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A SCHAUDER BASIS FOR $L_1(0,\infty)$ CONSISTING OF NON-NEGATIVE FUNCTIONS

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ABSTRACT. We construct a Schauder basis for L_1 consisting of non-negative functions and investigate unconditionally basic and quasibasic sequences of non-negative functions in L_p, $1 \le p < \infty$.

1. Introduction

In [5], Powell and Spaeth investigate non-negative sequences of functions in L_p, $1 \le p < \infty$, that satisfy some kind of basis condition, with a view to determining whether such a sequence can span all of L_p. They prove, for example, that there is no unconditional basis or even unconditional quasibasis (frame) for L_p consisting of non-negative functions. On the other hand, they prove that there are non-negative quasibases and non-negative M-bases for L_p. The most important question left open by their investigation is whether there is a (Schauder) basis for L_p consisting of non-negative functions. In Section 2, we show that there is basis for L_1 consisting of non-negative functions.

In Section 3, we discuss the structure of unconditionally basic non-negative normalized sequences in L_p, $1 \le p < \infty$. The main result is that such a sequence is equivalent to the unit vector basis of ℓ_p. We also prove that the closed span in L_p of any unconditional quasibasic sequence embeds isomorphically into ℓ_p.

We use standard Banach space theory, as can be found in [4] or [1]. Let us just mention that L_p is $L_p(0,\infty)$, but in as much as this space is isometrically isomorphic under an order preserving operator to $L_p(\mu)$ for any separable purely non-atomic measure μ, our choice of $L(0,\infty)$ rather than, for example,

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$L_p(0,1)$, is a matter of convenience. Again as a matter of convenience, in the last part of Section 3, we revert to using $L_p(0,1)$ as a model for L_p.

2. A Schauder basis for $L_1(0,\infty)$ consisting of non-negative functions

For $j=1,2,\dots$ let $\{h^j_{n,i}\}_{n=0,i=1}^{\infty\, 2^n}$ be the mean zero L_1 normalized Haar functions on the interval $(j-1,j)$. That is, for $n=0,1,\dots$, $i=1,2,\dots,2^n$,

$$h^j_{n,i}(t)=\begin{cases}2^n, & j-1+\frac{2i-2}{2^{n+1}}<t<j-1+\frac{2i-1}{2^{n+1}},\\ -2^n, & j-1+\frac{2i-1}{2^{n+1}}<t<j-1+\frac{2i}{2^{n+1}},\\ 0, & \text{otherwise.}\end{cases}$$

The system $\{h^j_{n,i}\}_{n=0,i=1,j=1}^{\infty\, 2^n\, \infty}$, in any order which preserves the lexicographic order of $\{h^j_{n,i}\}_{n=0,i=1}^{\infty\, 2^n}$ for each j, constitutes a basis for the subspace of $L_1(0,\infty)$ consisting of all functions whose restriction to each interval $(j-1,j)$ have mean zero. To simplify notation, for each j we shall denote by $\{h^j_i\}_{i=1}^\infty$ the system $\{h^j_{n,i}\}_{n=0,i=1}^{\infty\, 2^n}$ in its lexicographic order. We shall also denote by $\{h_i\}_{i=1}^\infty$ the union of the systems $\{h^j_i\}_{i=1}^\infty$, $j=1,2,\dots$, in any order that respects the individual orders of each of the $\{h^j_i\}_{i=1}^\infty$.

Let π be any permutation of the natural numbers and for each $i\in\mathbb{N}$ let F_i be the two dimensional space spanned by $2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|$ and h_i.

PROPOSITION 1. *$\sum_{i=1}^\infty F_i$ is an FDD of $\overline{\operatorname{span}}^{L_1}\{F_i\}_{i=1}^\infty$.*

Proof. The assertion will follow from the following inequality, which holds for all scalars $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$,

$$\begin{aligned}(1)\qquad &\frac12\sum_{i=1}^\infty|a_i|+\frac18\Big\|\sum_{i=1}^\infty b_ih_i\Big\|\\ &\le\Big\|\sum_{i=1}^\infty a_i\big(2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|\big)+\sum_{i=1}^\infty b_ih_i\Big\|\\ &\le 3\sum_{i=1}^\infty|a_i|+\Big\|\sum_{i=1}^\infty b_ih_i\Big\|.\end{aligned}$$

