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MINIMAL QUASI-COMPLETE INTERSECTION IDEALS

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Abstract. A quasi-complete intersection (q.c.i.) ideal of a local ring is an ideal with "free exterior Koszul homology"; the definition can also be understood in terms of vanishing of André-Quillen homology functors. Principal q.c.i. ideals are well understood, but few constructions are known to produce q.c.i. ideals of grade zero that are not principal. This paper examines the structure of q.c.i. ideals. We exhibit conditions on a ring R which guarantee that every q.c.i. ideal of R is principal. On the other hand, we give an example of a minimal q.c.i. ideal I which does not contain any principal q.c.i. ideals and is not embedded, in the sense that no faithfully flat extension of I can be written as a quotient of complete intersection ideals. We also describe a generic situation in which the maximal ideal of R is an embedded q.c.i. ideal that does not contain any principal q.c.i. ideals.

Introduction

This paper is concerned with a class of ideals referred to as *quasi-complete intersection (q.c.i.) ideals* in recent work of Avramov et al. [4]. As discussed there, the notion goes back to work of Rodicio [19], and appears in subsequent papers of A. Blanco, J. Majadas Soto and A. Rodicio Garcia. Q.c.i. ideals of local rings can be defined as ideals with "free exterior Koszul homology", see Definition 1.1. The class of q.c.i. ideals contains that of *complete intersection ideals* (i.e., ideals generated by a regular sequence), and inherits many of the homological change of rings properties of the latter.

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Consult [7], [4] for the connection between q.c.i. ideals and the vanishing of André–Quillen homology. In particular, an ideal I is a q.c.i. if and only if the homomorphism $R \to R/I$ satisfies the conclusion of the Quillen conjecture [17, Conjecture 5.6]. Not many examples or methods of constructing such homomorphisms are known, and a better understanding of q.c.i. ideals is of value as one tries to prove or disprove the conjecture.

Our goal is to understand the structure of q.c.i. ideals I of a commutative local noetherian ring $(R, \mathfrak{m})$; this notation identifies $\mathfrak{m}$ as the maximal ideal of the local ring R. Principal q.c.i. ideals are well understood: If $x \neq 0$ is an element of $\mathfrak{m}$, then the ideal (x) is q.c.i. if and only if x is either regular or else $\operatorname{ann}(x) \cong R/(x)$; in the last case we say, following Henriques and Şega [13], that x is an *exact zero-divisor*. (Such elements are studied also in [14] under a slightly different name.) Existence of exact zero-divisors is known for certain classes of small Artinian rings, and has found various uses, see [8], [5] and [13]. Another well-understood method of constructing q.c.i. ideals is by means of a pair of embedded complete intersection ideals; see Remark 2.3. The q.c.i. ideals I that can be obtained in this manner after possibly a faithfully flat extension are exactly the ones for which $\text{CI-dim}_R(R/I) < \infty$, where CI-dim denotes complete intersection dimension, as defined in [3]. We say that such q.c.i. ideals are *embedded.*

New q.c.i. ideals can be constructed from old ones by "composition" and "decomposition" of surjective q.c.i. homomorphisms, see [4, 8.8, 8.9]. In particular, if $I = (a_1, \dots, a_s)$ is an ideal in R with a_{i+1} an exact zero-divisor or a regular element on $R/(a_1, \dots, a_i)$ for all i with $0 \le i \le s-1$, then I is a q.c.i. ideal of R. Following the lead of [14] and [4, §3], we call such ideals *exact.* We say that a q.c.i. ideal J is *minimal* if J does not properly contain any non-zero q.c.i. ideal.

Rodicio [20, Conjecture 11] conjectured that all q.c.i. ideals of R are embedded. Although this statement holds under some additional conditions on the ring R, see [20, Proposition 23], in general it does not. A counterexample consisting of a principal non-embedded q.c.i. ideal is given in [4, Theorem 3.5]; one can further argue that this q.c.i. ideal is minimal, see Proposition 3.6.

Beyond the information mentioned so far, the literature seems to lack other relevant examples and methods of constructing q.c.i. ideals. In this paper, we further clarify the structure of q.c.i. ideals and in particular examine relations between the classes of q.c.i. ideals (principal, exact, minimal, embedded) introduced above.

Note that a minimal q.c.i. ideal is exact if and only if it is principal. In Section 3, we discuss some classes of rings for which every q.c.i. ideal is principal (thus exact) as follows.

Theorem 1. *Let $(R, \mathfrak{m})$ be an Artinian local ring which is not a complete intersection. Assume that one of the following holds*:

(1) $\mathfrak{m}^3 = 0$;
(2) $\mathfrak{m}^4 = 0$ *and* R *is Gorenstein.*

Then every q.c.i. ideal of R *is principal.*

Theorem 1 is part of Theorem 3.2, which studies, more generally, bounds on the minimal number of generators of a q.c.i. ideal.

On the other hand, Proposition 4.3 and Theorem 4.5 give the following theorems.

THEOREM 2. *There exists an Artinian local ring* $(R,\mathfrak{m})$ *with* $\mathfrak{m}^4 = 0$ *and elements* $f_1, f_2 \in \mathfrak{m}$ *that are linearly independent modulo* $\mathfrak{m}^2$ *and generate a minimal, non-embedded and non-principal (thus non-exact) q.c.i. ideal.*

While the example involved in the proof of this result is rather special, in Section 5 we exhibit many grade zero embedded q.c.i. ideals which are not exact.

THEOREM 3. *Let* k *be an algebraically closed field of characteristic different from* 2 *and let* P *denote the polynomial ring* $k[x_1,\dots,x_n]$.

If $n \geq 5$ *and* $\mathbf{f} = f_1,\dots,f_n$ *is a generic regular sequence of quadratic forms, then* $(x_1,\dots,x_n)/\mathbf{f}P$ *is an embedded q.c.i. ideal of* $R = P/\mathbf{f}P$ *that does not contain any principal q.c.i. ideal.*

Theorem 3 is part of Theorem 5.1. The meaning of the word "generic" is made precise through Theorem 5.2.

Preliminaries and general results on q.c.i. ideals are collected in Sections 1 and 2. In particular, Corollary 1.11 gives a necessary condition for the existence of exact zero-divisors: If $R = Q/\mathfrak{a}$ with $(Q,\mathfrak{n},k)$ a regular local ring and $\mathfrak{a} \subseteq \mathfrak{n}^2$, and R admits an exact zero-divisor, then a minimal generator of $\mathfrak{a}$ factors non-trivially.

1. Preliminaries

In this section, we present and discuss the notion of q.c.i. ideal. A criterion for checking that a 2-generated ideal of grade zero is q.c.i. is given in Lemma 1.7. Theorem 1.8 and Propositions 1.9 and 1.10 give some consequences of the q.c.i. property.

The following notation and conventions are used throughout the paper: Let $(R,\mathfrak{m},k)$ be a local ring: R is a commutative noetherian ring with unique maximal ideal $\mathfrak{m}$, and $k = R/\mathfrak{m}$. If M is a finitely generated R-module, we denote by $\nu(M)$ the minimal number of generators of M.

Let I be an ideal of R with $\nu(I) = n$ and set $S = R/I$. Let $\mathbf{f} = f_1,\dots,f_n$ be a generating set of I and let E denote the Koszul complex on $\mathbf{f}$.

DEFINITION 1.1. We say that I is a *quasi-complete intersection (q.c.i.) ideal* if $\mathrm{H}_1(E)$ is free over S and the canonical homomorphism of graded

S-algebras

$$\lambda_*^S \colon \Lambda_*^S \mathrm{H}_1(E) \longrightarrow \mathrm{H}_*(E) \tag{1.1}$$

is bijective, where Λ_*^S denotes the exterior algebra functor.

We refer to [7] for the interpretation of the notion of q.c.i. in terms of vanishing of André–Quillen homology functors.

1.2. *Principal q.c.i. ideals.* We say that an element x of R is an *exact zero-divisor* if

$$R \neq (0 :_R x) \cong R/(x) \neq 0.$$

If x is an exact zero-divisor, then there exists y such that $(0 :_R x) = (y)$ and $(0 :_R y) = (x)$. We say that x, y is an *exact pair of zero-divisors* and y is a *complementary zero-divisor of* x. Such pairs were first studied in [14] under the name of *exact pairs of elements*; the name exact zero-divisor was introduced in [13].

It follows directly from Definition 1.1 that a non-trivial principal ideal $I = (x)$ is q.c.i. if and only if x is either a non zero-divisor or an exact zero-divisor.

1.3. Recall that $\mathrm{grade}_R(I)$ denotes the maximal length of an R-regular sequence in I; this number is equal to the least integer i with $\mathrm{Ext}_R^i(R/I, R) \neq 0$. In view of [4, Lemma 1.4], the study of the structure of q.c.i. ideals may be reduced to the case when $\mathrm{grade}_R(I) = 0$.

If I is a q.c.i. ideal, then [4, Lemma 1.2] gives:

$$\mathrm{grade}_R(I) = \nu(I) - \nu\big(\mathrm{H}_1(E)\big). \tag{1.2}$$

1.4. Assume that $\nu(\mathrm{H}_1(E)) = n$. This assumption holds whenever I is a q.c.i. ideal with $\mathrm{grade}_R(I) = 0$ by (1.2); however, we do not want to assume that I is q.c.i. at this time.

