbf{G}_0(Y\cap V(\mathbf{x}))$ via the rule

$$[M]\mapsto\sum_{i=0}^{n}(-1)^i\big[H_i(\mathbf{x},M)\big],$$

where by $H_i(\mathbf{x},M)$ we mean the homology of the Koszul complex $K(\mathbf{x},M)$. The well-definedness of this map follows from the functoriality of the Koszul complex and the fact that $\mathbf{x}\subseteq\operatorname{Ann}(H_i(\mathbf{x},M))$ [Ser65, IV, pp. 6–7].

Since the Koszul complex $K(\mathbf{x},M)$ may be realized as the iterated tensor product $K(x_1,A)\otimes_A\cdots\otimes_A K(x_n,A)\otimes_A M$, we can use the associated spectral sequence to give a "factorization" of the map $\Phi_{\mathbf{x}}$:

LEMMA 1.3. *Let* $\mathbf{x}=(x_1,\dots,x_m)$, $\mathbf{y}=(y_1,\dots,y_n)$ *and put* $\mathbf{x}+\mathbf{y}=(x_1,\dots,x_m,y_1,\dots,y_n)$. *Fix some closed* $Y\subseteq\operatorname{Spec}A$. *There is then a commutative diagram*

$$\begin{tikzcd}
\mathbf{G}_0(Y) \arrow[r, "\Phi_{\mathbf{y}}"] \arrow[dr, "\Phi_{\mathbf{x}+\mathbf{y}}"'] & \mathbf{G}_0(Y\cap V(\mathbf{y})) \arrow[d, "\Phi_{\mathbf{x}}"] \\
& \mathbf{G}_0(Y\cap V(\mathbf{x}+\mathbf{y}))
\end{tikzcd}$$

Proof. For any $M\in\mathcal{M}(Y)$, the Koszul complex $K(\mathbf{x}+\mathbf{y},M)$ may be realized as the total complex of $K(\mathbf{x},A)\otimes_A K(\mathbf{y},M)$. By considering the spectral sequence of the double complex

$$E^2_{pq}=H_p\big(\mathbf{x},H_q(\mathbf{y},M)\big)\Rightarrow H_{p+q}(\mathbf{x}+\mathbf{y},M),$$

it's clear that the E^2 page is bounded and supported on $Y\cap V(\mathbf{x}+\mathbf{y})$, thereby giving rise to the relation

$$\begin{aligned}\sum_{i=0}^{m+n}(-1)^i\big[H_i(\mathbf{x}+\mathbf{y},M)\big]&=\sum_{p,q\geq0}(-1)^{p+q}\big[E^2_{pq}\big]\\&=\sum_{p=0}^{m}\sum_{q=0}^{n}(-1)^{p+q}\big[H_p\big(\mathbf{x},H_q(\mathbf{y},M)\big)\big]\end{aligned}$$

in $\mathbf{G}_0(Y\cap V(\mathbf{x}+\mathbf{y}))$. The right-hand side is, of course, $\Phi_{\mathbf{x}}(\Phi_{\mathbf{y}}([M]))$. □

LEMMA 1.4. *Let* $\mathbf{x}=(x_1,\dots,x_m)$ *and suppose that for some* i *and* k, $x_i^k\cdot M=0$. *Then* $\Phi_{\mathbf{x}}([M])=0$.

Proof. By Lemma 1.3, $\Phi_{\mathbf{x}}$ is just the composition of all of the $\Phi_{(x_j)}$, and the maps can be composed in any order. Thus, it suffices to show that

$\Phi_{(x_i)}([M])=0$. The x_i-adic filtration on M gives exact sequences

$$0 \to x_i^{p+1}M \to x_i^pM \to \frac{x_i^pM}{x_i^{p+1}M} \to 0.$$

If we denote by N_p, the rightmost term of this sequence, we see that the Koszul complex $K(x_i,N_p)$ has 0-differential, whence $\Phi_{(x_i)}([N_p])=0$. Thus, $\Phi_{(x_i)}([x_i^pM])=\Phi_{(x_i)}([x_i^{p+1}M])$ for all $p\geq 0$ and the conclusion follows. $\square$

