Illinois Journal of Mathematics
Volume 58, Number 3, Fall 2014, Pages 665–687
S 0019-2082

TRANSLATION COVERS AMONG TRIANGULAR BILLIARD SURFACES

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ABSTRACT. We identify all translation covers among triangular billiard surfaces. Our main tools are the holonomy field of Kenyon and Smillie and a geometric property of triangular billiard surfaces, which we call the fingerprint of a point, that is preserved under balanced translation covers.

1. Introduction

An unfolding construction, already described in [3] and furthered in [9], associates a flat surface called a *translation surface* to each rational-angled triangle; we call such a surface a *triangular billiard surface.* Informally, a compact translation surface is a finite union of polygons in the plane, with pairs of parallel edges identified in such a way as to produce a compact surface.

Triangular billiard surfaces are highly symmetric examples of translation surfaces; as such, they are of interest in the field—see Kenyon and Smillie [10] and Aurell and Itzykson [1], for example. Structure-preserving maps called *translation covers* between translation surfaces have been used by Vorobets [11], Hubert and Schmidt [6], [7], Kenyon and Smillie [10], Gutkin and Judge [5] and others, to gain information about the affine symmetry groups of the surfaces.

Beyond genus 1, translation covers between triangular billiard surfaces are rare; in fact, our main result is the following.

THEOREM 1. *Let $f: X \to Y$ be a nontrivial translation cover of triangular billiard surfaces, where X has genus greater than* 1. *Then each of X and Y is either a right triangular billiards surface or an isosceles triangular billiard surface, and f is of degree at most* 2.

Received July 12, 2012; received in final form May 30, 2015.
2010 *Mathematics Subject Classification.* Primary 57M50. Secondary 30F30, 37D50.

We give an explicit description of which X and Y are related by such covers in Lemma 3. To prove Theorem 1, we use two main tools: the holonomy field of Kenyon and Smillie [10], and what we call the *fingerprint* of a point P on a translation surface, which is an invariant which depends on the surface and a point on the surface and which changes in a natural way under certain translation covers called *balanced covers*. In particular, for a given triangular billiard surface Y, we use the holonomy field to narrow the search for possible translation coverings of Y to a finite set of covering surfaces. We then use information (gleaned from fingerprints of points) about the geometric configuration of singular points on these surfaces to identify the actual translation coverings of Y.

Our hope is that this strategy of pairing the global information of the holonomy field with the local information of the fingerprint can be applied to other questions involving translation coverings. For example, our main theorem can be seen as establishing a case of a more general conjecture (communicated to us by Eugene Gutkin), that aside from a few trivial cases, most translation coverings between polygonal billiard surfaces are induced from the situation where one of the underlying polygons tiles the other.

1.1. Outline. In Sections 2.1 and 2.2, we review the construction of triangular billiard surfaces and the basics of translation covers. In Section 2.3, we prove Lemma 3, which explicitly identifies all possible translation covers among triangular billiard surfaces. The goal of the remainder of the paper is to show that the list given in Lemma 3 is complete—this is Theorem 1. In Section 3, we discuss the holonomy field of Kenyon and Smillie. We offer a new elementary computation of the holonomy field of a given triangular billiard surface, and explain why such surfaces related by a translation cover have the same holonomy field. In Section 4, we define the fingerprint of a point and prove results about its behavior under balanced translation covers. In Section 5.1, we prove the main theorem restricted to balanced covers. After some combinatorial lemmas in Section 5.2, we complete the proof of the main theorem in Section 5.3.

2. Rational billiards and translation covers

2.1. The rational billiards construction. Let R be a polygonal region whose interior angles are rational multiples of π. Let D_{2Q} be the dihedral group of order $2Q$ generated by Euclidean reflections in the sides of R. Suppose a particle moves within this region at constant speed and with initial direction vector v, changing directions only when it reflects off the sides of R, with the angle of incidence equaling the angle of reflection. Every subsequent direction vector for the particle is of the form $\delta \cdot v$, for some element $\delta \in D_{2Q}$, where D_{2Q} acts on $\mathbb{R}^2$ via Euclidean reflections.

The rational billiards construction consists of a flat surface corresponding to this physical system. Consider the set $D_{2Q} \cdot R$ of $2Q$ copies of R transformed by the elements of D_{2Q}. For each edge e of R, we consider the corresponding element $\rho_e \in D_{2Q}$ which represents reflection across e. For each $\delta \in D_{2Q}$, we glue $\rho_e \delta \cdot R$ and $\delta \cdot R$ together along their copies of e. The result is a closed Riemann surface with flat structure induced by the tiling by $2Q$ copies of R. This construction is described by Fox and Kershner in [3].

In fact, this surface is an example of a compact *translation surface.* A compact translation surface can be defined as the result of gluing together a finite set of polygons in the plane along parallel edges in such a way that the result is a compact surface (see, e.g., [2]).

Equivalently, a translation surface can be defined as a real two-manifold. Recall that given two coordinate maps $\phi_1 : U_1 \to \mathbb{R}^2$ and $\phi_2 : U_2 \to \mathbb{R}^2$ defining homeomorphisms from open sets U_1 and U_2 of a manifold M into $\mathbb{R}^2$, the map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is called a *transition function.* A conical singularity P on a flat surface is a point such that, in the flat metric induced by the coordinate maps, the total angle ("cone angle") about P is not equal to 2π.

Definition 1. Let X be a flat surface with conical singularities. Let $\tilde{X}$ be the flat surface obtained by puncturing all singularities of X. If all transition functions of $\tilde{X}$ are translations, then X is a *translation surface.*

On a translation surface, the cone angles of conical singularities are always integer multiples of 2π. See [13] for an introduction to flat surfaces.

We focus on billiards in rational triangles. Let (a_1, a_2, a_3) be a triple of positive integers. We fix the notation $T(a_1, a_2, a_3)$ to refer to a triangle with internal angles $\frac{a_1\pi}{Q}$, $\frac{a_2\pi}{Q}$, and $\frac{a_3\pi}{Q}$, where $Q := a_1 + a_2 + a_3$ and $\gcd(a_1, a_2, a_3) = 1$. We use the notation $X(a_1, a_2, a_3)$ to refer to the translation surface arising from billiards in $T(a_1, a_2, a_3)$ via the Fox–Kershner construction. We call such a surface a *triangular billiards surface.* If the triangle is isosceles or right, we call the corresponding surface an *isosceles triangular billiards surface* or a *right triangular billiard surface.*

Definition 2. Note that the Fox–Kershner construction gives a natural "tiling by flips" of the surface by copies of T. A *billiards triangulation* is a triangulation τ of X whose triangles are the various elements of $D_{2Q} \cdot T$ described above.

Remark 1. Some data can be gained about τ via simple combinatorics. Letting $T := T(a_1, a_2, a_3)$, label the vertices of T as v_1, v_2, and v_3, where v_i corresponds to a_i. It is not hard to check that the total number of triangles in τ is $2Q$, that the number of vertices of τ corresponding to v_i is $\gcd(a_i, Q)$, and that each member of this set has a cone angle of $(\frac{a_i}{\gcd(a_i,Q)})2\pi$. A good reference for these matters is [1].

Definition 3. Let v_1, v_2, v_3 be the vertices of $T(a_1, a_2, a_3)$, and let $\pi_X : X(a_1, a_2, a_3) \to T(a_1, a_2, a_3)$ be the standard projection. A *vertex class* of $X(a_1, a_2, a_3)$ is any of the three sets $\pi_X^{-1}(v_1)$, $\pi_X^{-1}(v_2)$, or $\pi_X^{-1}(v_3)$. Note that for a given vertex class, either all the elements are singular or all are nonsingular; hence, we call a vertex class *singular* if its elements are singularities and *nonsingular* if its elements are nonsingular.

Clearly, a vertex class $\pi_X^{-1}(v_i)$ is nonsingular if and only if a_i divides Q. Furthermore, the sum of the cone angles of the elements of $\pi_X^{-1}(v_i)$ is $2a_i\pi$.

2.2. Translation covers. The natural map between translation surfaces is one which respects the translation structure:

Definition 4. A *translation cover* is a holomorphic (possibly ramified) cover of translation surfaces $f : X \to Y$ such that, for each pair of coordinate maps ϕ_X and ϕ_Y on X and Y, respectively, the map $\phi_Y \circ f \circ \phi_X^{-1}$ is a translation when ϕ_X and ϕ_Y are restricted to open sets not containing singular points. We say that f is *balanced* if f does not map singular points to nonsingular points.

