Illinois Journal of Mathematics
Volume 60, Number 2, Summer 2016, Pages 587–611
S 0019-2082

EXPONENTIAL CONVERGENCE FOR SOME SPDES WITH LÉVY NOISES

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ABSTRACT. In this paper, we generalize the Malliavin calculus for jump processes in the infinite-dimensional setting and obtain an integration by parts formula for jump processes on Hilbert spaces. By using this formula, we investigate derivative formula and exponential convergence for SPDEs driven by purely jump processes.

1. Introduction and main results

1.1. Introduction. Stochastic partial differential equations (SPDEs) driven by Lévy noises have been extensively studied in recent years; see [1], [2], [14], [17], [18], [19], [16], [21], [25] and references therein. For diffusions, the derivative formulas [7] (also called Bismut–Elwealthy–Li formula [13]) is a quite useful tool in various aspects such as functional inequalities [32], heat kernel estimates [7], strong Feller properties [10] and so on. Due to these various applications, many scholars have paid much attention to the analogous derivative formula for jump processes. For finite-dimensional jump processes, we refer to [3], [4], [20], [33], [35] etc. In most of these references, the Lévy measures of forced noises are always required to having absolutely continuous (parts) lower bounds w.r.t. the Lebesgue measure whose shift-invariance property plays an essential role. But in infinite-dimensional setting, there is no Lebesgue measure available. In [34], the authors investigated the strong Feller and coupling properties for linear SDEs driven by non-cylindrical Lévy processes on a Banach space equipped with a nice reference measure, which

Received September 22, 2016; received in final form January 20, 2017.
The first author is supported by NNSF (No.11501286), JSNSF (No. BK20150564) of China and Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (No. 2008DP173182).
2010 *Mathematics Subject Classification.* 60H07, 60H15.

has quasi-invariance property. By Galerkin's approximation, a derivative formula was established in [28] for semilinear SPDEs forced by by non-cylindrical Lévy processes. However, the representation of the formula in [28] is cumbersome. So one aim of this work is to give a succinct formulation of derivative formula for a modified transitional probability function.

As an application of the derivative formula, we will study the long-time asymptotic behaviors of SPDEs with jumps (see Theorem 2 below). For this topic, in finite-dimensional case, the algebraic or exponential convergence of Lévy processes and SDEs forced by Lévy processes were studied in [8], [26], [27], [15], [22] and references therein. When the nonlinear terms of SPDEs forced by cylindrical α-stable processes were bounded and Lipschitz continuous, exponential convergence were derived in [22], [24] and [23]. For stochastic Burgers equations, the exponential convergence was discussed in [12]. The exponential convergence for nonlinear SPDEs driven by non-cylindrical pure jump processes was also discussed in [28]. Another aim of this paper is to give the exponential convergence for SPDEs with Lévy noises.

1.2. Preliminaries. Let $(\mathbb{H},\langle\cdot,\cdot\rangle)$ be a separable Hilbert space and μ be a Gaussian measure on $\mathbb{H}$ with covariance operator Q, which is nonnegative, symmetric and with trace class. Its square root, denoted by $Q^{\frac{1}{2}}$, is a nonnegative and symmetric Hilbert–Schmidt operator. Let $\operatorname{Im} Q^{\frac{1}{2}}$ be the image space of $Q^{\frac{1}{2}}$, i.e., $\operatorname{Im} Q^{\frac{1}{2}} = \{Q^{\frac{1}{2}}x | x\in\mathbb{H}\}$. As is known, $\operatorname{Im} Q^{\frac{1}{2}}$ is a Hilbert space with the induced inner product

$$\langle x,y\rangle_0 := \big\langle Q^{-\frac{1}{2}}x, Q^{-\frac{1}{2}}y\big\rangle, \quad x,y\in \operatorname{Im} Q^{\frac{1}{2}},$$

where $Q^{-\frac{1}{2}}$ is the pseudo inverse of Q in the case that it is not one-to-one, that is, for $h\in \operatorname{Im} Q^{\frac{1}{2}}$,

$$Q^{-\frac{1}{2}}h = x, \quad \text{if } Q^{\frac{1}{2}}x = h \quad \text{and} \quad \|x\| = \inf\big\{\|y\| : Q^{\frac{1}{2}}y = h\big\}.$$

As is known, the Gaussian measure μ has quasi-invariant property under the shift $z\mapsto z+h$ for any $h\in \operatorname{Im} Q^{\frac{1}{2}}$ (see Theorem 2.21 in [9]), that is, $\mu(\cdot+h)$ and μ are mutually absolutely continuous. The Randon–Nikodym derivative of $\mu(\cdot+h)$ w.r.t. μ is

$$\varphi(z,h) := \frac{\mu(\mathrm{d}z+h)}{\mu(\mathrm{d}z)} = \exp\bigg\{\langle h,z\rangle_0 - \frac{1}{2}\langle h,h\rangle_0\bigg\}, \quad \mu\text{-a.s.} \tag{1}$$

Denote the eigenvectors of Q by $\{e_k\}_{k\in\mathbb{N}}$, which can consist of an orthonormal basis of $\mathbb{H}$.

Let $\mathbb{W}$ be the space of all càdlàg functions from $[0,\infty)$ to $\mathbb{H}$ vanishing at 0, which is endowed with the Skorohod topology. The σ-algebra $\mathcal{B}(\mathbb{W})$ is generated by all of the open sets of $\mathbb{W}$. Let $\mathbb{P}^1$ be a probability measure on $\mathbb{W}$ such that the coordinate process $L_t(w) = w(t)$ is a Lévy process with characteristic measure ν. Let $\mathbb{P}^2$ be another probability measure on $\mathbb{W}$ such that the coordinate process $Z_t(w) = w_t$ is also a Lévy process with Lévy measure ν_Z satisfying $\int_{\mathbb{H}} |z|^2 \nu_Z(\mathrm{d}z) < \infty$. Throughout this paper, for Lévy measure ν, we assume that

($\mathbf{H}_\nu$) There exists a Fréchet differentiable function $\rho : \mathbb{H} \to (0,\infty)$ with bounded derivative such that

$$\nu(\mathrm{d}z) = \rho(z)\mu(\mathrm{d}z), \qquad \lambda := \nu(\mathbb{H}) \in (0,+\infty), \quad \text{and} \quad \int_{\mathbb{H}} |z|^2 \nu(\mathrm{d}z) < \infty.$$

Define a probability space as

$$(\Omega, \mathcal{F}, \mathbb{P}) := \big(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W}) \times \mathcal{B}(\mathbb{W}), \mathbb{P}^1 \times \mathbb{P}^2\big).$$

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by L and Z. Denote the jump measure of L by $N(\mathrm{d}z, \mathrm{d}t)$. $\widetilde{N}(\mathrm{d}z, \mathrm{d}t) := N(\mathrm{d}z, \mathrm{d}t) - \nu(\mathrm{d}z)\,\mathrm{d}t$ is the martingale measure. Let $N_t := N([0,t] \times \mathbb{H})$ stand for the associated counting process.

In this paper, we consider the following stochastic equation on $\mathbb{H}$

$$\begin{cases} \mathrm{d}X_t = AX_t\,\mathrm{d}t + F(X_t)\,\mathrm{d}t + \mathrm{d}L_t + \mathrm{d}Z_t, \\ X_0 = x, \end{cases} \tag{2}$$

where $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ is an adjoint, unbounded and linear operator generating a C_0-semigroup $\{S_t\}_{t\geq 0}$ and $F : \mathbb{H} \to \mathbb{H}$ is bounded and Lipschitz continuous. Then the mild solution of Eq. (2) exists uniquely (see [21, Theorem 9.7]) and can be formulated as

$$\begin{aligned} X_t^x = S(t)x + \int_0^t S(t-s)F\big(X_s^x\big)\,\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}L_s \\ + \int_0^t S(t-s)\,\mathrm{d}Z_s. \end{aligned} \tag{3}$$

Let $\{P_t(x,\cdot)\}_{t\geq 0}$ and $\{P_t\}_{t\geq 0}$ be the transition probability measures and transition semigroups respectively.

Now we give some notations for later use. Let $\mathcal{B}(\mathbb{H})$ be the σ-algebra generated by all of the open subsets of $\mathbb{H}$. For $i = 1,2$, we employ $C_b^i(\mathbb{H},\mathbb{H})$ ($C_b^i(\mathbb{H})$) to denote the family of $\mathbb{H}$-valued (real-valued) ith Fréchet differentiable functions f such that f and its derivatives are bounded and continuous. Let $\mathcal{B}_b(\mathbb{H})$ be the Banach space of bounded Borel-measurable functions $f : \mathbb{H} \to \mathbb{R}$ with the supremum norm $\|f\|_\infty := \sup_{y\in\mathbb{H}} |f(y)|$. Let $\mathcal{P}(\mathbb{H})$ be the set of probabilities on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$. Recall that the total variation distance between two finite measures μ_1, μ_2 is defined by

$$\|\mu_1 - \mu_2\|_{\mathrm{Var}} := \frac{1}{2} \sup_{f\in\mathcal{B}_b(\mathbb{H}), \|f\|_\infty \leq 1} \big|\mu_1(f) - \mu_2(f)\big|.$$

The norm of a linear bounded operator $P : \mathbb{H} \to \mathbb{H}$ is defined by

$$\|P\| := \sup_{y\in\mathbb{H}, |y|=1} |Py|.$$

We define the operators $\{P_t^1\}_{t\geq 0}$ which were first introduced in [31] as

$$P_t^1 f(x) := \mathbb{E}\big\{f\big(X_t^x\big)I_{[N_t\geq 1]}\big\}, \quad x\in\mathbb{H}, t\geq 0, f\in\mathcal{B}_b(\mathbb{H}), \tag{4}$$

where $I_{[N_t\geq 1]}$ is an indicator function.

1.3. Main results. We list the hypotheses for Equ. (2):

- $(\mathbf{H}_A)$ A is a dissipative operator defined by

$$A = \sum_{k\geq 1}(-\gamma_k)e_k\otimes e_k, \tag{5}$$

  for $0<\gamma_1\leq\gamma_2\leq\cdots\leq\gamma_k\leq\cdots$ and $\gamma_k\to\infty$ as $k\to\infty$.
- $(\mathbf{H}_{SQ})$ $\operatorname{Im} S(t)\subset \operatorname{Im} Q$ and $\int_0^t \|Q^{-1}S(s)\|\,\mathrm{d}s<\infty$ hold for any $t>0$.

We have the following main results.

