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EMBEDDING OF GROUPS AND QUADRATIC EQUATIONS OVER GROUPS

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ABSTRACT. We prove that, for every integer $n \geq 2$, a finite or infinite countable group G can be embedded into a 2-generated group H in such a way that the solvability of quadratic equations of length at most n is preserved, that is, every quadratic equation over G of length at most n has a solution in G if and only if this equation, considered as an equation over H, has a solution in H.

1. Introduction

It is a classical result of Higman, B. Neumann, and H. Neumann [6] that every finite or infinite countable group can be embedded into a 2-generated group. In this note, we are concerned with such an emdedding that would preserve the solvability of every quadratic equation of bounded length.

We start with definitions. Let G be a finite or infinite countable group and let

$$G = \langle a_1, a_2, \dots \parallel R_1 = 1, R_2 = 1, \dots \rangle \tag{1.1}$$

be a presentation for G by means of generators $a_1, a_2, \dots$ and defining relations $R_1 = 1, R_2 = 1, \dots$, where $R_1, R_2, \dots$ are nonempty cyclically reduced words over the alphabet $\mathcal{A}^{\pm 1} := \{a_1^{\pm 1}, a_2^{\pm 1}, \dots\}$. If U is a word over $\mathcal{A}^{\pm 1}$ and the image of U in G is trivial, we write $U \overset{G}{=} 1$ or say that $U = 1$ in G.

Let $\mathcal{X}$ be a finite or infinite countable set, called a set of variables, $\mathcal{X}^{-1} := \{x^{-1} \mid x \in \mathcal{X}\}$, and $\mathcal{X}^{\pm 1} := \mathcal{X} \cup \mathcal{X}^{-1}$. Let $\mathcal{F}(\mathcal{X})$ denote the free group with the free base $\mathcal{X}$ and let $G * \mathcal{F}(\mathcal{X})$ denote the free product of G and $\mathcal{F}(\mathcal{X})$. Elements of $G * \mathcal{F}(\mathcal{X})$ can be regarded as words over the alphabet $\mathcal{Y}^{\pm 1}$, where $\mathcal{Y} := \mathcal{A} \cup \mathcal{X}$.

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A word $W = y_1 \dots y_\ell$ over $\mathcal{Y}^{\pm 1}$, where $y_1, \dots, y_\ell \in \mathcal{Y}^{\pm 1}$, is called *reduced* if $\ell > 0$, that is, W is not empty, and W contains no subwords of the form yy^{-1} or $y^{-1}y$, where $y \in \mathcal{Y}$. A word W over $\mathcal{Y}^{\pm 1}$ is *cyclically reduced* if W is reduced and every cyclic permutation of W is reduced. The *length* of a word $W = y_1 \dots y_\ell$ over $\mathcal{Y}^{\pm 1}$ is $\ell = |W|$ and the $\mathcal{X}$*-length* $|W|_\mathcal{X}$ of W is the number of all occurrences of letters of $\mathcal{X}^{\pm 1}$ in the word W. For example, $|a_1 x_1 x_2 a_2^{-1} x_1^{-1}|_\mathcal{X} = 3$ if $a_1, a_2 \in \mathcal{A}$ and $x_1, x_2 \in \mathcal{X}$.

An *equation* over G is a formal expression $W = 1$, where W is a cyclically reduced word over $\mathcal{Y}^{\pm 1}$ with $|W|_\mathcal{X} > 0$. The *length* of an equation $W = 1$ over G is the number $|W|_\mathcal{X}$. The *total length* of an equation $W = 1$ over G is $|W|$. An equation $W = 1$ over G is called *quadratic* if, for every letter $x \in \mathcal{X}$, the sum of the number of occurrences of x in W and the number of occurrences of x^{-1} in W is either 2 or 0.

We say that an equation $W = 1$ over G has a *solution* if there exists a homomorphism $\psi_W : G * \mathcal{F}(\mathcal{X}) \to G$ which is identical on G and which takes the word $W \in G * \mathcal{F}(\mathcal{X})$ to the identity, that is, $\psi_W|_G = \mathrm{id}_G$ and $\psi_W(W) = 1$ in G. Let $x_1, \dots, x_k$ be all letters of $\mathcal{X}$ that occur in W or in W^{-1}. A *solution tuple* to the equation $W = 1$, defined by a homomorphism $\psi_W : G * \mathcal{F}(\mathcal{X}) \to G$, is a tuple $(U_1, \dots, U_k)$, where $U_1, \dots, U_k$ are some words over $\mathcal{A}^{\pm 1}$, such that $\psi_W(x_j) = U_j$ in G for every $j = 1, \dots, k$. The *length* of a solution tuple $(U_1, \dots, U_k)$ to the equation $W = 1$ is the sum $\sum_{j=1}^{k} |U_j|$.

If $\mu : G \to H$ is a group monomorphism and $W = 1$ is an equation over G, then we can use μ and $W = 1$ to obtain an equation over H by replacing every letter $a_i^\varepsilon \in \mathcal{A}^{\pm 1}$, $\varepsilon = \pm 1$, that appears in $W = 1$ with $\mu(a_i^\varepsilon)$. This new equation over H is denoted by $\mu(W) = 1$.

THEOREM 1.1. *Let $n \geq 2$ be an integer and let G be a finite or infinite countable group. Then there exists an embedding $\mu_n : G \to H$ of G into a 2-generated group $H = \langle h_1, h_2 \rangle$, that preserves the solvability of every quadratic equation $W = 1$ over G of length $|W|_\mathcal{X} \leq n$, that is, for every equation $W = 1$ over G of length at most n, the equation $W = 1$ has a solution in G if and only if $\mu_n(W) = 1$ has a solution in H.*

We remark that the embedding $\mu_n : G \to H$ of Theorem 1.1 has additional properties that are of interest even in the case when G is already a 2-generated group. For example, a solution tuple to a quadratic equation $W = 1$ over G such that $|W|_\mathcal{X} \leq n$ may be arbitrarily long relative to the original alphabet $\mathcal{A}$ whereas the equation $\mu_n(W) = 1$ has a relatively short solution tuple in H with respect to the alphabet $\{h_1, h_2\}$. This and other technical properties of the embedding μ_n, that could be useful for potential future applications, are recorded in the following.

THEOREM 1.2. *The embedding $\mu_n : G \to H$ of Theorem 1.1 can be constructed in such a way that μ_n has the following properties.*

(a) *Fix an enumeration* $W_1 = 1, W_2 = 1, \dots$ *of all quadratic equations over* G *such that, for every* $i \ge 1$, $|W_i|_{\mathcal{X}} \le n$ *and* $W_i = 1$ *has a solution in* G. *Then there is a constant* $C > 0$ *such that, for every* $i \ge 1$, *there exists a solution tuple to the equation* $\mu_n(W_i) = 1$ *over* H *whose length, in generators* h_1, h_2 *of* H, *does not exceed* $Cn^4 i$.

(b) *Assume that the presentation* (1.1) *for* G *is recursively enumerable. Then defining relations of the* 2*-generated group* $H = \langle h_1, h_2 \rangle$ *can be recursively enumerated.*

(c) *Assume that the presentation* (1.1) *for* G *is decidable and there is an algorithm that detects whether a quadratic equation over* G *of length at most* n *has a solution in* G. *Then the* 2*-generated group* $H = \langle h_1, h_2 \rangle$ *has a decidable set of defining relations and the embedding* $\mu_n : G \to H$ *can be effectively constructed.*

As an example of a quadratic equation, consider the equation $xU_1x^{\varepsilon}U_2 = 1$, where $\varepsilon = \pm 1$ and U_1, U_2 are some reduced (or possibly empty if $\varepsilon = 1$) words over $\mathcal{A}^{\pm 1}$. Note that if $\varepsilon = -1$ then this equation has a solution if and only if the elements of G, represented by the words U_1, U_2^{-1}, are conjugate in G. If $\varepsilon = 1$, then this equation has a solution if and only if the element of G, represented by the word $U_1^{-1}U_2$, is a square in G, that is, there is a word T over $\mathcal{A}^{\pm 1}$ with $U_1^{-1}U_2 \overset{G}{=} T^2$. According to Theorem 1.1 applied with $n = 2$, if G is a finite or infinite countable group, then G embeds into a 2-generated group H, $\mu_2 : G \to H$, in which two elements of $\mu_2(G)$ are conjugate if and only if they are conjugate in G and every element of $\mu_2(G)$ is a square in H if and only if it is a square in G. This is reminiscent of an embedding result of Ol'shanskii and Sapir [13] that states that a finitely generated group G with the solvable conjugacy problem can be embedded into a finitely presented group K with the solvable conjugacy problem, $\sigma : G \to K$, in such a way that two elements of $\sigma(G)$ are conjugate in K if and only if they are conjugate in G.