The term "balanced" cover is due to Gutkin [4]. Balanced translation covers $f : X \to Y$ of translation surfaces are of particular interest because they imply an especially strong relationship between the affine symmetry groups of X and Y; in particular, these groups must share a subgroup which is of finite index in each (see [5] and [11]).

Definition 5. We say that X and Y are *translation equivalent* if there exists a degree 1 translation cover $f : X \to Y$.

The surface $X(4, 3, 3)$ is pictured at the top of Figure 1. The emphasized points labeled P_1 and P_2 form the singular vertex class corresponding to the vertex angle of $2\pi/5$ in $T(4, 3, 3)$. Each of these points has a cone angle of 4π, because 10 copies of $T(4, 3, 3)$ are developed about each point. To help visualize this, we follow Hubert and Schmidt [8] and use the dotted lines to represent slits with gluings indicated by labels a and b. The identification of parallel edges as indicated by the labeling scheme in the figure leads to two more singular points on $X(4, 3, 3)$, each of cone angle 6π. Also in Figure 1, we depict $X(3, 1, 1)$ and $X(5, 3, 2)$. These two surfaces are translation equivalent, and there exist degree two translation covers from $X(4, 3, 3)$ to each of them.

The following lemma demonstrates how we will use Remark 1 to analyze translation covers.

Lemma 1. *Suppose $f : X(a_1, a_2, a_3) \to X(b_1, b_2, b_3)$ is a translation cover of triangular billiard surfaces. Let $\pi_X : X(a_1, a_2, a_3) \to T(a_1, a_2, a_3)$ and $\pi_Y : X(b_1, b_2, b_3) \to T(b_1, b_2, b_3)$ be the canonical projections to triangles with vertices v_1, v_2, v_3 and w_1, w_2, w_3 respectively. Suppose that $P \in \pi_Y^{-1}(w_i)$,*

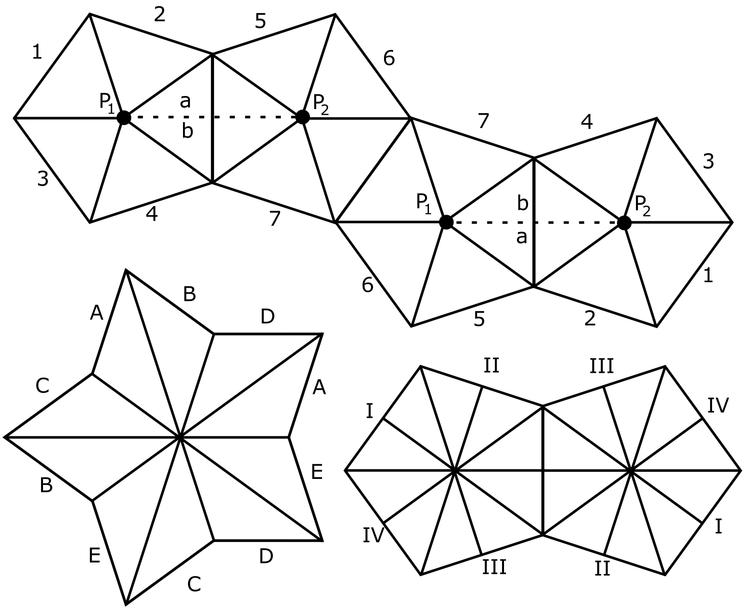

FIGURE 1. Above, the surface $X(4,3,3)$. Below, $X(3,1,1)$ (left) and $X(5,3,2)$ (right).

$P' \in \pi_X^{-1}(v_j)$, and $f(P') = P$ with a ramification index of m at P'. Then $\frac{mb_i}{\gcd(b_i, b_1+b_2+b_3)} = \frac{a_j}{\gcd(a_j, a_1+a_2+a_3)}$.

Proof. The cone angle at P' is m times the cone angle at P. Therefore, the result follows from Remark 1. $\square$

Of course, the translation structure of $X(a_1, a_2, a_3)$ depends on the chosen area and direction of $T(a_1, a_2, a_3)$. A translation surface X can represented as a pair (S, ω), where S is a Riemann surface and ω is a holomorphic 1-form on S which induces the translation structure of X. Using this language, suppose that (S, ω) is a triangular billiard surface arising from billiards in some $T(a_1, a_2, a_3)$, and that α is a nonzero complex number. The notation $X(a_1, a_2, a_3)$ does not distinguish the pairs (S, ω) and $(S, \alpha\omega)$. The following lemma shows that this ambiguity will not affect our classification of translation covers.

LEMMA 2. *Suppose that (S, ω) is a triangular billiard surface of genus greater than one, and let $\alpha \in \mathbb{C} \setminus \{0\}$. Then any translation cover $f : (S, \omega) \to (S, \alpha\omega)$ is of degree* 1.

Proof. This is a simple application of the Riemann–Hurwitz formula. Let (S, ω) have genus g, and let $\deg f = n$. The 1-form ω which gives (S, ω) its translation structure has $2g-2$ zeros (counting multiplicities). Clearly $\alpha\omega$ has the same zeros as ω. The Riemann–Hurwitz formula then gives us that

$$g = n(g-1) + 1 + \frac{R}{2}, \tag{1}$$

where R is the total ramification number of f. Since $R \geq 0$, Equation (1) is only satisfied if $n = 1$. □

2.3. The possible translation covers. Any isosceles triangle is naturally "tiled by flips" by a right triangle. The following lemma demonstrates how to use this tiling to create nontrivial translation covers in the category of triangular billiard surfaces. In fact, our main theorem is that the covers of Lemma 3 are the only nontrivial translation covers among triangular billiard surfaces.

LEMMA 3. *Let a and b be relatively prime positive integers, not both equal to one. The right triangular billiard surface $Y := X(a_1+a_2, a_1, a_2)$ is related to two isosceles triangular billiard surfaces X_1 and X_2 via translation covers $f_i : X_i \to Y$. For each i, if a_i is odd then $X_i = X(2a_j, a_i, a_i)$ and f_i has degree 2. If i is even then $X_i = X(a_j, \frac{a_i}{2}, \frac{a_i}{2})$ and f_i has degree 1.*

Proof. It suffices to prove the result for X_1 and f_1. Write $Q := 2a_1 + 2a_2$. We reflect the triangle $T = T(a_1+a_2, a_1, a_2)$ across the edge connecting the vertices corresponding to a_2 and a_1+a_2, to obtain its mirror image T'. By joining T and T' along the edge of reflection, we create an isosceles triangle $\tilde{T}$ which can be written as either $T(2a_2, a_1, a_1)$ (if a_1 is odd) or $T(a_2, \frac{a_1}{2}, \frac{a_1}{2})$ (if a_1 is even). Note that since (a_1+a_2, a_1, a_2) must be a reduced triple, a_1 and a_2 cannot both be even. It also follows that $\gcd(a_i, Q) \leq \gcd(2a_i, Q) = 2$.

Suppose a_1 is even. Consider the translation surface S (with boundary) obtained by developing T around its vertex corresponding to a_2. Since a_2 is odd we have $\gcd(a_2, Q) = 1$, so S is tiled (by reflection) by $2Q$ copies of T, and hence after appropriate identifications along the boundary we will have $X(a_1+a_2, a_1, a_2)$. Let $\tilde{S}$ be the surface obtained by developing $\tilde{T}$ around the corresponding vertex; it is tiled via reflection by Q copies of $\tilde{T}$, so appropriate boundary identifications will yield Y_1. Because $\tilde{T}$ is tiled via reflection by two copies of T, it follows that S and $\tilde{S}$ are translation equivalent. Finally, note that the boundary identifications are the same for S and $\tilde{S}$. Therefore, Y and X_1 are translation equivalent.

Now suppose that a_1 is odd and a_2 is even. We then have $\tilde{T} = T(2a_2, a_1, a_1)$. Since $\gcd(2a_2, Q) = 2$, we again have that $\tilde{S}$ is tiled by Q copies of $\tilde{T}$. Since a_2 is even, $\gcd(a_2, Q) = 2$, implying that S is tiled by Q copies of T. Thus if a_2 is even then there exists a degree two cover $f : \tilde{S} \to S$, ramified over a single point. Furthermore, in this case X_1 and Y are obtained by identifying appropriate edges of *two* copies of $\tilde{S}$ and S, respectively. It follows that if a_2 is even then there exists a ramified degree two cover $f : X_1 \to Y$.