THEOREM 1. *Let $F\in C_b^1(\mathbb{H},\mathbb{H})$ with ∇F Lipschitz continuous. Assume $(\mathbf{H}_\nu)$, $(\mathbf{H}_A)$ and $(\mathbf{H}_{SQ})$ hold. Then for $f\in C_b^2(\mathbb{H})$ and $\xi\in\mathbb{H}$,*

$$\begin{aligned}\nabla_\xi P_t^1 f(x) = -\mathbb{E}\bigg\{ & f\big(X_t^x\big)\frac{I_{[N_t\geq 1]}}{N_t} \\ & \times\int_0^t\int_{\mathbb{H}}\big(\langle z, Q^{-1}J_s\xi\rangle + \langle\nabla\log\rho(z), J_s\xi\rangle\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\bigg\},\end{aligned} \tag{6}$$

where $J_t\xi$ is the derivative of X_t^x w.r.t. the initial value x along the direction ξ.

REMARK 1. Compared with Theorem 1.2 in [28], this modified formula is much succinct. Meanwhile, the technical conditions of the Lévy measure ν are also relaxed. The price we pay here is the loss of strong Feller property of P_t. Fortunately, this formula can be applied to prove the following exponential convergence of P_t.

THEOREM 2. *Let F be a Lipschitz continuous function with Lipschitz constant $\|F\|_{\mathrm{Lip}}$. Assume $(\mathbf{H}_\nu)$, $(\mathbf{H}_A)$ and $(\mathbf{H}_{SQ})$ hold. If $\gamma_1>\|F\|_{\mathrm{Lip}}$ and $\lim_{t\to\infty}\frac{\int_0^t\|Q^{-1}S(s)\|^2\,\mathrm{d}s}{t}<\infty$, then there exists a constant $C>0$ such that for x and $y\in\mathbb{H}$,*

$$\big\|P_t(x,\cdot)-P_t(y,\cdot)\big\|_{\mathrm{Var}}\leq C\big(1+|x-y|\big)\exp\bigg\{-\frac{\lambda(\gamma_1-\|F\|_{\mathrm{Lip}})}{\lambda+\gamma_1-\|F\|_{\mathrm{Lip}}}t\bigg\}. \tag{7}$$

REMARK 2. According to Theorem 16.2 in [21], there is a unique invariant measure Ξ for (3). Integrating both sides of (7) w.r.t. $\Xi(\mathrm{d}y)$, we immediately have

$$\big\|P_t(x,\cdot)-\Xi\big\|_{\mathrm{Var}}\leq C\big(1+|x|\big)\exp\bigg\{-\frac{\lambda(\gamma_1-\|F\|_{\mathrm{Lip}})}{\lambda+\gamma_1-\|F\|_{\mathrm{Lip}}}t\bigg\}, \quad \forall x\in\mathbb{H}. \tag{8}$$

We should point out that the exponential convergence (8) is not the same with the one discussed in [21, Theorem 16.2], since here we use the total variation norm not the so-called Fortet-Mourier norm (see [21, Definition 16.2]).

EXAMPLE 1. Consider the following stochastic semilinear equation on $D = [0,T]^d$ with $d \geq 1$ and the Dirichlet boundary condition:

$$\begin{cases} \mathrm{d}X(t,\xi) = [\Delta X(t,\xi) + F(X(t,\xi))]\,\mathrm{d}t + \mathrm{d}Z_t(\xi), \\ X(0,\xi) = x(\xi), \\ X(t,\xi) = 0, \quad \xi \in \partial D, \end{cases} \tag{9}$$

where $\{Z_t\}$ is a square integrable Lévy processes valued on $\mathbb{H} := L^2(D)$ and $F : \mathbb{H} \to \mathbb{H}$ is Lipschitz continuous. It is clear that Δ with a Dirichlet boundary condition has the following eigenfunctions

$$e_k(\xi) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \sin(k_1\xi_1)\sin(k_2\xi_2)\cdots\sin(k_d\xi_d), \quad k \in \mathbb{N}^d, \xi \in D.$$

It is known that $\Delta e_k = -|k|^2 e_k$, i.e.

$$\gamma_k = |k|^2 = k_1^2 + k_2^2 + \cdots + k_d^2, \quad \text{for all } k \in \mathbb{N}^d.$$

We study the dynamics defined by (9) in the space $\mathbb{H} = L^2(D)$ with orthonormal basis $\{e_k\}_{k\in\mathbb{N}^d}$. For $0 < \delta < \frac{1}{2}$, the fractional power $(-\Delta)^\delta$ of $-\Delta$ is defined by

$$(-\Delta)^\delta = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{-\delta} S(t)\,\mathrm{d}t,$$

where Γ is the Euler function. Due to Proposition A.12 in [9], we have $S(t)\mathbb{H} \subset \mathcal{D}((-\Delta)^\delta)$ and for any $t > 0$,

$$\|(-\Delta)^\delta S(t)\| \leq C_\delta t^{-\delta}$$

for a suitable positive constant C_δ. If the operator Q is defined as $Q := ((-\Delta)^\delta)^{-1}$, then we have $S(t)\mathbb{H} \subset \operatorname{Im} Q$. Moreover,

$$\lim_{t\to\infty} \frac{\int_0^t \|Q^{-1}S(s)\|^2\,\mathrm{d}s}{t} \leq \lim_{t\to\infty} \frac{C_\delta^2 \int_0^t s^{-2\delta}\,\mathrm{d}s}{t} = \lim_{t\to\infty} \frac{C_\delta^2 t^{1-2\delta}}{t} = 0.$$

The rest of this paper is organized as follows: in Section 2, we shall investigate an integration by parts formula for jump processes in infinite-dimensional case; the proofs of the main results will be presented in Section 3.

2. Integration by parts formula

The Malliavin calculus has played an important role in many fields as one of powerful tools in infinite-dimensional analysis. An integration by parts formula, which plays an important role in Malliavin calculus, can be used to derive the derivative formula. In finite-dimensional case, the integration by part formulas for jump processes was studied in [6], [5], [4], [20], [30], [29] and so on. But so far, there are few references studying the formula for pure jump processes in infinite-dimensional case.

2.1. Girsanov's theorem. Denote

$$\mathbb{V}_0 = \Big\{ V : \Omega \times [0,T] \to \operatorname{Im} Q \ \Big| \ V \text{ is predictable and } \sup_{\omega\in\Omega}\sup_{t\le T} \big|Q^{-1}V(\omega,t)\big| < \infty \Big\},$$

and

$$\mathbb{V} = \Big\{ V : \Omega \times [0,T] \to \operatorname{Im} Q \ \Big| \ V \text{ is predictable and } \int_0^T \mathbb{E}\big(\big|Q^{-1}V(\cdot,t)\big| + \big|V(\cdot,t)\big|^2\big)\,\mathrm{d}t < \infty \Big\}.$$

In the following, we will drop "ω" in V for the sake of writing. For any $V \in \mathbb{V}$, define a perturbed random measure N^ε by

$$N^\varepsilon\big(\Gamma \times [0,t]\big) = \int_0^t \int_{\mathbb{H}} I_\Gamma\big(z + \varepsilon V(s)\big) N(\mathrm{d}z,\mathrm{d}s), \quad \Gamma \in \mathcal{B}(\mathbb{H}). \tag{10}$$

For $\varepsilon > 0$, let

$$\lambda^\varepsilon(t,z) := \varphi\big(z, \varepsilon V(t)\big)\frac{\rho(z+\varepsilon V(t))}{\rho(z)}, \quad \forall z \in \mathbb{H}, \forall t \in [0,T], \tag{11}$$

where φ is defined by (1). Let

$$\Theta_t^\varepsilon := \exp\Big\{ \int_0^t \int_{\mathbb{H}} \log \lambda^\varepsilon(s,z) N(\mathrm{d}z,\mathrm{d}s) - \int_0^t \int_{\mathbb{H}} \big(\lambda^\varepsilon(s,z) - 1\big)\nu(\mathrm{d}z)\,\mathrm{d}s \Big\}. \tag{12}$$

Recalling that ν is a finite measure, by Itô's formula we can easily check that $\{\Theta_t^\varepsilon, \mathcal{F}_t\}_{t\le T}$ is a uniformly integrative martingale. Moreover, Θ_t^ε can also be written as

$$\Theta_t^\varepsilon = 1 + \int_0^t \int_{\mathbb{H}} \Theta_{s-}^\varepsilon \big(\lambda^\varepsilon(s,z) - 1\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s). \tag{13}$$

By Girsanov's theorem (see [11, Theorem 12.21]), there exists a probability measure $\mathbb{P}^\varepsilon$ such that

$$\frac{\mathrm{d}\mathbb{P}^\varepsilon}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \Theta_t^\varepsilon, \quad t \le T. \tag{14}$$

Then we have the following results.

LEMMA 3. *Let* $(\mathbf{H}_\nu)$ *hold. Then for any* $V \in \mathbb{V}$, ν *is the characteristic measure of* $N^\varepsilon(\mathrm{d}z,\mathrm{d}t)$ *under* $\mathbb{P}^\varepsilon$.

Proof. For any bounded test function $\phi : [0,T] \times \mathbb{H} \to \mathbb{R}$, define

$$Y_t^\varepsilon := \exp\Big\{ \int_0^t \int_{\mathbb{H}} \phi(s,z) N^\varepsilon(\mathrm{d}z,\mathrm{d}s) \Big\}, \quad G_t^\varepsilon := Y_t^\varepsilon \Theta_t^\varepsilon. \tag{15}$$

By Itô's formula, we have

$$Y_t^\varepsilon = \exp\left\{\int_0^t \int_{\mathbb{H}} \phi(s,z) N^\varepsilon(\mathrm{d}z, \mathrm{d}s)\right\} = \exp\left\{\int_0^t \int_{\mathbb{H}} \phi\big(s, z+\varepsilon V(s)\big) N(\mathrm{d}z, \mathrm{d}s)\right\} = 1 + \int_0^t \int_{\mathbb{H}} Y_{s-}^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big) N(\mathrm{d}z, \mathrm{d}s), \tag{16}$$

and

$$\big[Y^\varepsilon, \Theta^\varepsilon\big]_t = \int_0^t \int_{\mathbb{H}} Y_{s-}^\varepsilon \Theta_{s-}^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big)\big(\lambda^\varepsilon(s,z) - 1\big) N(\mathrm{d}z, \mathrm{d}s). \tag{17}$$