It would be of interest to find out whether Theorem 1.1 generalizes to arbitrary equations of bounded length and whether one could drop the upper bound on the length of quadratic equations in Theorem 1.1. The first question seems to be technically relevant to the following interesting problem.

PROBLEM 1.3. For given integer $n > 0$, does there exist a real number $\lambda > 0$ such that if a presentation (1.1) satisfies the small cancelation condition $C'(\lambda)$, for every relation $R = 1$ of (1.1), $|R| > \lambda^{-1}$ and R is not a proper power, then every equation $W = 1$ over G of total length $|W| \le n$ has a solution in G if and only if the equation $W = 1$, considered as an equation over the free group $F(\mathcal{A})$, has a solution in $F(\mathcal{A})$?

We remark that for quadratic equations of total length $\le n$ this problem would likely have a positive solution and a proof would be analogous to the proof of Theorem 1.1 with additional consideration of contiguity subdiagrams between boundary paths of faces of type F3 and boundary paths of a surface diagram Δ. We also mention that the arguments of Frenkel and Klyachko [5], which are used to prove that a nontrivial commutator cannot be a proper power in a torsion-free group G that satisfies the small cancelation condition $C'(\lambda)$ with $\lambda \ll 1$, might be useful for making some progress in nonquadratic case.

At the suggestion of the referee, we mention that connections between compact surfaces and solutions of quadratic equations in free groups, free products and in hyperbolic groups were first studied by Culler [2] and Ol'shanskii [11]. Earlier work on quadratic equations in free groups and in free products was done by Edmunds [3], [4], Comerford and Edmunds [1], see also articles cited in [3], [4], [1]. The bound of Theorem 1.2(a) is reminiscent of bounds on the length of a minimal solution of quadratic equations in free groups obtained by Lysenok and Myasnikov [10] and by Kharlampovich and Vdovina [8].

2. Group presentations and diagrams

Fix an even integer $n \ge 2$. Since we consider quadratic equations $W = 1$ of length $|W|_{\mathcal{X}} \le n$, we may assume that the cardinality of $\mathcal{X}$ is n, $|\mathcal{X}| = n$. Since G is finite or countably infinite, we can choose an enumeration

$$W_1 = 1, W_2 = 1, \dots, \tag{2.1}$$

of all quadratic equations over G such that, for every $i \ge 1$, $|W_i|_{\mathcal{X}} \le n$ and $W_i = 1$ has a solution in G. Let $\bigcup_{i=1}^{\infty} \mathcal{X}_i$ be an infinite countable alphabet consisting of disjoint copies $\mathcal{X}_i, i = 1, 2, \dots,$ of $\mathcal{X}$. Let $W_i(\mathcal{X}_i)$ denote the word over the alphabet $\mathcal{A}^{\pm 1} \cup \mathcal{X}_i^{\pm 1}$ obtained by rewriting W_i so that every letter $b \in \mathcal{A}^{\pm 1}$ of W_i is unchanged and every letter y of W_i, such that $y \in \mathcal{X}^{\pm 1}$, is replaced with $\beta_i(y) \in \mathcal{X}_i^{\pm 1}$, where $\beta_i : \mathcal{X}^{\pm 1} \to \mathcal{X}_i^{\pm 1}$ is a bijection such that $\beta_i(\mathcal{X}) = \mathcal{X}_i$ and $\beta_i(x^{-1}) = \beta_i(x)^{-1}$ for every $x \in \mathcal{X}$.

Consider the following group presentation

$$\mathcal{G}_1 = \left\langle \bigcup_{i=1}^{\infty} \mathcal{X}_i \cup \mathcal{A} \;\middle\|\; R_1 = 1, R_2 = 1, \dots, W_1(\mathcal{X}_1) = 1, W_2(\mathcal{X}_2) = 1, \dots \right\rangle \tag{2.2}$$

whose generating set is $\bigcup_{i=1}^{\infty} \mathcal{X}_i \cup \mathcal{A}$ and whose defining relations are those of (1.1) and $W_i(\mathcal{X}_i) = 1, i = 1, 2, \dots$.

Lemma 2.1. *There is a natural embedding of the group G into the group $\mathcal{G}_1$ given by presentation (2.2), denoted $\nu_1 : G \to \mathcal{G}_1$. Furthermore, if $W = 1$ is an equation over G then $W = 1$ has a solution in G if and only if the equation $\nu_1(W) = 1$ has a solution in the group $\mathcal{G}_1$.*

Proof. Denote $\mathcal{X}_i = \{x_{i,1}, \dots, x_{i,n}\}$ for $i = 1, 2, \dots$. Since the equation $W_i(\mathcal{X}_i) = 1$ has a solution in G, there exists a homomorphism $\psi_i : G * \mathcal{F}(\mathcal{X}_i) \to G$ such that ψ_i is identical on G and $\psi_i(W_i(\mathcal{X}_i)) = 1$. Let $U_{i,1}, \dots, U_{i,n}$ be words over $\mathcal{A}^{\pm 1}$ such that $\psi_i(x_{i,j}) = U_{i,j}$ in G, $j = 1, \dots, n$. Then the map $\psi_\infty(x_{i,j}) := U_{i,j}$, where $1 \le j \le n$, $i = 1, 2, \dots$, and $\psi_\infty(a) := a$ for all $a \in \mathcal{A}$ induces a homomorphism $\psi_\infty : \mathcal{G}_1 \to G$ which is identical on G. Hence, the group G embeds in $\mathcal{G}_1$. The existence of this homomorphism $\psi_\infty : \mathcal{G}_1 \to G$ also implies that, for an arbitrary equation $W = 1$ over G, the equation $W = 1$ has a solution in G if and only if the equation $\nu_1(W) = 1$ over $\mathcal{G}_1$ has a solution in $\mathcal{G}_1$. $\square$

Denote $M := 24n$. For every $i \ge 1$, consider a word V_i over the alphabet $\{h_1, h_2\}$ defined by the formula

$$V_i = V_i(h_1, h_2) := h_1 h_2^{Mi+1} h_1 h_2^{Mi+2} \dots h_1 h_2^{M(i+1)-1} h_1 h_2^{M(i+1)} h_1. \tag{2.3}$$

The literal (or letter-by-letter) equality of two words U, V is denoted $U \equiv V$. In the following lemma, we establish a small cancelation condition for the words V_i, $i = 1, 2 \dots$.