PROPOSITION 1.5. *Let $(A,\mathfrak{m})$ be a Noetherian local ring and let M be a d-dimensional A-module. Let $\mathbf{x}=(x_1,\dots,x_m)$ and put $Y=\operatorname{Supp} M$. Suppose that* $\dim(Y\cap V(\mathbf{x}))=d-m$. *Then there exists a $(d-m)$-dimensional module $N\in\mathcal{M}(Y\cap V(\mathbf{x}))$ such that*

$$[N]=\Phi_{\mathbf{x}}\big([M]\big) \quad \textit{in } \mathbf{G}_0\big(Y\cap V(\mathbf{x})\big).$$

Proof. We put $\mathbf{y}=(x_1,\dots,x_{m-1})$. For the case of $m=1$ (i.e., $\mathbf{y}$ is an "empty" sequence), we shall declare $V(\mathbf{y})=\operatorname{Spec} A$ and $\Phi_{\mathbf{y}}$ to be the identity on $\mathbf{G}_0(Y)$. By induction, we may assume that there exists a $(d-m+1)$-dimensional module $N\in\mathcal{M}(Y\cap V(\mathbf{y}))$ such that

$$\Phi_{\mathbf{y}}\big([M]\big)=[N] \quad \text{in } \mathbf{G}_0\big(Y\cap V(\mathbf{y})\big).$$

From Lemma 1.3, $\Phi_{(x_m)}\circ\Phi_{\mathbf{y}}=\Phi_{\mathbf{x}}$, and hence,

$$\Phi_{\mathbf{x}}\big([M]\big)=\Phi_{(x_m)}\big([N]\big) \quad \text{in } \mathbf{G}_0\big(Y\cap V(\mathbf{x})\big).$$

Consider the module $\Gamma_{(x_m)}(N)=\{u\in N : x_m^ju=0 \text{ for some } j\}$. Since N is finitely-generated, there exists a $k>0$ such that $x_m^k\Gamma_{(x_m)}(N)=0$. From the exact sequence

$$0\to\Gamma_{(x_m)}(N)\to N\to N''\to 0,$$

we know that since

$$\dim\big(Y\cap V(\mathbf{y})\big)=d-m+1>\dim\big(Y\cap V(\mathbf{y},x_m)\big),$$

x_m lies outside of all primes $\mathfrak{p}\in Y\cap V(\mathbf{y})$ such that $\dim(A/\mathfrak{p})=d-m+1$. Thus, $\dim(\Gamma_{(x_m)}(N))<d-m+1$, thereby forcing $\dim N''=\dim N$. Using Lemma 1.4 and noting that x_m is, by definition, a non-zerodivisor on N'' gives the equation

$$\Phi_{\mathbf{x}}\big([M]\big)=\Phi_{(x_m)}\big([N]\big)=\Phi_{(x_m)}\big([N'']\big)=\big[N''/x_mN''\big]$$

in $\mathbf{G}_0(Y\cap V(\mathbf{x}))$ where N''/x_mN'' is a genuine $(d-m)$-dimensional module in $\mathcal{M}(Y\cap V(\mathbf{x}))$. $\square$

Remark 1.6. In the above proposition, we know that the class $[N] \in \mathbf{G}_0(Y \cap V(\mathbf{x}))$ is nontrivial: since $\dim N = \dim(Y \cap V(\mathbf{x}))$, $[N]$ will evaluate to a positive number under the map $\mathbf{G}_0(Y \cap V(\mathbf{x})) \to \mathbb{Z}$ given by

$$E \mapsto \sum_{\substack{\mathfrak{p} \in Y \cap V(\mathbf{x}) \\ \dim(A/\mathfrak{p}) = d-m}} \ell(E \otimes A_{\mathfrak{p}}).$$

See Corollary 1.8 below for a stronger statement.

Even if we omit the hypothesis that $\dim(Y \cap V(\mathbf{x})) = d - m$, we still have an equality

$$[N] = \Phi_{\mathbf{x}}\big([M]\big) \quad \text{in } \mathbf{G}_0\big(Y \cap V(\mathbf{x})\big).$$

for some $N \in \mathcal{M}(Y \cap V(\mathbf{x}))$. However, it can easily occur that $N = 0$ in this less restrictive case.