Finally, suppose that a_1 and a_2 are both odd. We have $\tilde{T} = T(2a_2, a_1, a_1)$, $\gcd(2a_2, Q) = 2$, and $\gcd(a_2, Q) = 1$. In this case, we have that S and $\tilde{S}$ are translation equivalent surfaces; however, X_1 is obtained from two copies of $\tilde{S}$ whereas Y is obtained from a single copy of S. Thus again, we have that X_1 is a double cover of Y, this time unramified. □

REMARK 2. Note that in addition to relating right and isosceles triangles, Lemma 3 also gives a way to construct covers between isosceles triangular billiard surfaces. In the language of Lemma 3, if a_2 is even, then $f_2^{-1} \circ f_1$ is a degree two translation cover of X_2 by X_1.

REMARK 3. If we allow $a_1 = a_2 = 1$ in the statement of Lemma 3, then we arrive at $Y = X_1 = X_2 = X(1,1,2)$. This is because $T(1,1,2)$ is the unique right isosceles triangle.

Because the location of singularities is such a major tool in analyzing translation surfaces, it is worth identifying the triangular billiard surfaces which have no singularities. As detailed in [1], there are only three of these surfaces: $X(1,1,2)$, $X(1,2,3)$, and $X(1,1,1)$. These are also the only three triangular billiard surfaces of genus 1; furthermore $X(1,2,3)$ and $X(1,1,1)$ are actually translation equivalent. Each of these surfaces admits balanced translation covers of itself by itself of arbitrarily high degree; this fact is related to the fact that $T(1,1,2)$, $T(1,2,3)$, and $T(1,1,1)$ are the only Euclidean triangles which tile the Euclidean plane by flips. Note that any such cover must be unramified, since about ramification points flat ramified covers are locally of the form $z \mapsto z^{1/n}$ for some $n > 1$, implying that the cone angle of the ramification point is greater than 2π; hence ramification points are singular.

3. The holonomy field

In order to prove that Lemma 3 provides a complete list of the translation covers between triangular billiard surfaces, we require some information about the holonomy of a triangular billiard surface.

DEFINITION 6. The rational *absolute holonomy* of a translation surface X is the image of the map $\mathrm{hol} : H_1(X; \mathbb{Q}) \to \mathbb{C}$ defined by $\mathrm{hol} : \sigma \mapsto \int_\sigma \omega$, where the 1-form ω is locally the differential dz in each chart not containing a singular point.

The following definition is due to Kenyon and Smillie [10].

DEFINITION 7. The *holonomy field* of a translation surface X, denoted k_X, is the smallest field k_X such that the rational absolute holonomy of X is contained in a two-dimensional vector space over k_X.

3.1. Calculation of the holonomy field. Kenyon and Smillie [10] calculate the holonomy field of $X(a_1, a_2, a_3)$ to be $k_X = \mathbb{Q}(\cos(2\pi/Q))$. We offer a more elementary proof of this result.

REMARK 4. We let U_n denote the nth Chebyshev polynomial of the second kind. We will use the following properties of Chebyshev polynomials.

(1) $\frac{\sin((n+1)\theta)}{\sin\theta} = U_n(\cos\theta)$.
(2) If n is even, then U_n is an even polynomial of degree n. If n is odd, then U_n is an odd polynomial of degree n.

REMARK 5. Let ϕ be the Euler totient function. It is well known that, for any positive integer Q, the degree of the number field $\mathbb{Q}(\cos(2\pi/Q))$ is equal to $\frac{1}{2}\phi(Q)$. Note that if Q is odd, then $\phi(Q) = \phi(2Q)$. It follows that, when Q is odd, we will have $\mathbb{Q}(\cos(2\pi/Q)) = \mathbb{Q}(\cos(\pi/Q))$.

The following is Proposition 2.5 of [2].

LEMMA 4 (Calta–Smillie). *If a translation surface X is obtained by identifying the edges of polygons in the plane by maps which are restrictions of translations, and if all the vertices of these polygons lie in a subgroup $\Lambda \subset \mathbb{R}^2$, then the holonomy of S is contained in Λ.*

LEMMA 5. *The holonomy field of $X = X(a_1, a_2, a_3)$ is $k_X = \mathbb{Q}(\cos(2\pi/Q))$.*

Proof. Let k_X denote the holonomy field of X. Let $\alpha = \frac{\pi}{Q}$. Let $T = T(a_1, a_2, a_3)$. Since $\gcd(a_1, a_2, a_3) = 1$, we can and do assume that a_1 is odd. Label the vertices of T corresponding to the angles $a_1\alpha$, $a_2\alpha$, and $a_3\alpha$ as P_1, P_2, and P_3. We scale and rotate T so that the $\overline{P_1P_2}$ side has edge vector $v = (1, 0)$, and so that the $\overline{P_1P_3}$ side has edge vector $w = (t\cos(a_1\alpha), t\sin(a_1\alpha))$, where by the Law of Sines we have $t = \frac{\sin(a_2\alpha)}{\sin(a_3\alpha)}$. The dihedral group D generated by reflections in the sides of T acts on the set $D \cdot T$ of $2Q$ distinct oriented triangles arising from billiards in T. We can construct X from this set by identifying the appropriate edges of the elements of $D \cdot T$. We may also view D as acting on the edge vectors of T (see Figure 2). Let $v_n = (\cos(2n\alpha), \sin(2n\alpha))$ and $w_n = (t\cos((2n+1)\alpha), t\sin((2n+1)\alpha))$. With this notation, we see that $D \cdot v$ is the set $\{v = v_0, v_1, \ldots, v_{Q-1}\}$. Recalling that a_1 is odd, we also see that $D \cdot w$ is the set $\{w_0, w_1, \ldots, w_{Q-1}\}$. Note that $w = w_{(a_1-1)/2}$.

Let $L = \mathbb{Q}(\cos(2\alpha))$. We will show that all the v_n and w_n are L-linear combinations of v_0 and v_1, and that furthermore L is the smallest such field.

To see that k_X contains L, we note that v_2 is the result of rotating v_1 by an angle of 2α. Consider a vector space over k_X with ordered basis $\{v_0, v_1\}$. In this basis, counterclockwise rotation by 2α is a linear transformation R with matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 2\cos(2\alpha) \end{smallmatrix}\right)$. Hence, $\cos(2\alpha)$ is in k_X, and we have $L \subseteq k_X$.

Since each v_i and each w_i is the result of repeated application of R to v_0 or w_0, it remains to be shown that w_0 is an L-linear combination of v_0 and v_1. Then Lemma 4 gives that $k_X \subseteq L$, completing the proof.

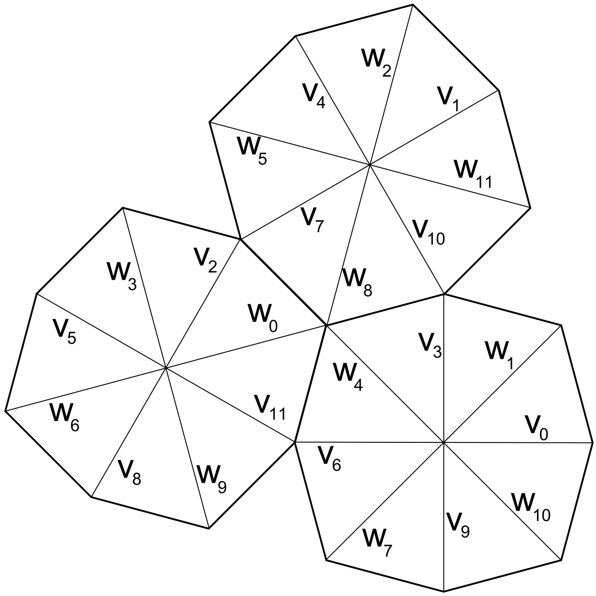

FIGURE 2. The sets $\{v_n\}$ and $\{w_n\}$ for $X(3,4,5)$, with $a_1 = 3$.

Let l and l' be the real numbers such that $lv_0 + l'v_1 = w_0$. Since v_0 and v_1 are reflections of each other across the line generated by w_0, we see that $v_0 + v_1$ is a real multiple of w_0. Hence, $l' = l$.