It follows from (16), (17) and Itô's formula that

$$\begin{aligned} G_t^\varepsilon &= 1 + \int_0^t \int_{\mathbb{H}} Y_{s-}^\varepsilon \, \mathrm{d}\Theta_s^\varepsilon + \int_0^t \int_{\mathbb{H}} \Theta_{s-}^\varepsilon \, \mathrm{d}Y_s^\varepsilon + \big[Y^\varepsilon, \Theta^\varepsilon\big]_t \\ &= 1 + \int_0^t \int_{\mathbb{H}} Y_{s-}^\varepsilon \, \mathrm{d}\Theta_s^\varepsilon + \int_0^t \int_{\mathbb{H}} \Theta_{s-}^\varepsilon Y_{s-}^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big) N(\mathrm{d}z, \mathrm{d}s) \\ &\quad + \int_0^t \int_{\mathbb{H}} Y_{s-}^\varepsilon \Theta_{s-}^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big)\big(\lambda^\varepsilon(s,z) - 1\big) N(\mathrm{d}z, \mathrm{d}s) \\ &= 1 + \int_0^t Y_{s-}^\varepsilon \, \mathrm{d}\Theta_s^\varepsilon + \int_0^t \int_{\mathbb{H}} G_{s-}^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big) \lambda^\varepsilon(s,z) N(\mathrm{d}z, \mathrm{d}s). \end{aligned}$$

By (13) and (11), we arrive at

$$\begin{aligned} \mathbb{E} G_t^\varepsilon &= 1 + \mathbb{E} \int_0^t \int_{\mathbb{H}} G_{s-}^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big) \lambda^\varepsilon(s,z) N(\mathrm{d}z, \mathrm{d}s) \\ &= 1 + \mathbb{E} \int_0^t \int_{\mathbb{H}} G_s^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big) \varphi\big(z, \varepsilon V(s)\big) \frac{\rho(z+\varepsilon V(s))}{\rho(z)} \rho(z) \mu(\mathrm{d}z) \, \mathrm{d}s \\ &= 1 + \mathbb{E} \int_0^t \int_{\mathbb{H}} G_s^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big) \varphi\big(z, \varepsilon V(s)\big) \rho\big(z+\varepsilon V(s)\big) \mu(\mathrm{d}z) \, \mathrm{d}s \\ &= 1 + \mathbb{E} \int_0^t \int_{\mathbb{H}} G_s^\varepsilon \big(e^{\phi(s,z+\varepsilon V(s))} - 1\big) \rho\big(z+\varepsilon V(s)\big) \mu\big(\mathrm{d}z + \varepsilon V(s)\big) \, \mathrm{d}s \\ &= 1 + \int_0^t \int_{\mathbb{H}} \big(e^{\phi(s,z)} - 1\big) \rho(z) \mu(\mathrm{d}z) \mathbb{E} G_s^\varepsilon \, \mathrm{d}s. \end{aligned}$$

Therefore,

$$\mathbb{E} G_t^\varepsilon = \exp\left\{\int_0^t \int_{\mathbb{H}} \big(e^{\phi(s,z)} - 1\big) \nu(\mathrm{d}z) \, \mathrm{d}s\right\}.$$

Combining this with (14) and (15), we have

$$\begin{aligned}\mathbb{E}^{\varepsilon}\exp\bigg\{\int_0^t\int_{\mathbb{H}}\phi(s,z)N^{\varepsilon}(\mathrm{d}z,\mathrm{d}s)\bigg\}\\ =\exp\bigg\{\int_0^t\int_{\mathbb{H}}\big(e^{\phi(s,z)}-1\big)\nu(\mathrm{d}z)\,\mathrm{d}s\bigg\}.\end{aligned}$$

That is, $\nu(\mathrm{d}z)$ is the characteristic measure of $N^{\varepsilon}(\mathrm{d}z,\mathrm{d}t)$ under $\mathbb{P}^{\varepsilon}$. □

LEMMA 4. *Let* $(\mathbf{H}_{\nu})$ *hold. Assume there exists a constant* $\delta>0$ *such that* $\rho(z)\geq\delta$ *for* $\forall z\in\mathbb{H}$. *Then for any* $V\in\mathbb{V}_0$,

$$\sup_{\varepsilon<1}\mathbb{E}\bigg\{\sup_{0\leq t\leq T}\bigg|\frac{\Theta_t^{\varepsilon}-1}{\varepsilon}\bigg|^2\bigg\}<\infty.$$

Proof. Since $V\in\mathbb{V}_0$, then one has

$$\sup_{\omega\in\Omega,t\in[0,T]}\big(\big|V(\omega,t)\big|+\big|Q^{-1}V(\omega,t)\big|\big)\leq C(T), \tag{18}$$

where $C(T)$, independent of ε, is a generic constant whose values might change from line to line. For any $t\in[0,T]$, $z\in\mathbb{H}$ and $\varepsilon\in(0,1)$, it follows from (1) and (18) that

$$\begin{aligned}\varphi^2\big(z,\varepsilon V(t)\big)&=\varphi\big(z,2\varepsilon V(t)\big)\exp\big\{\varepsilon^2\big\langle V(t),Q^{-1}V(t)\big\rangle\big\}\\ &\leq C(T)\varphi\big(z,2\varepsilon V(t)\big).\end{aligned}\tag{19}$$

Then by mean value theorem, we have

$$\begin{aligned}I_1:=&\int_{\mathbb{H}}\frac{\varphi^2(z,\varepsilon V(t))}{\rho(z)}\bigg|\frac{\rho(z+\varepsilon V(t))-\rho(z)}{\varepsilon}\bigg|^2\mu(\mathrm{d}z)\\ &\leq\frac{1}{\delta}\|\nabla\rho\|_{\infty}^2\big|V(t)\big|^2\int_{\mathbb{H}}\varphi^2\big(z,\varepsilon V(t)\big)\mu(\mathrm{d}z)\\ &\leq\frac{C(T)}{\delta}\|\nabla\rho\|_{\infty}^2\int_{\mathbb{H}}\varphi\big(z,2\varepsilon V(t)\big)\mu(\mathrm{d}z)\\ &\leq C(T).\end{aligned}\tag{20}$$

Also, there exists a constant $\varepsilon_1\in(0,\varepsilon)$ such that

$$\begin{aligned}I_2:=&\int_{\mathbb{H}}\bigg|\frac{\varphi(z,\varepsilon V(t))-1}{\varepsilon}\bigg|^2\rho(z)\mu(\mathrm{d}z)\\ =&\int_{\mathbb{H}}\varphi^2\big(z,\varepsilon_1V(t)\big)\big|\big\langle z-\varepsilon_1V(s),Q^{-1}V(t)\big\rangle\big|^2\rho(z)\mu(\mathrm{d}z)\\ \leq&\, C(T)\int_{\mathbb{H}}\varphi^2\big(z,\varepsilon_1V(t)\big)|z|^2\rho(z)\mu(\mathrm{d}z)\\ &+C(T)\int_{\mathbb{H}}\varphi^2\big(z,\varepsilon_1V(t)\big)\rho(z)\mu(\mathrm{d}z)\end{aligned}\tag{21}$$

$$\le C(T)\int_{\mathbb{H}} \varphi\big(z, 2\varepsilon_1 V(t)\big)|z|^2\rho(z)\mu(\mathrm{d}z) + C(T)\int_{\mathbb{H}} \varphi\big(z, 2\varepsilon_1 V(t)\big)\rho(z)\mu(\mathrm{d}z).$$

By triangle inequality and mean value theorem, one has

$$\begin{aligned}
I_3 &:= \int_{\mathbb{H}} \varphi\big(z, 2\varepsilon_1 V(t)\big)|z|^2\rho(z)\mu(\mathrm{d}z) \\
&= \int_{\mathbb{H}} |z|^2\rho(z)\mu\big(\mathrm{d}z + 2\varepsilon_1 V(t)\big) \\
&\le 2\int_{\mathbb{H}} \big|z + 2\varepsilon_1 V(t)\big|^2 \big|\rho\big(z + 2\varepsilon_1 V(t)\big) - \rho(z)\big| \mu\big(\mathrm{d}z + 2\varepsilon_1 V(t)\big) \\
&\quad + 2\int_{\mathbb{H}} \big|z + 2\varepsilon_1 V(t)\big|^2 \rho\big(z + 2\varepsilon_1 V(t)\big)\mu\big(\mathrm{d}z + 2\varepsilon_1 V(t)\big) \\
&\quad + 2\int_{\mathbb{H}} \big|2\varepsilon_1 V(t)\big|^2 \big|\rho\big(z + 2\varepsilon_1 V(t)\big) - \rho(z)\big| \mu\big(\mathrm{d}z + 2\varepsilon_1 V(t)\big) \\
&\quad + 2\int_{\mathbb{H}} \big|2\varepsilon_1 V(t)\big|^2 \rho\big(z + 2\varepsilon_1 V(t)\big)\mu\big(\mathrm{d}z + 2\varepsilon_1 V(t)\big) \\
&\le C(T)\|\nabla\rho\|_\infty \int_{\mathbb{H}} |z|^2\mu(\mathrm{d}z) + 2\int_{\mathbb{H}} |z|^2\rho(z)\mu(\mathrm{d}z) \\
&\quad + C(T)\|\nabla\rho\|_\infty + C(T)\int_{\mathbb{H}} \rho(z)\mu(\mathrm{d}z) \\
&\le C(T).
\end{aligned} \tag{22}$$

Similar arguments give that

$$I_4 := \int_{\mathbb{H}} \varphi\big(z, 2\varepsilon_1 V(t)\big)\rho(z)\mu(\mathrm{d}z) \le C(T). \tag{23}$$

It follows from (21), (22) and (23) that

$$\begin{aligned}
I_2 &= \int_{\mathbb{H}} \left| \frac{\varphi(z, \varepsilon V(t)) - 1}{\varepsilon} \right|^2 \rho(z)\mu(\mathrm{d}z) \\
&\le C(T)I_3 + C(T)I_4 \le C(T).
\end{aligned} \tag{24}$$

By (11) and triangle inequality, we have

$$\begin{aligned}
&\int_{\mathbb{H}} \left| \frac{\lambda^\varepsilon(t,z) - 1}{\varepsilon} \right|^2 \rho(z)\mu(\mathrm{d}z) \\
&\quad = \frac{1}{\varepsilon^2}\int_{\mathbb{H}} \left| \varphi\big(z, \varepsilon V(t)\big)\frac{\rho(z + \varepsilon V(t))}{\rho(z)} - 1 \right|^2 \rho(z)\mu(\mathrm{d}z) \\
&\quad \le 2I_1 + 2I_2 \le C(T).
\end{aligned} \tag{25}$$

