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WELL-POSEDNESS OF THE MARTINGALE PROBLEM FOR SUPERPROCESS WITH INTERACTION

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ABSTRACT. We consider the martingale problem for superprocess with interactive immigration mechanism. The uniqueness of the solution to this martingale problem is established using the strong uniqueness of the solution to a corresponding SPDE, which is obtained by an extended version of the Yamada–Watanabe argument.

1. Introduction

Let $M_F(\mathbb{R})$ be the collection of all finite Borel measures on $\mathbb{R}$. Let $q: M_F(\mathbb{R}) \to M_F(\mathbb{R})$ be the interactive immigration measure. Here, the word "interactive" means that the immigration measure q depends on the measure-valued process itself. Namely, we consider a continuous $M_F(\mathbb{R})$-valued process (μ_t) which solves the following martingale problem (MP): $\forall f \in C_b^2(\mathbb{R})$, the process

$$M_t^f = \langle \mu_t, f\rangle - \langle \mu_0, f\rangle - \int_0^t \left(\left\langle \mu_s, \frac{1}{2} f''\right\rangle + \langle q(\mu_s), f\rangle\right) ds \tag{1.1}$$

is a continuous martingale with quadratic variation process

$$\langle M^f\rangle_t = \gamma \int_0^t \langle \mu_s, f^2\rangle \, ds, \tag{1.2}$$

where the constant $\gamma > 0$ is the branching rate, the notation $C_b^k(\mathbb{R})$ (resp. $C_0^k(\mathbb{R})$) stands for the collection of all bounded (resp. compactly-supported) continuous functions on $\mathbb{R}$ with bounded derivatives up to kth order, and

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the notation $\langle \mu, f\rangle$ denotes the integral of the function f with respect to the measure μ. Such a process (μ_t) is called a super-Brownian motion (SBM) with interactive immigration. The *aim* of the present article is to prove the uniqueness of the solution to the MP (1.1, 1.2) under some conditions on q.

This martingale problem is studied by Shiga [12], and Fu and Li [3] using an equation driven by a Poisson random measure. Its solution is also constructed by Dawson and Li [1] using the excursion theory. They studied various properties of the process while leaving the uniqueness of the solution as an *open problem*.

In this paper, we prove the uniqueness of the solution to the MP under suitable conditions. The main idea is to relate the MP to a stochastic partial differential equation (SPDE), whose pathwise uniqueness of the solution can be established, satisfied by the distribution valued process (u_s) corresponding to the measure-valued process (μ_s). Such a connection is first studied by one of us [13] for the special case of $q = 0$. The proof of the pathwise uniqueness in [13] is done by relating the SPDE to a backward stochastic differential equation, while for the current setup the proof is done by an extended Yamada–Watanabe argument to SPDE which is inspired by Mytnik and Perkins [8] and Mytnik et al. [9]. When the spatial motion is interactive, that is, it is a diffusion process with diffusion and drift coefficients depending on the superprocess itself, the well-posedness of the MP has been studied by Donnelly and Kurtz [2] in their lookdown approach and thanks also to results of Kurzt [5] on filtered martingale problem (see also Theorem V.5.1 in Perkins [11]). Uniqueness for "historical" superprocesses with certain interactions was investigated by Perkins in [10].

We now proceed to presenting the main result of this paper. We first state the precise definition of the martingale problem. For $\nu_i \in M_F(\mathbb{R})$, let $v_i(x) = \nu_i((-\infty, x])$ for $x \in \mathbb{R}$ and $i = 1, 2$. Define distance ρ on $M_F(\mathbb{R})$ by

$$\rho(\nu_1, \nu_2) = \int_{\mathbb{R}} e^{-|x|} \big| v_1(x) - v_2(x)\big|\, dx.$$

It is easy to see that, under metric ρ, $M_F(\mathbb{R})$ is a Polish space whose topology coincides with that given by weak convergence of measures. Denote the collection of all continuous mappings from $\mathbb{R}_+$ to $M_F(\mathbb{R})$ by $\mathcal{X} \equiv C(\mathbb{R}_+, M_F(\mathbb{R}))$. Throughout the paper we use K to denote a non-negative constant whose value may change from line to line.

Definition 1.1. A probability measure Γ on $\mathcal{X}$ is a solution to MP (1.1, 1.2) if there exists a continuous $M_F(\mathbb{R})$-valued process μ_t on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ such that Γ is the probability measure induced on $\mathcal{X}$ by (μ_t), and for any $f \in C_0^2(\mathbb{R})$, the process M_t^f given by (1.1) is a continuous martingale with quadratic variation process given by (1.2).