1.1. Hilbert-Samuel multiplicity. Let $(A, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely-generated module. Suppose that $\mathfrak{a} \subseteq A$ is an ideal for which $\ell(M/\mathfrak{a}M) < \infty$. We define the Hilbert–Samuel multiplicity of M with respect to $\mathfrak{a}$ via

$$e_{\mathfrak{a}}(M) = \lim_{n\to\infty} \frac{d!}{n^d} \ell\big(M/\mathfrak{a}^n M\big) \quad (d = \dim M).$$

To obtain a function which is additive over exact sequences, we introduce, for each $r \le \dim M$, the modified multiplicity function:

$$e_{\mathfrak{a}}(M, r) = \lim_{n\to\infty} \frac{r!}{n^r} \ell\big(M/\mathfrak{a}^n M\big) = \begin{cases} e_{\mathfrak{a}}(M), & r = \dim M, \\ 0, & r > \dim M. \end{cases}$$

If $Y \subseteq \operatorname{Spec} A$ is a closed subset of dimension r and $Y \cap V(\mathfrak{a}) = \{\mathfrak{m}\}$, then $e_{\mathfrak{a}}(-, r)$ is additive over short exact sequences in $\mathcal{M}(Y)$ [Ser65, II, Prop. 10] and so defines a homomorphism $\mathbf{G}_0(Y) \to \mathbb{Z}$. For $Z = \{\mathfrak{m}\}$, $\mathbf{G}_0(Z)$ is simply the Grothendieck group on finite-length A-modules and $e_{\mathbf{0}}(-, 0)$ coincides with the length function $\ell : \mathbf{G}_0(Z) \to \mathbb{Z}$.

Theorem 1.7 ([Ser65, IV.3, Thm. 1]). *Let $(A, \mathfrak{m})$ be a Noetherian local ring and suppose that M is a finitely generated module. Put $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ and suppose that $\ell(M/\mathbf{x}M) < \infty$. Then*

$$e(M, r) = \sum_{p=0}^{r} (-1)^p \ell\big(H_p(\mathbf{x}, M)\big).$$

Corollary 1.8. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and let Y be a closed subset of* $\operatorname{Spec} A$. *Put* $\mathbf{x} = (x_1, x_2, \ldots, x_r)$, $\mathbf{x}' = (x_{r+1}, x_{r+2}, \ldots, x_{r+s})$

and suppose that $Y \cap V(\mathbf{x}+\mathbf{x}') = \{\mathfrak{m}\}$. *Then there is a commutative diagram*:

$$\begin{tikzcd} \mathbf{G}_0(Y) \arrow[rr, "e_{\mathbf{x}+\mathbf{x}'}(-,r+s)"] \arrow[d, "\Phi_{\mathbf{x}}"'] & & \mathbb{Z} \\ \mathbf{G}_0(Y \cap V(\mathbf{x})) \arrow[urr, "e_{\mathbf{x}'}(-,s)"] \arrow[r, "\Phi_{\mathbf{x}'}"] & \mathbf{G}_0(\{\mathfrak{m}\}) \arrow[ur, "\ell"'] & \end{tikzcd}$$

Proof. Commutativity of the lower triangle follows at once from Theorem 1.7. Since $\Phi_{\mathbf{x}'} \circ \Phi_{\mathbf{x}} = \Phi_{\mathbf{x}+\mathbf{x}'}$ by Lemma 1.3, the outer square commutes again by appealing to Theorem 1.7. The upper triangle is now forced to commute for formal reasons. □

2. Proof of the theorem

2.1. Dimension-one.

LEMMA 2.1. *Let* $(B, \mathfrak{m})$ *be a complete Noetherian local ring of dimension* 1. *Suppose that* B *is reduced and has algebraically closed residue field. Then for any fixed prime* $p \in \mathbb{Z}$, *there is a principal ideal* $I = (f)$ *such that* $e_I(B) = \ell(B/fB)$ *is relatively prime to* p.

Proof. By hypothesis, B has no embedded primes, so if $f \in B$ is such that $\dim(B/fB) = 0$, then f is necessarily a non-zerodivisor. Putting $I = (f)$, it's easily seen that $\ell(B/I^nB) = n\ell(B/fB)$, whence $e_I(B) = \ell(B/fB)$. We therefore want to find an f for which p does not divide $\ell(B/fB)$.

Denote by S the set of non-zerodivisors of B, and let $Q = S^{-1}B$ be the total quotient ring of B. For any $f, g \in S$, the short-exact sequence

$$0 \to B/fB \xrightarrow{g} B/fgB \to B/gB \to 0$$

permits us to define a monoid homomorphism $S \to \mathbb{Z}$ via $f \mapsto \ell(B/fB)$. This naturally extends to an "order" homomorphism $\mathrm{ord}_B : Q^\times \to \mathbb{Z}$ defined by $\mathrm{ord}_B(f/g) = \ell(B/fB) - \ell(B/gB)$ (cf. [Ful98, A.3]). It will suffice to show that ord_B is surjective, for if $\ell(B/fB) - \ell(B/gB) = 1$, then p cannot divide both terms.