Combining this with (13) and Burkholder's inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{t\le T}\left|\frac{\Theta_t^\varepsilon-1}{\varepsilon}\right|^2\right\}\\
&\quad=\mathbb{E}\left\{\sup_{t\le T}\left|\int_0^t\int_{\mathbb{H}}\frac{\Theta_{s-}^\varepsilon(\lambda^\varepsilon(s,z)-1)}{\varepsilon}\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\right|^2\right\}\\
&\quad\le C(T)\mathbb{E}\left\{\int_0^T\int_{\mathbb{H}}\sup_{s\le t}\left|\frac{\Theta_s^\varepsilon-1}{\varepsilon}\right|^2\left|\lambda^\varepsilon(t,z)-1\right|^2\rho(z)\mu(\mathrm{d}z)\,\mathrm{d}t\right\}\\
&\qquad+C(T)\mathbb{E}\left\{\int_0^T\int_{\mathbb{H}}\left|\frac{\lambda^\varepsilon(t,z)-1}{\varepsilon}\right|^2\rho(z)\mu(\mathrm{d}z)\,\mathrm{d}t\right\}\\
&\quad\le C(T)\int_0^T\mathbb{E}\sup_{s\le t}\left|\frac{\Theta_s^\varepsilon-1}{\varepsilon}\right|^2\mathrm{d}t+C(T).
\end{aligned}
$$

Then Gronwall's inequality implies

$$\mathbb{E}\left\{\sup_{0\le t\le T}\left|\frac{\Theta_t^\varepsilon-1}{\varepsilon}\right|^2\right\}\le C(T).$$

Furthermore, we have

$$\sup_{\varepsilon<1}\mathbb{E}\left\{\sup_{0\le t\le T}\left|\frac{\Theta_t^\varepsilon-1}{\varepsilon}\right|^2\right\}<\infty.$$

□

2.2. Malliavin derivatives. Let $G_t:=G(\{N(\mathrm{d}z,\mathrm{d}s)\}_{s\le t})$ be an $\mathbb{H}$-valued functional. Denote $G_t^\varepsilon:=G(\{N^\varepsilon(\mathrm{d}z,\mathrm{d}s)\}_{s\le t})$, where N^ε is defined in (10).

DEFINITION 5. A Poisson functional G_t is called to be Malliavin differentiable along some $V\in\mathbb{V}$, if for some $p\ge 1$ there exists a random variable denoted by D_VG_t with $\mathbb{E}|D_VG_t|^p<\infty$, such that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left|\varepsilon^{-1}\left(G_t^\varepsilon-G_t\right)-D_VG_t\right|^p=0. \tag{26}$$

REMARK 3. This notion appeared firstly in [4]. It is based on Bismut's approach about the Malliavin calculus with jumps in [6].

Before we move on, it is necessary for us to prove the Malliavin differentiability of the solution to (2).

PROPOSITION 6. *Let* $(\mathbf{H}_\nu)$ *hold. Assume* A *generates the* C_0*-semigroup* $\{S(t)\}_{t\ge 0}$ *and* $F\in C_b^2(\mathbb{H},\mathbb{H})$. *Then for any* $V\in\mathbb{V}$, X_t *is Malliavin differentiable along* V. *Moreover, the derivative satisfies the following equation*:

$$
\begin{cases}
\mathrm{d}D_VX_t=AD_VX_t\,\mathrm{d}t+\nabla F(X_t)D_VX_t\,\mathrm{d}t+\int_{\mathbb{H}}V(t)N(\mathrm{d}z,\mathrm{d}t),\\
D_VX_0=0.
\end{cases}\tag{27}
$$

Proof. It is easy to prove that there exists a unique solution to the linear equation (27). And the solution can be written as

$$D_V X_t = \int_0^t S(t-s)\nabla F(X_s) D_V X_s \,\mathrm{d}s + \int_0^t \int_{\mathbb{H}} S(t-s)V(s)N(\mathrm{d}z,\mathrm{d}s). \tag{28}$$

For each $\varepsilon > 0$, let L^ε be the perturbed process of L,

$$L_t^\varepsilon := \int_0^t \int_{\mathbb{H}} z N^\varepsilon(\mathrm{d}z,\mathrm{d}s), \quad \forall t \in [0,T].$$

Then by (10) we have

$$L_t^\varepsilon = L_t + \varepsilon \int_0^t \int_{\mathbb{H}} V(s)N(\mathrm{d}z,\mathrm{d}s), \quad \forall t \in [0,T]. \tag{29}$$

Let $\{X_t^\varepsilon\}_{t\le T}$ be the solution of the following equation:

$$X_t^\varepsilon = x + \int_0^t S(t-s)F\left(X_s^\varepsilon\right)\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}L_s^\varepsilon + \int_0^t S(t-s)\,\mathrm{d}Z_s. \tag{30}$$

Now we aim to prove

$$\lim_{\varepsilon\to 0} \mathbb{E}\left\{\sup_{s\le t}\left|\frac{X_s^\varepsilon - X_s}{\varepsilon} - D_V X_s\right|\right\} = 0. \tag{31}$$

In fact, it follows (3), (29) and (30) that

$$X_t^\varepsilon - X_t = \int_0^t S(t-s)\left(F\left(X_s^\varepsilon\right) - F(X_s)\right)\mathrm{d}s + \varepsilon \int_0^t \int_{\mathbb{H}} S(t-s)V(s)N(\mathrm{d}z,\mathrm{d}s). \tag{32}$$

Since A is a generator of C_0-semigroup on $\mathbb{H}$, then there exist constant $C_A > 0$ and $\kappa > 0$ such that $\|S(t)\| \le C_A e^{\kappa t}$ for $t \in [0,T]$. Therefore,

$$\begin{aligned}
\left|X_t^\varepsilon - X_t\right| &\le \int_0^t \left|S(t-s)\left(F\left(X_s^\varepsilon\right) - F(X_s)\right)\right|\mathrm{d}s \\
&\quad + \varepsilon \int_0^t \int_{\mathbb{H}} \left|S(t-s)V(s)\right| N(\mathrm{d}z,\mathrm{d}s) \\
&\le C_A \|\nabla F\|_\infty \int_0^t e^{\kappa(t-s)} \left|X_s^\varepsilon - X_s\right|\mathrm{d}s \\
&\quad + \varepsilon C_A \int_0^t \int_{\mathbb{H}} e^{\kappa(t-s)} \left|V(s)\right| N(\mathrm{d}z,\mathrm{d}s).
\end{aligned}$$

Then,

$$\sup_{s\le t}\left\{e^{-\kappa s}\left|X_s^\varepsilon - X_s\right|\right\} \le C_A\|\nabla F\|_\infty \int_0^t \sup_{r\le s}\left\{e^{-\kappa r}\left|X_r^\varepsilon - X_r\right|\right\}\mathrm{d}s + \varepsilon C_A \int_0^t\int_{\mathbb{H}} e^{-\kappa s}\left|V(s)\right| N(\mathrm{d}z,\mathrm{d}s).$$

By Gronwall's inequality, we have

$$\sup_{s\le t}\left|X_s^\varepsilon - X_s\right| \le \varepsilon C_A \exp\left\{\kappa t + C_A\|\nabla F\|_\infty t\right\}\int_0^t\int_{\mathbb{H}}\left|V(s)\right| N(\mathrm{d}z,\mathrm{d}s). \tag{33}$$

By (28), (32) and Taylor's formula, we arrive at

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\left(X_t^\varepsilon - X_t\right) - D_V X_t\right| \\
&\quad\le C_A\int_0^t e^{\kappa(t-s)}\left|\frac{1}{\varepsilon}\left(F\left(X_s^\varepsilon\right) - F(X_s)\right) - \nabla F(X_s) D_V X_s\right|\mathrm{d}s \\
&\quad\le C_A\int_0^t e^{\kappa(t-s)}\left\{\|\nabla F\|_\infty\left|\frac{1}{\varepsilon}\left(X_s^\varepsilon - X_s\right) - D_V X_s\right|\right. \\
&\qquad\left. + \frac{1}{\varepsilon}\left\|\nabla^2 F\right\|_\infty\left|X_s^\varepsilon - X_s\right|^2\right\}\mathrm{d}s.
\end{aligned}$$

Gronwall's inequality implies

$$\begin{aligned}
&\sup_{s\le t}\left|\frac{1}{\varepsilon}\left(X_s^\varepsilon - X_s\right) - D_V X_s\right| \\
&\quad\le C_A\left\|\nabla^2 F\right\|_\infty t\exp\left\{C\|\nabla F\|_\infty t + \kappa t\right\}\frac{1}{\varepsilon}\sup_{s\le t}\left|X_s^\varepsilon - X_s\right|^2.
\end{aligned}$$

Combining this with (33), we can obtain

$$\mathbb{E}\left\{\sup_{s\le t}\left|\frac{1}{\varepsilon}\left(X_s^\varepsilon - X_s\right) - D_V X_s\right|\right\} \le C_1\mathbb{E}\left\{\int_0^t\int_{\mathbb{H}}\left|V(s)\right| N(\mathrm{d}z,\mathrm{d}s)\right\}^2\varepsilon,$$

where C_1 is a constant independent of ε. In view of $V\in\mathbb{V}$ and ν is a finite measure, we have

$$\begin{aligned}
&\mathbb{E}\left\{\int_0^t\int_{\mathbb{H}}\left|V(s)\right| N(\mathrm{d}z,\mathrm{d}s)\right\}^2 \\
&\quad\le 2\nu(\mathbb{H})\int_0^t \mathbb{E}V^2(s)\,\mathrm{d}s + 2\nu(\mathbb{H})^2\mathbb{E}\left(\int_0^t \mathbb{E}V(s)\,\mathrm{d}s\right)^2 \\
&\quad\le 2\left(\nu(\mathbb{H}) + \nu(\mathbb{H})^2 T\right)\int_0^t \mathbb{E}V^2(s)\,\mathrm{d}s < \infty.
\end{aligned}$$

Therefore, there exists a constant $C_2 > 0$ such that

$$\mathbb{E}\left\{\sup_{s\le t}\left|\frac{1}{\varepsilon}\left(X_s^\varepsilon - X_s\right) - D_V X_s\right|\right\} \le C_2\varepsilon, \tag{34}$$

which yields (31). □

PROPOSITION 7 (Chain rule). *Let* $(\mathbf{H}_\nu)$ *hold. Assume* A *generates a* C_0*-semigroup* $\{S(t)\}_{t\ge 0}$ *and* $F \in C_b^2(\mathbb{H},\mathbb{H})$. *Then for any* $f \in C_b^2(\mathbb{H})$ *and* $V \in \mathbb{V}$, $f(X_t)$ *is Malliavin differentiable and*

$$D_V f(X_t) = \left\langle \nabla f(X_t), D_V X_t\right\rangle. \tag{35}$$

Proof. In view of Proposition 6, we have

$$\mathbb{E}\left|\left\langle \nabla f(X_t), D_V X_t\right\rangle\right| \le \|\nabla f\|_\infty \mathbb{E}|D_V X_t| < \infty.$$