Since $\nu(I) = n$, we have $E_1 \cong R^n$. Let $v_1, \ldots, v_n$ denote a basis of E_1 with $\partial(v_i) = f_i$ for each i. Consider a set of cycles

$$z_j = \sum_{i=1}^{n} a_{ij} v_i \tag{1.3}$$

with $a_{ij} \in R$ and $j = 1, \ldots, n$ such that the homology classes $\mathrm{cls}(z_j)$ minimally generate $\mathrm{H}_1(E)$. Set

$$A = (a_{ij}) \quad \text{and} \quad \Delta = \det(A)$$

and note that the map

$$\lambda := \lambda_n^S \colon \Lambda_n^S \mathrm{H}_1(E) \longrightarrow \mathrm{H}_n(E)$$

is described by

$$\lambda\big(\mathrm{cls}(z_1) \wedge \cdots \wedge \mathrm{cls}(z_n)\big) = \Delta v_1 \cdots v_n.$$

Note that $\Delta \in (0 :_R I)$. Since $\mathbf{f}$ is a minimal generating set for I, and each z_i is a syzygy in the free cover $E_1 \to I$, we have $a_{ij} \in \mathfrak{m}$ for all i, j. In particular, we have:

$$\Delta \in \mathfrak{m}^n. \tag{1.4}$$

LEMMA 1.5. *If I is a q.c.i. ideal with* $\operatorname{grade}_R(I) = 0$, *then the following hold*:

(1) $\mathrm{H}_1(E) \cong S^n$;
(2) $(0 :_R I) \cong S$;
(3) $(0 :_R I) = \Delta R$ *and* $(0 :_R \Delta) = I$.

Proof. Since $\operatorname{grade}_R(I) = 0$, we know that $\nu(\mathrm{H}_1(E)) = n$ by (1.2). Then (1) follows from the fact that $\mathrm{H}_1(E)$ is free over S, according to Definition 1.1.

(2) We have

$$S \cong \textstyle\bigwedge_n^S(S^n) \cong \bigwedge_n^S \mathrm{H}_1(E) \cong \mathrm{H}_n(E) \cong (0 :_R I),$$

where the third isomorphism is given by the map λ in 1.4. The first and the last isomorphism are general facts, and the second one is a consequence of (1).

(3) Using the description of the map λ in 1.4, we see that the isomorphism $S \xrightarrow{\cong} (0 :_R I)$ from the proof of (2) can be described by

$$1 \mapsto \Delta.$$

In particular, $(0 :_R I) = \Delta R$. The fact that this map is an isomorphism shows that $(0 :_R \Delta) = I$. □

REMARK 1.6. For any q.c.i. ideal I the module R/I has a Tate resolution T with $T_1 = R^n$, $T_0 = R$ and $d_1 = [f_1 \cdots f_n]$ (see [4, Construction 1.5, 1.6]). When $\operatorname{grade}_R(I) = 0$ it gives rise to an infinite in both directions exact sequence

$$\cdots \longrightarrow T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} T_0^* \xrightarrow{d_1^*} T_1^* \longrightarrow \cdots,$$

where d_0 is given by multiplication with Δ. Indeed, $\mathrm{H}^n(T^*) = 0$ for $n \geq 1$ and $\mathrm{H}^0(T^*) \simeq S$ by [4, Theorem 2.5(4)]; Lemma 1.5 gives exactness at T_0 and T_0^*.

As noted in 1.2, principal q.c.i. ideals admit a simple characterization. Based on Lemma 1.5 and Definition 1.1, the two-generated q.c.i. ideals can also be given a relatively simple characterization as follows.

LEMMA 1.7. *Let I be an ideal with $\nu(I) = 2$ and* $\operatorname{grade}_R(I) = 0$. *Then the following statements are equivalent*:

(1) *I is q.c.i.*
(2) $\mathrm{H}_1(E) \cong S^2$, $(0 :_R I) = \Delta R$, *and* $(0 :_R \Delta) = I$, *where Δ is defined as in* 1.4.
(3) *There exist elements a, b, c, d in $\mathfrak{m}$ with* (1.5) *an exact sequence*, *where*

$$R^4 \xrightarrow{d_3} R^3 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \xrightarrow{d_0} R \xrightarrow{d_1^{\mathrm{T}}} R^2, \tag{1.5}$$

with $d_0 = [ad - bc]$, $d_1 = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$,

$$d_2 = \begin{bmatrix} -f_2 & a & b \\ f_1 & c & d \end{bmatrix}, \quad \textit{and} \quad d_3 = \begin{bmatrix} -c & -d & a & b \\ f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{bmatrix}.$$

Proof. (1) $\implies$ (3): By Lemma 1.5 there are cycles $z_1 = av_1 + cv_2$ and $z_2 = bv_1 + dv_2$ in E_1, whose classes form a basis of $\mathrm{H}_1(E)$. Let T be the Tate resolution of R/I constructed with these cycles. Then Remark 1.6 gives the exact sequence in (1.5).

(3) $\implies$ (2): Most of the hypotheses of (2) follow immediately from (3). We only need to verify that $\mathrm{H}_1(E) \cong S^2$. The exactness of the complex (1.5) implies that $\nu(\mathrm{H}_1(E)) = 2$. Furthermore, the cycles z_1 and z_2 in 1.4 can be taken to be

$$z_1 = av_1 + cv_2 \quad \text{and} \quad z_2 = bv_1 + dv_2.$$

Consider the homomorphism

$$\varphi\colon R^2 \twoheadrightarrow \mathrm{H}_1(E)$$

given by $\varphi(t) = \mathrm{cls}(t_1 z_1 + t_2 z_2)$, where $t = [t_1, t_2]^{\mathrm{T}} \in R^2$. If $\varphi(t) = 0$, then there exists $t_0 \in R$ such that the element $[t_0, t_1, t_2]^{\mathrm{T}} \in R^3$ is in $\operatorname{Ker} d_2 = \operatorname{Im} d_3$. By looking at the matrix describing d_3, we conclude that t_1 and t_2 are in I. It follows that $\operatorname{Ker}(\varphi) \subseteq IR^2$. The reverse inclusion is clear, hence $\mathrm{H}_1(E) \cong S^2$.

(2) $\implies$ (1): Let v_1, v_2 be an R-module basis for E_1. It follows that

$$\mathrm{H}_2(E) = \left\{ rv_1v_2 \in \bigwedge^2 E_1 \mid r \in (0 :_R I) \right\}.$$

The hypothesis $\mathrm{H}_1(E) \cong S^2$ of (2) guarantees that there exist cycles z_1 and z_2 in E_1 such that $\mathrm{cls}(z_1)$ and $\mathrm{cls}(z_2)$ form a basis for the free S-module $\mathrm{H}_1(E)$. We know from 1.4 that the S-module homomorphism $\lambda_2^S : \bigwedge^2(\mathrm{H}_1(E)) \to \mathrm{H}_2(E)$ is given by $\lambda(r\,\mathrm{cls}(z_1) \wedge \mathrm{cls}(z_2)) = r\Delta(v_1v_2)$. The hypotheses $(0 :_R I) = \Delta R$, and $(0 :_R \Delta) = I$ of (2) ensure that λ_2 is an isomorphism of S-modules. $\square$

THEOREM 1.8. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring, $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal of Q, $(R, \mathfrak{m}, k)$ be the local ring with $R = Q/\mathfrak{a}$ and $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$, and J be an ideal of Q which contains $\mathfrak{a}$. If $I = J/\mathfrak{a}$ is a q.c.i. ideal of R, then*

$$\nu(J) = \nu(\mathfrak{a}) + \mathrm{grade}_R(I).$$

Proof. Set $H = \mathrm{H}_1(E)$ and $S = Q/J = R/I$. By [4, Theorem 5.3] or [18] (in view of [4, Remark 5.3]), we have an exact sequence:

$$(1.6)\quad 0 \to H/\mathfrak{m}H \to \pi_2(R) \xrightarrow{\pi_2(\varphi)} \pi_2(S) \xrightarrow{\delta} I/\mathfrak{m}I \to \pi_1(R) \xrightarrow{\pi_1(\varphi)} \pi_1(S) \to 0.$$

We refer to [4, 5.1, 5.2] for the definition of the modules of indecomposables $\pi_i(-)$. According to [4, 5.2] and the proof of [12, Proposition 3.3.4], we have canonical identifications $\pi_1(R) = \mathfrak{n}/\mathfrak{n}^2$ and $\pi_2(R) = \mathfrak{a}/\mathfrak{n}\mathfrak{a}$. In particular:

$$\operatorname{rank}_k \pi_1(R) = \nu(\mathfrak{n}) \quad \text{and} \quad \operatorname{rank}_k \pi_2(R) = \nu(\mathfrak{a}).$$

Choose $b_1, \ldots, b_s$ in J so that their images form a basis of the kernel of the induced map $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{n}S/(\mathfrak{n}S)^2$. The local ring $\overline{Q} = Q/(b_1, \ldots, b_s)$ is regular with maximal ideal $\overline{\mathfrak{n}} = \mathfrak{n}/(b_1, \ldots, b_s)$ and $S \cong \overline{Q}/\overline{J}$, where $\overline{J} = J/(b_1, \ldots, b_s) \subseteq \overline{\mathfrak{n}}^2$. We then have

$$\begin{aligned}\operatorname{rank}_k \pi_1(S) &= \nu(\overline{\mathfrak{n}}) = \nu(\mathfrak{n}) - s = \nu(\mathfrak{m}) - s,\\ \operatorname{rank}_k \pi_2(S) &= \nu(\overline{J}) = \nu(J) - s.\end{aligned}$$

The Euler characteristic of (1.6) computed using the expressions above and the relation $\nu(H) = \nu(I) - \operatorname{grade}_R(I)$ from (1.2) gives $\nu(J) = \nu(\mathfrak{a}) + \operatorname{grade}_R(I)$. ☐

Proposition 1.9. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring. Let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal and set $R = Q/\mathfrak{a}$. Let $F, G \in Q$ such that $FG \in \mathfrak{a}$, and let f, g denote the images of these elements in R.*

If f, g is an exact pair of zero-divisors, then $FG \notin \mathfrak{n}\mathfrak{a}$. Furthermore, if $\mathfrak{a}$ is generated by a regular sequence, then the converse holds.