Let $\widetilde{B}$ be the normalization of B inside of Q. Since B is complete, the map $B \to \widetilde{B}$ is finite. $\widetilde{B}$ is therefore a semilocal, one-dimensional ring, and since $k = B/\mathfrak{m}$ is algebraically closed, the residue field of each closed point of $\widetilde{B}$ is isomorphic to k. It follows that for every finite-length $\widetilde{B}$ module M, it does not matter whether we measure its length over B or $\widetilde{B}$: $\ell_B(M) = \ell_{\widetilde{B}}(M)$ (cf. [Ful98, A.1]). Since there is no ambiguity, we shall henceforth drop the subscripts.

Since B and $\widetilde{B}$ share the same total quotient ring Q, we can define another order function $\mathrm{ord}_{\widetilde{B}} : Q^\times \to \mathbb{Z}$ with $\mathrm{ord}_{\widetilde{B}}(f/g) = \ell(\widetilde{B}/f\widetilde{B}) - \ell(\widetilde{B}/g\widetilde{B})$. We claim that $\mathrm{ord}_B = \mathrm{ord}_{\widetilde{B}}$. Since S generates $Q^\times$ as an Abelian group, it will suffice to show that both order functions agree on S. Given a non-zerodivisor

$f \in B$, it will continue to be a non-zerodivisor in Q and hence also in $\widetilde{B}$. This gives rise to the following diagram of finitely-generated B modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, "f"'] & \widetilde{B} \arrow[r] \arrow[d, "f"'] & \widetilde{B}/B \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & \widetilde{B} \arrow[r] \arrow[d] & \widetilde{B}/B \arrow[r] \arrow[d] & 0 \\
 & B/fB & \widetilde{B}/f\widetilde{B} & C &
\end{tikzcd}$$

Since B is reduced and hence has no embedded primes, the collection of all non-zerodivisors S is just the complement of the union of all minimal primes of B. As $S^{-1}B = Q = S^{-1}\widetilde{B}$, we have that $S^{-1}(\widetilde{B}/B) = 0$, meaning that $\widetilde{B}/B$ vanishes at every minimal prime of B and hence is supported in dimension 0. As such,

$$0 \to K \to \widetilde{B}/B \xrightarrow{f} \widetilde{B}/B \to C \to 0$$

is an exact sequence of finite-length modules, whence $\ell(K) = \ell(C)$. From the snake lemma, we have

$$0 \to K \to B/fB \to \widetilde{B}/f\widetilde{B} \to C \to 0$$

which implies that $\ell(B/fB) = \ell(\widetilde{B}/f\widetilde{B})$ as desired.

Finally, we show that $\mathrm{ord}_B = \mathrm{ord}_{\widetilde{B}}$ is surjective. $\widetilde{B}$, by construction, is a one-dimensional, semilocal normal ring and hence the localization at each maximal prime is a DVR. In fact, since $\widetilde{B}$ is complete with respect to the linear topology defined by its Jacobson radical, it must decompose into a direct product of DVRs. By choosing an $h \in \widetilde{B}$ which generates one of the maximal ideals and lies outside of all of the others, we see that $\widetilde{B}/h\widetilde{B}$ is a field—that is, $\mathrm{ord}(h) = 1$. ☐

Example 2.2. Lemma 2.1 can fail if the residue field is not algebraically closed. Let $L = \mathbb{F}_2(t)$ where t is a transcendental element. Put $A = L[[X,Y]]/(X^2 + XY + Y^2)$. It is easily checked that A is a domain. Since A has infinite residue field, every $\mathfrak{m}$-primary ideal I has a principal reduction $(f) \subseteq I$ ([Mat86, 14.14]). We claim that $\ell_A(A/fA) = e_{(f)}(A) = e_I(A)$ is divisible by 2 for every $0 \neq f$.