LEMMA 2.2. *Let U be a subword of both words V_i and V_j, defined by* (2.3), *so $V_i \equiv V_{i,1} U V_{i,2}$ and $V_j \equiv V_{j,1} U V_{j,2}$. Then either $|U| < \frac{4}{M} \min\{|V_i|, |V_j|\}$ or $i = j$ and $V_{i,1} \equiv V_{j,1}$.*

Proof. Suppose that U is a subword of the word V_i, where $i = 1, 2, \dots$, and $|U| \ge \frac{4}{M}|V_i|$. Then

$$|U| \ge \frac{4}{M}|V_i| > 4(Mi + 2) > 2M(i+1) + 2. \tag{2.4}$$

Since every maximal power of h_2 in V_i is no longer than $M(i+1)$, it follows from (2.4) that U contains a subword of the form $h_1 h_2^k h_1$, where $Mi + 1 \le k \le M(i+1)$. Now our claim follows from the fact that each word $V_1, V_2, \dots$ contains a unique subword of the form $h_1 h_2^k h_1$, where $Mi + 1 \le k \le M(i+1)$. $\square$

Let $\bigcup_{i=1}^\infty \mathcal{X}_i = \{x_1, x_2, \dots\}$ be an enumeration of elements of $\bigcup_{i=1}^\infty \mathcal{X}_i$ compatible with the enumeration of sets $\mathcal{X}_i$, that is, if $x_j \in \mathcal{X}_k$, $x_{j'} \in \mathcal{X}_{k'}$ and $k < k'$, then $j < j'$. Using this enumeration, new generators h_1, h_2 and the words $V_i(h_1, h_2)$, we extend the presentation (2.2) as follows

$$\begin{aligned} \mathcal{G}_2 = \Big\langle \bigcup_{i=1}^\infty \mathcal{X}_i \cup \mathcal{A} \cup \{h_1, h_2\} \ \Big\| \ & R_1 = 1, R_2 = 1, \dots, W_1(\mathcal{X}_1) = 1, \\ & W_2(\mathcal{X}_2) = 1, \dots, x_i V_{2i}^{-1} = 1, a_i V_{2i+1}^{-1} = 1, i = 1, 2, \dots \Big\rangle. \end{aligned} \tag{2.5}$$

To study this group presentation and quadratic equations over $\mathcal{G}_2$, we will use diagrams over the presentation (2.5). We start with basic definitions.

Let Δ be a finite 2-complex and let $\Delta(i)$ denote the set of closures of i-cells of Δ, $i = 0, 1, 2$. The elements of $\Delta(i)$ are called *vertices*, *edges*, *faces* of Δ

if $i=0,1,2$, resp. We also consider the set $\vec{\Delta}(1)$ of oriented 1-cells of Δ. If $e \in \vec{\Delta}(1)$, then e^{-1} denotes e with opposite orientation. For every $e \in \vec{\Delta}(1)$, let e_-, e_+ denote the initial, terminal, resp., vertices of e. In particular, $(e^{-1})_- = e_+$ and $(e^{-1})_+ = e_-$. Note that $e \neq e^{-1}$.

A path $p = e_1 \dots e_\ell$ in Δ is a sequence of oriented edges $e_1, \dots, e_\ell$ of Δ with $(e_i)_+ = (e_{i+1})_-$, $i = 1, \dots, \ell-1$. The length of a path $p = e_1 \dots e_\ell$ is $|p| = \ell$. The initial vertex of p is $p_- := (e_1)_-$ and the terminal vertex of p is $p_+ := (e_\ell)_+$. A path p is called *closed* if $p_- = p_+$. A path p is called *reduced* if $|p| > 0$ and p contains no subpath of the form ee^{-1}, where e is an edge. A *cyclic* path is a closed path with no distinguished initial vertex. A path $p = e_1 \dots e_\ell$ is called *simple* if the vertices $(e_1)_-, \dots, (e_\ell)_-, (e_\ell)_+$ are all distinct. A closed path is *simple* if the vertices $(e_1)_-, \dots, (e_\ell)_-$ are all distinct.

A *diagram* Δ over presentation (2.5) is a connected finite 2-complex which is equipped with a labeling function

$$\varphi : \vec{\Delta}(1) \to \bigcup_{i=1}^{\infty} \mathcal{X}_i^{\pm 1} \cup \mathcal{A}^{\pm 1} \cup \{h_1^{\pm 1}, h_2^{\pm 1}, 1\}$$

such that, for every $e \in \vec{\Delta}(1)$, one has $\varphi(e^{-1}) = \varphi(e)^{-1}$, where $1^{-1} := 1$, and, for every face Π of Δ, if $\partial\Pi = e_1 \dots e_\ell$ is a boundary path of Π, where $e_1, \dots, e_\ell \in \vec{\Delta}(1)$, then the label $\varphi(\partial\Pi) := \varphi(e_1) \dots \varphi(e_\ell)$ of $\partial\Pi$ has one of the following three forms.

(F1) $\varphi(\partial\Pi) = 1^\ell$.

(F2) $\ell = 4$ and $\varphi(\partial\Pi)$ is a cyclic permutation of a word $y1y^{-1}1$, where $y \in \bigcup_{i=1}^{\infty} \mathcal{X}_i \cup \mathcal{A} \cup \{h_1, h_2\}$.

(F3) $\varphi(\partial\Pi)$ is a cyclic permutation of one of the words $R^{\pm 1}$, where $R = 1$ is a relation of the presentation (2.5).

A face Π of Δ is said to have *type F1, F2, F3* if $\varphi(\partial\Pi)$ has the form (F1), (F2), (F3), resp. The set of faces of type Fj is denoted $\Delta_j(2)$, $j = 1, 2, 3$.

An edge $e \in \vec{\Delta}(1)$ is called an *a-edge*, *x-edge*, *h-edge*, *1-edge* if $\varphi(e) \in \mathcal{A}^{\pm 1}$, $\varphi(e) \in \bigcup_{i=1}^{\infty} \mathcal{X}_i^{\pm 1}$, $\varphi(e) \in \{h_1^{\pm 1}, h_2^{\pm 1}\}$, $\varphi(e) = 1$, resp. An edge $e \in \vec{\Delta}(1)$ is termed *essential* if e is not a 1-edge.

We will say that Δ is a *surface diagram of type* (k, k') over (2.5) if Δ is a diagram over (2.5) and Δ, as a topological space, is homeomorphic to a compact (orientable or nonorientable) surface that has Euler characteristic k and contains k' punctures. This surface is called the *underlying* surface for Δ. In particular, Δ is called a *disk* diagram if Δ is a surface diagram of type $(1, 1)$, hence, the underlying surface for Δ is a disk.

If Δ is a surface diagram and the underlying surface is orientable, then a fixed orientation of the underlying surface makes it possible to define positive (=counterclockwise) and negative (=clockwise) orientation for boundaries of faces of Δ and for connected components of $\partial\Delta$. Regardless of whether the underlying surface is orientable or not, we always consider the boundary $\partial\Pi$

of a face Π of Δ or a connected component c of the boundary $\partial\Delta$ of Δ as a cyclic path which is called a *boundary path* of Π or a *boundary path* of Δ, resp. Note that $(\partial\Pi)^{-1}$ or c^{-1} are also boundary paths of Π or Δ, resp., with the opposite orientation.

Suppose that Δ is a surface diagram over (2.5). Making refinements of Δ by using faces of type F1, F2 if necessary (informally, we "thicken" boundary paths of faces of type F3 and $\partial\Delta$, this should be evident; more formal details can be found in [12]), we may assume that the following property holds for Δ.

(A) Suppose that each of c_1, c_2 is either a boundary path of a face of type F3 in Δ or a boundary path of Δ. Then c_1, c_2 are closed simple paths and either c_1 is a cyclic permutation of one of c_2, c_2^{-1} or c_1, c_2 have no common vertices.

Note that the property (A) implies that if an essential edge e of Δ belongs to a boundary path of a face of type F3 or e belongs to a boundary path of Δ, then e also belongs to a boundary path of a face of type F2.

From now on we always assume, unless stated otherwise, that a diagram is a surface diagram over (2.5) with the property (A).

Recall that the literal (or letter-by-letter) equality of the words U, V is denoted $U \equiv V$.