MP (1.1, 1.2) is well-posed if it has a unique solution.

For any $x \in \mathbb{R}$ and $\nu \in M_F(\mathbb{R})$, we define

$$\eta(x,\nu) = q(\nu)\big((-\infty, x]\big).$$

Here is the main result of this article.

THEOREM 1.2. (a) *Assume the following conditions*:

(I1) $\int_{\mathbb{R}}(1+x^2)\mu_0(dx) < \infty$;

(I2) *There exists a constant K such that for any $\nu \in M_F(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} (1+x^2) q(\nu)(dx) \le K.$$

Then, *MP* (1.1, 1.2) *has a solution.*

(b) *In addition to* (I1), (I2), *assume that η satisfies the following condition*

(I3) *There exists a constant K such that for any $y \in \mathbb{R}$, $\nu_1, \nu_2 \in M_F(\mathbb{R})$, we have*

$$\big|\eta(y,\nu_1) - \eta(y,\nu_2)\big| \le K\rho(\nu_1,\nu_2), \tag{1.3}$$

Then, *MP* (1.1, 1.2) *is well-posed.*

This paper is organized as follows. In Section 2, we establish the equivalency between the MP (1.1, 1.2) and a stochastic partial differential equation (SPDE). Then, in Section 3, we prove the strong uniqueness of the SPDE by a Yamada–Watanabe argument, which then gives the uniqueness to the MP (1.1, 1.2).

2. A related SPDE

A relationship between a super-Brownian motion and the SPDE satisfied by its corresponding distribution function valued process is established in Xiong [13]. In this section, we extend that result to the case when the system receives immigration with a rate depending on the current state of the system. In fact, our result follows from a more general result to be given below for a model with interactive location-dependent branching rate of the following form

$$\gamma(x,\nu) = \lambda^2\big(\nu(-\infty,x]\big), \quad \forall x \in \mathbb{R}, \nu \in M_F(\mathbb{R}), \tag{2.1}$$

where λ is a bounded measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$.

We do not know whether this change in branching rate has any significance to applications and we put it just for the sake of completeness and with the hope that somebody could be able to generalize it to more interesting cases.

From now on, we consider the following more general martingale problem (GMP): $\forall f \in C_0^2(\mathbb{R})$, the process M_t^f given by (1.1) is a continuous martingale with quadratic variation process

$$\langle M^f\rangle_t = \int_0^t \langle \mu_s, \gamma(\cdot,\mu_s) f^2\rangle\, ds, \tag{2.2}$$

where γ is given by (2.1).

DEFINITION 2.1. A probability measure Γ on $\mathcal{X}$ is a solution to GMP (1.1, 2.2) if there exists a continuous $M_F(\mathbb{R})$-valued process μ_t on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ such that Γ is the probability measure induced on $\mathcal{X}$ by (μ_t), and for any $f \in C_0^2(\mathbb{R})$, the process M_t^f given by (1.1) is a continuous martingale with quadratic variation process given by (2.2). We also refer to (μ_t) as a solution to the GMP.

GMP (1.1, 2.2) is well-posed if it has a unique solution.

Let $W(ds\,da)$ be a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $ds\,da$. Consider the following SPDE on the space of nondecreasing (in spatial variable) functions taking values in $[0,\infty)$: For $y \in \mathbb{R}$,

$$u_t(y) = F(y) + \int_0^t \int_0^{u_s(y)} \lambda(a) W(ds\,da) + \int_0^t \left(\frac{1}{2}\Delta u_s(y) + \eta(y, \mu_s)\right) ds, \tag{2.3}$$

where Δ is the one-dimensional Laplacian and $F(y) = \mu_0((-\infty, y])$.

Let $C_{b,m}(\mathbb{R})$ be the subset of $C_b(\mathbb{R})$ consisting of nondecreasing bounded continuous functions on $\mathbb{R}$.