If we denote by $\tilde{A}$ the normalization, the proof of Lemma 2.1 shows that $\ell_A(A/fA) = \ell_A(\tilde{A}/f\tilde{A})$. Since $X^2 + XY + Y^2 = 0$ in A, dividing by Y^2 gives the relation $(\frac{X}{Y})^2 + (\frac{X}{Y}) + 1 = 0$ in the fraction field. Thus, $\frac{X}{Y}$ lies in $\widetilde{A}$ and satisfies an irreducible polynomial over L. The residue field of $\widetilde{A}$ therefore contains a quadratic extension of L, and hence, 2 divides $\ell_A(M)$ for all Artinian $\tilde{A}$-modules M (cf. [Ful98, A.1.3]).

In particular, if $R = k[[T]]$ is any power-series subring of A over which A is finite, then we know [ZS75, VIII.10, Cor. 2] that $e_{(T)}(A)[L : k] = [K(A) : K(R)]$. Thus, the degree of the fraction field extension $[K(A) : K(R)]$ will always be divisible by 2.

REMARK 2.3. Strictly speaking, it is not absolutely necessary that A have algebraically closed residue field for the conclusion of Lemma 2.1 to hold. One simply needs the residue fields at closed points of A and $\widetilde{A}$ to all be isomorphic.

2.2. General case. We begin by recalling the Cohen–Gabber theorem from the Introduction but include some slight modifications to include the classically-known cases of equicharacteristic 0 and mixed-characteristic.

THEOREM 2.4 ([ILO14, IV.2.1.1] (Gabber)). *Let A be a complete, equidimensional, reduced local ring. Suppose that the residue fields $k(\mathfrak{p})$ for each minimal prime $\mathfrak{p} \subseteq A$ all have the same characteristic. Then there exists a subring $R \subseteq A$ such that*

(1) *R is a complete regular local ring.*
(2) *The map $R \to A$ is finite and generically étale.*
(3) *R and A share the same residue field.*

REMARK 2.5. As we mentioned in the Introduction, if the residue fields at each minimal prime all have characteristic zero, the classical Cohen structure theorem guarantees a finite, injective map $R = V[[X_1, \dots, X_m]] \to A$ where V is a field or DVR of characteristic 0 (cf. [Mat86, 29.4]). In this case being generically étale is automatic since the fraction field of R will have characteristic 0 and hence be perfect. The nontrivial case where A has equicharacteristic $p > 0$ is precisely what Gabber proved.

THEOREM 2.6. *Let $(A, \mathfrak{m})$ be a complete, reduced local ring of equidimension $d > 0$ with algebraically closed residue field. Suppose that the residue fields $k(\mathfrak{p})$ for each minimal prime $\mathfrak{p} \subseteq A$ all have the same characteristic. Then for each prime $p \in \mathbb{Z}$, there exists a parameter ideal $I = (f_1, f_2, \dots, f_d) \subseteq A$ such that $e_I(A)$ is relatively prime to p.*

Proof. The idea is to reduce to the one-dimensional case where the result is known by Lemma 2.1. By the Cohen–Gabber theorem, we can find a regular local subring R over which A is finite and generically étale. Choose $0 \neq h \in \mathfrak{m}$ such that $R_h \to A_h$ is étale. Let $d = \dim A$. We now construct parameters $f_1, \dots, f_{d-1}$ in R such that

(a) $R/(f_1, \dots, f_t)$ is a regular local ring for $0 \leq t \leq d-1$.
(b) $h, f_1, \dots, f_t$ is a regular sequence in R for $0 \leq t \leq d-1$.

When $t = 0$, this just amounts to saying that h is a non-zerodivisor on R, which is obvious. Assume that $f_1, \dots, f_t$ have been constructed with $t < d-1$. Put $\overline{R} = R/(f_1, \dots, f_t)$ and denote by $\mathfrak{q}$ its maximal ideal. By hypothesis, h is a non-zerodivisor on $\overline{R}$, so that $\overline{R}/h\overline{R}$ is a Cohen–Macaulay ring of positive dimension, and hence, $\mathfrak{q} \notin \operatorname{Ass}_{\overline{R}}(\overline{R}/h\overline{R})$. By prime avoidance [Eis95, 3.3], we have that $\mathfrak{q} \not\subset (\bigcup \operatorname{Ass}_{\overline{R}}(\overline{R}/h\overline{R})) \cup \mathfrak{q}^2$, meaning that we can choose some $\overline{f}_{t+1} \in \mathfrak{q} - \mathfrak{q}^2$ which is $\overline{R}/h\overline{R}$-regular. By lifting to an element $f_{t+1} \in R$, we we see that conditions (a) and (b) are satisfied.