The right inequality in (1) follows easily from the triangle inequality. As for the left inequality, notice that the conditional expectation projection onto the closed span of $\{\mathbf{1}_{(i-1,i)}\}_{i=1}^\infty$ is of norm one and the complementary projection, onto the closed span of $\{h_i\}_{i=1}^\infty$, is of norm 2. It follows that

$$\Big\|\sum_{i=1}^\infty a_i(2\mathbf{1}_{(\pi(i)-1,\pi(i))})+\sum_{i=1}^\infty b_ih_i\Big\|\ge\max\Big\{2\sum_{i=1}^\infty|a_i|,\frac12\Big\|\sum_{i=1}^\infty b_ih_i\Big\|\Big\}.$$

Since $\|\sum_{i=1}^{\infty} a_i|h_i|\| \le \sum_{i=1}^{\infty} |a_i|$, we get

$$\left\|\sum_{i=1}^{\infty} a_i\big(2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|\big)+\sum_{i=1}^{\infty} b_i h_i\right\| \ge \max\left\{\sum_{i=1}^{\infty}|a_i|, \frac{1}{4}\left\|\sum_{i=1}^{\infty} b_i h_i\right\|\right\}$$

from which the left-hand side inequality in (1) follows easily. □

Proposition 2. *Let π be any permutation of the natural numbers and for each $i \in \mathbb{N}$ let F_i be the two dimensional space spanned by $2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|$ and h_i. Then $\overline{\text{span}}^{L_1}\{F_i\}_{i=1}^{\infty}$ admits a basis consisting of non-negative functions.*

Proof. In view of Proposition 1, it is enough to show that each F_i has a two term basis consisting of non-negative functions and with uniform basis constant. Put $x_i = 2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|+h_i$ and $y_i = 2\mathbf{1}_{(\pi(i)-1,\pi(i))}+|h_i|-h_i$. Then clearly $x_i, y_i \ge 0$ everywhere and $\|x_i\| = \|y_i\| = 3$. We now distinguish two cases: If $\mathbf{1}_{(\pi(i)-1,\pi(i))}$ is disjoint from the support of h_i then, for all scalars a, b,

$$\|ax_i + by_i\| \ge \big\|a\big(|h_i|+h_i\big)+b\big(|h_i|-h_i\big)\big\| = |a|+|b|.$$

If the support of h_i is included in $(\pi(i)-1,\pi(i))$, let 2^{-s} be the size of that support, $s \ge 0$. Then for all scalars a, b,

$$\begin{aligned}\|ax_i + by_i\| &\ge \big\|a\big(|h_i|+h_i\big)+b\big(|h_i|-h_i\big)+2(a+b)\mathbf{1}_{\text{supp}(h_i)}\big\|\\ &= 2^{-s-1}(\big|\big(2^{s+1}+2\big)a+2b\big|+\big|\big(2^{s+1}+2\big)b+2a\big|\\ &\ge \max\big\{|a|,|b|\big\}.\end{aligned}$$

□

Theorem 1. $L_1(0,\infty)$, *and consequently any separable L_1 space, admits a Schauder basis consisting of non-negative functions.*

Proof. When choosing the order on $\{h_i\}$ we can and shall assume that $h_1 = h_{0,1}^1$; that is, the first mean zero Haar function on the interval $(0,1)$. Let π be any permutation of $\mathbb{N}$ such that $\pi(1)=1$ and for $i>1$, if $h_i = h_{n,k}^j$ for some n,k, and j then $\pi(i)>j$. It follows that except for $i=1$ the support of h_i is disjoint from the interval $(\pi(i)-1,\pi(i))$. It is easy to see that such a permutation exists. We shall show that under these assumptions $\sum_{i=1}^{\infty} F_i$ spans $L_1(0,\infty)$ and, in view of Proposition 2, this will prove the theorem for $L_1(0,\infty)$. First, since $\pi(1)=1$ we get that $3\mathbf{1}_{(0,1)} = 2\mathbf{1}_{(\pi(1)-1,\pi(1))}+|h_1| \in F_1$, and since all the mean zero Haar functions on $(0,1)$ are clearly in $\sum_{i=1}^{\infty} F_i$, we get that $L_1(0,1) \subset \sum_{i=1}^{\infty} F_i$.