DEFINITION 2.2. The SPDE (2.3) has a weak solution if there exists a continuous $C_{b,m}(\mathbb{R})$-valued process u_t on a stochastic basis such that for any $f \in C_0^2(\mathbb{R})$ and $t > 0$,

$$\begin{aligned}\langle u_t, f\rangle = \langle F, f\rangle &+ \int_0^t \int_0^\infty \int_{\mathbb{R}} f(y)\lambda(y) 1_{a \le u_s(y)}\, dy W(ds\,da) \\ &+ \int_0^t \left(\left\langle u_s, \frac{1}{2} f''\right\rangle + \langle \eta(\cdot, \mu_s), f\rangle\right) ds, \quad \text{a.s.,}\end{aligned}$$

where $\langle f, g\rangle = \int_{\mathbb{R}} f(x) g(x)\, dx$.

Similar to Theorem 2.2 in Xiong [13], we have the following lemma.

LEMMA 2.3. *$\{\mu_t\}$ is a solution to GMP (1.1, 2.2) if and only if $\{u_t\}$ defined by*

$$u_t(y) = \mu_t\big((-\infty, y]\big), \quad \forall y \in \mathbb{R}, \tag{2.4}$$

is a weak solution to SPDE (2.3).

Proof. Suppose that (u_t) is a solution to SPDE (2.3). For a non-decreasing continuous function g on $\mathbb{R}$, we define its generalized inverse as

$$g^{-1}(a) = \inf\{x : g(x) > a\}.$$

Then, for $f \in C_0^3(\mathbb{R})$, we have

$$\begin{aligned}\langle \mu_t, f\rangle &= -\langle u_t, f'\rangle \\ &= -\langle F, f'\rangle - \int_0^t \int_0^\infty \int_{\mathbb{R}} f'(y) 1_{a \le u_s(y)}\, dy \lambda(a) W(ds\,da)\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t \left\langle \frac{1}{2}\Delta u_s, f' \right\rangle ds - \int_0^t \int_{\mathbb{R}} \eta(y,\mu_s) f'(y)\, dy\, ds \\
&= \mu(f) + \int_0^t \left(\left\langle \mu_s, \frac{1}{2} f'' \right\rangle + \langle q(\mu_s), f \rangle \right) ds \\
&\quad + \int_0^t \int_0^\infty f\big(u_s^{-1}(a)\big) \lambda(a) W(ds\, da).
\end{aligned}
$$

Thus, M_t^f is a martingale with quadratic variation process

$$
\begin{aligned}
\langle M^f \rangle_t &= \int_0^t \int_0^\infty \lambda(a)^2 f\big(u_s^{-1}(a)\big)^2 \, da\, ds \\
&= \int_0^t \int_{\mathbb{R}} \lambda\big(u_s(x)\big)^2 f(x)^2 \, du_s(x)\, ds \\
&= \int_0^t \mu_s\big(\gamma(\cdot,\mu_s) f^2\big)\, ds.
\end{aligned}
$$

Thus, (μ_t) is a solution to GMP.

On the other hand, suppose that (μ_t) is a solution to GMP (1.1, 2.2). Let $f \in C_0^2(\mathbb{R})$ and $g(y) = \int_y^\infty f(x)\, dx$. Then,

$$
\begin{aligned}
\langle u_t, f \rangle &= \langle \mu_t, g \rangle \\
&= \langle \mu_0, g \rangle + \int_0^t \left(\left\langle \mu_s, \frac{1}{2} g'' \right\rangle + \langle q(\mu_s), g \rangle \right) ds + M_t^g \\
&= \langle F, f \rangle + \int_0^t \left(\left\langle u_s, \frac{1}{2} f'' \right\rangle + \langle \eta(\cdot,\mu_s), f \rangle \right) ds + M_t^g.
\end{aligned} \tag{2.5}
$$

Let $\mathcal{S}'(\mathbb{R})$ be the space of Schwarz distributions and define the $\mathcal{S}'(\mathbb{R})$-valued process N_t by $N_t(f) = M_t^g$ for any $f \in C_0^\infty(\mathbb{R})$. Then, N_t is an $\mathcal{S}'(\mathbb{R})$-valued continuous square-integrable martingale with

$$
\begin{aligned}
\langle N(f) \rangle_t &= \int_0^t \int_{\mathbb{R}} \gamma(y,\mu_s) g(y)^2 \mu_s(dy)\, ds \\
&= \int_0^t \int_{\mathbb{R}} \lambda^2\big(u_s(y)\big) g(y)^2 \mu_s(dy)\, ds \\
&= \int_0^t \int_0^\infty \lambda(a)^2 g\big(u_s^{-1}(a)\big)^2 \, da\, ds \\
&= \int_0^t \int_0^\infty \left(\lambda(a) \int_{\mathbb{R}} 1_{a \le u_s(y)} f(y)\, dy \right)^2 da\, ds.
\end{aligned}
$$