Since $R \to A$ is finite, $A' := A/(f_1, \dots, f_{d-1})$ is one-dimensional. Let $\mathfrak{N}$ be its nilradical and let $B = A'/\mathfrak{N}$. By Lemma 2.1, there is a parameter $f_d \in A$ such that $\ell(B/f_dB)$ is not divisible by our fixed prime p. Note that since B is reduced and has no embedded primes, f_d is B-regular.

Put $\mathbf{f} = (f_1, \dots, f_{d-1})$ and $\mathbf{f}' = (f_d)$. We claim that for the parameter ideal $I = \mathbf{f} + \mathbf{f}' = (f_1, \dots, f_d)$, $e_I(A)$ is equal to $\ell(B/f_dB)$ and hence is relatively prime to p. Put $Y = \operatorname{Spec} A$, $Y' = Y \cap V(\mathbf{f}) = \operatorname{Spec} A'$. From Corollary 1.8 we have a diagram

$$\begin{tikzcd}
\mathbf{G}(Y) \arrow[r, "e_I(-,d)"] \arrow[d, "\Phi_{\mathbf{f}}"] & \mathbb{Z} \\
\mathbf{G}(Y') \arrow[r, "\Phi_{\mathbf{f}'}"] & \mathbf{G}(\{\mathfrak{m}\}) \arrow[u, "\ell"']
\end{tikzcd}$$

By the definition of $\Phi_{\mathbf{f}}$ and the exact sequence $0 \to \mathfrak{N} \to A' \to B \to 0$, we have the relation

$$\Phi_{\mathbf{f}}\big([A]\big) = \big[A'\big] + \sum_{i=1}^{d-1} (-1)^i \big[H_i(\mathbf{f}, A)\big] = [B] + [\mathfrak{N}] + \sum_{i=1}^{d-1} (-1)^i \big[H_i(\mathbf{f}, A)\big]$$

in $\mathbf{G}_0(Y')$.

Since $R/(f_1, \dots, f_{d-1}, h)$ is Artinian, so too is $A/(f_1, \dots, f_{d-1}, h) = A'/hA'$. Thus, h lies outside of every minimal prime of the one-dimensional ring A'. In other words, if some finitely-generated A' module M is such that $M_h = 0$, then M is supported only on the closed point and hence is killed by some power of $\mathfrak{m}$. Since $R_h \to A_h$ is étale, then by base-change, so too is $(R/(f_1, \dots, f_{d-1}))_h \to A'_h$. But $R/(f_1, \dots, f_{d-1})$ is regular, meaning that A'_h is regular (and, in particular, reduced). In other words, $\mathfrak{N}_h = 0$. Similarly, since $R_h \to A_h$ is flat, the R-regular sequence $f_1, \dots, f_{d-1}$ must also be A_h-regular. In other words we obtain a vanishing of the Koszul homology: $H_i(\mathbf{f}, A)_h = 0$ for $i > 0$. Thus, some power of $\mathfrak{m}$—and hence some power of f_d—kills each of the these modules, so by Lemma 1.4,

$$\Phi_{\mathbf{f}'}\big([\mathfrak{N}]\big) = 0 \quad \text{and} \quad \Phi_{\mathbf{f}'}\Big(\big[H_i(\mathbf{f}, A)\big]\Big) = 0 \quad \text{for } i > 0.$$

Since f_d is a non-zerodivisor on B, we conclude that

$$e_I(A,d) = \ell \circ \Phi_{\mathbf{f}'} \circ \Phi_{\mathbf{f}}\big([A]\big) = \ell \circ \Phi_{\mathbf{f}'}\left([B] + [\mathfrak{N}] + \sum_{i=1}^{d-1} (-1)^i \big[H_i(\mathbf{f}, A)\big]\right)$$
$$= \ell(B/f_d B).$$ □

The proof of the sharpened Cohen–Gabber theorem now follows immediately:

THEOREM 2.7. *Let $(A, \mathfrak{m}, k)$ be a complete, equicharacteristic local domain of positive dimension whose residue field is algebraically closed. Then for any prime $p > 0$, there exists a regular subring $R = k[[X_1, \dots X_d]]$ such that $R \to A$ is finite and p is relatively prime to the generic degree $[K(A) : K(R)]$.*