Projecting v_0 and v_1 onto w_0, we see that

$$(2)\quad l = \frac{\|w_0\|}{\|v_0+v_1\|} = \frac{t}{2\cos\alpha} = \frac{\sin(a_2\alpha)\sin\alpha}{\sin(a_3\alpha)\sin(2\alpha)} = \frac{\sin(a_2\alpha)}{\sin\alpha}\frac{\sin\alpha}{\sin(a_3\alpha)}\frac{\sin\alpha}{\sin(2\alpha)}.$$

Applying Remark 4 to the last expression, we get

$$l = \frac{U_{a_2-1}(\cos\alpha)}{U_{a_3-1}(\cos\alpha)U_1(\cos\alpha)}. \tag{3}$$

If Q is even, we have that $(a_2 - 1)$ and $(a_3 - 1)$ have opposite parity, and thus by our Remark 4, $\frac{U_{a_2-1}(\cos\alpha)}{U_{a_3-1}(\cos\alpha)U_1(\cos\alpha)}$ is a rational function in $\cos^2\alpha$. Therefore $l \in \mathbb{Q}(\cos^2\alpha) = L$. If Q is odd, then already by Remark 5, $\mathbb{Q}(\cos\alpha) = L$, and since $\frac{U_{a_2-1}(\cos\alpha)}{U_{a_3-1}(\cos\alpha)U_1(\cos\alpha)}$ is a rational function in $\cos\alpha$, we again have that $l \in L$.

Hence $\mathrm{span}_L\{v_0, v_1\} = \Lambda$. Theorem 4 says that Λ contains the absolute holonomy of S. So L contains k_X, which completes the proof. $\square$

COROLLARY 1. *If $f : X \to Y$ is a translation cover of triangular billiard surfaces, then $k(X) = k(Y) = \mathbb{Q}(\cos(2\pi/Q))$.*

Proof. Since f is a translation cover, the billiards triangulation τ of Y by T_Y lifts to a triangulation of X by T_Y, in which each edge is mapped by f to an edge of τ. Hence, we may take the subgroup Λ referred to in Lemma 4 to be the same group for both X and Y. Then our calculation in Lemma 5 will be the same for X and Y. $\square$

3.2. Some elementary number theory. Since $k_X = \mathbb{Q}(\cos(2\pi/Q))$, we would like to know when distinct values of Q yield the same field k_X.

Letting ζ_Q denote a primitive Qth root of unity, we have that $\mathbb{Q}(\cos(2\pi/Q))$ is equal to $\mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$, which is a degree two subfield of the cyclotomic field $\mathbb{Q}(\zeta_Q)$, since it is the maximal subfield fixed by complex conjugation. In light of this, we list some classical results about these two fields as recorded in Washington's text [12].

Lemma 6. *If Q is odd, then $\mathbb{Q}(\zeta_Q) = \mathbb{Q}(\zeta_{2Q})$.*

Lemma 7 (Proposition 2.3 in [12]). *Assume that $Q \not\equiv 2 \bmod 4$. A prime p ramifies in $\mathbb{Q}(\zeta_Q)$ if and only if $p|Q$.*

Lemma 8 (Proposition 2.15 in [12]). *Let p be a prime, and assume that $Q \not\equiv 2 \bmod 4$. If $Q = p^m$ then $\mathbb{Q}(\zeta_Q)/\mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$ is ramified only at the prime above p and at the Archimedean primes. If Q is not a prime power, then $\mathbb{Q}(\zeta_Q)/\mathbb{Q}(\zeta_Q + \zeta_Q^{-1})$ is unramified except at the Archimedean primes.*

Remark 6. Washington's proofs of Lemmas 7 and 8 make clear that the results carry through to the case $Q \equiv 2 \bmod 4$ except that in that case, the prime 2 does not ramify in $\mathbb{Q}(\zeta_Q)$.

For a triangular billiard surface $X = X(a_1, a_2, a_3)$, it is tempting to define a "Q-value" for the surface by $Q_X := a_1 + a_2 + a_3$. Unfortunately this notion is not quite well-defined up to translation equivalence; as demonstrated in Lemma 3, the distinct triangles $T(a, a, b)$ and $T(2a, b, 2a + b)$ unfold to translation equivalent translation surfaces if (and only if) b is odd. However, the following lemma and its corollary show that this notion is well-defined up to a factor of 2.

Lemma 9. *Distinct cyclotomic fields have distinct maximal totally real subfields.*

Proof. This is an exercise in elementary algebraic number theory, and is presumably well known. Let k be the maximal totally real subfield of the cyclotomic fields $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_n)$ for positive integers $m, n > 2$. Let p be an odd prime dividing m. By Lemma 7, p ramifies in $\mathbb{Q}(\zeta_m)$. If m is a power of p, then p is totally ramified in $\mathbb{Q}(\zeta_m)$. Since $\mathbb{Q} \subset k \subset \mathbb{Q}(\zeta_m)$, if m is a power of p then p must ramify in k. If m is not a power of p, then Lemma 8 tells us that the extension $\mathbb{Q}(\zeta_m)/k$ is not ramified at the prime above p; thus again p must ramify in k. But also $\mathbb{Q} \subseteq k \subset \mathbb{Q}(\zeta_n)$, so p must ramify in $\mathbb{Q}(\zeta_n)$. By

Lemma 7, this implies that p divides n. Therefore m and n have the same odd prime divisors; furthermore, by Remark 6, these arguments extend to show that either 4 divides both m and n or it divides neither.

The degrees of $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_n)$ as field extensions of $\mathbb{Q}$ are $\phi(m)$ and $\phi(n)$ respectively, where ϕ is the Euler totient function. Since $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_n)$ are each degree 2 extensions of k, we have that $\phi(m) = \phi(n)$.

First suppose that m and n are congruent modulo 2. Let $m = \prod p_i^{e_i}$ and $n = \prod p_i^{f_i}$ be the prime factorizations of m and n. Then we have

$$1 = \frac{\phi(m)}{\phi(n)} = \frac{\prod (p_i - 1) p_i^{e_i - 1}}{\prod (p_i - 1) p_i^{f_i - 1}} = \prod p_i^{e_i - f_i}. \tag{4}$$

Therefore, $e_i = f_i$ for each i, and $m = n$. Hence, in this case $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_n)$.

If m and n are not congruent modulo 2, then we may assume that m is odd and n is congruent to 2 modulo 4. Since $\phi(m) = \phi(2m)$ when m is odd, we can repeat the calculation (4) with $2m$ and n, and get that $2m = n$. But it is well known that for any odd m, $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{2m})$. Therefore, in fact k is the maximal totally real subfield of only one cyclotomic field. □

COROLLARY 2. *Suppose that* $X_1 = X(a_1, a_2, a_3)$ *and* $X_2 = X(b_1, b_2, b_3)$ *are related by a translation cover, and that* $b_1 + b_2 + b_3 < a_1 + a_2 + a_3$. *Then* $b_1 + b_2 + b_3$ *is odd, and* $a_1 + a_2 + a_3 = 2(b_1 + b_2 + b_3)$.

Proof. By Corollary 1, X_1 and X_2 have the same holonomy field k. Write $Q_{X_1} = a_1 + a_2 + a_3$ and $Q_{X_2} = b_1 + b_2 + b_3$. Then k is the maximal totally real subfield of $\mathbb{Q}(\zeta_{Q_{X_1}})$ and of $\mathbb{Q}(\zeta_{Q_{X_2}})$. Hence by Lemma 8, $\mathbb{Q}(\zeta_{Q_{X_1}}) = \mathbb{Q}(\zeta_{Q_{X_2}})$. The result then follows directly from the proof of Lemma 9. □

4. The fingerprint

We have seen in Section 3 that we can use the holonomy field to reduce the list of possible translation covers of a given triangular billiard surface Y to a finite set: those surfaces X such that Q_X and Q_Y differ by at most factor of two. To decide which of these surfaces actually are related by translation covers, we will study the geometric configuration of singularities on a billiard surface by examining the shortest paths from points on the surface to singularities.

Consider a point P on a translation surface X, along with the set of S all shortest geodesic segments on X which connect P to a singularity. Let s_1 and s_2 be two of these segments. We say that s_1 and s_2 are *adjacent* if s_1 can be rotated continuously about P onto s_2 without first coinciding with any other elements of S.