Let $G : \mathbb{R}_+ \times \Omega \to L_{(2)}(H,H)$ be defined as

$$
G(s,\omega) f(a) = \lambda(a) \int_{\mathbb{R}} 1_{a \le u_s(x)} f(x)\, dx, \quad \forall f \in H,
$$

where $H = L^2(\mathbb{R})$ and $L_{(2)}(H,H)$ is the space consisting of all Hilbert–Schmidt operators on H. By Theorem 3.3.5 of Kallianpur and Xiong [4], on an extension of the original stochastic basis, there exists an H-cylindric Brownian motion B_t such that

$$N_t(f) = \int_0^t \langle G(s,\omega)f, dB_s\rangle_H.$$

Let $\{h_j\}$ be a complete orthonormal system (CONS) of the Hilbert space H and define random measure W on $\mathbb{R}_+ \times \mathbb{R}$ as

$$W([0,t]\times A) = \sum_{j=1}^{\infty} \langle 1_A, h_j\rangle B_t^{h_j}.$$

It is easy to show that W is a Gaussian white noise random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $ds\,da$. Furthermore,

$$N_t(f) = \int_0^t \int_0^\infty \int_{\mathbb{R}} \lambda(a) 1_{a\le u_s(x)} f(x)\,dx W(ds\,da).$$

Plugging back to (2.5) verifies that u_t is a solution to (2.3). □

PROPOSITION 2.4. *Assume* (I1), (I2), (2.1). *Then GMP* (1.1, 2.2) *has a solution.*

Proof. Let $t_i^n = \frac{i}{n}$, $i = 0,1,2,\ldots.$ Let $\pi^n(s) = t_i^n$ for $s \in [t_i^n, t_{i+1}^n)$. For each n, let μ_t^n be a solution to the approximating martingale problem: $\forall f \in C_0^2(\mathbb{R})$,

$$M_t^{n,f} = \langle \mu_t^n, f\rangle - \langle \mu_0, f\rangle - \int_0^t \left(\left\langle \mu_s^n, \frac{1}{2} f'' \right\rangle + \langle q(\mu^n_{\pi^n(s)}), f\rangle \right) ds \tag{2.6}$$

is a continuous martingale with quadratic variation process

$$\langle M^{n,f}\rangle_t = \int_0^t \langle \mu_s^n, \gamma(\cdot, \mu^n_{\pi^n(s)}) f^2\rangle\, ds. \tag{2.7}$$

The existence of a solution in each subinterval $[t_i^n, t_{i+1}^n]$ follows from classical theory of superprocesses (cf. Corollary 7.15 in Li [6]).

Let T be fixed and $t \le T$. Taking $f = 1$ in (2.6), we get

$$\langle \mu_t^n, 1\rangle = \langle \mu_0, 1\rangle + \int_0^t \langle q(\mu^n_{\pi^n(s)}), 1\rangle\, ds + M_t^{n,1}.$$

Hence, by our assumptions on q and γ, we have

$$\begin{aligned} a^n(t) &\equiv \mathbb{E}\sup_{s\le t} \langle \mu_t^n, 1\rangle^4 \\ &\le K_1 + K_2 \mathbb{E}\langle M^{n,1}\rangle_t^2 \\ &\le K_1 + K_2 \mathbb{E}\left(\int_0^t \langle \mu_s^n, \gamma(\cdot, \mu^n_{\pi^n(s)})\rangle\, ds \right)^2 \end{aligned}$$

$$\le K_1 + K_3 \int_0^t \mathbb{E}\langle \mu_s^n, 1\rangle^4 \, ds$$
$$\le K_1 + K_3 \int_0^t a^n(s)\, ds.$$

By classical moment bounds for superprocesses, we can easily get that $a^n(t)$ is finite for any n and $t > 0$. Therefore, we can apply Gronwall's inequality to see that $a^n(t) \le K_1 e^{K_3 t} \le K_4$ uniformly on $t \in [0, T]$, and $n \ge 1$.