Proof. Fix the prime p. By Theorem 2.6, we know that A admits a system of parameters $I = (f_1, \dots, f_d)$ (with $d > 0$) such that $e_I(A)$ is relatively prime to p. The Cohen structure theorem [Mat86, 28.3] says that k embeds into A as a coefficient field (i.e. $k \to A \to A/\mathfrak{m}$ is an isomorphism). We define $R = k[[T_1, \dots T_d]] \to A$ via $T_i \mapsto f_i$ and obtain a finite morphism which must be injective for dimensional reasons. By [ZS75, VIII.10, Cor. 2], we know that $[K(A) : K(R)] = e_I(A)$ and hence is relatively prime to p. □

REMARK 2.8. Example 2.2 shows that this theorem can fail in dimension one if the residue field is not algebraically closed. In view of Remark 2.3, the proofs of Theorems 2.6 and 2.7 will go through as long as we can be assured that after taking A and going modulo the $f_1, \dots, f_{d-1}$, the residue field at the maximal ideal of $B = (A/(f_1, \dots, f_{d-1}))_{\text{red}}$ is isomorphic to those of the normalization $\widetilde{B}$. Since the first $d-1$ parameters $f_1, \dots, f_{d-1}$ are selected in a fairly generic fashion, one can ask whether there is a Bertini-type result that would allow one to choose the f_i carefully enough so that the one-dimensional ring B has the desired properties. If there were, we would be able to relax the requirement that A have algebraically closed residue field as long as $\dim A \geq 2$.

2.3. The mixed-characteristic case. If $(A, \mathfrak{m}, k)$ is a complete, local domain of mixed-characteristic, one could ask for an analogue of Theorem 2.7 where R is a power-series over a DVR. In the presence of an additional regularity assumption, we have the following result.

THEOREM 2.9. *Let $(A, \mathfrak{m}, k)$ be a complete, local domain of mixed-characteristic having dimension $d > 0$ and whose residue field is algebraically closed. Let $q = \operatorname{char}(k)$ and assume A/qA is generically reduced. Fix a prime number $p > 0$. Then*

(a) *There exists a parameter ideal $I = (q, f_1, \dots, f_{d-1})$ such that $e_I(A)$ is relatively prime to p.*
(b) *There exists a regular subring $R = V[[X_1, \dots, X_{d-1}]]$ such that*

(1) *(V, qV, k) is a complete DVR with uniformizer q.*
(2) *$R \to A$ is finite.*
(3) *The generic degree $[K(A) : K(R)]$ is relatively prime to p.*

Proof. We begin by remarking that if $d = 1$, then the assumption that A/qA is generically reduced implies that A/qA is a field. Thus, A is a DVR with uniformizer q, thereby making statements (a) and (b) trivialities. We shall henceforth assume that $d > 1$.

(a) Since A is a domain, $\Phi_{(q)}([A]) = [A/qA]$ in $\mathbf{G}_0(V(q))$. Let $\mathfrak{N}$ be the nilradical of A/qA and put $C = (A/qA)/\mathfrak{N}$. Since A/qA is generically-reduced, $\mathfrak{N}_\mathfrak{q} = 0$ for all minimal primes of A/qA, meaning that $\dim(\mathfrak{N}) < \dim(A/qA) = d - 1$. As C is reduced and equidimensional, Theorem 2.6 gives us $f_1, \dots, f_{d-1} \in A$ which form a parameter system for A/qA and $e_{(f_1,\dots,f_{d-1})}(C)$ is relatively prime to p.

Put $\mathbf{f} = (f_1, \dots, f_{d-1})$ and let $I = (q) + \mathbf{f}$. It is clear that I is a parameter ideal for A; it will suffice to prove that $e_I(A) = e_\mathbf{f}(C)$. From the exact sequence

$$0 \to \mathfrak{N} \to A/qA \to C \to 0$$

we have that $e_\mathbf{f}(A/qA, d-1) = e_\mathbf{f}(C, d-1) = e_\mathbf{f}(C)$ as $\dim(\mathfrak{N}) < d - 1$. Now, by Corollary 1.8, it follows that

$$e_I(A) = \ell \circ \Phi_\mathbf{f} \circ \Phi_{(q)}\big([A]\big) = \ell \circ \Phi_\mathbf{f}\big([A/qA]\big) = e_\mathbf{f}(A/qA, d-1) = e_\mathbf{f}(C).$$