DEFINITION 8. A *fingerprint* of a point $P \in \tau$ is the data $\{\{\theta_i\}, \phi, d\}$, where $\{\theta_i\}$ contains the distinct angle measures separating adjacent pairs of shortest geodesic segments connecting P to singularities, ϕ is the total cone angle at

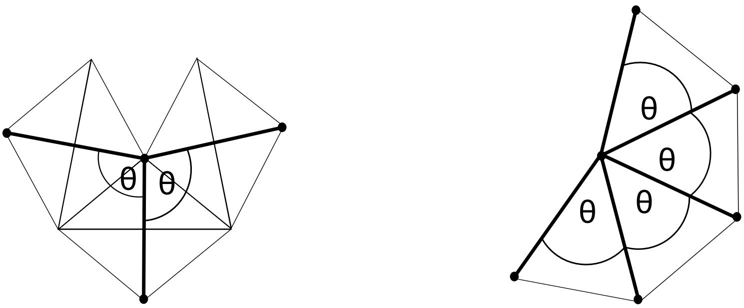

FIGURE 3. Type I fingerprints arising from isosceles triangles.

P, and d is the length of each of the shortest geodesic segments. We shall say that P has a *Type I fingerprint* if $\{\theta_i\}$ has one element, and that P has a *Type II fingerprint* if $\{\theta_i\}$ has two elements. We call $\{\theta_i\}$ the *angle set* of a fingerprint.

The left side of Figure 3 depicts part of the fingerprint of the point P (in bold) on $X(1,1,3)$ corresponding to the vertex of angle $\frac{3\pi}{5}$. Since the points corresponding to vertex angles of $\frac{\pi}{5}$ (circled) are nonsingular, the sides of the billiards triangulation of $X(1,1,3)$ which are geodesics connecting them to P are not part of the fingerprint of P.

For a given triangle T_X corresponding to a triangular billiard surface X of genus greater than 1, we can calculate the fingerprint of each element P of a vertex class on X. We describe this in the following theorem.

THEOREM 2. *Suppose T_X unfolds to a surface X of genus greater than* 1. *Let T_X have vertices v_1, v_2, and v_3. Denote the angle measure of v_i by α_i. Suppose $P \in \pi_X^{-1}(v_1)$. Then one of three situations exists*:

(1) *If v_1 is the apex of an isosceles triangle, then P has angle set $\{\theta\}$, where $\theta = \alpha_1$.*
(2) *If v_1 is not the apex of an isosceles triangle, and if for some $j \in \{2,3\}$ we have that both $\pi_X^{-1}(v_j)$ is singular and $\alpha_j > \frac{\pi}{6}$, then P has a Type I fingerprint with angle set $\{\theta\}$, where $\theta = 2\alpha_1$.*
(3) *If v_1 is not the apex of an isosceles triangle, and if for each $j \in \{2,3\}$ either $\alpha_j = \frac{\pi}{k}$ for some integer $k > 1$ or $\alpha_j < \frac{\pi}{6}$, then P has a Type II fingerprint with angle set $\{\theta_1, \theta_2\}$, where $\theta_1 = \pi - 2\alpha_2$ and $\theta_2 = \pi - 2\alpha_3$.*

Proof. (1) Suppose that v_1 is the apex of an isosceles triangle. Then the shortest geodesics connecting P to a singularity either correspond to the edges of T_X incident on v_1 or to the billiard path from v_1 to the midpoint of the opposite side and back. See Figure 3. The incident edges are longer than the billiard path precisely when $\alpha_2 = \alpha_3 < \frac{\pi}{6}$; in this situation $\alpha_1 > \frac{2\pi}{3}$ and so

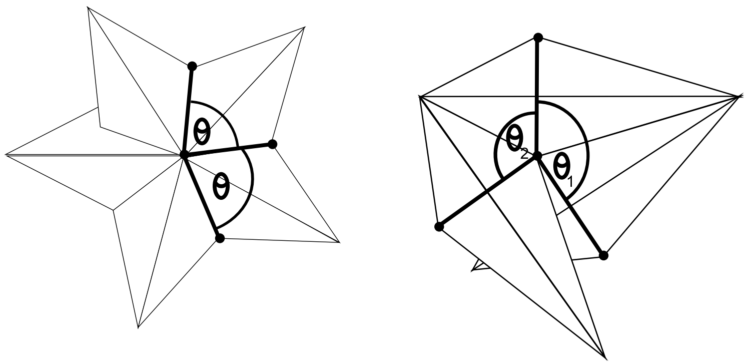

FIGURE 4. Parts of a Type I fingerprint (left) and a Type II fingerprint (right).

$\pi^{-1}(v_1)$ must be singular. Note that when $\alpha_2 = \alpha_3 = \frac{\pi}{6}$ we have that $\pi^{-1}(v_2)$ and $\pi^{-1}(v_3)$ are nonsingular, so the edges and the billiard path cannot both correspond to shortest geodesics connecting P to a singular point. In either case, it is evident that adjacent shortest geodesics form an angle of α_1.

(2) Now suppose that v_1 is not the apex of an isosceles triangle, and that for some $j \in \{2,3\}$ we have that both $\pi^{-1}(v_j)$ is singular and $\alpha_j > \frac{\pi}{6}$. Then the edge of T_X with endpoints v_1 and v_j is shorter than the shortest billiard path from v_1 to itself. Since v_1 is not the apex of an isosceles triangle, exactly one of the edges incident on v_1 corresponds to all of the shortest geodesics connecting P to singularities. Thus two such geodesics which are adjacent form an angle of $2\alpha_1$. See Figure 4.

(3) Finally, suppose that v_1 is not the apex of an isosceles triangle, and that for each $j \in \{2,3\}$ either $\alpha_j = \frac{\pi}{k}$ for some integer $k > 1$ or $\alpha_j < \frac{\pi}{6}$. It follows that for each side of T_X incident on v_1, either the side will not correspond to a geodesic connecting P to a singularity, or else the side will be longer than the shortest billiard path from v_1 to itself. Furthermore, we see that $\pi_X^{-1}(v_1)$ must be singular. For, either $\pi_X^{-1}(v_2)$ and $\pi_X^{-1}(v_3)$ are nonsingular, in which case $\pi_X^{-1}(v_1)$ must be singular since X has a singularity, or else $\alpha_2 + \alpha_3 < \frac{2\pi}{3}$; in this case, $\alpha_1 > \frac{\pi}{3}$ implies that $\pi_X^{-1}(v_1)$ is singular, since if $\alpha = \frac{\pi}{2}$ it is impossible to find α_2 and α_3 satisfying the conditions of case 3. The claim is now evident from Figure 5, which illustrates the fingerprint of the singularity on $X(3,4,5)$. (In the figure, the geodesics defining the fingerprint are the thicker lines, whereas the edges of the billiards triangulation are the thinner lines.) Let the angle set of the fingerprint of P be $\{\theta_1, \theta_2\}$. Each θ_i is an interior angle of a quadrilateral whose other three angles include two right angles and an angle which has twice the measure of an angle of the triangular

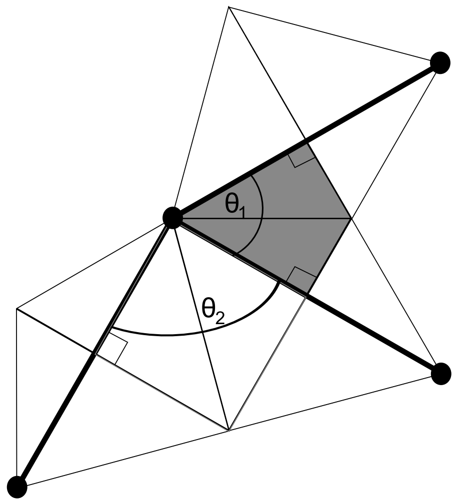

FIGURE 5. Part of a Type II fingerprint on $X(3,4,5)$. Nonsingular vertex points are circled.

billiard table T_X for X. Therefore, two of the angles of T have the form $\frac{1}{2}(2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \theta_i) = \frac{\pi - \theta_i}{2}$, and the third angle is $\frac{\theta_1+\theta_2}{2}$. □

COROLLARY 3. *Suppose the billiards triangulation of a triangular billiard surface X contains a point with a Type II fingerprint. Then X is uniquely determined by that fingerprint, up to an action of $O(2,\mathbb{R})$. Indeed, if the fingerprint has angle set $\{\theta_1, \theta_2\}$, then X is the billiards surface for the triangle of angles $\frac{\theta_1+\theta_2}{2}$, $\frac{\pi-\theta_1}{2}$, and $\frac{\pi-\theta_2}{2}$.*

The fingerprint is a useful tool for studying translation covers because it is nearly invariant under balanced covers. It seems worth noting that one can define the notion of a fingerprint on any translation surface; this makes particular sense for any polygonal billiard surface. Then the following invariance result will still hold, except that the cone angles may differ by a larger integer factor.