For any $f \in C_b^2(\mathbb{R})$ and $s < t$, we then have

$$\begin{aligned}
&\mathbb{E}\left|\langle \mu_t^n - \mu_s^n, f\rangle\right|^4 \\
&\le 2^4 \mathbb{E}\left|\int_s^t \left(\left\langle \mu_r^n, \frac{1}{2} f''\right\rangle + \langle q(\mu_{\pi^n(r)}^n), f\rangle\right) dr\right|^4 + 2^4 \mathbb{E}\left|M_t^{n,f} - M_s^{n,f}\right|^4 \\
&\le K_1 |t-s|^4 + K_2 \mathbb{E}\left(\langle M^{n,f}\rangle_t - \langle M^{n,f}\rangle_s\right)^2 \\
&\le K_1 |t-s|^4 + K_3 |t-s|^2 \\
&\le K_4 |t-s|^2.
\end{aligned}$$

It then follows from Kolmogorov's criteria that $\{\langle \mu^n, f\rangle : n \ge 1\}$ is tight in $C([0, T], \mathbb{R})$. This implies that $\{\mu^n : n \ge 1\}$ is tight in $C([0, T], M_F(\bar{\mathbb{R}}))$, where $\bar{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. Denote by (μ_t) a limit point.

Taking $f(x) = x^2$ in (2.6), we get

$$\mathbb{E}\langle \mu_t^n, x^2\rangle = \langle \mu_0, x^2\rangle + \mathbb{E}\int_0^t \left(\langle \mu_r^n, 1\rangle + \langle q(\mu_{\pi^n(r)}^n), x^2\rangle\right) dr \le K,$$

where the last inequality follows by the assumptions (I1) and (I2). This implies that μ_t is supported on $\mathbb{R}$ and hence $\mu \in C([0, T], M_F(\mathbb{R}))$ a.s. Passing (2.6, 2.7) to the limit, it is standard to show that (μ_t) is a solution to the GMP. □

REMARK 2.5. The above lemma finishes the proof of Theorem 1.2(a).

In the next section, we shall prove the uniqueness of the solution to SPDE (2.3). To this end, we need the following lemma.

LEMMA 2.6. *Let $\mu_0 \in M_F(R)$, and suppose that Conditions* (I2), (2.1) *hold. Let $\{\mu_t\}$ be arbitrary solution to GMP* (1.1, 2.2). *Then, for any $T > 0$, there exists $K_1 = K_1(T)$ such that*

$$\mathbb{E}\left[\sup_{t \le T} \langle \mu_t, 1\rangle^2\right] \le K_1. \tag{2.8}$$

Proof. Fix arbitrary $T > 0$. Choosing $f = 1$, using martingale inequalities for the martingale at (1.1) and our conditions on q and γ, we get

$$\begin{aligned}
\mathbb{E}\langle \mu_t, 1\rangle &= \langle \mu_0, 1\rangle + \mathbb{E}\int_0^t \langle q(\mu_s), 1\rangle \, ds \\
&\le \langle \mu_0, 1\rangle + KT \equiv K_2,
\end{aligned}$$

and

$$\mathbb{E}\Big[\sup_{t\le T}\langle\mu_t,1\rangle^2\Big]\le 3\langle\mu_0,1\rangle^2+3K^2T+3K\int_0^T\mathbb{E}\langle\mu_s,1\rangle\,ds$$
$$\le 3\langle\mu_0,1\rangle^2+3K^2T+3KK_2T\equiv K_1.\qquad\square$$

3. Uniqueness for SPDE

This section is devoted to the proof of the pathwise uniqueness for the solution to SPDE (2.3). By Lemma 2.3, the uniqueness for the solution to the GMP is then a direct consequence, and thus Theorem 1.2(b) will follow.

PROPOSITION 3.1. *Assume* (I1), (I2), (I3) *and* (2.1). *Then the pathwise uniqueness holds for SPDE* (2.3), *namely*, *if* (2.3) *has two solutions defined on the same stochastic basis with the same initial conditions*, *then the solutions coincide a.s.*

Proof. Let $\{u^1_t(y)\}$ and $\{u^2_t(y)\}$ be two solutions to SPDE (2.3) and $v_t=u^1_t-u^2_t$. Also let $\{\mu^1_t\},\{\mu^2_t\}$ be corresponding solutions of the martingale problem (1.1), (2.2), that is $u^i_t(x)=\mu^i_t((-\infty,x]),x\in\mathbb{R},i=1,2$.