LEMMA 2.3. *Let W be a nonempty word over the alphabet*

$$\bigcup_{i=1}^{\infty} \mathcal{X}_i^{\pm 1} \cup \mathcal{A}^{\pm 1} \cup \{h_1^{\pm 1}, h_2^{\pm 1}, 1\}$$

and let $\mathcal{G}_2$ be the group defined by presentation (2.5). *Then $W \overset{\mathcal{G}_2}{=} 1$ if and only if there is a surface diagram Δ of type* $(1,1)$, *called a* disk diagram, *over presentation* (2.5) *such that $\varphi(\partial\Delta) \equiv W$.*

Proof. The proof is straightforward, for details the reader is referred to [12], [7], see also [9]. As in [12], faces of type F1, F2 make it possible to "thicken" the diagram and turn its underlying topological space into a disk. $\square$

Suppose that Ψ is a finite graph on a compact surface S. Consider the following property of Ψ in which $m \geq 2$ is an integer parameter.

(B) If f is an oriented edge of Ψ with $f_- = f_+$, then the edge f does not bound a disk on S whose interior contains no vertices of Ψ. Furthermore, if $f_1, \dots, f_m$ are oriented edges of Ψ such that $(f_i)_- = (f_j)_-$ and $(f_i)_+ = (f_j)_+$ for all $i, j = 1, \dots, m$, then it is not true that each path $f_1 f_2^{-1}$, $f_2 f_3^{-1}, \dots,$ $f_{m-1} f_m^{-1}$ bounds a disk on S whose interior contains no vertices of Ψ.

We finish this section with a lemma about graphs on surfaces.

LEMMA 2.4. *Let S be a compact surface whose Euler characteristic is $\chi(S) = k$ and let Ψ be a finite graph on S that has the property* (B) *with* *parameter $m = 2$. If V_Ψ and E_Ψ denote the number of vertices and nonoriented edges of Ψ, resp., then $E_\Psi \le 3(V_\Psi - k)$.*

Proof. Note that the property (B) with parameter $m = 2$ can be stated less formally by saying that the partial cell decomposition of S, defined by the graph Ψ, contains no 1- and 2-gons whose interiors contain no vertices of Ψ. Preserving this condition, that is, preserving the property (B) with parameter $m = 2$, we will draw as many new edges in Ψ as possible and obtain a graph Ψ' with $V_{\Psi'} = V_\Psi$, $E_{\Psi'} \ge E_\Psi$. Note that Ψ' is connected and if c is a connected component of $\partial\Delta$ then there is a closed simple path $e_{c,1} \dots e_{c,k_c}$, where $e_{c,1}, \dots, e_{c,k_c}$ are edges of Ψ', such that $e_{c,1} \dots e_{c,k_c}$ and c bound an annulus A_c whose interior contains no vertices of Ψ'. Hence, taking A_c out of S and adding back the cycle $e_{c,1} \dots e_{c,k_c}$ for every connected component c of $\partial\Delta$, we obtain a surface S' such that $\chi(S') = \chi(S) = k$. In addition, it follows from definitions that $S' \setminus \Psi'$ is a collection of open disks. Indeed, if a connected component of $S' \setminus \Psi'$ were different from a disk, then one could draw an additional edge in Ψ' without creating a 1- or 2-gon, contrary to the maximality of Ψ'. Hence, the graph Ψ' defines a cell decomposition of S' and

$$V_{\Psi'} - E_{\Psi'} + F_{\Psi'} = \chi\big(S'\big) = k, \tag{2.6}$$

where $F_{\Psi'}$ is the number of faces of the cell decomposition of S' defined by Ψ'. Since there are no 1- and 2-gons in this decomposition, every face has 3 edges in its boundary path which implies that $3F_{\Psi'} \le 2E_{\Psi'}$ or $F_{\Psi'} \le \frac{2}{3}E_{\Psi'}$. Hence, it follows from (2.6) that $V_{\Psi'} - \frac{1}{3}E_{\Psi'} \ge k$ or $E_{\Psi'} \le 3(V_{\Psi'} - k)$. Since $V_{\Psi'} = V_\Psi$, $E_{\Psi'} \ge E_\Psi$, our claim is proved. ☐

3. Contiguity subdiagrams

As in Section 2, let Δ be a surface diagram over presentation (2.5) with property (A). Consider a relation $\sim_2$ on the set $\Delta_2(2)$ of faces of type F2 so that $\Pi_1 \sim_2 \Pi_2$ if and only if there is an essential edge e such that e belongs to $(\partial\Pi_1)^{\pm 1} := \partial\Pi_1 \cup \partial\Pi_1^{-1}$ and e belongs to $(\partial\Pi_2)^{\pm 1}$. It is easy to see that this relation is reflexive and symmetric on $\Delta_2(2)$. The transitive closure of this relation $\sim_2$ is an equivalence relation on $\Delta_2(2)$ which we denote by $\sim$. Let $[\Pi]_\sim$ denote the equivalence class of a face Π of type F2 relative to this equivalence relation. For every $\Pi \in \Delta_2(2)$, we consider a minimal subcomplex $\mathrm{B}_\Pi = \mathrm{B}([\Pi]_\sim)$ of Δ that contains all faces of $[\Pi]_\sim$. It follows from definitions that there exists a surface diagram A_Π of type (1,1) (meaning that A_Π is a disk) or of type (0,1) (meaning that A_Π is an annulus) and a continuous cellular map $\mu_\Pi : \mathrm{A}_\Pi \to \mathrm{B}_\Pi$ such that μ_Π preserves dimension of cells, φ-labels of edges, and $\mu_\Pi(\mathrm{A}_\Pi) = \mathrm{B}_\Pi$. We also require that A_Π consists of faces of type F2 and their number $|\mathrm{A}_\Pi(2)|$ equals the number $|\mathrm{B}_\Pi(2)|$ of faces in B_Π. Note that μ_Π need not be injective and this is the reason we consider an "ideal" preimage A_Π of the subcomplex B_Π.

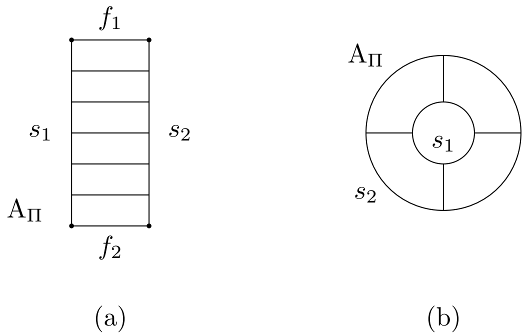

FIGURE 1

If A_Π is a disk, then $\partial A_\Pi = s_1 f_1 s_2 f_2$, where f_1, f_2 are essential edges with $\varphi(f_1) = \varphi(f_2)^{-1} \neq 1$, and s_1, s_2 are simple paths consisting of 1-edges with $|s_1| = |s_2| = |A_\Pi(2)|$, see Figure 1(a). In this case, we say that B_Π is a *band* between the edges e_1, e_2 and that $\partial B_\Pi = u_1 e_1 u_2 e_2$, where $e_i = \mu_\Pi(f_i)$, $u_i = \mu_\Pi(s_i)$, $i = 1, 2$, is a *standard boundary path* of the band B_Π. Clearly, e_1, e_2 are essential edges with $\varphi(e_1) = \varphi(f_1) = \varphi(e_2)^{-1} \neq 1$ and $|u_1| = |s_1| = |u_2|$ but u_1, u_2 need not be simple paths. If $\varphi(e_1)^{\pm 1} = y$, where $y \in \bigcup_{i=1}^{\infty} \mathcal{X}_i \cup \mathcal{A} \cup \{h_1, h_2\}$, then we may also specify that B_Π is a y-band.

Since we neither fix a base vertex for ∂B_Π, nor fix an orientation for B_Π, it follows that if $\partial B_\Pi = u_1 e_1 u_2 e_2$ is a standard boundary path for a band B_Π, then $u_2 e_2 u_1 e_1$ and $u_2^{-1} e_1^{-1} u_1^{-1} e_2^{-1}$ are also standard boundary paths for B_Π. We also observe that a standard boundary path of a band B need not be the topological boundary of B but it can be turned into the topological boundary (of a deformed space) by an arbitrarily small deformation of B which pushes B into its interior.