Assume by induction that $L_1(0,j) \subset \sum_{i=1}^{\infty} F_i$. Let l be such that $\pi(l) = j+1$. By our assumption on π, the support of h_l is included in $(0,j)$, and so by the induction hypothesis, $|h_l| \in \sum_{i=1}^{\infty} F_i$. Since also $2\mathbf{1}_{(j,j+1)}+|h_l| \in \sum_{i=1}^{\infty} F_i$ we get that $\mathbf{1}_{(j,j+1)} \in \sum_{i=1}^{\infty} F_i$. Since the mean zero Haar functions on $(j,j+1)$ are also in $\sum_{i=1}^{\infty} F_i$ we conclude that $L_1(0,j+1) \subset \sum_{i=1}^{\infty} F_i$.

This finishes the proof for $L_1(0,\infty)$. Since every separable L_1 space is order isometric to one of the spaces ℓ_1^k, $k=1,2,\dots$, $\ell_1, L_1(0,\infty)$, $L_1(0,\infty)\bigoplus_1 \ell_1^k$, $k=1,2,\dots$, or $L_1(0,\infty)\bigoplus_1 \ell_1$, and since the discrete L_1 spaces ℓ_1^k, $k=1,2,\dots$, and ℓ_1 clearly have non-negative bases, we get the conclusion for any separable L_1 space. □

3. Unconditional non-negative sequences in L_p

Here we prove the following theorem.

THEOREM 2. *Suppose that $\{x_n\}_{n=1}^\infty$ is a normalized unconditionally basic sequence of non-negative functions in L_p, $1\le p<\infty$. Then $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_p.*

Proof. First, we give a sketch of the proof, which should be enough for experts in Banach space theory. By unconditionality, we have for all coefficients a_n that $\|\sum_n a_n x_n\|_p$ is equivalent to the square function $\|(\sum_n |a_n|^2 x_n^2)^{1/2}\|_p$, and, by nonnegativity of x_n, is also equivalent to $\|\sum_n |a_n| x_n\|_p$. Thus by trivial interpolation when $1\le p\le 2$, and by extrapolation when $2<p<\infty$, we see that $\|\sum_n a_n x_n\|_p$ is equivalent to $\|(\sum_n |a_n|^p x_n^p)^{1/p}\|_p = (\sum_n |a_n|^p)^{1/p}$.

We now give a formal argument for the benefit of readers who are not familiar with the background we assumed when giving the sketch. Let K be the unconditional constant of $\{x_n\}_{n=1}^\infty$. Then

$$K^{-1}\left\|\sum_{n=1}^N a_n x_n\right\|_p \le B_p\left\|\left(\sum_{n=1}^N |a_n|^2 x_n^2\right)^{1/2}\right\|_p \le B_p\left\|\sum_{n=1}^N |a_n| x_n\right\|_p \le B_p K\left\|\sum_{n=1}^N a_n x_n\right\|_p, \tag{2}$$

where the first inequality is obtained by integrating against the Rademacher functions (see, e.g., [4, Theorem 2.b.3]). The constant B_p is Khintchine's constant, so $B_p=1$ for $p\le 2$ and B_p is of order $\sqrt{p}$ for $p>2$. If $1\le p\le 2$, we get from (2)

$$K^{-1}\left\|\sum_{n=1}^N a_n x_n\right\|_p \le \left\|\left(\sum_{n=1}^N |a_n|^p x_n^p\right)^{1/p}\right\|_p \le K\left\|\sum_{n=1}^N a_n x_n\right\|_p. \tag{3}$$

Since $\|(\sum_{n=1}^N |a_n|^p x_n^p)^{1/p}\|_p = (\sum_{n=1}^N |a_n|^p)^{1/p}$, this completes the proof when $1\le p\le 2$. When $2<p<\infty$, we need to extrapolate rather than do (trivial) interpolation. Write $1/2=\theta/1+(1-\theta)/p$. Then

$$
\begin{aligned}
(4)\qquad (KB_p)^{-1}\Big\|\sum_{n=1}^N a_nx_n\Big\|_p &\le \Big\|\Big(\sum_{n=1}^N |a_n|^2x_n^2\Big)^{1/2}\Big\|_p\\
&\le \Big\|\sum_{n=1}^N |a_n|x_n\Big\|_p^{\theta}\Big\|\Big(\sum_{n=1}^N |a_n|^px_n^p\Big)^{1/p}\Big\|_p^{1-\theta}\\
&\le K\Big\|\sum_{n=1}^N a_nx_n\Big\|_p^{\theta}\Big(\sum_{n=1}^N |a_n|^p\Big)^{(1-\theta)/p}, \quad \text{so that}
\end{aligned}
$$

$$
\big(K^2B_p\big)^{(-1)/(1-\theta)}\Big\|\sum_{n=1}^N a_nx_n\Big\|_p \le \Big(\sum_{n=1}^N |a_n|^p\Big)^{1/p} \le K\Big\|\sum_{n=1}^N a_nx_n\Big\|_p. \qquad \square
$$