For simplicity of notation, given functions $G(\cdot,\cdot)$ on $\mathbb{R}_+\times\mathbb{R}_+$ and η on $M_F(\mathbb{R})\times\mathbb{R}$, we write

$$\bar G_s(a,y)=G\big(a,u^1_s(y)\big)-G\big(a,u^2_s(y)\big)$$

and

$$\bar\eta_s(y)=\eta\big(y,\mu^1_s\big)-\eta\big(y,\mu^2_s\big).$$

Then,

$$v_t(y)=\int_0^t\int_{\mathbb{R}_+}\bar G_s(a,y)W(ds\,da)+\int_0^t\bigg(\frac12\Delta v_s(y)-\bar\eta_s(y)\bigg)\,ds,$$

where $G(a,u)=\lambda(a)1_{a\le u}$. Let $\Phi\in C_0^\infty(\mathbb{R})^+$ be such that $\operatorname{supp}(\Phi)\subset(-1,1)$ and the total integral is 1. Let $\Phi_m(x)=m\Phi(mx)$. Then,

$$\begin{aligned}\big\langle v_t,\Phi_m(x-\cdot)\big\rangle&=\int_0^t\int_{\mathbb{R}_+}\int_{\mathbb{R}}\bar G_s(a,y)\Phi_m(x-y)\,dyW(ds\,da)\\&\quad+\int_0^t\bigg\langle v_s,\frac12\Delta\Phi_m(x-\cdot)\bigg\rangle\,ds\\&\quad-\int_0^t\big\langle\bar\eta_s,\Phi_m(x-\cdot)\big\rangle\,ds.\end{aligned}$$

Next, we apply a modified Yamada–Watanabe argument. We will follow closely the argument from [9]. First, we define a sequence of functions ϕ_k as follows. Let $\{a_k\}$ be a decreasing positive sequence defined recursively by

$$a_0 = 1 \quad \text{and} \quad \int_{a_k}^{a_{k-1}} z^{-1}\,dz = k, \quad k \geq 1.$$

Let ψ_k be non-negative functions in $C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(\psi_k) \subset (a_k, a_{k-1})$ and

$$\int_{a_k}^{a_{k-1}} \psi_k(z)\,dz = 1 \quad \text{and} \quad \psi_k(z) \leq 2(kz)^{-1}, \quad \forall z \in \mathbb{R}.$$

Let

$$\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x)\,dx, \quad \forall z \in \mathbb{R}.$$

Then, $\phi_k(z) \uparrow |z|$ and $|z|\phi_k''(z) \leq 2k^{-1}$.

Applying Itô's formula, we get

$$\begin{aligned}
&\phi_k\big(\langle v_t, \Phi_m(x-\cdot)\rangle\big)\\
&\quad= \int_0^t \int_{\mathbb{R}_+} \phi_k'\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big) \int_{\mathbb{R}} \bar{G}_s(a,y)\Phi_m(x-y)\,dy W(ds\,da)\\
&\qquad+ \int_0^t \phi_k'\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\Big\langle v_s, \frac{1}{2}\Delta\Phi_m(x-\cdot)\Big\rangle\,ds\\
&\qquad- \int_0^t \phi_k'\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\langle \bar{\eta}_s, \Phi_m(x-\cdot)\rangle\,ds\\
&\qquad+ \frac{1}{2}\int_0^t \int_{\mathbb{R}_+} \phi_k''\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\left|\int_{\mathbb{R}} \bar{G}_s(a,y)\Phi_m(x-y)\,dy\right|^2 da\,ds.
\end{aligned}$$

Let

$$J(x) = \int_{\mathbb{R}} e^{-|y|}\varrho(x-y)\,dy,$$

where ϱ is the mollifier given by

$$\varrho(x) = K\exp\big(-1/\big(1-x^2\big)\big)1_{|x|<1},$$

and K is a constant such that $\int_{\mathbb{R}} \varrho(x)\,dx = 1$. Then, for any $m \in \mathbb{Z}_+$, there are positive constants c_m and C_m such that

$$c_m e^{-|x|} \leq \big|J^{(m)}(x)\big| \leq C_m e^{-|x|}, \quad \forall x \in \mathbb{R}, \tag{3.1}$$

(cf. Mitoma [7], (2.1)). Then

$$\mathbb{E}\int_{\mathbb{R}} \phi_k\big(\langle v_t, \Phi_m(x-\cdot)\rangle\big)J(x)\,dx = I_1^{m,k} + I_2^{m,k} + \frac{1}{2}I_3^{m,k}, \tag{3.2}$$

where

$$I_1^{m,k} = \mathbb{E}\int_0^t \int_{\mathbb{R}} \phi_k'\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\Big\langle v_s, \frac{1}{2}\Delta\Phi_m(x-\cdot)\Big\rangle J(x)\,dx\,ds,$$

$$I_2^{m,k} = -\mathbb{E}\int_0^t \int_{\mathbb{R}} \phi_k'\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\langle \bar{\eta}_s, \Phi_m(x-\cdot)\rangle J(x)\,dx\,ds,$$

and

$$I_3^{m,k} = \mathbb{E}\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}_+} \phi_k''\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big) \times \left|\int_{\mathbb{R}} \bar{G}_s(a,y)\Phi_m(x-y)\,dy\right|^2 da J(x)\,dx\,ds.$$