Proof. Assume f, g is an exact pair of zero-divisors. We apply Theorem 1.8 to the q.c.i. ideal $I = J/\mathfrak{a}$ with $J = \mathfrak{a} + (F)$. The proposition yields that the ideals $\mathfrak{a}$ and $\mathfrak{a} + (F)$ have the same minimal number of generators. Fix a minimal generating set of $\mathfrak{a}$. Then one of the minimal generators of $\mathfrak{a}$ can be written as a linear combination of the remaining minimal generators of $\mathfrak{a}$ and F, and hence there exists an element $H \in Q$ such that $FH \in \mathfrak{a} \smallsetminus \mathfrak{n}\mathfrak{a}$. Since $FH \in \mathfrak{a}$, we see that $h \in (0 \colon f) = (g)$, where h denotes the image of H in R. Hence $H = YG + X$ with $X \in \mathfrak{a}$ and $Y \in Q$. Thus $FH - YFG \in \mathfrak{n}\mathfrak{a}$, and we conclude $FG \notin \mathfrak{n}\mathfrak{a}$, since $FH \notin \mathfrak{n}\mathfrak{a}$.

Assume now that $\mathfrak{a}$ is generated by a regular sequence. Assume that $FG \notin \mathfrak{n}\mathfrak{a}$. In particular, FG is minimal generator of $\mathfrak{a}$ and can be completed to a minimal generating set for $\mathfrak{a}$, say $a_1, a_2, \ldots, a_r, FG$. Since $\mathfrak{a}$ can be generated by a regular sequence, its minimal generating set $a_1, a_2, \ldots, a_r, FG$ is itself a regular sequence. It follows that $a_1, a_2, \ldots, a_r, F$ is a regular sequence as well. Using this information, one can easily argue that $(0 :_R f) = (g)$, and similarly $(0 :_R g) = (f)$. ☐

Let k be a field. We let $P = k[x_1, \ldots, x_e]$ denote the polynomial ring in n variables of degree 1, and we set $\mathfrak{p} = (x_1, \ldots, x_e)P$. We take $Q = k[[x_1, \ldots, x_e]]$ to be the power series ring, with maximal ideal $\mathfrak{n} = (x_1, \ldots, x_e)Q$. If $h \in Q$, we denote by h^* the initial form of h (which can be regarded as both an element of P and of Q).

PROPOSITION 1.10. *Let* $\mathfrak{b}$ *be a homogeneous ideal of* P *and set* $\mathfrak{a} = \mathfrak{b}Q$, *where* $Q = k[[x_1, \dots, x_e]]$. *If* $y \in \mathfrak{a}$, *then* $y^* \in \mathfrak{b}$. *Furthermore, if* $\mathfrak{b}$ *is generated by homogeneous polynomials of the same degree, then the following hold*:

(a) *If* $y \in \mathfrak{a}$, *then* $y - y^* \in \mathfrak{n}\mathfrak{a}$.

(b) *If* F, G *are elements of* Q *such that their images* f, g *in* $Q/\mathfrak{a}$ *form an exact pair of zero-divisors, then* $F^*G^* \notin \mathfrak{n}\mathfrak{a}$.

Proof. For each integer i one has canonical isomorphisms

$$P/\mathfrak{p}^i \cong Q/\mathfrak{n}^i$$

which allow one to translate the statements to a graded setting, where they are clear. If $y \in \mathfrak{a}$, it follows that $y^* \in \mathfrak{b} + \mathfrak{p}^i$ for each i, hence $y^* \in \mathfrak{b}$.

Assume now that $\mathfrak{b}$ is generated by homogeneous polynomials of the *same* degree.

(a) If $y \in \mathfrak{a}$, then $y - y^* \in \mathfrak{n}\mathfrak{a} + \mathfrak{n}^i$ for all $i > \deg(y^*)$, hence $y - y^* \in \mathfrak{n}\mathfrak{a}$.

(b) Assume that F, G are such that f, g form a pair of exact zero-divisors. By Proposition 1.9, we know that $FG \notin \mathfrak{n}\mathfrak{a}$. Part (a) gives then that $FG - (FG)^* \in \mathfrak{n}\mathfrak{a}$, hence $F^*G^* = (FG)^* \notin \mathfrak{n}\mathfrak{a}$. □

COROLLARY 1.11. *Let* $(Q, \mathfrak{n}, k)$ *be a regular local ring and* $\mathfrak{a} \subseteq \mathfrak{n}^2$. *If* $R = Q/\mathfrak{a}$ *contains an exact zero-divisor, then* $\mathfrak{a}$ *has a minimal generator* fg *with* $f, g \in \mathfrak{n}$. *Furthermore, if* Q *is a power series ring over* k *and* $\mathfrak{a}$ *is generated by homogeneous polynomials of the same degree, then* f *and* g *can be chosen to be homogeneous polynomials.*

2. Embedded q.c.i. ideals

In this section, we define the notion of embedded q.c.i. ideal. We spell out a known characterization of such ideals in Remark 2.3. We are mainly interested in finding a procedure for checking that a given q.c.i. ideal is **not** embedded. This is achieved in Lemma 2.7, by using the terminology of homotopy Lie algebra. The approach used here expands the one in the proof of [4, Theorem 3.5].

2.1. A *quasi-deformation* is a pair $R \to R' \leftarrow Q$ of homomorphisms of local rings, with $R \to R'$ faithfully flat and $R' \leftarrow Q$ surjective with kernel generated by a Q-regular sequence. By definition, the *CI-dimension* of an R-module M, denoted $\text{CI-dim}_R M$, is finite if $\text{pd}_Q(R' \otimes_R M)$ is finite for some quasi-deformation; see [3].

If M is a finitely generated R-module, then its nth *betti number* is the integer

$$\beta_n^R(M) = \text{rank}_k\big(\text{Tor}_n^R(M, k)\big).$$

2.2. Consider the following conditions concerning an ideal I of the local ring R:

(1) $\operatorname{CI-dim}_R(R/I) < \infty$ and $\mathrm{H}_1(E)$ is a free R/I-module.
(2) I is a q.c.i. ideal.
(3) The betti numbers of the R-module R/I have polynomial growth and $\mathrm{H}_1(E)$ is a free R/I-module.

Soto [20, Proposition 23] shows that the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) always hold, and that the three statements are equivalent for certain classes of rings, for which the asymptotic behavior of betti numbers is well understood.

Rodicio also conjectured that (1) $\Longleftrightarrow$ (2) always holds. As discussed in the Introduction, [4, Theorem 3.5] provides a counterexample with I a principal ideal.

In what follows, we say that an ideal of a ring Q is a *complete intersection* ideal if it can be generated by a Q-regular sequence.

REMARK 2.3. The following statements are equivalent:

(1) I is a q.c.i. ideal and $\operatorname{CI-dim}_R(R/I) < \infty$.
(2) There exists a faithfully flat extension $R \to R'$, a local ring Q and complete intersection ideals $\mathfrak{a} \subseteq \mathfrak{b}$ of Q such that $R' = Q/\mathfrak{a}$ and $R'/IR' = Q/\mathfrak{b}$.

The implication (1) $\Longrightarrow$ (2) is given by [4, Lemma 2.7] and the converse follows from [4, Lemma 1.3, Lemma 1.4].

To simplify the terminology and better convey the structural property described in Remark 2.3(2), we introduce the following definition:

DEFINITION 2.4. We say that a q.c.i. ideal I of R is *embedded* if $\operatorname{CI-dim}_R(R/I) < \infty$.

2.5. *Complexity.* If M is a finitely generated R-module, the *complexity of* M, denoted $\operatorname{cx}_R(M)$, is the least integer d such that there exists a polynomial $f(t)$ of degree $d-1$ such that $\beta_i^R(M) \le f(i)$ for all $i \ge 1$.

If I is a q.c.i. ideal of R, then (1.2) and the minimality of Tate's resolution (see [4, Construction 1.5, 1.6]) yield

$$\nu(I) - \operatorname{grade}_R(I) = \operatorname{rank}_R \mathrm{H}_1(E) = \operatorname{cx}_R(R/I). \tag{2.1}$$

Next, we extend an argument used in the proof of [4, Theorem 3.5].