Meanwhile, by Taylor's formula we obtain

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\left(f\left(X_t^\varepsilon\right) - f(X_t)\right) - \left\langle \nabla f(X_t), D_V X_t\right\rangle\right| \\ &\quad \le \|\nabla f\|_\infty \left|\frac{1}{\varepsilon}\left(X_t^\varepsilon - X_t\right) - D_V X_t\right| + \frac{1}{2\varepsilon}\left\|\nabla^2 f\right\|_\infty \left|X_t^\varepsilon - X_t\right|^2.\end{aligned}$$

Combining this with (33) and (34), we arrive at

$$\lim_{\varepsilon\to 0}\mathbb{E}\left|\frac{1}{\varepsilon}\left(f\left(X_t^\varepsilon\right) - f(X_t)\right) - \left\langle \nabla f(X_t), D_V X_t\right\rangle\right| = 0. \qquad \square$$

2.3. Integration by parts formula. Now we are ready to give the following integration by parts formula.

THEOREM 8. *Let* $(\mathbf{H}_\nu)$ *hold. Assume the operator* A *generates a strongly continuous semigroups* $\{S(t)\}_{t\ge 0}$ *and* $F \in C_b^2(\mathbb{H},\mathbb{H})$. *Then for any* $f \in C_b^2(\mathbb{H})$ *and* $V \in \mathbb{V}$, *we have*

$$\mathbb{E}\left\{D_V f(X_t)\right\} = -\mathbb{E}\left\{f(X_t) M_t\right\}, \quad t \le T, \tag{36}$$

with

$$M_t = \int_0^t \int_{\mathbb{H}} \left(\left\langle z, Q^{-1}V(s)\right\rangle + \left\langle \nabla \log\rho(z), V(s)\right\rangle\right)\widetilde{N}(\mathrm{d}z,\mathrm{d}s).$$

REMARK 4. From the proof below, we will see that the formula hold for Poisson functionals which are Malliavin differentiable along some $V \in \mathbb{V}$.

Proof. We give the proof in three steps.

Step 1. Assume $V \in \mathbb{V}_0$ and $\rho \ge \delta$ for some $\delta > 0$. By Lemma 3, for any $\varepsilon \in (0,1)$, we have

$$\mathbb{E}f(X_t) = \mathbb{E}\left\{f\left(X_t^\varepsilon\right)\Theta_t^\varepsilon\right\}.$$

Then,

$$\frac{1}{\varepsilon}\mathbb{E}\left(f\left(X_t^\varepsilon\right)\Theta_t^\varepsilon - f(X_t)\right) = 0.$$

Furthermore,

$$\frac{1}{\varepsilon}\mathbb{E}\big(f\big(X_t^\varepsilon\big)-f(X_t)\big)+\frac{1}{\varepsilon}\mathbb{E}f\big(X_t^\varepsilon\big)\big(\Theta_t^\varepsilon-1-\varepsilon M_t\big)+\mathbb{E}f\big(X_t^\varepsilon\big)M_t=0, \tag{37}$$

where

$$M_t=\int_0^t\int_{\mathbb{H}}\big(\big\langle z,Q^{-1}V(s)\big\rangle+\big\langle\nabla\log\rho(z),V(s)\big\rangle\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s).$$

For the first and third terms of (37), by Proposition 7, we obtain

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\big(f\big(X_t^\varepsilon\big)-f(X_t)\big)&=\mathbb{E}D_Vf(X_t),\\ \lim_{\varepsilon\to0}\mathbb{E}\big\{f\big(X_t^\varepsilon\big)M_t\big\}&=\mathbb{E}\big\{f(X_t)M_t\big\}.\end{aligned} \tag{38}$$

It follows from (11) and (12) that

$$\lim_{\varepsilon\to0}\Theta_s^\varepsilon=1,\quad\forall s\in(0,T].$$

Observe that ν is a finite measure, then also by (11) and (12) we arrive at

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big(\Theta_t^\varepsilon-1\big)&=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{H}}\Theta_{s-}^\varepsilon\big(\lambda^\varepsilon(s,z)-1\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\\ &=\int_0^t\int_{\mathbb{H}}\lim_{\varepsilon\to0}\Theta_{s-}^\varepsilon\frac{1}{\varepsilon}\big(\lambda^\varepsilon(s,z)-1\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\\ &=\int_0^t\int_{\mathbb{H}}\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big(\lambda^\varepsilon(s,z)-1\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\\ &=\int_0^t\int_{\mathbb{H}}\big(\big\langle z,Q^{-1}V(s)\big\rangle+\big\langle\nabla\log\rho(z),V(s)\big\rangle\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s).\end{aligned}$$

Combining this with Lemma 4, we obtain

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\big|\Theta_t^\varepsilon-1-\varepsilon M_t\big|=0.$$

Then

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big|\mathbb{E}f(X_t)\big(\Theta_t^\varepsilon-1-\varepsilon M_t\big)\big|\le\|f\|_\infty\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}\big|\Theta_t^\varepsilon-1-\varepsilon M_t\big|=0. \tag{39}$$

Letting $\varepsilon\to0$ in (37), by (38) and (39), we derive (36).

Step 2. Assume $V\in\mathbb{V}_0$. For $n\ge1$, let $L^n:=\{L_t^n\}_{t\ge0}$ be a purely jump Lévy process with characteristic measure $\frac{1}{n}\mu$ and jump measure $N^n(\mathrm{d}z,\mathrm{d}t)$. Assume that L^n, L and Z are independent. Now, $L+L^n$ is a jump process with Lévy measure $\nu+\frac{1}{n}\mu$. Moreover, its jump measure is $N_n(\mathrm{d}z,\mathrm{d}t):=N(\mathrm{d}z,\mathrm{d}t)+N^n(\mathrm{d}z,\mathrm{d}t)$. Let $\widetilde{N_n}(\mathrm{d}z,\mathrm{d}t)$ be the associated martingale measure, that is, $\widetilde{N_n}(\mathrm{d}z,\mathrm{d}t):=N_n(\mathrm{d}z,\mathrm{d}t)-(\rho(z)+\frac{1}{n})\mu(\mathrm{d}z)\,\mathrm{d}t$. Let D_V^n be the derivative operator associated with $N_n(\mathrm{d}z,\mathrm{d}t)$, which is defined as in Definition 5.

Let $\{X_t^n\}_{t\le T}$ be the solution to the following equation:

$$\begin{cases} \mathrm{d}X_t^n = AX_t^n\,\mathrm{d}t + F(X_t^n)\,\mathrm{d}t + \mathrm{d}L_t + \mathrm{d}L_t^n + \mathrm{d}Z_t, \\ X_0 = x. \end{cases} \tag{40}$$

Then it is easy to see that

$$\lim_{n\to\infty} \mathbb{E}\Big\{\sup_{s\le t}\big|X_s^n - X_s\big|^2\Big\} = 0, \quad \forall t \in [0,T]. \tag{41}$$

According to Proposition 6, the derivative $D_V^n X_t^n$ exists and satisfies

$$D_V^n X_t^n = \int_0^t S(t-s)\nabla F\big(X_s^n\big) D_V^n X_s^n \,\mathrm{d}s + \int_0^t \int_{\mathbb{H}} V(s) N_n(\mathrm{d}z, \mathrm{d}s). \tag{42}$$

Moreover, we have $\mathbb{E}\{\sup_{s\le t}|D_V^n X_s|^2\} < \infty$. By Step 1,

$$\mathbb{E}D_V^n f\big(X_t^n\big) = -\mathbb{E}\big\{f\big(X_t^n\big)M_t^n\big\}, \tag{43}$$

where

$$M_t^n = \int_0^t \int_{\mathbb{H}} \biggl(\bigl\langle z, Q^{-1}V(s)\bigr\rangle + \biggl\langle \frac{\nabla\rho(z)}{\rho(z)+\frac{1}{n}}, V(s)\biggr\rangle\biggr)\widetilde{N_n}(\mathrm{d}z, \mathrm{d}s).$$

Then it follows from (28) and (42) that

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{s\le t}\big|D_V^n X_s - D_V X_s\big|\Big\} \\
&\quad\le \mathbb{E}\int_0^t \big|S(t-s)\nabla F\big(X_s^n\big)D_V^n X_s^n - S(t-s)\nabla F(X_s)D_V X_s\big|\,\mathrm{d}s \\
&\qquad + \mathbb{E}\int_0^t \big|S(t-s)V(s)\big| N^n(\mathrm{d}z, \mathrm{d}s) \\
&\quad\le \mathbb{E}\int_0^t \big|S(t-s)\nabla F\big(X_s^n\big)D_V^n X_s^n - S(t-s)\nabla F\big(X_s^n\big)D_V X_s\big|\,\mathrm{d}s + \frac{C(t)}{n} \\
&\qquad + \mathbb{E}\int_0^t \big|S(t-s)\nabla F\big(X_s^n\big)D_V X_s - S(t-s)\nabla F(X_s)D_V X_s\big|\,\mathrm{d}s \\
&\quad\le C(t)\int_0^t \mathbb{E}\Big(\sup_{r\le s}\big|D_V^n X_r^n - D_V X_r\big|\Big)\,\mathrm{d}s + \frac{C(t)}{n} \\
&\qquad + C(t)\Big(\mathbb{E}\sup_{s\le t}\big|X_s^n - X_s\big|^2\Big)^{\frac{1}{2}},
\end{aligned}$$

where $C(t)$ is a generic constant independent of n. Gronwall's inequality yields

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\Big\{\sup_{s\le t}\big|D_V^n X_s - D_V X_s\big|\Big\} \\
&\qquad \le C(t)\lim_{n\to\infty}\bigg\{\frac{1}{n} + \Big(\mathbb{E}\sup_{s\le t}\big|X_s^n - X_s\big|^2\Big)^{\frac{1}{2}}\bigg\} = 0.
\end{aligned} \tag{44}$$

By Proposition 7, the triangle inequality and Hörder inequality, one can derive

$$\begin{aligned}\lim_{n\to\infty}\big|\mathbb{E}D_V^n f\big(X_t^n\big)-\mathbb{E}D_V f(X_t)\big| &\le \lim_{n\to\infty}\mathbb{E}\big|\big\langle\nabla f\big(X_t^n\big),D_V^n X_t\big\rangle-\big\langle\nabla f(X_t),D_V X_t\big\rangle\big|\\ &\le \lim_{n\to\infty}\mathbb{E}\big|\big\langle\nabla f\big(X_t^n\big),D_V^n X_t\big\rangle-\big\langle\nabla f\big(X_t^n\big),D_V X_t\big\rangle\big|\\ &\quad+\lim_{n\to\infty}\mathbb{E}\big|\big\langle\nabla f\big(X_t^n\big),D_V X_t\big\rangle-\big\langle\nabla f(X_t),D_V X_t\big\rangle\big|\\ &\le\|\nabla f\|_\infty\lim_{n\to\infty}\mathbb{E}\big|D_V^n X_t^n-D_V X_t\big|\\ &\quad+\big\|\nabla^2 f\big\|_\infty\big\{\mathbb{E}|D_V X_t|^2\big\}^{\frac12}\lim_{n\to\infty}\big\{\mathbb{E}\big|X_t^n-X_t\big|^2\big\}^{\frac12}=0.\end{aligned}\tag{45}$$

Meanwhile, by the same argument, one can also have

$$\lim_{n\to\infty}\mathbb{E}\big\{f\big(X_t^n\big)M_t^n\big\}=\mathbb{E}\big\{f(X_t)M_t\big\}.\tag{46}$$

Now, using (45), (46) and letting $n\to\infty$ in (43), we obtain (36).