THEOREM 3. *Suppose that $f : X \to Y$ is a balanced translation cover, that $P' \in X$ and $P \in Y$ are vertices of billiards triangulations on their respective surfaces, and that $f(P') = P$. Then the fingerprints of P' and P have the same angle sets and shortest geodesic lengths, and their cone angles are either equal or differ only by a factor of two.*

Proof. Let d and d' be the lengths of the shortest geodesics which connect P and P', respectively, to a singularity. Since f is a translation cover, the image under f of any geodesic on X is a geodesic of equal or lesser length on Y;

since f is balanced, any geodesic with singular endpoints on X is mapped by f onto a geodesic with singular endpoints on Y. Thus, $d \leq d'$. Conversely, since f is a translation cover, the preimage of any geodesic of length d with singular endpoints is a union of geodesics of length d with singular endpoints. Thus, $d' \leq d$, and we see that the fingerprints of P and P' have the same geodesic lengths.

Define $B_\delta(P)$ to be the set of all points of Y that are of distance less than or equal to δ from P. Let $B'_\delta(P')$ be the connected component of $f^{-1}(B)$ containing P'. Since Y is a translation surface, and f is a translation cover (possibly ramified above P with index m), we can and do choose δ to be a sufficiently small positive number such that B_δ and $B'_\delta(P')$ are simply connected closed metric balls centered at P and P', respectively.

Consider a pair of adjacent geodesics e_1 and e_2, each of length d, connecting P to singularities. Label the angle between them θ. Let l_1 and l_2 be the intersections of e_1 and e_2 with $B_\delta(P)$.

The union of l_1 and l_2 with a portion of the boundary of $B_\delta(P)$ bounds a wedge-shaped region W which does not contain in its interior any portion of a shortest geodesic connecting P to a singular point. Since f is a translation cover, the intersection of the $f^{-1}(W)$ with $B'_\delta(P')$ is m copies of W. Let W' be one of these copies. Then W' is bounded by part of the boundary of B' and two geodesics (say, l'_1 and l'_2) which extend to be length d geodesics e'_1 and e'_2 connecting P' to singular points. Since f is a translation cover, the angle measure between l'_1 and l'_2 is θ; therefore the angle measure between e'_1 and e'_2 is θ.

Suppose that e'_1 and e'_2 are not adjacent. Then there is some geodesic e'_3 of length d connecting P' to a singularity. The intersection of this geodesic with W' is a geodesic l'_3 in the interior of W'. But then $f(l'_3)$ is a geodesic in the interior of W which extends to the image of e'_3—a length d geodesic connecting P to a singular point. This contradicts the adjacency of e_1 and e_2. Hence, in fact e'_1 and e'_2 are adjacent, and it follows that the fingerprints of P and P' have the same angle sets.

Finally, we claim that $m \leq 2$. Let v and v' be the vertices of the triangles T_Y and T_X corresponding to P and P'. By Remark 1, the cone angle at P is completely determined by $\angle v$. But Theorem 2 tells us that the measure of $\angle v$ is determined, up to a factor of 2, by the angle set of the fingerprint of P. Hence, since the fingerprints of P and P' have the same angle set, we see that $m \in \{1, 2\}$. ☐

COROLLARY 4. *Fingerprint type is invariant under balanced translation covers.*

COROLLARY 5. *Any rational triangular billiard surface with a Type II singularity cannot be a part of any composition of nontrivial balanced covers.*

Proof. By Theorem 3. Suppose we have $f : X \to Y$ a balanced cover with either X or Y possessing a singularity with a Type II fingerprint. By Corollary 4, X and Y must both have singularities with Type II fingerprints. Since a Type II fingerprint identifies the triangular billiards table of a surface, X and Y must be the same surface. □

As we shall see, the preceding results about fingerprints allow us to quickly classify all balanced covers in the category of triangular billiards surfaces. However, to extend our results to unbalanced covers, we shall refine our use of the fingerprint by considering *punctured surfaces*. By deleting a collection of singular points from a surface X to obtain a surface $\tilde{X}$, we will arrive at a set of shortest geodesics connecting a point P in $\tilde{X}$ to the remaining singularities; in general this set will be different than the set of geodesics determining the fingerprint of P on X. However, since simply deleting singularities of X does not affect the flat metric on open sets not previously containing the deleted points, we can get important information about X by considering the fingerprint of P on $\tilde{X}$.

LEMMA 10. *Let X be a triangular billiard surface with more than one singular vertex class. Let $\tilde{X}$ be the surface obtained from X by puncturing either one entire singular vertex class or two entire singular vertex classes such that neither deleted class corresponds to an obtuse angle of the triangular billiard table and such that at least one singular vertex class remains. Let $\pi_X^{-1}(v_i)$ be a singular vertex class not deleted. Let $P \in \pi_X^{-1}(v_i)$. If P has Type II fingerprint on $\tilde{X}$ with angle set $\{\theta_1, \theta_2\}$, then X arises from billiards in the triangle with angles $\frac{\pi-\theta_1}{2}$, $\frac{\pi-\theta_2}{2}$, and $\frac{\theta_1+\theta_2}{2}$. If P has a Type I fingerprint on $\tilde{X}$ with angle set $\{\theta_1\}$, then the measure of $\angle v_i$ is in $\{\theta_1, \frac{1}{2}\theta_1\}$.*

Proof. If none of the punctured points were endpoints of geodesics defining the fingerprint of P, then P has the same fingerprint on $\tilde{X}$ as on X, and we are done.

Suppose a singular vertex class has been punctured which contained endpoints of such geodesics. Then there is a new "closest" vertex class to P; call it C. If C does not contain P then the shortest geodesics connecting P to C are edges of the billiards triangulation of X. If C does contain P then, since a vertex class corresponding to an obtuse angle of the billiard table must be singular (by Remark 1) and we have assumed that no such classes have been deleted, it follows that the shortest geodesics from P to C correspond to a single reflection in the original dynamical system. Thus, the same reasoning holds as in Theorem 2.

The only potential difficulty would be if the new "closest" vertex class was the one containing P, for in that case, since the shortest geodesics from P to elements of its own class pass through more than one triangle, we must consider the possibility that our punctures obstruct these geodesics. However, since the shortest geodesics are perpendicular to the sides of the triangles opposite P, this is only a problem if the vertex class punctured is $\pi_X^{-1}(v_j)$ with $\angle v_j = \frac{\pi}{2}$. But such a class is nonsingular, so it would not have been punctured. $\square$

5. All translation covers

5.1. Balanced covers. In this subsection, we shall prove Theorem 1 for balanced covers using the simple geometric idea of the fingerprint as our main tool (this is Lemma 13).

Most points on a billiard surface have no more than three shortest geodesics connecting them to singular points. We define an *exceptional point* on a billiard surface X to be a point which has more than three shortest geodesic paths to singularities.

LEMMA 11. *Let X be a triangular billiard surface of genus greater than* 1. *A point P on X is exceptional if and only if P is a vertex of the billiards triangulation of X not corresponding to a right angle.*

Proof. If P is a vertex of τ, then Theorem 2 shows that it is exceptional unless it corresponds to a right angle on T_X. If P does correspond to a right angle, then clearly there are two shortest paths to singularities (corresponding to one side of T_X incident on $\pi_X(P)$) unless T_X is isosceles; but then $X = X(1,1,2)$, which has no singularities.

Suppose that P is in the interior of a triangle of τ. Then for each vertex of that triangle, P is strictly closer to that vertex than to any other element of its vertex class. Hence, P cannot have more than three shortest paths to singularities.