On the other hand, if A_Π is an annulus, then $\partial A_\Pi = s_1 \cup s_2$, where s_1, s_2 are cyclic simple paths consisting of 1-edges, $|s_1| = |s_2| = |A_\Pi(2)|$, see Figure 1(b). In this case, we say that B_Π is an annulus and that $\partial B_\Pi = u_1 \cup u_2$, where $u_i = \mu_\Pi(s_i)$, $i = 1, 2$, are boundary paths of the annulus B_Π.

Note that if B is a band and $\partial B = u_1 e_1 u_2 e_2$ is a standard boundary path of B, then each of the essential edges e_1, e_2 belongs either to a boundary path of Δ or to a boundary path of a face of type F3. If, say, e_i belongs to c_i, where $i = 1, 2$ and c_i is a boundary path of Δ or is a boundary path of a face of type F3, then we say that $B([\Pi]_\sim)$ is a band between c_1 and c_2.

Let B be a band between edges e_1 and e_2. Let $o_1 \in e_1$, $o_2 \in e_2$ be interior points of edges e_1, e_2 and let $\ell(B)$ be a simple arc such that $\ell(B)$ is contained in B, the boundary points of $\ell(B)$ are o_1, o_2 and the intersection of $\ell(B)$ with every face Π of B consists of a single arc which is properly embedded in Π and the boundary points of the arc are interior points of essential edges of $\partial\Pi$.

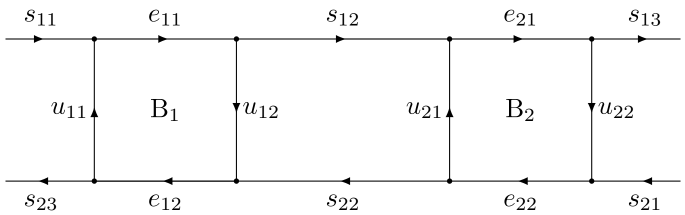

FIGURE 2

Such an arc $\ell(\mathrm{B})$ is called a *connecting line* for B. It follows from definitions that if B is a band between edges e_1 and e_2, then a connecting line $\ell(\mathrm{B})$ for B connects interior points of e_1, e_2 through faces of B of type F2.

Let s be either a subpath of $\partial\Pi$ (where Π is a face of type F3 in Δ) or a subpath of $\partial\Delta$ such that s consists of h-edges and s is maximal with respect to this property. Such s is called an *h-section* of Δ.

Suppose that s_1, s_2 are h-sections of Δ, not necessarily distinct, and $\mathrm{B}_1, \mathrm{B}_2$ are bands between s_1, s_2, perhaps $\mathrm{B}_1 = \mathrm{B}_2$, whose standard boundary paths are $\partial \mathrm{B}_i = u_{i1}e_{i1}u_{i2}e_{i2}$, $i = 1, 2$, where e_{i1}, e_{i2} are essential edges of $\partial\mathrm{B}_i$. Also, assume that e_{11}, e_{21} are edges of s_1 so that $s_1 = s_{11}e_{11}s_{12}e_{21}s_{13}$ and e_{22}, e_{12} are edges of s_2 so that $s_2 = s_{21}e_{22}s_{22}e_{12}s_{23}$, see Figure 2.

Note that the path $p = u_{11}e_{11}s_{12}e_{21}u_{22}e_{22}s_{22}e_{12}$ is closed. Furthermore, assume that there exists a connected subcomplex Γ' of Δ such that Γ' contains $\mathrm{B}_1, \mathrm{B}_2, p$, Γ' has no faces of type F3 with h-edges, and the path p is nullhomotopic in Γ'. Then we consider a minimal (relative to the inclusion relation) such subcomplex Γ whose boundary path $\partial\Gamma$ (up to arbitrarily small deformation; this time we skip introduction of an "ideal" disk diagram whose image is Γ) can be written in the form $\partial\Gamma = u_{11}(e_{11}s_{12}e_{21})u_{22}(e_{22}s_{22}e_{12})$. Note that if $\mathrm{B}_1 = \mathrm{B}_2$, then $\Gamma := \mathrm{B}_1$ and $\partial\Gamma = \partial\mathrm{B}_1 = u_{11}e_{11}u_{12}e_{12}$. Such a subcomplex Γ of Δ is unique and is called a *contiguity subdiagram* between h-sections s_1 and s_2 defined by the bands B_1, B_2. Denote $\Gamma \wedge s_1 := e_{11}s_{12}e_{21}$ and $\Gamma \wedge s_2 := e_{22}s_{22}e_{12}$ and call these paths *contiguity arcs* of Γ. If $\mathrm{B}_1 = \mathrm{B}_2$, then $\Gamma \wedge s_1 := e_{11}$ and $\Gamma \wedge s_2 := e_{12}$. Since Γ contains no faces of type F3 with h-edges, s_1, s_2 are h-sections and u_{11}, u_{12} consist of 1-edges, it follows that $\varphi(e_{11}s_{12}e_{21}) \equiv \varphi(e_{22}s_{22}e_{12})^{-1}$ and, by definitions and property (A), there exists a simple path t, $|t| > 0$, that connects $(u_{11})_- \in s_2$ with $(u_{11})_+ \in s_1$ and consists of 1-edges. A factorization of $\partial\Gamma$ of the form

$$\partial\Gamma = u_{11}(e_{11}s_{12}e_{21})u_{22}(e_{22}s_{22}e_{12})$$

is called a *standard boundary path* of the contiguity subdiagram Γ.

A contiguity subdiagram Γ between h-sections s_1, s_2 is called *maximal* if there is no contiguity subdiagram Γ' between s_1, s_2 such that $\Gamma \wedge s_i$ is a subpath of $\Gamma' \wedge s_i$, for both $i = 1, 2$, and $|\Gamma \wedge s_1| + |\Gamma \wedge s_2| < |\Gamma' \wedge s_1| + |\Gamma' \wedge s_2|$.

In the following lemma, we record simple facts about bands and contiguity subdiagrams.

LEMMA 3.1. *Suppose that e is an edge of an h-section of a surface diagram Δ and* B *is an h-band in Δ. Then the following are true.*

(a) *There is an h-band one of whose essential edges is e.*
(b) *There is a unique maximal contiguity subdiagram Γ that contains* B.
(c) *There is a unique maximal contiguity subdiagram one of whose contiguity arcs contains e.*

Proof. (a) Suppose that e belongs to a boundary path of Π, where Π is a face of type F3 in Δ. Then it follows from property (A) that if o is an interior point of e then a regular neighborhood N of o in Δ consists of two parts separated by the arc $N \cap e$, one of which is in Π and the other of which is in a face Π' of type F2. Then $\mathrm{B}_{\Pi'}$ is a desired h-band. If e is on $\partial\Delta$ then, again by property (A), there is a face Π'' of type F2 whose boundary path contains e. Then $\mathrm{B}_{\Pi''}$ is a desired h-band.

(b) Let B be a band between h-sections s_1, s_2. Then there exists a contiguity subdiagram Γ between s_1 and s_2 that contains B. For example, $\Gamma = \mathrm{B}$. If Γ_1, Γ_2 are two contiguity subdiagrams between s_1 and s_2 that contain B, then it is easy to check that there is also a contiguity subdiagram Γ_0 that contains both Γ_1 and Γ_2. This implies the uniqueness of a maximal contiguity subdiagram that contains B.

(c) This follows from parts (a)–(b). $\square$

Let Δ be a surface diagram over presentation (2.5) of type (k, k'). Consider the set $\mathcal{C}_h$ of all maximal contiguity subdiagrams between h-sections in Δ. It follows from Lemma 3.1 that, for every edge e of an h-section s of Δ, there is a unique maximal contiguity subdiagram $\Gamma \in \mathcal{C}_h$ whose contiguity arc contains e, that is, e belongs to $\Gamma \wedge s$.

