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A NON-COMMUTATIVE WIENER–WINTNER THEOREM

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ABSTRACT. For a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ and a positive ergodic homomorpsism $\alpha : \mathcal{L}^1(\mathcal{M},\tau) \to \mathcal{L}^1(\mathcal{M},\tau)$ such that $\tau \circ \alpha = \tau$ and α does not increase the norm in $\mathcal{M}$, we establish a non-commutative counterpart of the classical Wiener–Wintner ergodic theorem.

1. Introduction and preliminaries

The celebrated Wiener–Wintner theorem [12] is by far one of the most deep and fruitful results of the classical ergodic theory. It may be stated as follows.

THEOREM 1.1. *Let (Ω,μ) be a probability space, and let $T:\Omega\to\Omega$ be an ergodic measure preserving transformation. Then for any function $f \in L^1(\Omega,\mu)$ there exists a set Ω_f of full measure such that, given $\omega \in \Omega_f$, the averages*

$$a_n(f,\lambda)(\omega) = \frac{1}{n}\sum_{k=0}^{n-1} \lambda^k f\big(T^k\omega\big)$$

converge for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

The aim of this article is to establish a non-commutative extension of Theorem 1.1. We follow the path of "simple inequality" as it is outlined in [1]. This means that our argument relies on a non-commutative Van der Corput's inequality. Note that such an inequality was established in [10].

Let H be a Hilbert space, $B(H)$ the algebra of all bounded linear operators in H, $\|\cdot\|_\infty$ the uniform norm in $B(H)$, $\mathbb{I}$ the unit of $B(H)$. Let $\mathcal{M} \subset B(H)$ be a semifinite von Neumann algebra with a faithful normal semifinite trace τ. We denote by $P(\mathcal{M})$ the complete lattice of all projections in $\mathcal{M}$ and set $e^\perp = \mathbb{I} - e$ whenever $e \in P(\mathcal{M})$.

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A densely-defined closed operator x in H is said to be *affiliated* with the algebra $\mathcal{M}$ if $x'x \subset xx'$ for every $x' \in B(H)$ such that $x'x = xx'$ for all $x \in \mathcal{M}$. An operator x affiliated with $\mathcal{M}$ is called *τ-measurable* if for each $\varepsilon > 0$ there exists such $e \in P(\mathcal{M})$ with $\tau(e^{\perp}) \leq \varepsilon$ that the subspace eH belongs to the domain of x. (In this case $xe \in \mathcal{M}$.) Let $\mathcal{L} = \mathcal{L}(\mathcal{M}, \tau)$ be the set of all τ-measurable operators affiliated with the algebra $\mathcal{M}$. The topology t_τ defined in $\mathcal{L}$ by the family

$$V(\varepsilon, \delta) = \big\{x \in \mathcal{L} : \|xe\|_\infty \leq \delta \text{ for some } e \in P(\mathcal{M}) \text{ with } \tau\big(e^{\perp}\big) \leq \varepsilon\big\};$$
$$\varepsilon > 0, \delta > 0$$

of (closed) neighborhoods of zero is called a *measure topology.*

THEOREM 1.2 ([11]; see also [9]). *$(\mathcal{L}, t_\tau)$ is a complete metrizable topological $*$-algebra.*

For a positive self-adjoint operator $x = \int_0^\infty \lambda \, de_\lambda$ affiliated with $\mathcal{M}$, one can define

$$\tau(x) = \sup_n \tau\left(\int_0^n \lambda \, de_\lambda\right) = \int_0^\infty \lambda \, d\tau(e_\lambda).$$

If $1 \leq p < \infty$, then the non-commutative *L^p-space* associated with $(\mathcal{M}, \tau)$ is defined as

$$\mathcal{L}^p = \mathcal{L}^p(\mathcal{M}, \tau) = \big\{x \in \mathcal{L} : \|x\|_p = \big(\tau\big(|x|^p\big)\big)^{1/p} < \infty\big\},$$

where $|x| = (x^*x)^{1/2}$, the absolute value of x. Naturally, $\mathcal{L}^\infty = \mathcal{M}$.

Let $\alpha : \mathcal{L}^1 \to \mathcal{L}^1$ be a positive linear map such that $\alpha(x) \leq \mathbb{I}$ and $\tau(\alpha(x)) \leq \tau(x)$ for every $x \in \mathcal{L}^1 \cap \mathcal{M}$ with $0 \leq x \leq \mathbb{I}$ (see [13]). Note that, as it is shown in [3, Proposition 1.1], such an α can be uniquely extended to a positive linear contraction on $\mathcal{M}$. Therefore, α satisfies the hypotheses of [5, Lemma 1.1] and as such can be uniquely extended to a positive linear contraction on $\mathcal{L}^p$, $1 \leq p < \infty$.