Step 3. Assume $V\in\mathbb{V}$. For $n\ge 1$, define

$$V_n(t)=V(t)I_{[0,n]}\big(\big|Q^{-1}V(t)\big|\big),\quad t\in[0,T].$$

Then $V_n\in\mathbb{V}_0$. By Step 2, we have

$$\mathbb{E}D_{V_n}f(X_t)=-\mathbb{E}\big\{f(X_t)M_t^{(n)}\big\},\tag{47}$$

with

$$M_t^{(n)}=\int_0^t\int_{\mathbb{H}}\big(\big\langle z,Q^{-1}V_n(s)\big\rangle+\big\langle\nabla\log\rho(z),V_n(s)\big\rangle\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s).$$

It is easy to check that

$$\lim_{n\to\infty}\mathbb{E}\big|M_t^n-M_t\big|=0.\tag{48}$$

Meanwhile, observe that

$$\begin{aligned}\mathbb{E}|D_{V_n}X_t-D_V X_t| &\le\mathbb{E}\int_0^t\big|S(t-s)\nabla F(X_s)(D_{V_n}X_s-D_V X_s)\big|\,\mathrm{d}s\\ &\quad+\mathbb{E}\int_0^t\int_{\mathbb{H}}\big|S(t-s)\big(V_n(s)-V(s)\big)\big|N(\mathrm{d}z,\mathrm{d}s)\\ &\le C_A\|\nabla F\|_\infty\int_0^t e^{(t-s)\kappa}\mathbb{E}|D_{V_n}X_s-D_V X_s|\,\mathrm{d}s\\ &\quad+C_A\lambda\int_0^t e^{(t-s)\kappa}\mathbb{E}\big|V_n(s)-V(s)\big|\,\mathrm{d}s.\end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned}&\mathbb{E}|D_{V_n}X_t-D_V X_t|\\ &\quad\le C_A\lambda\exp\big\{\big(C_A\|\nabla F\|_\infty+\kappa\big)t\big\}\int_0^t\mathbb{E}\big|V_n(s)-V(s)\big|\,\mathrm{d}s,\end{aligned}$$

which tends to 0 as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \left|\mathbb{E}D_{V_n} f(X_t) - \mathbb{E}D_V f(X_t)\right| \le \|\nabla f\|_\infty \lim_{n\to\infty} \mathbb{E}|D_{V_n} X_t - D_V X_t| = 0. \tag{49}$$

With the help of (48) and (49), we finish the proof by letting $n \to \infty$ in (47). □

3. Proofs of main results

We use the notation J_{st} with $s \le t$ for the derivative flow between times s and t, that is, for every $\xi \in \mathbb{H}$, $J_{st}\xi$ is the solution of

$$\begin{cases} \mathrm{d}J_{st}\xi = AJ_{st}\xi\,\mathrm{d}t + \nabla F(X_t)J_{st}\xi\,\mathrm{d}t, \\ J_{ss}\xi = \xi. \end{cases} \tag{50}$$

Note that we have the important cocycle property $J_{st} = J_{rt}J_{sr}$ for $r \in [s,t]$. By (27) and (50), we can derive

$$D_V X_t = \int_0^t \int_{\mathbb{H}} J_{st}V(s)N(\mathrm{d}z, \mathrm{d}s). \tag{51}$$

In the following discussions, for the sake of writing, let us denote $J_t := J_{0t}$. To summarize, $J_t\xi$ is the effect on X_t of an infinitesimal perturbation of the initial condition along the direction ξ and $D_V X_t$ is the effect on X_t of an infinitesimal perturbation of the Poisson jump measure $N(\mathrm{d}z, \mathrm{d}t)$ along the direction V.

3.1. Proof of Theorem 1.

We will give the proof in two steps.

Step1: Let us first assume $F \in C_b^2(\mathbb{H}, \mathbb{H})$. For each $\xi \in \mathbb{H}$, by (50) we have

$$J_t\xi = S(t)\xi + \int_0^t S(t-s)\nabla F(X_s)J_s\xi\,\mathrm{d}s. \tag{52}$$

In view of $(\mathbf{H}_A)$, we have $\|S(t)\| \le e^{-\gamma_1 t}$ for each $t \ge 0$. Then

$$|J_t\xi| \le e^{-\gamma_1 t}|\xi| + \|\nabla F\|_\infty \int_0^t e^{-\gamma_1(t-s)}|J_s\xi|\,\mathrm{d}s.$$

Gronwall's inequality implies

$$|J_t\xi| \le \exp\bigl\{\bigl(-\gamma_1 + \|\nabla F\|_\infty\bigr)t\bigr\}|\xi|. \tag{53}$$

Observe that

$$\begin{aligned} Q^{-1}J_t\xi &= Q^{-1}S(t)\xi + Q^{-1}\int_0^t S(t-s)\nabla F(X_s)J_s\xi\,\mathrm{d}s \\ &= Q^{-1}S(t)\xi + \int_0^t Q^{-1}S(t-s)\nabla F(X_s)J_s\xi\,\mathrm{d}s, \end{aligned} \tag{54}$$

where we use $(\mathbf{H}_{SQ})$ in the second equality. Then one has

$$\begin{aligned}\int_0^t \left|Q^{-1}J_s\xi\right| \mathrm{d}s \le& \int_0^t \left|Q^{-1}S(s)\xi\right| \mathrm{d}s \\ &+ \|\nabla F\|_\infty \int_0^t \int_0^s \left\|Q^{-1}S(s-r)\right\| |J_r\xi| \,\mathrm{d}r\,\mathrm{d}s \\ &\le \big(1+\exp\big\{\big(-\gamma_1+\|\nabla F\|_\infty\big)t\big\}t\big)|\xi| \int_0^t \left\|Q^{-1}S(s)\right\| \mathrm{d}s < \infty,\end{aligned}$$

which together with (53) show $\{J_s\xi\}_{s\le T} \in \mathbb{V}$. Now set $V(s) = J_s\xi$ in (51). Then we obtain $D_V X_t = N_t J_t \xi$ with $N_t := N([0,t]\times\mathbb{H})$. Therefore,

$$\frac{I_{[N_t\ge 1]}}{N_t} D_V X_t = J_t\xi I_{[N_t\ge 1]}, \tag{55}$$

where $I_{[N_t\ge 1]}$ is an indicator function and we let $\frac{0}{0}=0$ for convention. Let $\beta : [0,\infty) \to [0,\infty)$ be a smooth function satisfying $\beta(y) = y^2, y \in [0, \frac{1}{2}]$ and $\beta(y) \equiv 1, y \in [1,\infty)$. By Proposition 7 and the fact $D_V N_t = 0$, we have

$$D_V\left\{\frac{I_{[N_t\ge 1]}}{N_t}\right\} = D_V\left\{\frac{\beta(N_t)}{N_t}\right\} = \frac{\beta'(N_t)N_t - \beta(N_t)}{N_t^2} D_V N_t = 0. \tag{56}$$

It follows from (55), (56) and Theorem 8 that

$$\begin{aligned}\nabla_\xi P_t^1 f(x) &= \nabla_\xi \mathbb{E}\big\{f\big(X_t^x\big)I_{[N_t\ge 1]}\big\} \\ &= \mathbb{E}\big\langle \nabla f\big(X_t^x\big), J_t\xi I_{[N_t\ge 1]}\big\rangle \\ &= \mathbb{E}\left\langle \nabla f\big(X_t^x\big), \frac{I_{[N_t\ge 1]}}{N_t} D_V X_t^x \right\rangle \\ &= \mathbb{E}\left\{ D_V f\big(X_t^x\big) \frac{I_{[N_t\ge 1]}}{N_t} \right\} \\ &= \mathbb{E}\left\{ D_V\left(f\big(X_t^x\big) \frac{I_{[N_t\ge 1]}}{N_t} \right) \right\} \\ &= -\mathbb{E}\Bigg\{ f\big(X_t^x\big) \frac{I_{[N_t\ge 1]}}{N_t} \\ &\quad \times \int_0^t \int_{\mathbb{H}} \big(\big\langle z, Q^{-1}J_s\xi\big\rangle + \big\langle \nabla \log\rho(z), J_s\xi\big\rangle\big) \widetilde{N}(\mathrm{d}z,\mathrm{d}s) \Bigg\}.\end{aligned} \tag{57}$$

Step 2: Assume $F \in C_b^1(\mathbb{H},\mathbb{H})$ and ∇F is Lipschitz continuous. We aim to construct approximation sequence $\{F_k\}_{k\ge 1} \subset C_b^2(\mathbb{H},\mathbb{H})$ such that $F_k \to F$ and $\nabla F_k \to \nabla F$ in pointwise sense as $k\to\infty$. In fact, for $k\ge 1$, we take a sequence of non-negative, twice differential function $\{g_k\}_{k\ge 1}$ such that

$$\mathrm{Supp}\{g_k\} \subset \left\{ y \in \mathbb{R}^k : |y|_{\mathbb{R}^k} \le \frac{1}{k} \right\}, \qquad \int_{\mathbb{R}^k} g_k(y)\,\mathrm{d}y = 1.$$