For every $\Gamma \in \mathcal{C}_h$, we pick a connecting line $\ell(\mathrm{B})$, where $\mathrm{B} = \mathrm{B}(\Gamma)$ is a band that defines Γ. Denote $\ell(\Gamma) := \ell(\mathrm{B})$ and call $\ell(\Gamma)$ a *connecting line* of Γ. For every face Π of type F3, whose boundary path $\partial\Pi$ contains h-edges, we pick a vertex v_Π in the interior of Π. Then we connect each point in $(\bigcup_{\Gamma \in \mathcal{C}_h} \ell(\Gamma)) \cap \partial\Pi$ to v_Π by drawing simple arcs in Π such that the arcs' pairwise intersections are $\{v_\Pi\}$ and each arc intersects $\partial\Pi$ only at its endpoint different from v_Π. The union of all such arcs and connecting lines $\ell(\Gamma)$, $\Gamma \in \mathcal{C}_h$, is a graph on Δ, denoted Ψ_h, whose vertex set is the union of the set $\{v_\Pi \mid \Pi \in \Delta_3(2), \partial\Pi \text{ has } h\text{-edges}\}$ and the set of those boundary points of connecting lines $\ell(\Gamma)$, $\Gamma \in \mathcal{C}_h$, that belong to $\partial\Delta$. Note that the set of nonoriented edges of Ψ_h is in bijective correspondence with the set $\mathcal{C}_h$ of maximal contiguity subdiagrams and that each edge of Ψ_h is obtained from $\ell(\Gamma)$, where $\Gamma \in \mathcal{C}_h$, by extending $\ell(\Gamma)$ into a face Π of type F3 whenever a point of $\partial\ell(\Gamma)$ belongs to $\partial\Pi$.

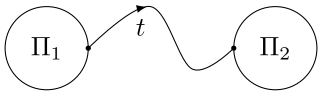

FIGURE 3

Now we will define reduced diagrams over the presentation (2.5). We say that a pair of distinct faces Π_1, Π_2 of type F3 with h-edges in a surface diagram Δ over (2.5) forms a *reducible pair* if there is a simple path t such that t connects some vertices $t_- \in \partial\Pi_1$, $t_+ \in \partial\Pi_2$, t consists of 1-edges, $|t| > 0$, and the label $\varphi(\partial\Gamma)$ of the boundary path $\partial\Gamma = t\partial\Pi_2 t^{-1}\partial\Pi_1$ of the subdiagram Γ, consisting of t, Π_1, Π_2, is equal to 1 in the free group whose free base is the alphabet $\bigcup_{i=1}^{\infty} \mathcal{X}_i \cup \mathcal{A} \cup \{h_1, h_2\}$, see Figure 3.

It is easy to see that if Π_1, Π_2 form a reducible pair in Δ, then one can perform a surgery on Δ that replaces the subdiagram Γ, whose boundary path is $\partial\Gamma = t\partial\Pi_2 t^{-1}\partial\Pi_1$, by a subdiagram that consists of faces of type F1–F2. If Δ' is obtained from Δ by this surgery, then $\varphi(\partial\Delta')$ is identical to $\varphi(\partial\Delta)$ (in fact, the surgery does not affect the boundary of Δ) and $|\Delta'_3(2)| = |\Delta_3(2)| - 2$. Hence, by induction on the number $|\Delta_3(2)|$ of faces of type F3, every diagram Δ can be turned into a diagram $\bar\Delta$ without reducible pairs and with no change in $\varphi(\partial D)$. A diagram Δ will be called *reduced* if Δ contains no reducible pairs.

LEMMA 3.2. *Suppose that Δ is a reduced surface diagram of type (k, k'), there are no h-edges contained in $\partial\Delta$, Δ contains a face of type F3 whose boundary path has h-edges, and the graph Ψ_h is defined as above. Then there exists a vertex in Ψ_h whose degree is positive and is at most*

$$\max\{12(1-k), 12\}.$$

Proof. Let v_Π be a vertex of Ψ_h, let f be an oriented edge of Ψ_h such that $f_- = f_+ = v_\Pi$ and f bounds a disk on Δ. It follows from the definition of relations in (2.5) that if e_1, e_2 are h-edges of $\partial\Pi$, then either $\varphi(e_1), \varphi(e_2) \in \{h_1, h_2\}$ or $\varphi(e_1), \varphi(e_2) \in \{h_1^{-1}, h_2^{-1}\}$. On the other hand, let $\Gamma \in \mathcal{C}_h$ be the contiguity subdiagram that f passes through and let B denote the bond that contains the connecting line $\ell(\Gamma)$. If e_3, e_4 are h-edges of ∂B, then it follows from the fact that f bounds a disk on Δ that $\varphi(e_3) = \varphi(e_2)^{-1}$, hence, the inclusions $e_3, e_4 \in \partial\Pi$ are impossible. Thus, there is no 1-gon in the partial cell decomposition of Δ defined by Ψ_h.

Now assume that the property (B) fails for Ψ_h with parameter $m = 3$. This means that there are three distinct edges f_1, f_2, f_3 in Ψ_h such that

$$(f_1)_- = (f_2)_- = (f_3)_- = v_\Pi, \qquad (f_1)_+ = (f_2)_+ = (f_3)_+ = v_{\Pi'},$$

where Π, Π' are some faces of type F3 with h-edges, such that both paths $f_1 f_2^{-1}$, $f_2 f_3^{-1}$ bound disks on Δ whose interiors contain no vertices of Ψ_h. Let f_i be the extension of the connecting line $\ell(\Gamma_i)$, where $\Gamma_i \in \mathcal{C}_h$, $i = 1, 2, 3$, and s, s' be h-sections of the faces Π, Π', resp. Then it is not difficult to check that either Γ_1, Γ_2 or Γ_2, Γ_3 are contained in a contiguity subdiagram Γ between s and s', contrary to the maximality of contiguity subdiagrams $\Gamma_1, \Gamma_2, \Gamma_3$. This contradiction proves that the property (B) holds for Ψ_h with $m=3$.

Consider those pairs $\{f, f'\}$ of oriented edges of Ψ_h for which the property (B) with $m=2$ fails. Note that the property (B) with $m=3$ for Ψ_h implies that every oriented edge e of Ψ_h is contained in at most one such pair $\{f, f'\}$. For each such pair $\{f, f'\}$, we remove edges $(f')^{\pm 1}$ (or $f^{\pm 1}$) from Ψ_h. Doing this results in a graph $\widehat{\Psi}_h$ which, as follows from definitions, has the property (B) with $m=2$. Therefore, Lemma 2.4 applies to $\widehat{\Psi}_h$ and yields that $E_{\widehat{\Psi}_h} \le 3(V_{\widehat{\Psi}_h} - k)$, where $V_{\widehat{\Psi}_h}, E_{\widehat{\Psi}_h}$ denote the number of vertices, nonoriented edges, resp., in $\widehat{\Psi}_h$. Note that $V_{\Psi_h} = V_{\widehat{\Psi}_h}$ and $E_{\Psi_h} \le 2E_{\widehat{\Psi}_h}$. Hence, $E_{\Psi_h} \le 6(V_{\Psi_h} - k)$. If d is the minimal positive degree of a vertex in V_{Ψ_h}, then it is easy to see from definitions that $d > 0$ and $dV_{\Psi_h} \le 2E_{\Psi_h}$. Thus, $dV_{\Psi_h} \le 12(V_{\Psi_h} - k)$ and

$$d \le 12\left(1 - \frac{k}{V_{\Psi_h}}\right) \le \max\bigl\{12(1-k), 12\bigr\},$$