Let $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$. If $1 \leq p \leq \infty$, $x \in \mathcal{L}^p$, $\lambda \in \mathbb{C}_1$, we denote

$$a_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(x), \tag{1}$$

$$a_n(x, \lambda) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k \alpha^k(x). \tag{2}$$

There are several generally distinct types of "pointwise" (or "individual") convergence in $\mathcal{L}$ each of which, in the commutative case with finite measure, reduces to the almost everywhere convergence. We deal with the so-called *almost uniform* (a.u.) and *bilateral almost uniform* (b.a.u.) convergences for which $x_n \to x$ a.u. (b.a.u.) means that for every $\varepsilon > 0$ there exists such $e \in P(\mathcal{M})$ that $\tau(e^{\perp}) \leq \varepsilon$ and $\|(x - x_n)e\|_\infty \to 0$ ($\|e(x - x_n)e\|_\infty \to 0$, respectively). Clearly, a.u. convergence implies b.a.u. convergence.

In [13] the following non-commutative ergodic theorem was established.

THEOREM 1.3. *For every* $x \in \mathcal{L}^1$, *the ergodic averages* (1) *converge b.a.u. to some* $\widehat{x} \in \mathcal{L}^1$.

REMARK 1.1. B.a.u. convergence of the averages (2) for $x \in \mathcal{L}^1$ ($\mathcal{L}^p$, $1 < p < \infty$) and a *fixed* $\lambda \in \mathbb{C}_1$ was proved in [4] ([3], respectively).

2. Non-commutative Wiener–Wintner property

Now we turn our attention to a study of the "simultaneous" on $\mathbb{C}_1$ individual convergence of the averages (2). We begin with the following definition [7].

DEFINITION 2.1. Let $(X, \|\cdot\|)$ be a normed space. A sequence $a_n : X \to \mathcal{L}$ of additive maps is called *bilaterally uniformly equicontinuous in measure (b.u.e.m.) at* $0 \in X$ if for every $\varepsilon > 0$, $\delta > 0$ there exists $\gamma > 0$ such that for every $x \in X$ with $\|x\| < \gamma$ there is $e_x \in P(\mathcal{M})$ for which

$$\tau\big(e_x^{\perp}\big) \le \varepsilon \quad \text{and} \quad \sup_n \big\|e_x a_n(x) e_x\big\|_\infty \le \delta.$$

A proof of the next fact can be found in [7].

PROPOSITION 2.1. *For any* $1 \le p < \infty$, *the sequence* $\{a_n\}$ *given by* (1) *is b.u.e.m. at* $0 \in \mathcal{L}^p$.

LEMMA 2.1. *If* $1 \le p < \infty$, *then, given* $\varepsilon > 0$, $\delta > 0$, *there exists* $\gamma > 0$ *such that for every* $x \in \mathcal{L}^p$ *with* $\|x\|_p \le \gamma$ *there is* $e \in P(\mathcal{M})$ *satisfying*

$$\tau\big(e^{\perp}\big) \le \varepsilon \quad \textit{and} \quad \sup_n \big\|e a_n(x,\lambda) e\big\|_\infty \le \delta \quad \textit{for all } \lambda \in \mathbb{C}_1.$$

Proof. Fix $\varepsilon > 0$, $\delta > 0$. By Proposition 2.1, there exists $\gamma > 0$ such that for each $\|x\|_p < \gamma$ it is possible to find $e \in P(\mathcal{M})$ such that

$$\tau\big(e^{\perp}\big) \le \frac{\varepsilon}{4} \quad \text{and} \quad \sup_n \big\|e a_n(x) e\big\|_\infty \le \frac{\delta}{24}.$$

Fix $x \in \mathcal{L}^p$ with $\|x\|_p < \gamma$. We have $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_j \in \mathcal{L}^p_+$ and $\|x_j\|_p \le \|x\|_p$ for each $j = 1, 2, 3, 4$.

If $1 \le j \le 4$, then $\|x_j\|_p < \gamma$, so there is $e_j \in P(\mathcal{M})$ satisfying

$$\tau\big(e_j^{\perp}\big) \le \frac{\varepsilon}{4} \quad \text{and} \quad \sup_n \big\|e_j a_n(x_j) e_j\big\|_\infty \le \frac{\delta}{24}.$$

Let $e = \bigwedge_{j=1}^4 e_j$. Then we have

$$\tau\big(e^{\perp}\big) \le \varepsilon \quad \text{and} \quad \sup_n \big\|e a_n(x_j) e\big\|_\infty \le \frac{\delta}{24}, \quad j = 1, 2, 3, 4.$$

Now, fix $\lambda \in \mathbb{C}_1$. For $1 \le j \le 4$ denote

$$a_n^{(R)}(x_j,\lambda) = \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{Re}\big(\lambda^k\big)\alpha^k(x_j) + a_n(x_j) = \frac{1}{n}\sum_{k=0}^{n-1}\big(\mathrm{Re}\big(\lambda^k\big)+1\big)\alpha^k(x_j),$$

$$a_n^{(I)}(x_j,\lambda) = \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{Im}\big(\lambda^k\big)\alpha^k(x_j) + a_n(x_j) = \frac{1}{n}\sum_{k=0}^{n-1}\big(\mathrm{Im}\big(\lambda^k\big)+1\big)\alpha^k(x_j).$$