Identifying $\mathbb{R}^k$ with $\operatorname{span}\{e_1,\dots,e_k\}$, we define

$$F_k(x)=\int_{\mathbb{R}^k} g_k(y-\Pi_k x)F\left(\sum_{i=1}^k y_i e_i\right)\mathrm{d}y, \tag{58}$$

where $\Pi_k:\mathbb{H}\to\operatorname{span}\{e_1,\dots,e_k\}$ is the projection operator. Then F_k is a twice differentiable function with bounded and continuous derivatives (see [10, P127]). Furthermore, we have $\sup_{k\geq 1}\|\nabla F_k\|_\infty\leq\|\nabla F\|_\infty$ and $\sup_{k\geq 1}\|\nabla^2 F_k\|_\infty\leq\|\nabla F\|_{\mathrm{Lip}}$ where $\|\nabla F\|_{\mathrm{Lip}}$ denotes the smallest Lipschitz constant of ∇F. Now, for any $k\geq 1$, let us consider the following equation:

$$\begin{cases}\mathrm{d}X_t^k=AX_t^k\,\mathrm{d}t+F_k(X_t^k)\,\mathrm{d}t+\mathrm{d}L_t+\mathrm{d}Z_t,\\ X_0^k=x.\end{cases}$$

Let $\{X_t^{k,x}\}_{t\leq T}$ be its solution. Then it is easy to prove that

$$\lim_{k\to\infty}\left|X_t^{k,x}-X_t\right|=0,\quad\text{and}\quad\lim_{k\to\infty}\int_0^t\left|X_s^{k,x}-X_s\right|\mathrm{d}s=0,\quad\forall t\geq 0. \tag{59}$$

Let $\{J_t^k\xi\}_{t\geq 0}$ be the derivative flow w.r.t. the initial value along ξ. Then

$$J_t^k\xi=S(t)\xi+\int_0^t S(t-s)\nabla F^k\big(X_s^{k,x}\big)J_s^k\xi\,\mathrm{d}s. \tag{60}$$

Furthermore,

$$\sup_{k\geq 1}\left|J_t^k\xi\right|\leq\exp\big\{-\big(\gamma_1-\|\nabla F\|_\infty\big)t\big\}|\xi|. \tag{61}$$

Now, by (52), (60) the triangle inequality we have

$$\begin{aligned}\left|J_t^k\xi-J_t\xi\right|&\leq\int_0^t e^{-\gamma_1(t-s)}\left|\nabla F_k\big(X_s^{k,x}\big)J_s^k\xi-\nabla F\big(X_s^x\big)J_s\xi\right|\mathrm{d}s\\
&\leq\int_0^t e^{-\gamma_1(t-s)}\left|\nabla F_k\big(X_s^{k,x}\big)J_s^k\xi-\nabla F_k\big(X_s^{k,x}\big)J_s\xi\right|\mathrm{d}s\\
&\quad+\int_0^t e^{-\gamma_1(t-s)}\left|\nabla F_k\big(X_s^{k,x}\big)J_s\xi-\nabla F_k\big(X_s^x\big)J_s\xi\right|\mathrm{d}s\\
&\quad+\int_0^t e^{-\gamma_1(t-s)}\left|\nabla F_k\big(X_s^x\big)J_s\xi-\nabla F\big(X_s^x\big)J_s\xi\right|\mathrm{d}s\\
&\leq\|\nabla F\|_\infty\int_0^t e^{-\gamma_1(t-s)}\left|J_s^k\xi-J_s\xi\right|\mathrm{d}s\\
&\quad+\sup_{k\geq 1}\left\|\nabla^2 F_k\right\|_\infty\int_0^t e^{\kappa(t-s)}\left|X_s^{k,x}-X_s^x\right|\exp\big\{\|\nabla F\|_\infty s\big\}|\xi|\,\mathrm{d}s\\
&\quad+C_A^2\int_0^t\left\|\nabla F_k\big(X_s^x\big)-\nabla F\big(X_s^x\big)\right\|\exp\big\{\|\nabla F\|_\infty s\big\}|\xi|\,\mathrm{d}s.\end{aligned}$$

Using Gronwall's inequality and (59), one arrive at

$$\lim_{k\to\infty}\left|J_t^k\xi - J_t\xi\right| \le C\lim_{k\to\infty}\left\{\int_0^t \left|X_s^{k,x}-X_s^x\right|\mathrm{d}s + \int_0^t \left\|\nabla F_k\big(X_s^x\big)-\nabla F\big(X_s^x\big)\right\|\mathrm{d}s\right\} = 0, \tag{62}$$

where C is a constant depending on A, F and t. In view of (53), (59) and (62), one has

$$\begin{aligned}\lim_{k\to\infty}\int_0^t \left|Q^{-1}J_s^k\xi - Q^{-1}J_s\xi\right|\mathrm{d}s &= \lim_{k\to\infty}\int_0^t\left|\int_0^s Q^{-1}S(s-r)\nabla F\big(X_r^{k,x}\big)J_r^k\xi\,\mathrm{d}r\right.\\ &\qquad \left. - \int_0^s Q^{-1}S(s-r)\nabla F\big(X_r^x\big)J_r\xi\,\mathrm{d}r\right|\mathrm{d}s\\ &\le t\lim_{k\to\infty}\int_0^t \left\|Q^{-1}S(t-s)\right\|\left|\nabla F\big(X_s^{k,x}\big)J_s^k\xi - \nabla F\big(X_s^x\big)J_s\xi\right|\mathrm{d}s\\ &= t\int_0^t \left\|Q^{-1}S(t-s)\right\|\lim_{k\to\infty}\left|\nabla F\big(X_s^{k,x}\big)J_s^k\xi - \nabla F\big(X_s^x\big)J_s\xi\right|\mathrm{d}s = 0,\end{aligned}\tag{63}$$

where in the second equality we use the conditions $\int_0^t \|Q^{-1}S(s)\|\,\mathrm{d}s < \infty$ and $F\in C_b^1(\mathbb{H},\mathbb{H})$.

Now let us define

$$P_t^{k,1}f(x) := \mathbb{E}\big\{f\big(X_t^{k,x}\big)I_{[N_t\ge 1]}\big\},\quad \forall f\in C_b^2(\mathbb{H}).$$

By Step 1, we have

$$\begin{aligned}\nabla_\xi P_t^{k,1}f(x) &= -\mathbb{E}\left\{f\big(X_t^{k,x}\big)\frac{I_{[N_t\ge 1]}}{N_t}\right.\\ &\qquad\left.\times\int_0^t\int_{\mathbb{H}}\big(\big\langle z, Q^{-1}J_s^k\xi\big\rangle + \big\langle\nabla\log\rho(z), J_s^k\xi\big\rangle\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\right\}.\end{aligned}\tag{64}$$

Observe that

$$\lim_{k\to\infty}\left|\nabla_\xi P_t^{k,1}f(x) - \nabla_\xi P_t^1 f(x)\right| \le \|\nabla f\|_\infty \lim_{k\to\infty}\mathbb{E}\left|J_t^k\xi - J_t\xi\right| = 0,$$

and

$$\begin{aligned}&\lim_{k\to\infty}\mathbb{E}\left|\int_0^t\int_{\mathbb{H}}\big(\big\langle z, Q^{-1}J_s^k\xi - Q^{-1}J_s\xi\big\rangle + \big\langle\nabla\log\rho(z),\big(J_s^k\xi - J_s\xi\big)\big\rangle\big)\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\right|\\ &\quad\le C\lim_{k\to\infty}\left\{\int_0^t\left|Q^{-1}J_s^k\xi - Q^{-1}J_s\xi\right|\mathrm{d}s + \int_0^t\left|J_s^k\xi - J_s\xi\right|\mathrm{d}s\right\} = 0.\end{aligned}$$

We finish the proof by letting $k\to\infty$ in (64).

3.2. Proof of Theorem 2. We divide the proof into two steps.

Step 1: Assume $F \in C_b^2(\mathbb{H}, \mathbb{H})$. First, we recall that

$$|J_t \xi| \le \exp\{-(\gamma_1 - \|\nabla F\|_\infty)t\}|\xi|. \tag{65}$$

By direct computation,

$$\mathbb{E}\left\{\frac{I_{[N_t \ge 1]}}{(N_t)^2}\right\} = \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n^2 n!} \le \frac{6e^{-\lambda t}}{(\lambda t)^2} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n+2}}{(n+2)!} \le \frac{6}{(\lambda t)^2}. \tag{66}$$

By Hölder inequality, we have

$$\begin{aligned}
&\int_0^t \mathbb{E}\big|Q^{-1} J_s \xi\big|^2 \,\mathrm{d}s \\
&\quad = \int_0^t \mathbb{E}\left|Q^{-1}S(s)\xi + \int_0^s Q^{-1}S(s-r)\nabla F(X_r) J_r \xi \,\mathrm{d}r\right|^2 \mathrm{d}s \\
&\quad \le 2\int_0^t \big|Q^{-1}S(s)\xi\big|^2 \,\mathrm{d}s + 2\int_0^t \mathbb{E}\left|\int_0^s Q^{-1}S(s-r)\nabla F(X_r) J_r \xi \,\mathrm{d}r\right|^2 \mathrm{d}s \\
&\quad \le 2\|\nabla F\|_\infty^2 |\xi|^2 \\
&\qquad \times \int_0^t \left\{ s \int_0^s \big\|Q^{-1}S(s-r)\big\|^2 \exp\{-2(\gamma_1 - \|\nabla F\|_\infty)r\} \,\mathrm{d}r \right\} \mathrm{d}s \\
&\qquad + 2|\xi|^2 \int_0^t \big\|Q^{-1}S(s)\big\|^2 \,\mathrm{d}s \\
&\quad = 2\|\nabla F\|_\infty^2 |\xi|^2 \Gamma_t + 2|\xi|^2 \int_0^t \big\|Q^{-1}S(s)\big\|^2 \,\mathrm{d}s,
\end{aligned} \tag{67}$$

where

$$\Gamma_t := \int_0^t \left\{ s \int_0^s \big\|Q^{-1}S(s-r)\big\|^2 \exp\{-2(\gamma_1 - \|\nabla F\|_\infty)r\} \,\mathrm{d}r \right\} \mathrm{d}s.$$

Since $\lim_{t\to\infty} \frac{\int_0^t \|Q^{-1}S(s)\|^2 \,\mathrm{d}s}{t} < \infty$, then $\lim_{t\to\infty} \|Q^{-1}S(t)\|^2 < \infty$. Moreover,

$$\begin{aligned}
\lim_{t\to\infty} \frac{\Gamma_t}{t^2} &= \lim_{t\to\infty} \frac{\int_0^t \{s \int_0^s \|Q^{-1}S(s-r)\|^2 \exp\{-2(\gamma_1 - \|\nabla F\|_\infty)r\} \,\mathrm{d}r\} \,\mathrm{d}s}{t^2} \\
&\le \frac{1}{2} \lim_{t\to\infty} \int_0^t \big\|Q^{-1}S(t-s)\big\|^2 \exp\{-2(\gamma_1 - \|\nabla F\|_\infty)s\} \,\mathrm{d}s \\
&= \lim_{t\to\infty} \frac{\int_0^t \|Q^{-1}S(s)\|^2 \exp\{2(\gamma_1 - \|\nabla F\|_\infty)s\} \,\mathrm{d}s}{2\exp\{2(\gamma_1 - \|\nabla F\|_\infty)t\}} \\
&\le \frac{\lim_{t\to\infty} \|Q^{-1}S(t)\|^2}{4(\gamma_1 - \|\nabla F\|_\infty)} < \infty.
\end{aligned}$$