For (b), we begin by noting that we are guaranteed such a DVR $V \subset A$ by the Cohen structure theorem (see, for example, [Mat86, 29.3]). By part (a), we can find a parameter ideal $I = (q, f_1, \dots, f_{d-1}) \subset A$ such that $e_I(A)$ is relatively prime to p. The map $R = V[[X_1, \dots, X_{d-1}]] \to A$ is defined by sending the X_i to f_i. As in the proof of Theorem 2.7, this map is finite and injective. By [HS06, 11.2.6], we have $e_I(A) = [K(A) : K(R)]$. $\square$

To illustrate why it is necessary to require that A/qA be generically reduced, we present the following example:

EXAMPLE 2.10. Let $q \in \mathbb{Z}$ be a prime and let (V, qV, k) be a DVR with algebraically closed residue field k. Let $n > 0$ and put $A = V[[X, Y]]/(qX - Y^n)$. We claim that for any parameter ideal of the form $I = (q, f)$, $e_I(A)$ will be divisible by n. Consequently, if $R \subset A$ is finite with R a power-series over a complete DVR, then $[K(A) : K(R)]$ will be divisible by n.

First, note that $A/qA = k[[X, Y]]/(Y^n)$, so the A/qA-module isomorphism $(Y^i)/(Y^{i+1}) \cong k[[X, Y]]/(Y) \cong A/(q, Y)$ and the exact sequence

$$0 \to \big(Y^i\big)/\big(Y^{i+1}\big) \to k\big[[X, Y]\big]/\big(Y^{i+1}\big) \to k\big[[X, Y]\big]/\big(Y^i\big) \to 0$$

together show that $[A/qA] = n[A/(q, Y)]$ in $\mathbf{G}_0(V(q))$. By Corollary 1.8,

$$e_I(A) = \ell \circ \Phi_{(f)} \circ \Phi_{(q)}\big([A]\big) = \ell \circ \Phi_{(f)}\big([A/qA]\big) = n \cdot \ell \circ \Phi_{(f)}\big(\big[A/(q, Y)\big]\big).$$

Acknowledgments. I originally proposed this as one of several open problems at a 2016 Summer REU that I jointly organized with K. Tucker and W. Zhang at UIC. I would like to thank the students for their spirit and enthusiasm. In particular, Example 2.2 is based on an observation of J. Mundinger which showed that any automorphism of $\mathbb{F}_2[[X, Y]]$ will take $X^2 + XY + Y^2$ to itself (mod $\mathfrak{m}^3$). I would also like to thank the anonymous referee for their thoughtful comments and for catching a few mistakes in the original draft. The final section, concerning generalizations to the case of mixed-characteristic, was also added in response to a question raised by the referee.

References

[Eis95] D. Eisenbud, *Commutative algebra: With a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer, New York, 1995. MR 1322960

[Ful98] W. Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer, Berlin, 1998. MR 1644323

[HH90] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon–Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116. MR 1017784

[HH00] M. Hochster and C. Huneke, *Localization and test exponents for tight closure*, Michigan Math. J. **48** (2000), 305–329. Dedicated to William Fulton on the occasion of his 60th birthday.

[HS06] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006. MR 2266432

[KS] K. Kurano and K. Shimomoto, *An elementary proof of Cohen–Gabber theorem in the equal characteristic $p > 0$ case*; available at `arXiv:1510.03573` [math.AC].

[Mat86] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid. MR 0879273

[ILO14] Moreau, V. Pilloni, M. Raynaud, J. Riou, B. Stroh, M. Temkin and W. Zheng, *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, Séminaire à l'École Polytechnique 2006–2008. [Seminar of the polytechnic school 2006–2008]*, Société Mathématique de France, Paris, 2014. With the collaboration of Frédéric Déglise, Alban Moreau, Vincent Pilloni, Michel Raynaud, Joël Riou, Benoît Stroh, Michael Temkin and Weizhe Zheng, Astérisque No. 363–364 (2014). MR 3309086

[ZS75] Z. Oscar and P. Samuel, *Commutative algebra. Vol. II*, Graduate Texts in Mathematics, vol. 29, Springer, New York, 1975. Reprint of the 1960 edition. MR 0120249

[Ser65] J.-P. Serre, *Algèbre locale. Multiplicités, Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel*, 2nd ed., Lecture Notes in Mathematics, vol. 11, Springer, Berlin, 1965. MR 0201468

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