2.6. *The homotopy Lie algebra.* It is known that there exists a graded Lie algebra over k, denoted $\pi^*(R)$ such that the universal enveloping algebra of $\pi^*(R)$ is equal to the algebra $\operatorname{Ext}_R^*(k,k)$ with Yoneda products, see [1, §10] for details. We let $\zeta^*(R)$ denote the center of $\pi^*(R)$.

LEMMA 2.7. *If I is an embedded q.c.i. ideal, then* $\nu(I) - \operatorname{grade}_R(I) \le \operatorname{rank}_k \zeta^2(R)$.

Proof. By [6, 5.3], $\operatorname{Ext}_R(R/I,k)$ is a finitely generated module over the symmetric algebra $\mathcal{P}$ of $\zeta^2(R)$, and its Krull dimension equals $\operatorname{cx}_R(R/I)$ by

[3, Theorem 5.3]. Now (2.1) and elementary properties of Krull dimension give

$$\begin{aligned}\nu(I) - \operatorname{grade}_R(I) &= \operatorname{cx}_R(R/I) = \dim_{\mathcal{P}} \operatorname{Ext}_R(R/I,k)\\ &\leq \dim \mathcal{P} = \operatorname{rank}_k \zeta^2(R).\end{aligned}$$

□

3. Loewy length and minimal generation of q.c.i. ideals

If the local ring $(R,\mathfrak{m},k)$ is Artinian, then its *Loewy length* is defined as the number

$$\ell\ell(R) = \inf\{l \geq 0 \mid \mathfrak{m}^l = 0\}.$$

In this section we show that the number of generators of a q.c.i. ideal of R can be bounded in terms of $\ell\ell(R)$.

We say that R is a *complete intersection* ring if $\widehat{R} = Q/\mathfrak{a}$ for a regular local ring Q and a complete intersection ideal $\mathfrak{a}$.

3.1. If I is a q.c.i. ideal of R, then the following statements are equivalent (see for example [4, Proposition 7.7] and [4, Corollary 7.6]):

(1) R is Gorenstein, respectively complete intersection;
(2) R/I is Gorenstein, respectively complete intersection.

The main result of this section is as follows. Note that properties (2) and (5) below yield immediately the statement of Theorem 1 in the Introduction.

THEOREM 3.2. *Let $(R,\mathfrak{m},k)$ be a local Artinian ring. Let $I \subset R$ be a non-trivial q.c.i. ideal and set $l = \ell\ell(R)$. The following then hold*:

(1) $\nu(I) \leq l-1$;
(2) *If R/I is not a complete intersection, then $\nu(I) \leq l-2$*;
(3) *If $\nu(I) = l-2$ and $I \cap \mathfrak{m}^2 \subseteq \mathfrak{m}I$, then $\nu(\mathfrak{m}/I) \leq \nu(\mathfrak{m}^{l-1})$*;
(4) *If R/I is Gorenstein, not a complete intersection, and $I \cap \mathfrak{m}^2 \subseteq \mathfrak{m}I$, then $\nu(I) \leq l-3$*;
(5) *If R/I is Gorenstein, not a complete intersection, then $l \geq 4$. If $l = 4$, then $\nu(I) = l-3 = 1$.*

REMARK 3.3. We can argue that the bounds in the theorem are sharp, by pointing out extremal examples.

For (1), consider the ring $R = k[[X_1,\dots,X_n]]/(X_1^2,\dots,X_n^2)$. The ideal $I = (x_1,\dots,x_n)$ is an embedded q.c.i. with $\nu(I) = n$ and $l = n+1$.

For (2) and (3), consider for example the ring R and the ideal I in Section 4, for which $\nu(I) = 2$, $l = 4$, $\nu(\mathfrak{m}) = 5$ and $\nu(\mathfrak{m}^3) = 3$.

For (4), any generic Gorenstein algebra with $\mathfrak{m}^4 = 0$ and $\nu(\mathfrak{m}) \geq 3$ works, since such a ring is known (see [13, Remark 4.3]) to have an exact zero-divisor, so that one can take I with $\nu(I) = 1$.

Proof. Set $n = \nu(I)$. We use the notation in 1.4. By Lemma 1.5 we have $(0:_R I) = \Delta R$ and $(0:_R \Delta) = I$. Also, (1.4) gives $\Delta \in \mathfrak{m}^n$.

(1) Note that $\Delta \neq 0$, hence $\mathfrak{m}^n \neq 0$.

(2) Assume that $n = l-1$. Since $\Delta \in \mathfrak{m}^{l-1}$, we have $\Delta \in (0 :_R \mathfrak{m})$. On the other hand, we have $(0 :_R \mathfrak{m})I = 0$, hence $(0 :_R \mathfrak{m}) \subseteq (0 :_R I) = \Delta R$. It follows that $(0 :_R \mathfrak{m}) = \Delta R$. In particular, R is Gorenstein. We conclude $I = (0 :_R \Delta) = \mathfrak{m}$. Hence, $I = \mathfrak{m}$ is a q.c.i. ideal. Using 3.1, we conclude that R is a complete intersection, a contradiction.

(3) Since $I \cap \mathfrak{m}^2 \subseteq \mathfrak{m}I$, the ideal $\mathfrak{m}$ has a minimal generating set $f_1, \dots, f_n, h_1, \dots, h_t$ such that $f_1, \dots, f_n$ minimally generate I. Assuming $n = l-2$, we have $\Delta \in \mathfrak{m}^{l-2}$, and thus $h_i\Delta \in \mathfrak{m}^{l-1}$ for all i. Note that the elements $h_i\Delta$ of $\mathfrak{m}^{l-1}$ are linearly independent. Indeed, if $\sum c_i h_i \Delta = 0$ for some constants c_i, not all zero, then it would follow $\sum_i c_i h_i \in (0 :_R \Delta) = I = (f_1, \dots, f_n)$, a contradiction. It follows that $t \leq \operatorname{rank}_k(\mathfrak{m}^{l-1}) = \nu(\mathfrak{m}^{l-1})$.

(4) By (2), we know that $n \leq l-2$. Assume $n = l-2$. Then (3) gives that $\nu(\mathfrak{m}/I) \leq 1$. Note that R/I is Gorenstein by 3.1. The ring R/I is thus a Gorenstein ring of embedding dimension 1; it is thus a complete intersection, and hence R is a complete intersection by 3.1.

(5) By (2), we have $l \geq 3$. Assume $l = 3$ and $\nu(I) = 1$. If $I = (f)$, then (4) shows that $f \in \mathfrak{m}^2$. Since $f\mathfrak{m} = 0$, it follows that $\mathfrak{m} \subseteq (0 : f) = (\Delta)$. Thus $\mathfrak{m}$ is 1-generated, and it follows that R is a complete intersection, a contradiction.

Assume now that $l = 4$. If I is not principal, then it can be minimally generated by two elements. Let $I = (f_1, f_2)$. By (3), we may assume that one of these elements is in $\mathfrak{m}^2$. Assume $f_1 \in \mathfrak{m}^2$ and note that $f_2 \notin \mathfrak{m}^3$.

Let $\mathfrak{m}^3 = (\delta)$ be the socle of R. For every $x \in \mathfrak{m}$ we have $xf_1 \in \mathfrak{m}^3$, and therefore $xf_1 = \alpha_x \delta$ where α_x is either zero or a unit in R. If $\alpha_x = 0$, then we take $y_x = 0$; if α_x is a unit we use the fact that there exists a non-zero multiple of f_2 in the socle to find y_x such that $y_x f_2 = xf_1$. Since $f_2 \notin \mathfrak{m}^3$, we have $y_x \in \mathfrak{m}$.

In either case, there exists $y_x \in \mathfrak{m}$ such that $xf_1 = y_x f_2$. With the notation in 1.4, the elements

$$xv_1 - y_x v_2$$

are cycles in the Koszul complex E. Since $\nu(I) = 2$, we have that $\nu(\mathrm{H}_1(E)) = 2$. Let z_1 and z_2 be the two cycles in 1.4 whose classes generate $\mathrm{H}_1(E)$, with

$$z_j = a_{1j}v_1 + a_{2j}v_2.$$

It follows that for every $x \in \mathfrak{m}$, the element $xv_1 - y_x v_2$ is a linear combination of z_1, z_2 and the boundary $f_2v_1 - f_1v_2$. Consequently, $\mathfrak{m} = (a_{11}, a_{12}, f_2)$. The ring R/I is then Gorenstein and has embedding dimension at most 2. It is thus a complete intersection, and thus R is a complete intersection, a contradiction. □

DEFINITION 3.4. We say that a q.c.i. ideal I is *minimal* if I does not properly contain any non-zero q.c.i. ideal.

REMARK 3.5. If I is a minimal q.c.i. ideal, then $\operatorname{grade}_R(I)=0$, because every regular element generates a q.c.i. ideal.

The results proved so far allow us to show that certain q.c.i. ideals are minimal.