Then $0 \le \mathrm{Re}(\lambda^k)+1 \le 2$ and $\alpha^k(x_j) \ge 0$ for every k entail

$$0 \le e a_n^{(R)}(x_j,\lambda)e \le 2e a_n(x_j)e \quad \text{for all } n.$$

Therefore

$$\sup_n \big\| e a_n^{(R)}(x_j,\lambda)e\big\|_\infty \le \frac{\delta}{12}$$

and, similarly,

$$\sup_n \big\| e a_n^{(I)}(x_j,\lambda)e\big\|_\infty \le \frac{\delta}{12}.$$

This implies that, given $1 \le j \le 4$, we have

$$\begin{aligned}\sup_n &\big\| e a_n(x_j,\lambda)e\big\|_\infty \\ &= \sup_n \big\| e\big(a_n^{(R)}(x_j,\lambda) + i a_n^{(I)}(x_j,\lambda) - a_n(x_j) - i a_n(x_j)\big)e\big\|_\infty \le \frac{\delta}{4},\end{aligned}$$

and we conclude that

$$\begin{aligned}\sup_n &\big\| e a_n(x,\lambda)e\big\|_\infty \\ &= \sup_n \big\| e\big(a_n(x_1,\lambda) - a_n(x_2,\lambda) + i a_n(x_3,\lambda) - i a_n(x_4,\lambda)\big)e\big\|_\infty \le \delta\end{aligned}$$

for every $\lambda \in \mathbb{C}_1$. □

Definition 2.2. Let $1 \le p < \infty$. We say that $x \in \mathcal{L}^p$ satisfies *Wiener–Wintner (bilaterally Wiener–Wintner) property* and we write $x \in WW$ ($x \in bWW$, respectively) if, given $\varepsilon > 0$, there exists a projection $e \in P(\mathcal{M})$ with $\tau(e^\perp) \le \varepsilon$ such that the sequence

$$\big\{a_n(x,\lambda)e\big\} \quad \big(\big\{e a_n(x,\lambda)e\big\}, \text{ respectively}\big) \text{ converges in } \mathcal{M} \text{ for all } \lambda \in \mathbb{C}_1.$$

Note that $WW \subset bWW$, while in the commutative case these sets coinside.

Let (Ω,μ) be a probability space, and let $T : \Omega \to \Omega$ be a measure preserving transformation. Then $f \in L^1(\Omega,\mu) \cap WW$ would imply that for every $m \in \mathbb{N}$ there exists Ω_m with $\mu(\Omega \setminus \Omega_m) \le \frac{1}{m}$ such that the averages $a_n(f,\lambda)(\omega) = \frac{1}{n}\sum_{k=0}^{n-1}\lambda^k f(T^k\omega)$ converge for all $\omega \in \Omega_m$ and $\lambda \in \mathbb{C}_1$. Then, with $\Omega_f = \bigcup_{m=1}^\infty \Omega_m$, we have $\mu(\Omega_f) = 1$, while the averages $a_n(f,\lambda)(\omega)$ converge for all $\omega \in \Omega_f$ and $\lambda \in \mathbb{C}_1$.

Therefore Definition 2.2 presents a proper generalization of the classical Wiener–Wintner property; see [1, p. 28]. In an attempt to clarify what happens in the non-commutative situation without imposing any additional conditions on τ and α, we suggest the following.

PROPOSITION 2.2. *Let* $1 \le p < \infty$ *and* $x \in \mathcal{L}^p \cap WW$ $(x \in \mathcal{L}^p \cap bWW)$. *Then*

(1) *for every* $\lambda \in \mathbb{C}_1$ *there is such* $x_\lambda \in \mathcal{L}^p$ *that*

$$a_n(x,\lambda) \to x_\lambda \quad \textit{a.u.} \ \big(a_n(x,\lambda) \to x_\lambda \ \textit{b.a.u., respectively}\big),$$

(2) *if* $e \in P(\mathcal{M})$ *is such that* $\{a_n(x,\lambda)e\}$ $(\{ea_n(x,\lambda)e\})$ *converges in* $\mathcal{M}$ *for all* $\lambda \in \mathbb{C}_1$, *then, given* $\lambda \in \mathbb{C}_1$ *and* $\nu > 0$, *there is a projection* $e_\lambda \in P(\mathcal{M})$ *such that* $e_\lambda \le p$, $\tau(e - e_\lambda) \le \nu$, *and*

$$\big\|\big(a_n(x,\lambda) - x_\lambda\big)e_\lambda\big\|_\infty \to 0 \quad \big(\big\|e_\lambda\big(a_n(x,\lambda) - x_\lambda\big)e_\lambda\big\|_\infty \to 0, \ \textit{respectively}\big).$$

Proof. We will provide a proof for the b.a.u. convergence. Same argument is applicable in the case of a.u. convergence.