Now suppose that P is in the interior of an edge of τ. Then P is on an edge of two triangles, so P has two shortest paths of length L_1 to members of the opposite vertex class. If P is the midpoint of the edge, then it has equidistant unique shortest paths of length L_2 to members of the other two vertex classes; but if $L_1 = L_2$ then one easily shows that the opposite vertex class corresponds to a right angle of T_X, so that the class is nonsingular. Hence, P cannot have more than two shortest paths to singularities in this case. $\square$

LEMMA 12. *A balanced translation cover $f: X \to Y$ maps the exceptional points of X onto the exceptional points of Y.*

Proof. Balanced covers preserve fingerprints, so this follows from Lemma 11 and Theorem 2. $\square$

LEMMA 13 (Theorem 1 restricted to balanced covers). *Let X and Y be triangular billiard surfaces such that the genus of X is greater than* 1. *Suppose* *that $f : X \to Y$ is a nontrivial balanced translation cover. Then f is of the form described in Lemma 3.*

Proof. Let $f : X \to Y$ be a balanced cover between triangular billiard surfaces. If X has three exceptional points with distinct fingerprints, then T_X is not isosceles and so by Theorem 2 we know all the angles of T_X; the same reasoning holds for Y. Since balanced covers preserve fingerprints up to cone angle, we see that $X = Y$.

If X has only two distinct fingerprints for its exceptional points then so does Y, and by Theorem 2 we know that T_X and T_Y are a pair of triangles described in Lemma 3. □

5.2. Combinatorial lemmas.

LEMMA 14. *Let $f : X(a_1, a_2, a_3) \to X(b_1, b_2, b_3)$ be a degree n translation cover of triangular billiard surfaces. Then*

$$\sum_{a_i \nmid (a_1+a_2+a_3)} a_i \geq n \sum_{b_i \nmid (b_1+b_2+b_3)} b_i. \tag{5}$$

Proof. The sum of the cone angles of the singular points of $X(a_1, a_2, a_3)$ is at least n times the sum of the cone angles of the singular points of $X(b_1, b_2, b_3)$. By Remark 1, the result follows. □

LEMMA 15. *Let $f : X = X(a_1, a_2, a_3) \to Y = X(b_1, b_2, b_3)$ be a translation cover of triangular billiard surfaces such that the genus of X is greater than 1. If $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ and f is not a composition of covers from Lemma 3, then f is of degree 1.*

Proof. Write $Q_X = Q_Y = Q = a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. Let n be the degree of f, and suppose that $n \geq 2$. Lemma 14 then gives $\sum_{b_i \nmid Q} b_i \leq \frac{Q}{n}$. Hence, since $n \geq 2$, we have

$$\sum_{b_i | Q} b_i \geq \frac{Q}{2}. \tag{6}$$

Writing $q_i = \frac{Q}{b_i}$, we have the equivalent expression

$$\sum_{b_i | Q} \frac{1}{q_i} \geq \frac{1}{2}. \tag{7}$$

Note that if $b_i | Q$ then q_i is an integer. Of course, Equation (6) is always satisfied if $T(b_1, b_2, b_3)$ is a right triangle. If $T(b_1, b_2, b_3)$ is not a right triangle, the equation is rarely satisfied. Thus, we will reduce the problem to two cases.

CASE 1. The triangle $T(b_1, b_2, b_3)$ is not a right triangle.

In this case, recalling that $\gcd(b_1, b_2, b_3) = 1$, we show that there are only three possibilities for the b_i which satisfy Equation (6).

If all three b_i divide Q then Y is nonsingular. The only non-right triangle which unfolds to a nonsingular surface is $T(1,1,1)$; but since this is also the only triangle with $Q = 3$, if $Y = X(1,1,1)$ then $X = X(1,1,1)$, contradicting our assumption that X has a singularity.

Hence, we can assume for this case that $b_3 \nmid Q$. Therefore to satisfy Equation (7), we seek integers $q_1, q_2 > 2$ such that

$$\frac{1}{q_1} + \frac{1}{q_2} > \frac{1}{2}. \tag{8}$$

Without loss of generality, we assume $q_1 \le q_2$. If $q_1 \ge 4$, Equation (8) is impossible. If $q_1 = 3$, then Equation (8) is satisfied if $q_2 \le 5$. Thus the remaining candidates for Y are $X(3,4,5)$ and $X(3,5,7)$. By Lemma 14, $X(3,4,5)$ admits at most a degree two cover; by Lemma 1 the degree two covers satisfying the hypotheses of the lemma could only be $f : X(2,5,5) \to X(3,4,5)$ or $X(1,1,10) \to X(3,4,5)$. However, these maps would have to be balanced covers, and $X(3,4,5)$ has a singularity with a Type II fingerprint. Thus by Corollary 5 these maps do not exist. Similarly, the only feasible cover of $X(3,5,7)$ of degree greater than 1 is $f : X(1,7,7) \to X(3,5,7)$; again, this would be a balanced cover, and $X(3,5,7)$ has a singularity with a Type II fingerprint.

Case 2. The triangle $T(b_1, b_2, b_3)$ is a right triangle.

Assume that $b_1 = \frac{Q}{2}$. If b_2 divides Q, then by Lemma 3, the surface $Y = X(b_2 + b_3, b_2, b_3)$ is translation equivalent to the surface $X(b_3, \frac{b_2}{2}, \frac{b_2}{2})$. But then this is equivalent to the situation where $Q_X = 2Q_Y$; we will address this situation in the proof of Theorem 1.

Thus for the remainder of this proof we will assume that neither b_2 nor b_3 divides Q. Here Lemma 14 implies that the degree of f is at most two. The sum of the cone angles of the singularities of Y is $2\pi(b_2 + b_3)$. Thus if $n = 2$ then the sum of the cone angles of the singularities of X is $4\pi(b_2 + b_3) = 2\pi Q = 2\pi(a_1 + a_2 + a_3)$. Therefore $T(a_1, a_2, a_3)$ must be either $T(b_2, b_2, 2b_3)$ or $T(2b_2, b_3, b_3)$. Both these possibilities are accounted for by the covers of Lemma 3. $\square$

Lemma 16. *Let $f : X \to Y$ be a translation cover of triangular billiard surfaces. Let m be the smallest integer such that all singularities of Y have cone angle at least $2m\pi$. Suppose that $\deg f < m$. Then for each vertex class C_i on X, $f(C_i)$ consists entirely of singular points or entirely of nonsingular points.*

Proof. Let m be as above and assume that $\deg(f) < m$. Suppose for contradiction that for some j, $f(C_j)$ contains singular points and nonsingular points. Each member of C_j has the same cone angle, and this cone angle must be at least $2m\pi$, since some of the members are mapped by a translation cover to a singularity of cone angle $2m\pi$. Thus, for those elements of C_j which are mapped to nonsingular points, the definition of a ramified cover requires that f be locally of degree at least m, which contradicts our assumption that $\deg(f) < m$. This completes the proof. □

5.3. Proof of the main theorem. Now we can prove Theorem 1, which for our ease we restate in the following way.

THEOREM 1. *Suppose $f : X \to Y$ is a translation cover of triangular billiard surfaces of degree greater than* 1, *with the genus of X greater than* 1. *Then f is of degree* 2, *and is a composition of one or two of the covers f_i described in Lemma* 3.

Proof. Suppose $X := X(a_1, a_2, a_3)$, $Y := X(b_1, b_2, b_3)$, and $f : X \to Y$ is a translation cover of degree $\deg f > 1$. Assume that the genus of X is greater than 1. Write $Q_X := a_1 + a_2 + a_3$ and $Q_Y := b_1 + b_2 + b_3$. Let v_1, v_2, v_3 and w_1, w_2, w_3 be the corresponding vertices of $T(a_1, a_2, a_3)$ and $T(b_1, b_2, b_3)$ respectively. By Corollary 1, X and Y have the same holonomy field k. By Corollary 2, we have $Q_Y \in \{2Q_X, Q_X, \frac{1}{2}Q_X\}$. If $Q_Y = 2Q_X$, then by Lemma 14, we must have $\sum_{b_i \nmid Q_Y} b_i \leq \frac{Q_X}{2} = \frac{Q_Y}{4}$. But then we would have $\sum_{b_i | Q_Y} b_i \geq \frac{3}{4} Q_Y$, which is only the case for the following surfaces with even Q-value: $X(1, 1, 2)$, $X(1, 2, 3)$, $X(3, 4, 5)$. Of course, $Q_X \geq 3$, so $Y \neq X(1, 1, 2)$. If $Y = X(1, 2, 3)$ then $X = X(1, 1, 1)$, which is of genus 1, a contradiction. If $Y = X(3, 4, 5)$, then Y has a singularity with cone angle 10π. But, no surface X with $Q_X = 6$ could have a cone angle of at least 10π.