PROPOSITION 3.6. *Let $R=Q/\mathfrak{a}$ be an Artinian local ring, where $(Q,\mathfrak{n},k)$ is a regular local ring and $\mathfrak{a}\subseteq\mathfrak{n}^2$. If R is not a complete intersection, $\ell\ell(R)=3$ and $\mathfrak{a}\cap\mathfrak{n}^3\subseteq\mathfrak{a}\mathfrak{n}$, then any q.c.i. ideal of R is minimal.*

In particular, the ideal I of [4, Theorem 3.5] *is a minimal q.c.i. ideal.*

Proof. By Theorem 3.2(2), any q.c.i. ideal of R is principal. Let $I=(h)$ with $h\in\mathfrak{m}$ be a q.c.i. ideal. If $J\subseteq I$ is another q.c.i. ideal with $J\neq I$ then $J=(f)$ and $f=ah$ with $a\in\mathfrak{m}$. In particular, $f\in\mathfrak{m}^2$. If g is a complementary zero-divisor of f, and F and G are the liftings of these elements in Q, 1.9 shows that FG is a minimal generator of $\mathfrak{a}$. Since $FG\in\mathfrak{n}^3$, this contradicts the hypothesis that $\mathfrak{a}\cap\mathfrak{n}^3\subseteq\mathfrak{a}\mathfrak{n}$. $\square$

4. A non-principal, non-embedded, minimal q.c.i. ideal

In this section we establish Theorem 2 in the Introduction, which is obtained by putting together information from Proposition 4.3 and Theorem 4.5. The relevant example is described below.

EXAMPLE 4.1. Let $X=\{X_1,X_2,\dots,X_5\}$ be a set of indeterminates, $\mathfrak{c}$ be the ideal of $\mathbb{Z}[X]$ generated by the elements:

$$\begin{array}{ccccc} X_1^2-X_2X_3, & & X_2^2-X_3X_5, & & X_3^2-X_1X_4, \\ X_4^2, & X_5^2, & X_3X_4, & X_2X_5, & X_4X_5. \end{array} \tag{4.1}$$

Let A be the ring $\mathbb{Z}[X]/\mathfrak{c}$. We denote the image of the variable X_i in A by x_i. Let f_1 and f_2 be the elements

$$f_1=x_1+x_2+x_4 \quad\text{and}\quad f_2=x_2+x_3+x_5$$

of A. Let E be the Koszul complex

$$E:0\to A\xrightarrow{\begin{bmatrix}-f_2\\ f_1\end{bmatrix}}A^2\xrightarrow{\begin{bmatrix}f_1 & f_2\end{bmatrix}}A.$$

LEMMA 4.2. *The following statements hold for the data of Example 4.1:*

(a) *The elements listed in (4.1) form a Gröbner basis for the ideal $\mathfrak{c}$.*

(b) *The ring A is a free $\mathbb{Z}$-module, with basis*

$$1,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5,\ x_2x_3,\ x_2x_4,\ x_3x_5,$$
$$x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_5.$$

(c) *The ring A is a free $\mathbb{Z}$-module with basis* 1, x_3, x_4, x_5.

(d) *The homology $H_\bullet(E)$ is free as a module over $\mathbb{Z}$.*

(e) *The homology $H_\bullet(E)$ is free as a module over $A/(f_1, f_2)$. Furthermore the elements* $\mathrm{cls}(1)$ *in* H_0, $\mathrm{cls}(\theta_1)$, *and* $\mathrm{cls}(\theta_2)$ *in* H_1, *and* $\mathrm{cls}(\Delta)$ *in* H_2 *form a basis for $H_\bullet(E)$ over $A/(f_1, f_2)$, where*

$$\theta_1 = \begin{bmatrix} x_1 - x_2 \\ -x_3 + x_4 + 2x_5 \end{bmatrix} \quad \textit{and} \quad \theta_2 = \begin{bmatrix} x_4 \\ x_2 - x_3 - x_4 \end{bmatrix} \quad \textit{in } E_1 \quad \textit{and}$$

$$\Delta = \det \begin{bmatrix} x_1 - x_2 & x_4 \\ -x_3 + x_4 + 2x_5 & x_2 - x_3 - x_4 \end{bmatrix} \quad \textit{in } E_2.$$

Proof. (a) One may use Buchsberger's algorithm to check that the listed generators already form a Gröbner basis. When using this algorithm, there is no need to check the S-polynomial for a pair of monomials and there is no need to check the S-polynomial for two polynomials whose leading terms are relatively prime. Thus, one need only check the S-polynomial for the pair X_3X_4 and $\underline{X_3^2} - X_1X_4$ and the S-polynomial for the pair X_2X_5 and $\underline{X_2^2} - X_3X_5$. Both S-polynomials reduce in the appropriate manner. (We have underlined the leading terms.) Hence the listed generators are already Gröbner basis.

(b) Every leading coefficient of the Gröbner basis is a unit in $\mathbb{Z}$, hence A is a free $\mathbb{Z}$-module. Furthermore, there is no difficulty using the Gröbner basis to show that the listed elements form a basis.

(c) In a similar manner, $X_2 + X_3 + X_5$, $X_1 - X_3 + X_4 - X_5$, X_5^2, X_4X_5, X_3X_5, X_4^2, X_3X_4, X_3^2 is a Gröbner basis for the ideal of $\mathbb{Z}[X_1, X_2, X_3, X_4, X_5]$ which defines $A/(f_1, f_2)$ and therefore $A/(f_1, f_2)$ is a free $\mathbb{Z}$-module with the listed basis.

(d) We treat the entire calculation as a calculation of free $\mathbb{Z}$-modules. We made the calculation by hand and also by using Macaulay2 [10]. The arXiv version of this paper includes an Appendix which contains the Macaulay2 commands that we used for the computer computation, so that the reader can easily reproduce this computation. We have already observed that the homology $H_0(E) = A/(f_1, f_2)$ is free as a $\mathbb{Z}$-module. One computes (see the Appendix on the arXiv) that the module of 1-cycles $Z_1(E)$ is a free $\mathbb{Z}$-module of rank 20 and one identifies a basis for this module. Similarly one computes that the module of 1-boundaries $B_1(E)$ is a free $\mathbb{Z}$-module of rank 12 and one identifies a basis for this module. By comparing the two bases, one sees that $B_1(E)$ is a direct summand of $Z_1(E)$ as $\mathbb{Z}$-modules and one now knows that $H_1(E)$ is a free $\mathbb{Z}$-module with basis represented by the 1-cycles $\theta_1, \theta_2, x_3\theta_1, x_3\theta_2, x_4\theta_1, x_4\theta_2, x_5\theta_1, x_5\theta_2$. In a similar manner, one computes that $Z_2(E)$ (which is equal to $H_2(E)$) is the free $\mathbb{Z}$-module with basis $\Delta, x_3\Delta, x_4\Delta, x_5\Delta$.

(e) The argument of (d) exhibits natural surjections of $A/(f_1, f_2)$-modules:

$$(A/(f_1, f_2))^2 \twoheadrightarrow H_1(E) \quad \text{and} \quad A/(f_1, f_2) \twoheadrightarrow H_2(E).$$

All four modules are free $\mathbb{Z}$-modules; therefore, rank considerations over $\mathbb{Z}$ show that the surjections are isomorphisms. □

For the remainder of the section, fix a field k and set

$$B = k \otimes_{\mathbb{Z}} A \quad \text{and} \quad I = (f_1, f_2)B,$$

where A is the ring in Example 4.1.

PROPOSITION 4.3. *The following hold*:

(1) *B is an artinian local ring with Hilbert series $H_B(z) = 1 + 5z + 7z^2 + 3z^3$.*
(2) *The algebra B is Koszul.*
(3) *The ideal I is a q.c.i. and $H_{B/I}(z) = 1 + 3z$.*

Proof. These assertions follow from Lemma 4.2 because $B = k \otimes_{\mathbb{Z}} A$. Indeed, (1) follows from (b), (2) follows from (a) and the last part of (3) follows from (c).

To see that I is a q.c.i. we use (d) and note that each short exact sequence

$$0 \to Z_i(E) \to E_i \to B_{i-1}(E) \to 0 \quad \text{and} \quad 0 \to B_i(E) \to Z_i(E) \to H_i(E) \to 0$$

is split exact over $\mathbb{Z}$ and remains split exact after the functor $k \otimes_{\mathbb{Z}} -$ has been applied. The isomorphism $H_\bullet(k \otimes_{\mathbb{Z}} E) \simeq k \otimes_{\mathbb{Z}} H_\bullet(E)$ has been established. The calculation of Lemma 4.2(e) continues to hold over $k \otimes_{\mathbb{Z}} A$ and therefore, $(f_1, f_2)(k \otimes_{\mathbb{Z}} A)$ is a q.c.i. ideal of $k \otimes_{\mathbb{Z}} A$. □

We use next the notation of 2.6 regarding homotopy Lie algebras. We use the recipe in [16, Corollary 1.3] (see also [1, Example 10.2.2]) to compute the graded Lie algebra $\pi^*(B)$. This technique is explained in significant detail in [2, Section 3]. We may apply the technique because $\operatorname{Ext}_B^*(k, k)$ is generated as a k-algebra in degree 1 since B is a Koszul algebra.