(1) Let $x \in \mathcal{L}^p \cap bWW$ and $\lambda \in \mathbb{C}_1$. Then for every $\varepsilon > 0$ there exists $e \in P(\mathcal{M})$ with $\tau(e^\perp) \le \varepsilon$ for which

$$\big\|e\big(a_m(x,\lambda) - a_n(x,\lambda)\big)e\big\|_\infty \to 0 \quad \text{as } m,n \to \infty.$$

Then, as it is noticed in [2, Proposition 1.3], $a_n(x,\lambda) \to x_\lambda$ b.a.u. for some $x_\lambda \in \mathcal{L}$, which clearly implies that $a_n(x,\lambda) \to x_\lambda$ *bilaterally in measure*, meaning that, given $\varepsilon > 0$, $\delta > 0$, there exists $N \in \mathbb{N}$ such that for every $n \ge \mathbb{N}$ there is $e_n \in P(\mathcal{M})$ with $\tau(e_n^\perp) \le \varepsilon$ satisfying $\|e_n(a_n(x,\lambda) - x_\lambda)e_n\|_\infty \le \delta$. Since the measure topology coincides with the bilateral measure topology on $\mathcal{L}$ (see [4, Theorem 2.2]), $a_n(x,\lambda) \to x_\lambda$ in measure. Then, as $\|a_n(x,\lambda)\|_p \le \|x\|_p$ for all n, [4, Theorem 1.2] implies that $x_\lambda \in \mathcal{L}^p$.

(2) Let $e \in P(\mathcal{M})$ be such that the sequence $\{ea_n(x,\lambda)e\}$ converges in $\mathcal{M}$ for all $\lambda \in \mathbb{C}_1$. By part (1), given $\lambda \in \mathbb{C}_1$ and $\nu > 0$, there is $f_\lambda \in P(\mathcal{M})$ with $\tau(f_\lambda^\perp) \le \nu$ such that $\|f_\lambda(a_n(x,\lambda) - x_\lambda)f_\lambda\|_\infty \to 0$ as $n \to \infty$. Then $e_\lambda = e \wedge f_\lambda$ satisfies the required conditions. □

REMARK 2.1. It is desirable to have the following: if $x \in WW$ ($x \in bWW$), then, given $\varepsilon > 0$, there exists such $e \in P(\mathcal{M})$ with $\tau(e^\perp) \le \varepsilon$ that $\|(a_n(x,\lambda) - x_\lambda)e\|_\infty \to 0$ ($\|e(a_n(x,\lambda) - x_\lambda)e\|_\infty \to 0$, respectively) for all $\lambda \in \mathbb{C}_1$; see Remark 5.1 below.

THEOREM 2.1. *For each* $1 \le p < \infty$ *the set* $X = \mathcal{L}^p \cap bWW$ *is closed in* $\mathcal{L}^p$.

Proof. Take x in the $\|\cdot\|_p$-closure of X and fix $\varepsilon > 0$. By Lemma 2.1, one can find sequences $\{x_m\} \subset X$ and $\{f_m\} \subset P(\mathcal{M})$ in such a way that

$$\tau\big(f_m^\perp\big) \le \frac{\varepsilon}{3 \cdot 2^m} \quad \text{and} \quad \sup_n \big\|f_m a_n(x - x_m, \lambda) f_m\big\|_\infty \le \frac{1}{m}$$

for all $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}_1$. If we let $f = \bigwedge_{m=1}^{\infty} f_m$, then

$$\tau\big(f^{\perp}\big) \le \frac{\varepsilon}{3} \quad \text{and} \quad \sup_n \big\| f a_n(x - x_m, \lambda) f \big\|_{\infty} \le \frac{1}{m},$$

$m \in \mathbb{N}$, $\lambda \in \mathbb{C}_1$. Also, since $\{x_m\} \subset bWW$, one can construct $g \in P(\mathcal{M})$ such that

$$\tau\big(g^{\perp}\big) \le \frac{\varepsilon}{3} \quad \text{and} \quad \big\{g a_n(x_m, \lambda) g\big\} \quad \text{converges in } \mathcal{M} \text{ for all } m \in \mathbb{N}, \lambda \in \mathbb{C}_1.$$

Next, there exists $h \in P(\mathcal{M})$ with $\tau(h^{\perp}) \le \frac{\varepsilon}{3}$ for which $\{\alpha^k(x)h\}_{k=0}^{\infty} \subset \mathcal{M}$ so that $\{h a_n(x, \lambda) h\} \subset \mathcal{M}$ for all $\lambda \in \mathbb{C}_1$. Now, if $e = f \wedge g \wedge h$, then we have $\tau(e^{\perp}) \le \varepsilon$,

$$\sup_n \big\| e a_n(x - x_m, \lambda) e \big\|_{\infty} \le \frac{1}{m},$$

$$\big\{e a_n(x_m, \lambda) e\big\} \quad \text{converges in } \mathcal{M}, \quad \text{and} \quad \big\{e a_n(x, \lambda) e\big\} \subset \mathcal{M}$$

for all $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}_1$.