If $Q_Y = Q_X$, then we are done by Lemma 15. Thus, appealing to Corollary 2, we shall assume for the remainder of the proof that $Q_X = 2Q_Y$ and that Q_Y is odd.

CASE 1. Y has three singular vertex classes.

In this case, Lemma 14 implies that f can only be a degree 2 balanced cover. Thus we are done by Lemma 13.

CASE 2. Y has no singular vertex classes.

In this case, since Q_Y is odd, we must have $Y = X(1, 1, 1)$. There are only two surfaces with a Q-value of 6: they are $X(1, 1, 4)$ and $X(1, 2, 3)$, and each of these surfaces covers $X(1, 1, 1)$ as described in Lemma 3.

CASE 3. Y has one singular vertex class.

In this case we have, without loss of generality, $b_1 | Q_Y$, $b_2 | Q_Y$, and $b_3 \nmid Q_Y$. Since b_1 and b_2 are divisors of the odd number $Q_Y := b_1 + b_2 + b_3$, b_3 must also be odd. Therefore $\frac{b_3}{\gcd(b_3,Q)} \geq 3$. The cone angle at each of the singularities of Y corresponding to b_3 is $\frac{b_3}{\gcd(b_3,Q)} 2\pi \geq 6\pi$.

Lemma 14 eliminates all possible Y for $\deg f \geq 4$ except $Y = X(3,5,7)$. But, again by Lemma 14, the only possible degree 4 covering surface would be $X(1,1,28)$, and such a cover would have to be balanced, contradicting Lemma 13.

If $\deg f = 2$*:* Lemma 16 tells us that if $\deg f = 2$ then for each $j = 1,2,3$, we have that $f(\pi_X^{-1}(v_j)) \cap \pi_Y^{-1}(w_3)$ is either empty or all of $f(\pi_X^{-1}(v_j))$.

Suppose that $Y = X(3,5,7)$. Lemma 16 restricts the possible degree 2 covers to surfaces of the form $X(14, a_2, a_3)$, where each of a_2 and a_3 is either a divisor of 30 or twice a divisor of 30. The only possibility this leaves is $X(15,14,1)$. But any translation cover $X(15,14,1) \to X(3,5,7)$ would have to be balanced, so Lemma 13 applies.

Now suppose that $Y \neq X(3,5,7)$. Let C be the singular vertex class of Y. We must have $\frac{b_3}{Q} > \frac{1}{2}$, and so by Remark 1 C must correspond to an obtuse angle θ of the billiard table. Let $\tilde{X}$ be the surface obtained from X by puncturing all singular vertex classes of X which are not contained in $f^{-1}(C)$. Since $\frac{b_3}{Q} > \frac{1}{2}$ and f is degree 2, the sum of the angles of the billiard table corresponding to the vertex classes in the f-preimage of C must be obtuse. Thus we can apply Lemma 10 to $\tilde{X}$. The restriction of f to $\tilde{X}$ is balanced. Since Y has only one singular vertex class, elements of C must have Type II fingerprints unless $T(b_1, b_2, b_3)$ is isosceles. If the fingerprints are Type II, then Proposition 3 and Lemma 10 demonstrate that X and Y are translation equivalent. So the only possibility is that the fingerprints are Type I. In that case Y is an isosceles triangular billiard surface. Let C' be a vertex class on X that is in $f^{-1}(C)$, and write $\theta = \frac{b_3\pi}{Q}$. The billiard table angle that C' corresponds to is either θ or $\frac{\theta}{2}$. If the angle is θ, then X and Y are translation equivalent. If the angle is $\theta \setminus 2$, then there is another vertex class on X which is also mapped to C. But then that vertex class would also correspond to an angle of $\theta \setminus 2$, and we would have that X is an isosceles triangular billiards surface which covers Y as described in Lemma 3.

If $\deg(f) = 3$*:* Then Lemma 14 allows only the following possibilities for Y: the surfaces

$$Y_n = \begin{cases} X(3, n, 2n-3), & 3 \nmid n, \\ X(1, \frac{n}{3}, \frac{2n}{3} - 1), & 3 | n. \end{cases}$$

Note that $\gcd(2n-3, 3n) \in \{1,3\}$. First suppose that $\gcd(2n-3, 3n) = 1$. Then $Q = 3n$ (thus n is odd), $3 \nmid n$, and we have $Y_n = X(3, n, 2n-3)$. We have that $n \geq 5$ and hence that $2n - 3 \geq 7$. On Y_n, there is only one singular vertex class and the cone angle of each singular point is $(2n-3)2\pi$. Thus Lemma 16 applies here. Since Y_n is never isosceles, its singular point has a Type II fingerprint. Let $\tilde{X}$ be the surface obtained from X by deleting all singularities of X which f maps to nonsingular points, and let $\tilde{f}$ be the restriction of f to $\tilde{X}$. By Lemma 16, the elements of $X-\tilde{X}$ are the union of entire vertex classes. Thus a Type II fingerprint on $\tilde{X}$ will uniquely identify the triangular billiards table used to generate X, by Lemma 10. Because $\tilde{f}$ is a balanced map, each singular point of $\tilde{X}$ must have the same Type II fingerprint (on $\tilde{X}$) as its $\tilde{f}$-image on Y. But, a Type II fingerprint uniquely identifies the triangle used to generate the surface (this works for $\tilde{X}$ as well); hence X and Y_n are the same billiard surface, and Lemma 2 says that a triple cover is impossible.

Now suppose that $\gcd(2n-3,3n)=3$. Then the cone angle of the singular point on Y_n is $\frac{2n-3}{3}2\pi$. If $n>6$ then $\frac{2n-3}{3}>3$, so that again we can apply Lemma 16 and Lemma 10, and the same fingerprint argument goes through. The remaining cases are $n=3,6$. We have $Y_3=X(1,1,1)$ and $Y_6=X(1,2,3)$, neither of which have singularities.

Case 4. Y has two singular vertex classes.

Assume $b_1|Q$ and $b_2,b_3\nmid Q$. Since Q is odd, we have $\frac{b_1}{Q}\le\frac{1}{3}$, and so Lemma 14 implies that $\deg(f)\le 3$. But, if $\deg(f)=3$, Lemma 14 also implies that f is balanced, contradicting the result of Lemma 13 that balanced covers are of degree at most 2. Thus $\deg(f)=2$.

Note that b_2 and b_3 must have the same parity.

Case 4.1. Both b_2 and b_3 are odd.

Then $\frac{b_i}{\gcd(b_i,Q)}\ge 3$, so by Lemma 16, each vertex class of X maps to all singular points or all nonsingular points.

If one vertex class of X maps to nonsingular points: Say the vertex class C_1 corresponding to a_1 maps to nonsingular points. Then $a=2b_1$, and $2b_1|2Q$, so C_1 is nonsingular, so f is balanced.

If two vertex classes of X map to nonsingular points: Let them be C_1 and C_2, corresponding to a_1 and a_2. If C_1 is singular, then by Lemma 16 we have $a_1=2d$ for some $d|Q$. But since $a_3=2(b_2+b_3)$, this would mean that all the a_i are even, contradicting the fact that $\gcd(a_1,a_2,a_3)=1$.

Case 4.2. Both b_2 and b_3 are even.

If one vertex class of X maps to nonsingular points: Let it be C_1. We have $a_2+a_3=2(b_2+b_3)$, so a_1 must be even. But also a_2 and a_3 must be even, since $2|\frac{b_i}{\gcd(b_i,Q)}$ and $\frac{b_i}{\gcd(b_i,Q)}|\frac{a_j}{\gcd(a_j,Q)}$ for each $i,j\in\{2,3\}$. Again, this is a contradiction.

If two vertex classes of X map to nonsingular points: Let them be C_1 and C_2. We have that $a_3=2(b_2+b_3)$ is even. If C_1 is singular then again we have that a_1 (and hence a_2) is even, once more contradicting that $\gcd(a_1,a_2,a_3)=1$. Hence C_1 and C_2 are nonsingular, and f is balanced. $\square$

Acknowledgments. The author thanks Thomas A. Schmidt for many helpful discussions. Thanks are also due to the referee for several useful critiques.

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