as desired. □

4. Proofs of theorems

Proof of Theorem 1.1. First, we observe that the group $\mathcal{G}_2$, given by presentation (2.5), can also be presented by generators and relations in the following form

$$\langle h_1, h_2 \parallel \widehat{R}_1 = 1, \widehat{R}_2 = 1, \dots, \widehat{W}_1 = 1, \widehat{W}_2 = 1, \dots \rangle, \tag{4.1}$$

where, for every possible $i = 1, 2, \dots$, the defining words $\widehat{R}_i$, $\widehat{W}_i$ result from rewriting of the words R_i, $W_i(\mathcal{X}_i)$, resp., of presentation (2.5) so that letters $a_{j_1}^{\varepsilon_1}$, $x_{j_2}^{\varepsilon_2}$, where $a_{j_1} \in \mathcal{A}$, $x_{j_2} \in \bigcup_{i'=1}^{\infty} \mathcal{X}_{i'}$, $\varepsilon_1, \varepsilon_2 = \pm 1$, are replaced with the words $V_{2j_1+1}^{\varepsilon_1}$, $V_{2j_2}^{\varepsilon_2}$ over $\{h_1^{\pm 1}, h_2^{\pm 1}\}$, see (2.3).

Now we will show that the group G given by the presentation (1.1) naturally embeds into the group $\mathcal{G}_2$ given by (2.5). Assume that U_0 is a cyclically reduced word over $\mathcal{A}^{\pm 1}$ and $U_0 = 1$ in $\mathcal{G}_2$. By Lemma 2.3, there is a disk diagram Δ_0 over (2.5) such that $\varphi(\partial \Delta_0) \equiv U_0$. Without loss of generality, we may assume that Δ_0 is reduced. Note that a boundary path of Δ_0 contains no h-edges. If Δ_0 contains no face of type F3 whose boundary path has h-edges then, turning h-edges into 1-edges by relabeling, we may assume that Δ_0 contains no h-edges. Hence, we may suppose that Δ_0 is a disk diagram over the presentation (2.2). Then it follows from Lemmas 2.1, 2.3 that $U_0 = 1$ in G. Thus, if U_0 is not trivial in G, then Δ_0 must contain a face of type F3 with h-edges. Therefore, Lemma 3.2 applies to Δ_0 and yields the existence of a vertex v_Π, where Π is a face of type F3 with h-edges, whose degree d in the graph Ψ_h is positive and is at most $\max\{12(1-k),12\}=12$ as $k=\chi(\Delta_0)=1$. It follows from the definition of the graph Ψ_h and Lemmas 2.4, 3.1 that there are $d\le 12$ maximal contiguity subdiagrams $\Gamma_1,\dots,\Gamma_d$ between an h-section q of Π and some h-sections of Δ_0 so that every edge of q is contained in exactly one of the contiguity arcs $\Gamma_i\wedge q$, $i=1,\dots,d$. Therefore, there is an index i^* such that $|\Gamma_{i^*}\wedge q|\ge\frac{1}{12}|q|$. Since $\partial\Delta_0$ contains no h-edges, it follows that Γ_{i^*} is a contiguity subdiagram between q and q', where q' is an h-section of a face Π'. Denote $q_\Pi:=\Gamma_{i^*}\wedge q$ and $q_{\Pi'}:=\Gamma_{i^*}\wedge q'$. Since $\varphi(q_\Pi)\equiv\varphi(q_{\Pi'})^{-1}$ and $|q_\Pi|\ge\frac{1}{12}|q|>\frac{4}{M}|q|$ as $n\ge 2$ and $M=24n\ge 48$, it follows from Lemma 2.2 that $\varphi(q)\equiv\varphi(q')^{-1}$. Hence, by the definition of relations in (2.5) and by the definition of a contiguity subdiagram, we have that $\varphi(\partial\Pi)\equiv\varphi(\partial\Pi')^{-1}$ and the faces Π, Π' form a reducible pair. This contradiction to the fact that Δ_0 is reduced proves that $U_0\overset{G}{=}1$ and, therefore, G naturally embeds in $\mathcal{G}_2$, as claimed. Let $\nu_2:G\to\mathcal{G}_2$ denote this embedding.

Consider a quadratic equation $W=1$ over G of length $\ell\le n$. We need to prove that the equation $W=1$ has a solution in the group G given by (1.1) if and only if the equation $\nu_2(W)=1$ has a solution in the group $\mathcal{G}_2$ given by (2.5).

First, assume that $W=1$ has a solution in G. By Lemma 2.1, the equation $\nu_1(W)=1$ has a solution in the group $\mathcal{G}_1$ given by (2.2). Since G naturally embeds in $\mathcal{G}_2$, it follows from the definition of presentations (2.2), (2.5) that there is a homomorphism $\mathcal{G}_1\to\mathcal{G}_2$ which is identical on G. Hence, we may conclude that the equation $\nu_2(W)=1$ has a solution in the group $\mathcal{G}_2$, as desired.

Conversely, suppose that the equation $\nu_2(W)=1$ has a solution in the group $\mathcal{G}_2$. Our goal is to show that $W=1$ has a solution in G. Let

$$W\equiv t_1^{\varepsilon_1}U_1t_2^{\varepsilon_2}U_2\dots t_\ell^{\varepsilon_\ell}U_\ell,$$

where $t_1,\dots,t_\ell\in\bigcup_{i=1}^\infty\mathcal{X}_i$, $\varepsilon_1,\dots,\varepsilon_\ell\in\{\pm1\}$, and $U_1,\dots,U_\ell$ are some reduced or empty words over $\mathcal{A}^{\pm1}$. Since $\nu_2(W)=1$ has a solution in $\mathcal{G}_2$, there are nonempty words $T_1,\dots,T_\ell$ over the alphabet $\bigcup_{i=1}^\infty\mathcal{X}_i^{\pm1}\cup\mathcal{A}^{\pm1}\cup\{h_1^{\pm1},h_2^{\pm1},1\}$ such that

$$T_1^{\varepsilon_1}U_1T_2^{\varepsilon_2}U_2\dots T_\ell^{\varepsilon_\ell}U_\ell\overset{\mathcal{G}_2}{=}1.$$

Note that we would use the letter 1 for the trivial element of $\mathcal{G}_2$. By Lemma 2.3, there is a disk diagram Δ over presentation (2.5) such that

$$\varphi(\partial\Delta)\equiv T_1^{\varepsilon_1}U_1T_2^{\varepsilon_2}U_2\dots T_\ell^{\varepsilon_\ell}U_\ell.$$

Since $W=1$ is a quadratic equation, there is a permutation

$$\tau:\{1,\dots,\ell\}\to\{1,\dots,\ell\}$$

such that $\tau^2=1$, $\tau(i)\ne i$ and $t_i=t_{\tau(i)}$ for every $i\in\{1,\dots,\ell\}$. Hence, we may assume that $T_i\equiv T_{\tau(i)}$ for every $i\in\{1,\dots,\ell\}$. Denote

$$\partial\Delta=r_1^{\varepsilon_1}u_1r_2^{\varepsilon_2}u_2\dots r_\ell^{\varepsilon_\ell}u_\ell,$$

where r_i, u_i are paths of $\partial\Delta^{\pm 1}$ such that $\varphi(r_i) \equiv T_i$, $\varphi(u_i) \equiv U_i$ for every $i = 1, \dots, \ell$. Now we construct a surface diagram $\widetilde{\Delta}$ from Δ by attaching the path r_i to $r_{\tau(i)}$ for every $i = 1, \dots, \ell$. Note that $\chi(\widetilde{\Delta}) = 1 - \frac{\ell}{2}$ and $\widetilde{\Delta}$ has k' connected components in its boundary $\partial\widetilde{\Delta}$, $1 \le k' \le \ell$. Thus, $\widetilde{\Delta}$ is a surface diagram of type $(1 - \frac{\ell}{2}, k')$.