It remains to show that, for a fixed $\lambda \in \mathbb{C}_1$, the sequence $\{e a_n(x, \lambda) e\}$ converges in $\mathcal{M}$. So, fix $\delta > 0$ and pick m_0 such that $\frac{1}{m_0} \le \frac{\delta}{3}$. Since the sequence $\{e a_n(x_{m_0}, \lambda) e\}$ converges in $\mathcal{M}$, there exists N such that

$$\big\| e\big(a_{n_1}(x_{m_0}, \lambda) - a_{n_2}(x_{m_0}, \lambda)\big) e \big\|_{\infty} \le \frac{\delta}{3}$$

whenever $n_1, n_2 \ge N$. Therefore, given $n_1, n_2 \ge N$, we can write

$$\begin{aligned}
&\big\| e\big(a_{n_1}(x, \lambda) - a_{n_2}(x, \lambda)\big) e \big\|_{\infty} \\
&\quad \le \big\| e a_{n_1}(x - x_{m_0}, \lambda) e \big\|_{\infty} + \big\| e a_{n_2}(x - x_{m_0}, \lambda) e \big\|_{\infty} \\
&\qquad + \big\| e\big(a_{n_1}(x_{m_0}, \lambda) - a_{n_2}(x_{m_0}, \lambda)\big) e \big\|_{\infty} \\
&\quad \le \delta.
\end{aligned}$$

This implies that the sequence $\{e a_n(x, \lambda) e\}$ converges in $\mathcal{M}$ for all $\lambda \in \mathbb{C}_1$, hence $x \in X$ and X is closed in $\mathcal{L}^p$. □

Let $\mathcal{K}$ be the $\|\cdot\|_2$-closure of the linear span of the set

$$E = \big\{x \in \mathcal{L}^2 : \alpha(x) = \mu x \text{ for some } \mu \in \mathbb{C}_1\big\}. \tag{3}$$

PROPOSITION 2.3. $\mathcal{K} \subset bWW$.

Proof. By Theorem 2.1, it is sufficient to show that $\sum_{j=1}^{m} a_j x_j \in bWW$ whenever $a_j \in \mathbb{C}$ and $x_j \in E$, $1 \le j \le m$. For this, one will verify that $E \subset WW$.

If $x \in E$, then $\alpha(x) = \mu x$, $\mu \in \mathbb{C}_1$. Fix $\varepsilon > 0$ and find $e \in P(\mathcal{M})$ with $\tau(e^{\perp}) \le \varepsilon$ such that $xe \in \mathcal{M}$. Then, given $\lambda \in \mathbb{C}_1$, we have

$$a_n(x, \lambda) = xe \frac{1}{n} \sum_{k=0}^{n-1} (\lambda\mu)^k.$$

Therefore, since the averages $\frac{1}{n}\sum_{k=0}^{n-1}(\lambda\mu)^k$ converge, we conclude that the sequence $\{a_n(x,\lambda)e\}$ converges in $\mathcal{M}$, whence $x \in WW$. □

3. Spectral characterization of $\mathcal{K}^\perp$

The space $\mathcal{L}^2$ equipped with the inner product $(x,y)_\tau = \tau(x^*y)$ is a Hilbert space such that $\|x\|_2 = \|x\|_\tau = (x,x)_\tau^{1/2}$, $x \in \mathcal{L}^2$.

From now on we shall assume that τ and α satisfy the following additional conditions: τ is a state, α is a homomorphism, and $\tau \circ \alpha = \tau$. Notice that then $\|\alpha(x)\|_2 = \|x\|_2$ and $|\tau(x)| \le \|x\|_2$ for every $x \in \mathcal{L}^2$.

Proposition 3.1. *If $x \in \mathcal{L}^2$, then the sequence $\{\gamma_x(l)\}_{-\infty}^{\infty}$ given by*

$$\gamma_x(l) = \begin{cases} \tau(x^*\alpha^l(x)), & \text{if } l \ge 0,\\ \overline{\tau(x^*\alpha^{-l}(x))}, & \text{if } l < 0 \end{cases}$$

is positive definite.