LEMMA 4.4. *$\zeta^2(B)$ is a 1-dimensional vector space.*

Proof. Since the algebra B is Koszul with Hilbert series described above, we have that the Poincaré series $\mathrm{P}_k^B(z)$ (which is defined to be $\sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^B(k, k) z^i$) is equal to

$$\mathrm{P}_k^B(z) = \frac{1}{1 - 5z + 7z^2 - 3z^3} = 1 + 5z + 18z^2 + 58z^3 + \cdots .$$

The ranks of the vector spaces $\pi^i(B)$, denoted ε_i and called the *deviations of* B, may be read from this series using the techniques of [1, Remark 7.1.1 and Theorem 10.2.1(2)]:

$$\begin{aligned}
\operatorname{rank}_k \pi^1(B) &= 5, \\
\operatorname{rank}_k \pi^2(B) &= 18 - \binom{5}{2} = 8, \\
\operatorname{rank}_k \pi^3(B) &= 58 - 8 \cdot 5 - \binom{5}{3} = 8
\end{aligned}$$

and so on. At any rate, $\pi^1(B)$ has basis t_1, t_2, t_3, t_4, t_5 with the following relations:

$$\begin{aligned}
&[t_1, t_2] = [t_1, t_3] = [t_2, t_4] = [t_1, t_5] = 0,\\
&[t_2, t_3] = t_1^{(2)},\\
&[t_1, t_4] = t_3^{(2)},\\
&[t_3, t_5] = t_2^{(2)}.
\end{aligned}$$

The following elements of $\pi^2(B)$ are then linearly independent and hence form a basis for $\pi^2(B)$:

$$\begin{aligned}
&u_1 = t_1^{(2)}, \qquad u_2 = t_2^{(2)}, \qquad u_3 = t_3^{(2)}, \qquad u_4 = t_4^{(2)}, \qquad t_5 = t_5^{(2)},\\
&u_6 = [t_2, t_5], \qquad u_7 = [t_3, t_4], \qquad u_8 = [t_4, t_5].
\end{aligned}$$

Computing the brackets $[t_i, u_j]$ and using the Jacobi identities and the relations $[t_i, t_i^{(2)}] = 0$, we see that the following elements form a basis for $\pi^3(B)$:

$$\begin{aligned}
v_1 &= \big[t_1, t_4^{(2)}\big] = -\big[t_4, t_3^{(2)}\big] = \big[t_3, [t_3, t_4]\big],\\
v_2 &= \big[t_2, t_5^{(2)}\big] = -\big[t_5, [t_2, t_5]\big],\\
v_3 &= \big[t_3, t_5^{(2)}\big] = -\big[t_5, t_2^{(2)}\big] = \big[t_2, [t_2, t_5]\big],\\
v_4 &= \big[t_4, t_5^{(2)}\big] = -\big[t_5, [t_4, t_5]\big],\\
v_5 &= \big[t_5, t_4^{(2)}\big] = -\big[t_4, [t_4, t_5]\big],\\
v_6 &= \big[t_4, [t_2, t_5]\big] = -\big[t_2, [t_4, t_5]\big],\\
v_7 &= \big[t_3, [t_4, t_5]\big] = -\big[t_5, [t_3, t_4]\big],\\
v_8 &= \big[t_3, t_4^{(2)}\big] = -\big[t_4, [t_3, t_4]\big].
\end{aligned}$$

Unless listed above, all the other brackets $[t_i, u_j]$ are zero. (The signs which pertain to the Lie bracket in a graded Lie algebra may be found in [1, Remark 10.1.2].)

Now let us take an element ξ in $\pi^2(B)$:

$$\xi = C_1 u_1 + \cdots + C_8 u_8.$$

If ξ is a central element in $\pi^2(B)$, then we need to have $[t_i, \xi] = 0$ for all i. For $i = 5$, we have:

$$0 = [t_5, \xi] = -C_2 v_3 + C_4 v_5 - C_6 v_2 - C_7 v_7 - C_8 v_4$$

and this yields $C_2 = C_4 = C_6 = C_7 = C_8 = 0$. Then for $i = 4$, we have:

$$0 = [t_4, \xi] = -C_3 v_1 + C_5 v_4 + C_6 v_6 - C_7 v_8 - C_8 v_5$$

which yields $C_3 = C_5 = 0$. On the other hand, note that $[t_i, u_1] = 0$ for all i. Thus $\zeta^2(B)$ is the vector space generated by $t_1^{(2)}$. □

THEOREM 4.5. *The ideal I of B is a non-principal, non-embedded, minimal q.c.i. ideal.*

Proof. The proof that I is a q.c.i. ideal in B is contained in Proposition 4.3.

Apply Lemma 2.7 to see that the q.c.i. ideal I of B is not an embedded q.c.i. ideal. Indeed, according to Lemma 4.4, we have: $\operatorname{rank}_k \zeta^2(B) = 1 < 2 - 0 = \nu(I) - \operatorname{grade}_B(I)$.

It remains to show that I is a minimal q.c.i. ideal. If $I' \subsetneqq I$ were another q.c.i. ideal, then it follows from Proposition 3.2(2) that $\nu(I') \leq 2$. We treat the cases $\nu(I') = 1$ and $\nu(I') = 2$ separately.

We first show that $\nu(I') = 1$ is not possible; that is, we prove that I does not contain any exact zero-divisors from B. Since B is Artinian, we can write $B = Q/\mathfrak{a}$, where Q is the power series ring $k[[X]]$ and $\mathfrak{a} = \mathfrak{c}Q$, with $\mathfrak{c}$ as in Example 4.1. In light of Corollary 1.11, it suffices to show that the ideal $(X_1 + X_2 + X_4, X_2 + X_3 + X_5)$ of Q does not contain any homogeneous minimal generators of the ideal $\mathfrak{a}$ that factor non-trivially. Suppose that a, b, c, d, e, f, g are elements of k with the product

$$\big[a(X_1 + X_2 + X_4) + b(X_2 + X_3 + X_5)\big][cX_1 + dX_2 + eX_3 + fX_4 + gX_5] \tag{4.2}$$

equal to a minimal generator of $\mathfrak{a}$. The ideal $\mathfrak{a}$ is generated by homogeneous forms of degree 2; so the element of (4.2) is a minimal generator of $\mathfrak{a}$ if and only if this element is in $\mathfrak{a}$ and this occurs if and only if the following seven expressions vanish

$$\begin{aligned} &ac + bd + ae + be,\\ &ad + bd + be + bg,\\ &ac + be + af,\\ &ac + bc + ad,\\ &bc + ae,\\ &bc + ag,\\ &ad + af + bf. \end{aligned} \tag{4.3}$$

The first expression in (4.3) is obtained by setting the coefficient of X_1^2 plus the coefficient of X_2X_3 in (4.2) equal to zero; the fourth expression is obtained by setting the coefficient of X_1X_2 in (4.2) equal to zero; and so on. We observe that if the seven expressions of (4.3) are zero, then the product (4.2) is also zero. Indeed, Macaulay2 [10] shows that in polynomial ring $\mathbb{Z}[a, b, c, d, e, f, g]$, the ideal $((a, b)(c, d, e, f, g))^2$ is contained in the ideal generated by the elements of (4.3). This inclusion of ideals passes to every field. This completes the proof that I does not contain any exact zero-divisors.

Now suppose that $I' \subseteq I$ is a q.c.i. with $\nu(I') = 2$. According to Lemma 1.5, or Lemma 1.7, there are elements a, b, c, d and a', b', c', d' in the maximal ideal $\mathfrak{m}_B$ of B such that $(0 :_B I) = \Delta B$, $(0 :_B \Delta) = I$, $(0 :_B I') = \Delta' B$ and $(0 :_B \Delta') = I'$ with $\Delta = ad - bc$ and $\Delta' = a'd' - b'c'$. The inclusion $I' \subseteq I$ yields $I'\Delta \subset I\Delta = 0$ and $\Delta \in (0 :_B I') = \Delta' B$. It follows that $\Delta = \alpha\Delta'$ for some

$\alpha \in B$. The element Δ is explicitly calculated in the proof of Proposition 4.3. This element of B is homogeneous of degree two. All four elements a', b', c', d' are in $\mathfrak{m}_B$; so, Δ' is in $\mathfrak{m}_B^2$. The element Δ is not in $\mathfrak{m}_B^3$; hence $\alpha \notin \mathfrak{m}_B$. Thus, α is a unit; the ideals ΔB and $\Delta' B$ of B are equal and $I = (0 :_B \Delta) = (0 :_B \Delta') = I'$. This completes the argument that I is a 2-generated minimal q.c.i. ideal in B. $\square$

REMARK 4.6. The ring B is an embedded deformation in the sense that $B = Q/\mathfrak{a}' \otimes_Q Q/(\theta)$ with θ regular on $Q/\mathfrak{a}'$, for $\theta = X_1^2 - X_2X_3$ and $\mathfrak{a}'$ equal to $(X_2^2 - X_3X_5, X_3^2 - X_1X_4, X_4^2, X_5^2, X_3X_4, X_2X_5, X_4X_5)$. (This is a Macaulay2 calculation made over the field $\mathbb{Q}$.) Nonetheless the q.c.i. ideal I of B is not an embedded q.c.i. ideal.