Let $c_1, \dots c_{k'}$ be connected components of $\partial\widetilde{\Delta}$. Note that each c_j is a product of some paths in the set $\{u_1^{\delta_1}, \dots, u_\ell^{\delta_\ell}\}$, where $\delta_1, \dots, \delta_\ell \in \{\pm 1\}$, and each $u_j^{\delta_j}$ occurs in one of $c_1, \dots c_{k'}$ exactly once. If $\widetilde{\Delta}$ contains a reducible pair of faces, then we remove this pair by the surgery described above and obtain a surface diagram $\widetilde{\Delta}'$ with unchanged boundary paths and $|\widetilde{\Delta}_3'(2)| = |\widetilde{\Delta}_3(2)| - 2$, where $|\widetilde{\Delta}_3(2)|$ is the number of faces of type F3 in Δ. It is not difficult to check that there exists a disk diagram Δ' such that

$$\partial\Delta' = \big(r_1'\big)^{\varepsilon_1} u_1' \big(r_2'\big)^{\varepsilon_2} u_2' \dots \big(r_\ell'\big)^{\varepsilon_\ell} u_\ell',$$

where r_i', u_i' are paths of $\partial(\Delta')^{\pm 1}$ such that $\varphi(r_i') \equiv \varphi(r_{\tau(i)}') \equiv T_i'$, $\varphi(u_i') \equiv \varphi(u_i)$ for every $i = 1, \dots, \ell$. Moreover, the diagram $\widetilde{\Delta}'$ can be obtained from Δ' in the same manner as $\widetilde{\Delta}$ was obtained from Δ, in particular, $|\widetilde{\Delta}_3'(2)| = |\Delta_3'(2)|$. Hence, by induction on the number $|\Delta_3(2)|$ of faces of type F3 in Δ, we may assume that the surface diagram $\widetilde{\Delta}$ is reduced.

Suppose that $\widetilde{\Delta}$ contains no faces of type F3 with h-edges. Then Δ also has this property, hence we can turn h-edges of Δ into 1-edges by relabeling and obtain thereby a disk diagram $\bar{\Delta}$ from Δ with no h-edges. Such a diagram $\bar{\Delta}$ could be regarded as a diagram over presentation (2.2). The existence of such $\bar{\Delta}$ over (2.2) means that the equation $\nu_1(W) = 1$ has a solution in the group $\mathcal{G}_1$ given by (2.2). By Lemma 2.1, the equation $W = 1$ has a solution in G, as required.

Hence, we may assume that Δ contains faces of type F3 with h-edges. Clearly, $\widetilde{\Delta}$ also has this property and we may consider the graph $\Psi_h = \Psi_h(\widetilde{\Delta})$ on $\widetilde{\Delta}$ as defined before. Since $\partial\widetilde{\Delta}$ contains no h-edges, Lemma 3.2 applies to the graph Ψ_h on $\widetilde{\Delta}$ and yields the existence of a vertex v_Π, where Π is a face of $\widetilde{\Delta}$, whose positive degree is at most

$$\max\big\{12(1-k), 12\big\} = \max\{6\ell, 12\} = 6\ell \le 6n$$

as $\ell \ge 2$. As above, it follows from the definition of the graph Ψ_h and Lemmas 2.4, 3.1 that there are $d \le 6n$ maximal contiguity subdiagrams $\Gamma_1, \dots, \Gamma_d$ between an h-section q of Π and some h-sections of $\widetilde{\Delta}$ so that every edge of q is contained in exactly one of the contiguity arcs $\Gamma_i \wedge q$, $i = 1, \dots, d$. Therefore, there is an index i^* such that $|\Gamma_{i^*} \wedge q| \ge \frac{1}{6n}|q|$. Let Γ_{i^*} be a contiguity subdiagram between q and q', where q' is an h-section of a face Π'. Denote $q_\Pi := \Gamma_{i^*} \wedge q$ and $q_{\Pi'} := \Gamma_{i^*} \wedge q'$. Since $\varphi(q_\Pi) \equiv \varphi(q_{\Pi'})^{-1}$ and $|q_\Pi| \ge \frac{1}{6n}|q| = \frac{4}{M}|q|$ as $M = 24n$, it follows from Lemma 2.2 that $\varphi(q) \equiv \varphi(q')^{-1}$. Hence, by the definition of relations in (2.5) and by the definition of a contiguity subdiagram, we have that $\varphi(\partial\Pi) \equiv \varphi(\partial\Pi')^{-1}$ and the faces Π, Π' form a reducible pair in $\widetilde{\Delta}$. This contradiction to the fact that $\widetilde{\Delta}$ is reduced proves that it is impossible that Δ contains faces of type F3 with h-edges. Hence, the equation $W = 1$ has a solution in G, as desired.

Thus, the group $\mathcal{G}_2$ has all of the required properties of the group H of the statement of Theorem 1.1 and the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. (a) Let $W_1 = 1, W_2 = 1, \dots$ be the enumeration, fixed in (2.1), of all quadratic equations over G such that, for every $i \geq 1$, $|W_i|_{\mathcal{X}} \leq n$ and $W_i = 1$ has a solution in G. Recall that the enumeration $\bigcup_{i=1}^{\infty} \mathcal{X}_i = \{x_1, x_2, \dots\}$ has the property that if $x_j \in \mathcal{X}_k$, $x_{j'} \in \mathcal{X}_{k'}$ and $k < k'$ then $j < j'$. This property implies, for every $W_i = 1$, that if $x_{k_1}, \dots, x_{k_\ell}$ are the letters of $\bigcup_{i=1}^{\infty} \mathcal{X}_i$ that appear in $W_i(\mathcal{X}_i)^{\pm 1}$, then $k_1, \dots, k_\ell \leq ni$. Thus, in view of the relations $x_j^{-1} V_{2j} = 1$ of the presentation (2.5), it follows that $(V_{2k_1}, \dots, V_{2k_\ell})$ is a solution tuple to the equation $\nu_2(W_i) = 1$ over $\mathcal{G}_2$. Since $|V_k| \leq (M(k+1)+1)M$ and $\ell \leq n$, we further obtain that

$$\sum_{j'=1}^{\ell} |V_{2k_{j'}}| \leq nM\big(M(2ni+1)+1\big) \leq 3n^2M^2 i = Cn^4 i,$$

where $C = 3 \cdot 24^2$ as $M = 24n$.

(b) Since the presentation (1.1) of G is recursively enumerable, it follows that the set of all words U over $\mathcal{A}^{\pm 1}$ such that $U \overset{G}{=} 1$ is also recursively enumerable. More generally, we can analogously obtain that all quadratic equations $W = 1$ over G of length $\leq n$ that have solutions in G can be recursively enumerated. The last observation means that we can create a recursive enumeration (2.1). Now we can use constructions of (2.5), (4.1) and see that defining relations of the presentation (4.1) for $\mathcal{G}_2$ can be recursively enumerated as well.

(c) The existence of an algorithm that detects whether a quadratic equation $W = 1$ over G of length $\leq n$ has a solution enables us to effectively write down all quadratic equations $W = 1$ over G of length $\leq n$ that have solutions in G. Hence, we can effectively create an enumeration (2.1) and, using constructions of (2.5), (4.1), write down all relations of the form $\widehat{W}_1 = 1, \widehat{W}_2 = 1, \dots$ in the presentation (4.1). Since the presentation (1.1) of G is decidable, we can also effectively write down all relations of the form $\widehat{R}_1 = 1, \widehat{R}_2 = 1, \dots$ in the presentation (4.1). Hence, the presentation (4.1) is decidable. Since the map $a_i \to V_{2i+1}$, $i = 1, 2, \dots$, extends to the embedding $\mu_n : G \to H$ and the set of defining relations of presentation (4.1) is recursive, we see that the embedding $\mu_n : G \to H$ can be effectively constructed. Theorem 1.2 is proven. □

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