Proof. If $\mu_0,\dots,\mu_m \in \mathbb{C}$, then, taking into account that positivity of α implies that $\alpha(y)^* = \alpha(y^*)$, $y \in \mathcal{L}^2$, we have

$$0 \le \left\|\sum_{k=0}^m \mu_k\alpha^k(x)\right\|_2^2 = \left(\sum_{j=0}^m \mu_j\alpha^j(x), \sum_{i=0}^m \mu_i\alpha^i(x)\right)_\tau$$
$$= \sum_{i,j=0}^m \mu_i\bar{\mu}_j\tau\big(\alpha^j(x^*)\alpha^i(x)\big).$$

If $i \ge j$, we can write

$$\tau\big(\alpha^j(x^*)\alpha^i(x)\big) = \tau\big(\alpha^j\big(x^*\alpha^{i-j}(x)\big)\big) = \tau\big(x^*\alpha^{i-j}(x)\big) = \gamma_x(i-j),$$

and if $i < j$, we have

$$\tau\big(\alpha^j(x^*)\alpha^i(x)\big) = \overline{\tau\big(\alpha^i(x^*)\alpha^j(x)\big)} = \overline{\tau\big(x^*\alpha^{j-i}(x)\big)} = \gamma_x(i-j).$$

Therefore

$$\sum_{i,j=0}^m \gamma_x(i-j)\mu_i\bar{\mu}_j \ge 0$$

for any $\mu_0,\dots,\mu_m \in \mathbb{C}$, hence $\{\gamma_x(l)\}$ is positive definite. □

Consequently, given $x \in \mathcal{L}^2$, by Herglotz–Bochner theorem, there exists a positive finite Borel measure σ_x on $\mathbb{C}_1$ such that

$$\tau\big(x^*\alpha^l(x)\big) = \gamma_x(l) = \widehat{\sigma}_x(l) = \int_{\mathbb{C}_1} e^{2\pi i l t}\, d\sigma_x(t), \quad l = 1,2,\dots. \tag{4}$$

Lemma 3.1. $\alpha(\mathcal{K}^\perp) \subset \mathcal{K}^\perp$.

Proof. Since $\alpha : \mathcal{L}^2 \to \mathcal{L}^2$ is an isometry, we have $\|\alpha\| = 1$. Therefore, $\|\alpha^*\| = 1$ as well, so that $\|\alpha^*(x)\|_2 \le \|x\|_2$, $x \in \mathcal{L}^2$.

Let $x \in E$, that is, $x \in \mathcal{L}^2$ and $\alpha(x) = \mu x$ for some $\mu \in \mathbb{C}_1$. Then we have

$$\big\|\alpha^*(x) - \bar{\mu}x\big\|_2^2 = \big\|\alpha^*(x)\big\|_2^2 - \bar{\mu}\big(\alpha^*(x), x\big)_\tau - \mu\big(x, \alpha^*(x)\big)_\tau + \|x\|_2^2 \le 0,$$

and it follows that $\alpha^*(x) = \bar{\mu}x$.

Now, if $y \in \mathcal{K}^\perp$, then $(\alpha(y), x)_\tau = (y, \alpha^*(x))_\tau = \bar{\mu}(y, x)_\tau = 0$, which implies that $\alpha(y) \perp E$, hence $\alpha(y) \in \mathcal{K}^\perp$. $\square$

PROPOSITION 3.2. *If $x \in \mathcal{K}^\perp$, then the measure σ_x is continuous.*

Proof. We need to show that $\sigma_x(\{t\}) = 0$ for every $t \in \mathbb{C}_1$. It is known [6, p. 42] that

$$\sigma_x\big(\{t\}\big) = \lim_{n\to\infty} \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l t}\widehat{\sigma}_x(t),$$

which is equal to

$$\lim_{n\to\infty} \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l t}\tau\big(x^*\alpha^l(x)\big) = \lim_{n\to\infty} \tau\left(x^*\left(\frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l t}\alpha^l(x)\right)\right).$$

Therefore, it is sufficient to verify that

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l t}\alpha^l(x)\right\|_2 = 0. \tag{5}$$

By the Mean Ergodic theorem applied to $\widetilde{\alpha} : \mathcal{L}^2 \to \mathcal{L}^2$ given by $\widetilde{\alpha}(x) = e^{2\pi i t}\alpha(x)$, we conclude that

$$\frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l t}\alpha^l(x) \to \bar{x} \quad \text{in } \mathcal{L}^2.$$

Since $x \in \mathcal{K}^\perp$, by Lemma 3.1, we have $\alpha^l(x) \in \mathcal{K}^\perp$ for each l, which implies that $\bar{x} \in \mathcal{K}^\perp$. Besides,

$$\alpha(\bar{x}) = \|\cdot\|_2 - \lim_{n\to\infty} \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l t}\alpha^{l+1}(x) = e^{-2\pi i t}\bar{x},$$

so that $\bar{x} \in \mathcal{K}$. Therefore $\bar{x} = 0$, and (5) follows. $\square$

4. Non-commutative Van der Corput's inequality

It was shown in [10] that the extremely useful Van der Corput's "Fundamental Inequality" (see [1]) can be fully extended to any $*$-algebra:

THEOREM 4.1 ([10]). *If $n \geq 1$, $0 \leq m \leq n-1$ are integers and $a_0, \dots, a_{n-1+m}$ are elements of a $*$-algebra such that $a_n = \dots = a_{n-1+m} = 0$, then*

$$\left(\sum_{k=0}^{n-1} a_k^*\right)\left(\sum_{k=0}^{n-1} a_k\right) \leq \frac{n-1+m}{m+1}\sum_{k=0}^{n-1} a_k^* a_k + \frac{2(n-1+m)}{m+1}\sum_{l=1}^{m}\frac{m-l+1}{m+1}\operatorname{Re}\sum_{k=0}^{n-1} a_k^* a_{k+l}.$$

COROLLARY 4.1. *If in Theorem 4.1, $a_0, \dots, a_{n-1+m}$ are elements of a C^*-algebra with the norm $\|\cdot\|$, then*

$$\left\|\sum_{k=0}^{n-1} a_k\right\|^2 \leq \frac{n-1+m}{m+1}\left\|\sum_{k=0}^{n-1} a_k^* a_k\right\| + \frac{2(n-1+m)}{m+1}\sum_{l=1}^{m}\frac{m-l+1}{m+1}\left\|\sum_{k=0}^{n-1} a_k^* a_{k+l}\right\|,$$

which implies that

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} a_k\right\|^2 \leq \frac{n-1+m}{(m+1)n}\left\|\frac{1}{n}\sum_{k=0}^{n-1} a_k^* a_k\right\| + \frac{2(n-1+m)}{(m+1)n}\sum_{l=1}^{m}\frac{m-l+1}{m+1}\left\|\frac{1}{n}\sum_{k=0}^{n-1} a_k^* a_{k+l}\right\|,$$

and further

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} a_k\right\|^2 < \frac{2}{m+1}\left\|\frac{1}{n}\sum_{k=0}^{n-1} a_k^* a_k\right\| + \frac{4}{m+1}\sum_{l=1}^{m}\left\|\frac{1}{n}\sum_{k=0}^{n-1} a_k^* a_{k+l}\right\|. \tag{6}$$

5. Proof of the main result

We will assume now that α is ergodic on $\mathcal{L}^2$, that is, $\alpha(x) = x$, $x \in \mathcal{L}^2$, implies that $x = c \cdot \mathbb{I}$, $c \in \mathbb{C}$.

PROPOSITION 5.1. *If $x \in \mathcal{L}^2$, then $a_n(x) \to \tau(x) \cdot \mathbb{I}$ a.u.*

Proof. By the Mean Ergodic theorem, $a_n(x) \to \bar{x}$ in $\mathcal{L}^2$. Therefore, $\alpha(a_n(x)) \to \alpha(\bar{x})$ in $\mathcal{L}^2$, so $\alpha(\bar{x}) = \bar{x}$, and the ergodicity of α implies that $\bar{x} = c(x) \cdot \mathbb{I}$. Then, since τ is also continuous in $\mathcal{L}^2$, we have $\tau(a_n(x)) \to \tau(\bar{x}) = c(x)$, hence $c(x) = \tau(x)$ because $\tau(a_n(x)) = \tau(x)$ for each n. It is known ([5], [7]) that $a_n(x) \to \widehat{x} \in \mathcal{L}^2$ a.u., which implies that $a_n(x) \to \widehat{x}$ in measure. Since $\|\cdot\|_2$-convergence entails convergence in measure, we conclude that $\widehat{x} = \bar{x} = \tau(x) \cdot \mathbb{I}$. $\square$

LEMMA 5.1. *If $a, b \in \mathcal{L}$ and $e \in P(\mathcal{M})$ are such that $ae, be \in \mathcal{M}$, then*

$$(ae)^* be = e a^* b e.$$

Proof. We have

$$\big((ae)^*be\big)^* = (be)^*ae \subset (be)^*\big(ea^*\big)^* \subset \big(ea^*be\big)^*,$$

which, since $((ae)^*be)^* \in B(H)$, implies that $((ae)^*be)^* = (ea^*be)^*$, hence the required equality. $\square$

Now we can prove our main result, a non-commutative Wiener–Wintner theorem.

THEOREM 5.1. *Let $\mathcal{M}$ be a von Neumann algebra, τ a faithful normal tracial state on $\mathcal{M}$. Let $\alpha : \mathcal{L}^1 \to \mathcal{L}^1$ be a positive ergodic homomorphism such that $\tau \circ \alpha = \tau$ and $\|\alpha(x)\|_\infty \le \|x\|_\infty$, $x \in \mathcal{M}$. Then $\mathcal{L}^1 = bWW$, that is, for every $x \in \mathcal{L}^1$ and $\varepsilon > 0$ there exists such a projection $e \in P(\mathcal{M})$ that*

$$\tau\big(e^\perp\big) \le \varepsilon \quad \textit{and} \quad \big\{ea_n(x,\lambda)e\big\} \quad \textit{converges in } \mathcal{M} \textit{ for all } \lambda \in \mathbb{C}_1.$$

Proof. Since $\mathcal{L}^2$ is dense in $\mathcal{L}^1$, $\mathcal{L}^2 = \mathcal{K} \oplus \mathcal{K}^\perp$, and $\mathcal{K} \subset bWW$ (Proposition 2.3), by Theorem 2.1, it remains to show that $\mathcal{K}^\perp \subset bWW$. (In fact, we will show that $\mathcal{K}^\perp \subset WW$.)