As stated, Theorem 2 gives no information when $p=2$ because every normalized unconditionally basic sequence in a Hilbert space is equivalent to the unit vector basis of ℓ_2. However, if we extrapolate slightly differently in the above argument (writing $1/2=\theta/1+(1-\theta)/\infty$) we see that, no matter what p is, $\|\sum_{n=1}^N a_nx_n\|_p$ is also equivalent to $\|\max_n |a_n|x_n\|_p$. From this one can deduce, for example, that only finitely many Rademachers can be in the closed span of $\{x_n\}_{n=1}^\infty$; in particular, $\{x_n\}_{n=1}^\infty$ cannot be a basis for L_p even when $p=2$. However, the proof given in [5] that a normalized unconditionally basic sequence of non-negative functions $\{x_n\}_{n=1}^\infty$ in L_p cannot span L_p actually shows that only finitely many Rademachers can be in the closed span of $\{x_n\}_{n=1}^\infty$. This is improved in our last result, which shows that the closed span of an unconditionally non-negative quasibasic sequence in $L_p(0,1)$ cannot contain any strongly embedded infinite dimensional subspace (a subspace X of $L_p(0,1)$ is said to be strongly embedded if the $L_p(0,1)$ norm is equivalent to the $L_r(0,1)$ norm on X for some – or, equivalently, for all – $r<p$; see e.g. [1, p. 151]). The main work for proving this is contained in Lemma 1.

Before stating Lemma 1, we recall that a quasibasis for a Banach space X is a sequence $\{f_n,g_n\}_{n=1}^\infty$ in $X\times X^*$ such that for each x in X the series $\sum_n\langle g_n,x\rangle f_n$ converges to x. (In [5], a sequence $\{f_n\}_{n=1}^\infty$ in X is a called a quasibasis for X provided there exists such a sequence $\{g_n\}_{n=1}^\infty$. Since the sequence $\{g_n\}_{n=1}^\infty$ is typically not unique, we prefer to specify it up front.) The quasibasis $\{f_n,g_n\}_{n=1}^\infty$ is said to be unconditional provided that for each x in X the series $\sum_n\langle g_n,x\rangle f_n$ converges unconditionally to x. One then gets from the uniform boundedness principle (see, e.g., [5, Lemma 3.2]) that there is a constant K so that for all x and all scalars a_n with $|a_n|\le 1$, we have $\|\sum_n a_n\langle g_n,x\rangle f_n\|\le K\|x\|$. A sequence $\{f_n,g_n\}_{n=1}^\infty$ in $X\times X^*$ is said to be [unconditionally] quasibasic provided $\{f_n,h_n\}_{n=1}^\infty$ is an [unconditional] quasibasis for the closed span $[f_n]$ of $\{f_n\}_{n=1}^\infty$, where h_n is the restriction of g_n to $[f_n]$.

LEMMA 1. *Suppose that $\{f_n, g_n\}_{n=1}^\infty$ is an uncondtionally quasibasic sequence in $L_p(0,1)$, $1<p<\infty$, with each f_n non-negative. If $\{y_n\}_{n=1}^\infty$ is a normalized weakly null sequence in the closed linear span $[f_n]$ of $\{f_n\}_{n=1}^\infty$, then $\|y_n\|_1 \to 0$ as $n\to\infty$.*

Proof. If the conclusion is false, we get a normalized weakly null sequence $\{y_n\}_{n=1}^\infty$ in $[f_n]$ and a $c>0$ so that for all n we have $\|y_n\|_1 > c$.

By passing to a subsequence of $\{y_n\}_{n=1}^\infty$, we can assume that there are integers $0=m_1<m_2<\cdots$ so that for each n,

$$\sum_{k=1}^{m_n} \big|\langle g_k, y_n\rangle\big| \|f_k\|_p < 2^{-n-3}c \quad \text{and}$$
$$\left\| \sum_{k=m_{n+1}+1}^{\infty} \big|\langle g_k, y_n\rangle\big| f_k \right\|_p < 2^{-n-3}c. \tag{5}$$

Effecting the first inequality in (5) is no problem because $y_n\to 0$ weakly, but the second inequality perhaps requires a comment. If y_n satisfies the first inquality in (5), from the unconditional convergence of the expansion of y_n and the nonnegativity of all f_k we get that $\|\sum_{k=N}^\infty |\langle g_k, y_n\rangle| f_k\|_p \to 0$ as $n\to\infty$, which allows us to select m_{n+1} to satisfy the second inequality in (5).