Therefore, there exists $C_1 > 0$ independent of t and F, such that

$$\left(\sup_{t\geq 1}\frac{\Gamma_t}{t^2}\right)+\left(\sup_{t\geq 1}\frac{\int_0^t\|Q^{-1}S(s)\|^2\,\mathrm{d}s}{t^2}\right)\leq \frac{C_1}{\gamma_1-\|\nabla F\|_\infty}. \tag{68}$$

Recalling that ν is finite measure and by Theorem 1, (65)–(68) one has for each $t\geq 1$

$$\begin{aligned}
&\left|\nabla_\xi P_t^1 f(x)\right|\\
&\quad=\left|-\mathbb{E}\left\{f\left(X_t^x\right)\frac{I_{[N_t\geq 1]}}{N_t}\int_0^t\int_{\mathbb{H}}\left(\left\langle z,Q^{-1}J_s\xi\right\rangle+\left\langle\nabla\log\rho(z),J_s\xi\right\rangle\right)\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\right\}\right|\\
&\quad\leq\|f\|_\infty\left\{\left\{\int_{\mathbb{H}}|z|^2\nu(\mathrm{d}z)\mathbb{E}\frac{I_{[N_t\geq 1]}}{(N_t)^2}\int_0^t\mathbb{E}\left|Q^{-1}J_s\xi\right|^2\mathrm{d}s\right\}^{\frac{1}{2}}\right.\\
&\qquad\left.+2\|\nabla\rho\|_\infty\int_0^t\mathbb{E}|J_s\xi|\,\mathrm{d}s\right\}\\
&\quad\leq\|f\|_\infty\left\{\left\{\int_{\mathbb{H}}|z|^2\nu(\mathrm{d}z)\mathbb{E}\frac{I_{[N_t\geq 1]}}{(N_t)^2}\int_0^t\mathbb{E}\left|Q^{-1}J_s\xi\right|^2\mathrm{d}s\right\}^{\frac{1}{2}}+\frac{2\|\nabla\rho\|_\infty|\xi|}{\gamma_1-\|\nabla F\|_\infty}\right\}\\
&\quad\leq\|f\|_\infty|\xi|\left\{\left\{12\int_{\mathbb{H}}|z|^2\nu(\mathrm{d}z)\frac{\|\nabla F\|_\infty^2\Gamma_t+\int_0^t\|Q^{-1}S(s)\|^2\,\mathrm{d}s}{\lambda^2t^2}\right\}^{\frac{1}{2}}\right.\\
&\qquad\left.+\frac{2\|\nabla\rho\|_\infty}{\gamma_1-\|\nabla F\|_\infty}\right\}\\
&\quad\leq\|f\|_\infty|\xi|\frac{\{12C_1\int_{\mathbb{H}}|z|^2\nu(\mathrm{d}z)(\|\nabla F\|_\infty^2+1)\}^{\frac{1}{2}}+2\|\nabla\rho\|_\infty}{(\lambda\wedge 1)((\gamma_1-\|\nabla F\|_\infty)\wedge 1)}.
\end{aligned}$$

Therefore, for each $x,y\in\mathbb{H}$ and $t\geq 1$

$$\begin{aligned}
&\left|P_tf(x)-P_tf(y)\right|\\
&\quad=\left|P_t^1f(x)-P_t^1f(y)\right|+\left|\mathbb{E}\left\{f\left(X_t^x\right)I_{[N_t=0]}\right\}-\mathbb{E}\left\{f\left(X_t^y\right)I_{[N_t=0]}\right\}\right|\\
&\quad\leq\|f\|_\infty|x-y|\frac{\{12C_1\int_{\mathbb{H}}|z|^2\nu(\mathrm{d}z)(\|\nabla F\|_\infty^2+1)\}^{\frac{1}{2}}+2\|\nabla\rho\|_\infty}{(\lambda\wedge 1)((\gamma_1-\|\nabla F\|_\infty)\wedge 1)}\\
&\qquad+2\|f\|_\infty e^{-\lambda t}.
\end{aligned} \tag{69}$$

Step 2: Assume F is Lipschitz continuous. Then there exist $\{F_n\}_{n\geq 1}\subset C_b^2(\mathbb{H},\mathbb{H})$ such that $F_n\to F$ as $n\to\infty$ in pointwise sense and $\sup_{n\geq 1}\|\nabla F_n\|_\infty\leq\|F\|_{\mathrm{Lip}}$. Now let us consider the following equation:

$$\begin{cases}\mathrm{d}Y_t^n=AY_t^n\,\mathrm{d}t+F_n(Y_t^n)\,\mathrm{d}t+\mathrm{d}L_t+\mathrm{d}Z_t,\\ Y_0^n=x.\end{cases} \tag{70}$$

Let $\{P_{n,t}\}_{t\geq 0}$ be the transition semigroup of the solution to Eq. (70). Then it is easy to prove that

$$\lim_{n\to\infty} P_{n,t}f(y) = P_t f(y), \quad \forall y \in \mathbb{H}, \forall f \in C_b^2(\mathbb{H}).$$

Now it follows from (69) that for any $t \geq 1$ and $f \in C_b^2(\mathbb{H})$

$$\begin{aligned}
&\big|P_{n,t}f(x) - P_{n,t}f(y)\big| \\
&\quad \leq \|f\|_\infty |x-y| \frac{\{12C_1 \int_{\mathbb{H}} |z|^2 \nu(\mathrm{d}z)(\|\nabla F_n\|_\infty^2 + 1)\}^{\frac{1}{2}} + 2\|\nabla\rho\|_\infty}{(\lambda \wedge 1)(\gamma_1 - \|\nabla F_n\|_\infty)} \\
&\qquad + 2\|f\|_\infty e^{-\lambda t} \\
&\quad \leq \|f\|_\infty |x-y| \frac{\{12C_1 \int_{\mathbb{H}} |z|^2 \nu(\mathrm{d}z)(\|F\|_{\mathrm{Lip}}^2 + 1)\}^{\frac{1}{2}} + 2\|\nabla\rho\|_\infty}{(\lambda \wedge 1)((\gamma_1 - \|F\|_{\mathrm{Lip}}) \wedge 1)} \\
&\qquad + 2\|f\|_\infty e^{-\lambda t}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\big|P_t f(x) - P_t f(y)\big| \leq C\|f\|_\infty \big\{|x-y| + e^{-\lambda t}\big\}, \quad \forall t \geq 1, \tag{71}$$

for some constant $C > 0$ independent of t. Observe that for each $x, y \in \mathbb{H}$,

$$\begin{aligned}
\mathbb{E}\big|X_t^x - X_t^y\big| &\leq \big|S(t)(x-y)\big| + \mathbb{E}\int_0^t \big|S(t-s)\big(F\big(X_s^x\big) - F\big(X_s^y\big)\big)\big|\,\mathrm{d}s \\
&\leq e^{-\gamma_1 t}|x-y| + \|F\|_{\mathrm{Lip}} \int_0^t e^{-\gamma_1(t-s)}\mathbb{E}\big|X_s^x - X_s^y\big|\,\mathrm{d}s.
\end{aligned}$$

Then Gronwall's inequality yields

$$\mathbb{E}\big|X_t^x - X_t^y\big| \leq \exp\big\{\big(-\gamma_1 + \|F\|_{\mathrm{Lip}}\big)t\big\}|x-y|, \quad \forall x, y \in \mathbb{H}.$$

Combining this with (71) and using the Markov property, we have for $t > s$

$$\begin{aligned}
&\big|P_t f(x) - P_t f(y)\big| \\
&\quad \leq \mathbb{E}\big|P_s f\big(X_{t-s}^x\big) - P_s f\big(X_{t-s}^y\big)\big| \\
&\quad \leq 2C\|f\|_\infty \big\{\mathbb{E}\big|X_{t-s}^x - X_{t-s}^y\big| + e^{-\lambda s}\big\} \\
&\quad \leq 2C\|f\|_\infty \big\{\exp\big\{-\big(\gamma_1 - \|F\|_{\mathrm{Lip}}\big)(t-s)\big\}|x-y| + e^{-\lambda s}\big\}.
\end{aligned} \tag{72}$$

Let $t > \frac{(\gamma_1 - \|F\|_{\mathrm{Lip}} + \lambda)}{\gamma_1 - \|F\|_{\mathrm{Lip}}}$ and take $s = \frac{(\gamma_1 - \|F\|_{\mathrm{Lip}})t}{\gamma_1 - \|F\|_{\mathrm{Lip}} + \lambda}$ in (72), then

$$\begin{aligned}
&\big|P_t f(x) - P_t f(y)\big| \\
&\quad \leq C\|f\|_\infty \big(1 + |x-y|\big) \exp\bigg\{-\frac{\lambda(\gamma_1 - \|F\|_{\mathrm{Lip}})}{\lambda + \gamma_1 - \|F\|_{\mathrm{Lip}}} t\bigg\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\big\|P_t(x,\cdot)-P_t(y,\cdot)\big\|_{\mathrm{Var}} &:= \frac{1}{2}\sup_{f\in\mathbb{B}_b(\mathbb{H}),\|f\|_\infty\le 1}\big|P_tf(x)-P_f(y)\big| \\ &= \frac{1}{2}\sup_{f\in C_b^2(\mathbb{H}),\|f\|_\infty\le 1}\big|P_tf(x)-P_f(y)\big| \\ &\le C\big(1+|x-y|\big)\exp\bigg\{-\frac{\lambda(\gamma_1-\|F\|_{\mathrm{Lip}})}{\lambda+\gamma_1-\|F\|_{\mathrm{Lip}}}t\bigg\}.\end{aligned}$$

The proof is completed by noting that the inequality trivially holds with a suitable constant $C>0$ for $t\le\frac{(\gamma_1-\|F\|_{\mathrm{Lip}}+\lambda)}{(\gamma_1-\|F\|_{\mathrm{Lip}})}$. The proof is finished.

Acknowledgment. The authors are grateful to the referee's valuable suggestions.

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