On the other hand, there is an elementary argument that B does not have the form $B = Q/\mathfrak{a}'' \otimes_Q Q/(\theta_1, \theta_2)$ with θ_1, θ_2 a regular sequence on $Q/\mathfrak{a}''$. The betti numbers of B, as a Q-module are $(b_0, \dots, b_5) = (1, 8, 20, 23, 13, 3)$. (Again, this is a Macaulay2 calculation, made over $\mathbb{Q}$.) If B were equal to $Q/\mathfrak{a}'' \otimes_Q Q/(\theta_1, \theta_2)$ with θ_1, θ_2 a regular sequence on $Q/\mathfrak{a}''$, then the betti numbers of $Q/\mathfrak{a}''$ would have to be $(b_0, b_1, b_2, b_3) = (1, 6, 7, 3)$. However, the Euler characteristic forbids these numbers from being the betti numbers of a module because no module has rank equal to -1.

5. Generic complete intersections of quadrics

The main result of this section is Theorem 5.1, which describes when an Artinian complete intersection defined by generic quadratic forms has exact zero-divisors, thereby establishing Theorem 3 in the Introduction. Theorem 5.1 is a consequence of Theorem 5.2, Proposition 1.9, and Corollary 1.11 and its proof is given at the end of the section.

THEOREM 5.1. *Let P be the polynomial ring $k[x_1, \dots, x_n]$ for some algebraically closed field k of characteristic not equal to 2 and let $A = P/(f_1, \dots, f_n)$.*

(1) *Assume $n \leq 4$. If $f_1, \dots, f_n$ is any regular sequence of quadratic forms in P, then A contains a homogeneous linear exact zero-divisor.*
(2) *Assume $5 \leq n$. If $f_1, \dots, f_n$ is a generic regular sequence of quadratic forms in P, then A does not contain any exact zero-divisor.*

For the purposes of Theorem 5.1, a regular sequence $\mathbf{f} = f_1, \dots, f_n$ is said to be *generic* if it is an element of the open set $\mathcal{I}$ below.

THEOREM 5.2. *Let P be the polynomial ring $k[x_1, \dots, x_n]$ for some algebraically closed field k of characteristic not equal to 2, and let $\mathbb{A}$ be the affine space*

$$\mathbb{A} = \{\mathbf{f} = (f_1, \dots, f_n) \mid \textit{such that each } f_i \textit{ is a quadratic form in } R\},$$

and $\mathcal{I}$ be the following subset of $\mathbb{A}$:

$$\mathcal{I} = \left\{ \mathbf{f} = (f_1, \dots, f_n) \in \mathbb{A} \;\middle|\; \begin{array}{l} f_1, \dots, f_n \textit{ is a regular sequence and every} \\ \textit{non-zero element of the } k\textit{-vector space which} \\ \textit{is spanned by } f_1, \dots, f_n \textit{ is } \textbf{\textit{irreducible}} \textit{ in } P \end{array} \right\}.$$

Then the following statements hold.

(1) *The set $\mathcal{I}$ is open in $\mathbb{A}$.*
(2) *If $n \le 4$, then $\mathcal{I}$ is empty.*
(3) *If $5 \le n$, then $\mathcal{I}$ is non-empty.*

Proof of (1) *from Theorem* 5.2. Each f_h in the definition of $\mathbb{A}$ is a homogeneous form in P of degree 2; consequently, the affine space $\mathbb{A}$ of Theorem 5.2 has dimension $n\binom{n+1}{2}$. The subset $\mathcal{I}$ of $\mathbb{A}$ is the complement of $X \cup Y$ where

$$X = \left\{ \mathbf{f} = (f_1, \dots, f_n) \in \mathbb{A} \;\middle|\; \begin{array}{l} \text{there exist elements } b_1, \dots, b_n \text{ in } k, \\ \text{not all of which are zero, such that} \\ \sum_{i=1}^n b_i f_i \text{ is } \textbf{reducible} \end{array} \right\} \tag{5.1}$$

and

$$Y = \{\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{A} \mid f_1, \dots, f_n \text{ is not a regular sequence}\}.$$

We show in Observation 5.3 that Y is a closed subset of $\mathbb{A}$ and in Observation 5.4 that X is a closed subset of $\mathbb{A}$. □

Observation 5.3. *Let $P = \mathbf{k}[x_1, \dots, x_n]$. Fix a sequence of degrees $\mathbf{d} = (d_1, \dots, d_n)$. Consider sequences of forms $\mathbf{f} = (f_1, \dots, f_n)$ from P, where f_i is homogeneous of degree d_i. Let $\mathbb{A}$ be the space of coefficients for $\mathbf{f}$. Then there exists a closed set $Y \subseteq \mathbb{A}$ such that the coefficients of $\mathbf{f}$ are in Y if and only if $\mathbf{f}$ is not a regular sequence.*

Proof. The polynomials of $\mathbf{f}$ form a regular sequence if and only if the following inclusion of ideals

$$(x_1, \dots, x_n)^N \subseteq (f_1, \dots, f_n)$$

holds, for $N = \sum d_i - n + 1$. The above inclusion of ideals holds if and only if various statements about vector spaces hold; namely,

$$\begin{aligned} &(x_1, \dots, x_n)^N \subseteq (f_1, \dots, f_n) \\ &\quad\iff\quad (x_1, \dots, x_n)_N \subseteq (f_1, \dots, f_n)_N \\ &\quad\iff\quad (x_1, \dots, x_n)_N = (f_1, \dots, f_n)_N \quad \text{since } (f_1, \dots, f_n)_N \subseteq (x_1, \dots, x_n)_N \\ &\quad\iff\quad \dim(f_1, \dots, f_n)_N = \dim(x_1, \dots, x_n)_N. \end{aligned}$$

Let T be the matrix which expresses a generating set for $(f_1, \dots, f_n)_N$ in terms of the monomial basis for $(x_1, \dots, x_n)_N$. The vector space $(f_1, \dots, f_n)_N$ is generated by $\{m_{N-d_j,i} f_j\}$ where, for each fixed d, $\{m_{d,i}\}$ is the set of monomials in $x_1,\dots,x_n$ of degree d. Express each $m_{N-j,i}f_j$ in terms of the basis $\{m_{N,i}\}$. We have:

$$[m_{N,1},\dots m_{N,\text{last}}]T=[m_{N-d_1,1}f_1,\dots,m_{N-d_n,\text{last}}f_n].$$

We have shown that $\mathbf{f}$ is not a regular sequence if and only $I_{\text{row size}}(T)=0$; this is a closed condition on the coefficients of $\mathbf{f}$. $\square$

Observation 5.4. *Retain the notation and hypotheses of Theorem 5.2 and (5.1). Then X is a closed subset of $\mathbb{A}$.*

Proof. The coordinate ring for $\mathbb{A}$ is $S=k[\{z_{i,j;h}\mid 1\le i\le j\le n \text{ and } 1\le h\le n\}]$. The point $\mathbf{a}=(\{a_{i,j;h}\})$ in affine space $\mathbb{A}^{n\binom{n+1}{2}}$ corresponds to the element $\mathbf{f_a}=(f_1,\dots,f_n)$ in $\mathbb{A}$ with $f_h=\sum_{i\le j}a_{i,j;h}x_ix_j$. We describe an ideal J of S so that every polynomial of J vanishes at the point $\mathbf{a}$ of affine space $\mathbb{A}^{n\binom{n+1}{2}}$ if and only if $\mathbf{f_a}$ is in X.

We work in the polynomial ring

$$T=k\big[x_1,\dots,x_n,\{z_{i,j;h}\mid 1\le i\le j\le n \text{ and } 1\le h\le n\},w_1,\dots,w_n\big].$$

Let $\mathbf{F}$ be the n-tuple $(F_1,\dots,F_n)$, where $F_h=\sum_{i\le j}z_{i,j;h}x_ix_j$, F be the polynomial $F=\sum_{h=1}^n F_iw_i$, H be the $n\times n$ matrix $H=(\frac{\partial^2F}{\partial x_i\,\partial x_j})$, and $G_1,\dots,G_\alpha$ be a set of generators for the ideal $I_3(H)$. Each G_ℓ is a tri-homogeneous polynomial in T with degree 0 in the x's, degree 3 in the z's, and degree 3 in the w's. For each large N, let $\mu_{N,1},\dots,\mu_{N,\binom{N+n-1}{N}}$ be a list of the monomials in $\{w_1,\dots,w_n\}$ of degree N, M_N be the matrix which expresses each $\mu_{N-3,i}G_\ell$ (as $\mu_{N-3,i}$ roams over the monomials of degree of $N-3$ in $\{w_1,\dots,w_n\}$ and $1\le\ell\le\alpha$) in terms of the monomials $\{\mu_{N,1},\dots,\mu_{N,\binom{N+n-1}{N}}\}$ of degree N in $\{w_1,\dots,w_n\}$:

$$[\mu_{N-3,1}G_1,\dots,\mu_{N-3,\binom{N+n-4}{N-3}}G_\alpha]=[\mu_{N,1},\dots,\mu_{N,\binom{N+n-1}{N}}]M_N.$$

Notice that each entry of each matrix M_N is a cubic form in $S=k[\{z_{i,j;h}\}]$. Let J_N be the ideal in S generated by the $\binom{N+n-1}{N}$ minors of M_N. Let J be the ideal $\sum_N J_N$ of S.