Since $\|y_n\|_1 > c$, from (5) we also have for every n that

$$\left\| \sum_{k=m_n+1}^{m_{n+1}} \big|\langle g_k, y_n\rangle\big| f_k \right\|_1 \ge \left\| \sum_{k=m_n+1}^{m_{n+1}} \langle g_k, y_n\rangle f_k \right\|_1 \ge c/2. \tag{6}$$

Since L_p has an unconditional basis, by passing to a further subsequence we can assume that $\{y_n\}_{n=1}^\infty$ is unconditionally basic with, say, constant K_p. Set $s=p\wedge 2$. Then L_p has type s (see [1, Theorem 6.2.14]), so for some constant K_p' we have for every N the inequality

$$\left\| \sum_{n=1}^{N} y_n \right\|_p \le K_p' N^{1/s}. \tag{7}$$

On the other hand, letting $\delta_k = \operatorname{sign}\langle g_k, y_n\rangle$ when $m_n+1\le k\le m_{n+1}$, $n=1,2,3,\ldots$, we have

$$\begin{aligned} KK_p &\left\| \sum_{n=1}^{N} y_n \right\|_p \\ &\ge K_p \left\| \sum_{n=1}^{N} \sum_{k=1}^{\infty} \delta_k \langle g_k, y_n\rangle f_k \right\|_p \end{aligned} \tag{8}$$

$$
\begin{aligned}
&\geq \left\| \sum_{n=1}^{N} \sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n\rangle| f_k \right\|_p - \left\| \sum_{n=1}^{N} \sum_{k\notin[m_n+1, m_{n+1}]} \delta_k \langle g_k, y_n\rangle f_k \right\|_p \\
&\geq \left\| \sum_{n=1}^{N} \sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n\rangle| f_k \right\|_1 - \left\| \sum_{n=1}^{N} \sum_{k\notin[m_n+1, m_{n+1}]} |\langle g_k, y_n\rangle| f_k \right\|_p \\
&\geq \sum_{n=1}^{N} \left\| \sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n\rangle| f_k \right\|_1 \\
&\quad - \sum_{n=1}^{N} \left(\sum_{k=1}^{m_n} |\langle g_k, y_n\rangle| \|f_k\|_p + \left\| \sum_{k=m_{n+1}+1}^{\infty} |\langle g_k, y_n\rangle| f_n \right\|_p \right) \\
&\geq Nc/2 - c/4 \quad \text{by (6) and (5)}
\end{aligned}
$$

This contradicts (7). □

THEOREM 3. *Let $\{f_n, g_n\}_{n=1}^{\infty}$ be an unconditional quasibasic sequence in $L_p(0,1)$, $1 \leq p < \infty$, with each f_n non-negative. Then the closed span $[f_n]$ of $\{f_n\}_{n=1}^{\infty}$ embeds isomorphically into ℓ_p.*

Proof. The case $p = 1$ is especially easy: Assume, as we may, that $\|f_n\|_1 = 1$. There is a constant K so that for each y in $[f_n]$

$$\|y\|_1 \leq \left\| \sum_{n=1}^{\infty} |\langle g_n, y\rangle| f_n \right\|_1 \leq K\|y\|_1, \tag{9}$$

hence the mapping $y \mapsto \{\langle g_k, y\rangle\}_{k=1}^{\infty}$ is an isomorphism from $[f_n]$ into ℓ_1.

So in the sequel assume that $p > 1$. From Lemma 1 and standard arguments (see, e.g., [1, Theorem 6.4.7]), we have that every normalized weakly null sequence in $[f_n]$ has a subsequence that is an arbitrarily small perturbation of a disjoint sequence and hence the subsequence is $1+\epsilon$-equivalent to the unit vector basis for ℓ_p. This implies that $[f_n]$ embeds isomorphically into ℓ_p (see [3] for the case $p > 2$ and [2, Theorems III.9, III.1, and III.2] for the case $p < 2$). □

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