So, let $x \in \mathcal{K}^\perp$ and fix $\varepsilon > 0$. Since $\{x^*\alpha^l(x)\}_{l=0}^\infty \subset \mathcal{L}^2$, due to Proposition 5.1, one can construct a projection $e \in P(\mathcal{M})$ in such a way that

$$\begin{gathered}\tau\big(e^\perp\big) \le \varepsilon, \qquad \big\{\alpha^k(x)e\big\} \subset \mathcal{M} \quad \text{for all } k,\\ ea_n\big(x^*x\big)e \to \tau\big(x^*x\big)e = \|x\|_2 e \quad \text{in } \mathcal{M}, \quad \text{and}\\ ea_n\big(x^*\alpha^l(x)\big)e \to \tau\big(x^*\alpha^l(x)\big)e = \widehat{\sigma}_x(l)e \quad \text{in } \mathcal{M} \text{ for every } l.\end{gathered}$$

Now, if $a_k = \lambda^k\alpha^k(x)e$, $k = 0, 1, 2, \ldots$, then, employing Lemma 5.1, we obtain

$$a_k^* a_{k+l} = \lambda^l e\alpha^k\big(x^*\alpha^l(x)\big)e, \quad k, l = 0, 1, 2, \ldots.$$

At this moment we apply inequality (6) to the sequence $\{a_k\} \subset \mathcal{M}$ yielding, in view of (1) and (2),

$$\sup_{\lambda\in\mathbb{C}_1} \big\|a_n(x,\lambda)e\big\|_\infty^2 \le \frac{2}{m+1}\big\|ea_n\big(x^*x\big)e\big\|_\infty + \frac{4}{m+1}\sum_{l=1}^m \big\|ea_n\big(x^*\alpha^l(x)\big)e\big\|_\infty.$$

Therefore, for a fixed m, we have

$$\limsup_n \sup_{\lambda\in\mathbb{C}_1} \big\|a_n(x,\lambda)e\big\|_\infty^2 \le \frac{2}{m+1}\|x\|_2^2 + \frac{4}{m+1}\sum_{l=1}^m \big|\widehat{\sigma}_x(l)\big|.$$

Since the measure σ_x is continuous by Proposition 3.2, Wiener's criterion of continuity of positive finite Borel measure [6, p. 42] yields

$$\lim_{m\to\infty} \frac{1}{m+1}\sum_{l=1}^m \big|\widehat{\sigma}_x(l)\big|^2 = 0,$$

which entails

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{l=1}^{m} \left|\widehat{\sigma}_x(l)\right| = 0.$$

Thus, we conclude that

$$\lim_{n\to\infty} \sup_{\lambda\in\mathbb{C}_1} \left\|a_n(x,\lambda)e\right\|_\infty = 0, \tag{7}$$

whence $x \in WW$. □

Note that (7) can be referred to as non-commutative Bourgain's uniform Wiener–Wintner ergodic theorem.

REMARK 5.1. As we have noticed (Proposition 2.2), for a fixed $\lambda \in \mathbb{C}_1$ and every $x \in \mathcal{L}^1$, the averages $a_n(x,\lambda)$ converge b.a.u. to some $x_\lambda \in \mathcal{L}^1$. It can be verified [8] that x_λ is a scalar multiple of $\mathbb{I}$. If we assume additionally that α is *weakly mixing* in $\mathcal{L}^2$, that is, 1 is its only eigenvalue there, then it is easy to see that the b.a.u. limit of $\{a_n(x,\lambda)\}$ with $x \in \mathcal{L}^2$ is zero unless $\lambda = 1$. Since $\mathcal{L}^2$ is dense in $\mathcal{L}^1$, one can employ an argument similar to that of Theorem 2.1 to show that $a_n(x,\lambda) \to 0$ b.a.u. for every $x \in \mathcal{L}^1$ if $\lambda \neq 1$. Therefore if α is weakly mixing, we can replace, in Theorem 5.1,

$$\left\{ea_n(x,\lambda)e\right\} \quad \text{converges in } \mathcal{M} \text{ for all } \lambda \in \mathbb{C}_1$$

by

$$\left\|ea_n(x,\lambda)e\right\|_\infty \to 0 \quad \text{if } \lambda \neq 0 \quad \text{and}$$
$$\left\|e\big(a_n(x) - x_1\big)e\right\|_\infty \to 0 \quad \text{for some } x_1 \in \mathcal{L}^1;$$

see Proposition 2.2 and Remark 2.1.

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