Let $\mathbf{a}\in\mathbb{A}^{n\binom{n+1}{2}}$. We claim that $\mathbf{f_a}$ is in X if and only if $\mathbf{a}\in V(J)$. Let $\mathbf{x}$ be the variables $(x_1,\dots,x_n)$ and $\mathbf{w}$ be the variables $(w_1,\dots,w_n)$. Observe that

$$\begin{aligned}
&(5.2)\qquad \mathbf{f_a} \text{ is in } X\\
&\qquad\iff \exists\mathbf{b}\in\mathbb{A}^n \text{ with } \mathbf{b}\ne 0 \text{ and } F(\mathbf{x},\mathbf{a},\mathbf{b}) \text{ is reducible}\\
&(5.3)\qquad\iff \exists\mathbf{b}\in\mathbb{A}^n \text{ with } \mathbf{b}\ne 0 \text{ and } \operatorname{rank} H(\mathbf{a},\mathbf{b})\le 2\\
&(5.4)\qquad\iff \exists\mathbf{b}\in\mathbb{A}^n \text{ with } \mathbf{b}\ne 0 \text{ and } I_3\big(H(\mathbf{a},\mathbf{b})\big)=0
\end{aligned}$$

$$\iff \begin{cases} \text{the ideal } I_3(H(\mathbf{a},\mathbf{w})) \text{ of the polynomial} \\ \text{ring } k[w_1,\dots,w_n] \text{ is not primary to the maximal} \\ \text{ideal } (w_1,\dots,w_n) \end{cases} \tag{5.5}$$

$$\iff (w_1,\dots,w_n)^N \nsubseteq I_3\big(H(\mathbf{a},\mathbf{w})\big) = 0, \text{ for any } N \tag{5.6}$$

$$\iff \text{every } \binom{N+n-1}{N} \text{ minor of } M_N(\mathbf{a}) \text{ is zero for all } N \tag{5.7}$$

$$\iff \mathbf{a} \text{ is in } V(J). \tag{5.8}$$

We explain the various equivalences. The point of (5.2) is that if $\mathbf{f_a}$ is the n-tuple $(f_1,\dots,f_n)$ in $\mathbb{A}$, then $F(\mathbf{x},\mathbf{a},\mathbf{b})$ is the element $b_1f_1+\cdots+b_nf_n$ in the vector space spanned by $f_1,\dots,f_n$. Hence, (5.2) is the definition of the set X.

(5.3) The matrix $H(\mathbf{a},\mathbf{b})$ is the Hessian of the polynomial $b_1f_1+\cdots+b_nf_n$ in $P=k[x_1,\dots,x_n]$. Lemma 5.5 shows that a quadratic form in P is irreducible if and only if its Hessian has rank at least 3.

(5.4) This is obvious.

(5.5) This is the critical translation where we are able to remove the words "$\exists\mathbf{b}$". If $\mathcal{S}$ is a set of homogeneous polynomials in $k[w_1,\dots,w_n]$, with k algebraically closed, then the homogeneous Nullstellensatz guarantees that the polynomials of $\mathcal{S}$ have a common non-trivial solution in k if and only if the ideal generated by the elements of $\mathcal{S}$ is not primary to the irrelevant ideal $(w_1,\dots,w_n)$.

(5.6) This is obvious.

(5.7) We turn (5.6) into a vector space calculation. We look at our favorite basis for $k[w_1,\dots,w_n]_N$ and we express the elements of the subspace $[I_3(H(\mathbf{a},\mathbf{w}))]_N$ in terms of the basis for the entire space $[(x_1,\dots,x_n)^N]_N$. The subspace is equal to the entire space if and only if the transition matrix has rank equal to the dimension of the entire vector space. We use the formulation that the subspace $[I_3(H(\mathbf{a},\mathbf{w}))]_N$ is a proper subspace of $[(x_1,\dots,x_n)^N]_N$ if and only if every maximal minor of the transition matrix $M_N(\mathbf{a})$ is zero. $\square$

Lemma 5.5 is well known; it can be seen, for example, by writing f in diagonal form and using [9, Proposition 11.2]. We include a short proof for the reader's convenience. Recall that the polynomial f in $k[x_1,\dots,x_n]$, where k is a field, is called *absolutely irreducible* if f is irreducible in $\bar{k}[x_1,\dots,x_n]$, where $\bar{k}$ is the algebraic closure of k.

LEMMA 5.5. *Let f be a quadratic form in the polynomial ring $P=k[x_1,\dots,x_n]$, where k is a field of characteristic not equal to* 2, *and $H(f)$ be the $n\times n$ matrix with $\frac{\partial^2 f}{\partial x_i\,\partial x_j}$ in the position row i and column j. Then f is absolutely irreducible if and only if* $3\le \operatorname{rank} H(f)$.

Proof. We pass to the algebraic closure $\bar{k}$ of k. Neither statement "f is absolutely irreducible" nor "$3 \le \operatorname{rank} H(f)$" is affected. Notice that $\operatorname{rank} H(f)$ is invariant under change of variables. Also, the ability, or lack of ability, to factor f into a product of two linear forms is invariant under change of variables. Thus, we may change variables at will.

If f factors into $\ell_1\ell_2$, then we may change variables and assume that $f = x_1x_2$ or $f = x_1^2$. In either event, $\operatorname{rank} H(f) \le 2$. Now we assume that $\operatorname{rank} H(f) \le 2$. It follows that the vector space $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ has dimension at most two; so, after a change of variables, $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) = (x_1, x_2)$. (This is the point where we use the hypothesis that the characteristic of k is not two.) It follows that f is a homogeneous polynomial in two variables; hence, f is reducible now that we have passed to $\bar{k}$. $\square$

Proof of (2) *from Theorem* 5.2. There is nothing to show for $n \le 2$. Fix $\mathbf{a} \in \mathbb{A}^{n\binom{n+1}{2}}$ for n equal to 3 or 4. We use (5.5) to show that $f_{\mathbf{a}}$ is in X. The matrix $H(\mathbf{a}, \mathbf{w})$ is an $n \times n$ symmetric matrix with entries which are linear forms in the polynomial ring $k[w_1, \dots, w_n]$. Observe that

$$\operatorname{grade}_{k[w_1,\dots,w_n]}(I_3(H(\mathbf{a},\mathbf{w})) \le \begin{cases} 1 < n & \text{for } n = 3, \\ 3 < n & \text{for } n = 4\text{; see [15, Theorem 1].} \end{cases}$$

It follows that $I_3(H(\mathbf{a}, \mathbf{w}))$ is not primary to $(w_1, \dots, w_n)$; and therefore, $\mathbf{f_a}$ is in X. $\square$

Proof of (3) *from Theorem* 5.2. Fix $n \ge 5$. Recall that $\mathcal{I} = (\mathbb{A} \setminus X) \cup (\mathbb{A} \setminus Y)$ for X (and Y) given in (and near) (5.1). We know that $\mathbb{A} \setminus X$ is open and $\mathbb{A} \setminus Y$ is open and non-empty. We must show that $\mathbb{A} \setminus X$ is non-empty. Again, we apply (5.5). That is, we prove the result by exhibiting an $n \times n$ symmetric matrix $W_n = (w_{ij})$ of linear forms from $k[w_1, \dots, w_n]$ such that $I_3(W_n)$ is primary to the ideal $(w_1, \dots, w_n)$. We take

$$w_{ij} = \begin{cases} w_{i+j-3} & \text{for } 4 \le i + j \le n + 3, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that each w_i is in the radical of $I_3(W_n)$ for $n \ge 5$. $\square$

Proof of Theorem 5.1. (1) Let $f_1, \dots, f_n$ be any regular sequence of quadratic forms from P with $n \le 4$. Assertion (2) of Theorem 5.2 ensures that some minimal generator of the ideal $(f_1, \dots, f_n)$ factors in a nontrivial manner in P. The factors represent a pair of exact zero-divisors in $A = P/(f_1, \dots, f_n)$, according to Proposition 1.9.

(2) Let $\mathbf{f} = (f_1, \dots, f_n)$, with $5 \le n$, be an element of the dense open subset $\mathcal{I}$ of $\mathbb{P}$, as described in Theorem 5.2. The definition of $\mathcal{I}$ ensures that $\mathbf{f}$ is a regular sequence and that every minimal generator of the ideal $(f_1, \dots, f_n)$ is irreducible in P. The ring $A = P/(f_1, \dots, f_n)$ is Artinian (hence complete) and

we may apply Corollary 1.11 to conclude that every pair of exact zero-divisors in A gives rise to a non-trivial factorization in P of a minimal generator of the ideal $(f_1, \dots, f_n)$. No such factorizations exist in P; consequently, no exact zero-divisors exist in A. □

Remark. In [14, Definition 3.1] a local ring $(R, \mathfrak{m}, k)$ is called *exact* if $\mathfrak{m} = (x_1, \dots, x_n)$ where x_i is an exact zero-divisor in $R/(x_1, \dots, x_{i-1})$ for $i = 1, \dots, n$. The resolution of k given by [14, Theorem 1.8] then yields $\mathrm{cx}_R(k) = n$, so R is a complete intersection by [11, Theorem 2.3]. This sharpens one implication in [14, Theorem 2.3].

In this terminology, (1) in Theorem 5.1 becomes the statement that Artinian complete intersections of n quadrics are exact when $n \leq 4$.

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