Now we estimate $I_1^{m,k}$. First, denote by Δ_x the Laplacian acting with respect to x. Since $v_s(\cdot)$ is locally integrable and Φ_m is smooth with compact support we have, for all $x\in\mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}} v_s(y)\Delta_y\Phi_m(x-y)\,dy &= \int_{\mathbb{R}} v_s(y)\Delta_x\Phi_m(x-y)\,dy \\ &= \Delta_x\int_{\mathbb{R}} v_s(y)\Phi_m(x-y)\,dy \\ &= \Delta_x\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big), \quad \forall m\geq 1.\end{aligned}$$

Then by using $\phi_k''=\psi_k\geq 0$, integration by parts and the chain rule, we have

$$\begin{aligned}2I_1^{m,k} &= \mathbb{E}\int_0^t\int_{\mathbb{R}} \phi_k'\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\Delta_x\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)J(x)\,dx\,ds \\ &= -\mathbb{E}\int_0^t\int_{\mathbb{R}} \psi_k\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\left(\frac{\partial}{\partial x}\langle v_s, \Phi_m(x-\cdot)\rangle\right)^2 \\ &\qquad \times J(x)\,dx\,ds \\ &\qquad - \mathbb{E}\int_0^t\int_{\mathbb{R}} \phi_k'\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\frac{\partial}{\partial x}\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)J'(x)\,dx\,ds \\ &\leq -\mathbb{E}\int_0^t\int_{\mathbb{R}} \frac{\partial}{\partial x}\big(\phi_k\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)\big)J'(x)\,dx\,ds \\ &= \mathbb{E}\int_0^t\int_{\mathbb{R}} \phi_k\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big)J''(x)\,dx\,ds.\end{aligned}$$

Use $\phi_k(z)\leq |z|$ to get

$$\phi_k\big(\langle v_s, \Phi_m(x-\cdot)\rangle\big) \leq \big|\langle v_s, \Phi_m(x-\cdot)\rangle\big| \leq \langle |v_s|, \Phi_m(x-\cdot)\rangle.$$

Therefore,

$$2I_1^{m,k} \leq \mathbb{E}\int_0^t\int_{\mathbb{R}} \langle |v_s|, \Phi_m(x-\cdot)\rangle \big|J''(x)\big|\,dx\,ds, \quad \forall k,m\geq 1. \tag{3.3}$$

Since for each t, $v_t(\cdot)$ is the difference of two non-decreasing functions, we have that, almost surely, the number of discontinuities of $v_t(\cdot)$ is at most countable for any time t. Therefore, we get

$$\lim_{m\to\infty}\langle v_s, \Phi_m(x-\cdot)\rangle = v_s(x), \quad \text{for Lebesgue-a.e. } x, \forall s\geq 0, \text{almost surely,} \tag{3.4}$$

and

$$\lim_{m\to\infty}\big\langle |v_s|,\Phi_m(x-\cdot)\big\rangle = \big|v_s(x)\big|, \tag{3.5}$$
$$\text{for Lebesgue-a.e. } x, \forall s\geq 0, \text{almost surely.}$$

This, almost sure boundedness of $|v_s(x)|$, on $(s,x)\in[0,t]\times\mathbb{R}$, and integrability of $J''(\cdot)$ implies, by the dominated convergence theorem, that

$$\begin{aligned}\lim_{m\to\infty}\int_0^t \big\langle |v_s|,\Phi_m(x-\cdot)\big\rangle \big|J''(x)\big|\,dx\,ds \\ = \int_0^t\int_{\mathbb{R}}\big|v_s(x)\big|\times\big|J''(x)\big|\,dx\,ds, \quad \text{a.s.}\end{aligned} \tag{3.6}$$

Moreover, by (2.8) we easily get that

$$\big\{\big\langle |v_s|,\Phi_m(x-\cdot)\big\rangle, m\geq 1, x\in\mathbb{R}, s\leq t\big\} \tag{3.7}$$

is uniformly integrable. This and (3.6) imply

$$\begin{aligned}\lim_{m\to\infty}\mathbb{E}\int_0^t \big\langle |v_s|,\Phi_m(x-\cdot)\big\rangle \big|J''(x)\big|\,dx\,ds \\ = \mathbb{E}\int_0^t\int_{\mathbb{R}}\big|v_s(x)\big|\times\big|J''(x)\big|\,dx\,ds.\end{aligned} \tag{3.8}$$

(3.3) and (3.8) imply

$$\limsup_{k,m\to\infty} 2I_1^{m,k} \leq \mathbb{E}\int_0^t\int_{\mathbb{R}}\big|v_s(x)\big|\times\big|J''(x)\big|\,dx\,ds. \tag{3.9}$$

Now, by (3.1) we conclude that for some constant K,

$$\limsup_{m,k\to\infty} I_1^{m,k} \leq K\mathbb{E}\int_0^t\int_{\mathbb{R}}\big|v_s(x)\big|J(x)\,dx\,ds. \tag{3.10}$$

It is easy to show that

$$\begin{aligned} I_3^{m,k} &\leq \mathbb{E}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}_+}\phi_k''\big(\big\langle v_s,\Phi_m(x-\cdot)\big\rangle\big) \\ &\qquad \times\int_{\mathbb{R}}\big(\bar{G}_s(a,y)\big)^2\Phi_m(x-y)\,dy\,da J(x)\,dx\,ds \\ &\leq \mathbb{E}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}_+}\phi_k''\big(\big\langle v_s,\Phi_m(x-\cdot)\big\rangle\big) \\ &\qquad \times\Big(\sup_{a\in\mathbb{R}_+}\big|\lambda(a)\big|^2\Big)\int_{\mathbb{R}}\big|v_s(y)\big|\Phi_m(x-y)\,dy J(x)\,dx\,ds.\end{aligned}$$

Now use (3.4), (3.5), (3.7) to get

$$\begin{aligned}\limsup_{m\to\infty} I_3^{m,k} &\leq K\mathbb{E}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}_+}\phi_k''\big(v_s(x)\big)\big|v_s(x)\big|J(x)\,dx\,ds \\ &= O\big(k^{-1}\big),\end{aligned} \tag{3.11}$$

where the last inequality follows since $k|z|\phi''(z)$ is bounded. Also, using $|\phi_k'(z)| \le 1$, and (1.3) we easily get that there are non-negative constants K_1, K such that

$$\begin{aligned}\limsup_{m,k\to\infty}\big|I_2^{m,k}\big| &\le K\int_0^t \mathbb{E}\rho\big(\mu_s^1,\mu_s^2\big)\,ds \\ &\le K_1\mathbb{E}\int_0^t\int_{\mathbb{R}}\big|v_s(x)\big|J(x)\,dx\,ds,\end{aligned}\tag{3.12}$$

where for the last inequality we applied (3.1). Use (3.4) and (3.7) to get

$$\lim_{m\to\infty}\mathbb{E}\int_{\mathbb{R}}\phi_k\big(\big\langle v_t,\Phi_m(x-\cdot)\big\rangle\big)J(x)\,dx = \mathbb{E}\int_{\mathbb{R}}\phi_k\big(v_t(x)\big)J(x)\,dx.\tag{3.13}$$

Since $\phi_k(z)\uparrow|z|$, we obtain by the monotone convergence

$$\begin{aligned}&\lim_{k\to\infty}\lim_{m\to\infty}\mathbb{E}\int_{\mathbb{R}}\phi_k\big(\big\langle v_t,\Phi_m(x-\cdot)\big\rangle\big)J(x)\,dx \\ &\quad = \mathbb{E}\int_{\mathbb{R}}\big|v_t(x)\big|J(x)\,dx.\end{aligned}\tag{3.14}$$

Now, put together (3.2), (3.10), (3.11), (3.12), (3.14) to get

$$\mathbb{E}\int_{\mathbb{R}}\big|v_t(x)\big|J(x)\,dx \le K_2\mathbb{E}\int_0^t\int_{\mathbb{R}}\big|v_s(x)\big|J(x)\,dx\,ds$$

for some constant K_2. Then the Grönwall lemma implies that

$$\mathbb{E}\int_{\mathbb{R}}\big|v_t(x)\big|J(x)\,dx = 0$$

and the uniqueness follows. □

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