REPRODUCING KERNEL HILBERT SPACES GENERATED BY THE BINOMIAL COEFFICIENTS

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ABSTRACT. We study a reproducing kernel Hilbert space of functions defined on the positive integers and associated to the binomial coefficients. We introduce two transforms, which allow us to develop a related harmonic analysis in this Hilbert space. Finally, we mention connections with the theory of discrete analytic functions, statistics, and with the quantum case.

1. Introduction

We show that the binomial coefficients form an orthonormal basis (ONB) in a naturally formed reproducing kernel Hilbert space $\mathcal{H}(K)$. We find explicit properties and formulas for the kernel, and we identify two pairs of transforms, again with explicit formulas. With the binomial functions forming an ONB in $\mathcal{H}(K)$, one naturally then gets transforms as isometric isomorphisms between $\mathcal{H}(K)$ and $\ell^2(\mathbb{Z}_+)$. We write down two such transforms, and for each we compute its inverse. Our work is motivated by applications to discrete analytic functions [2]. This is followed up here, and we outline additional applications to combinatorial probability theory. In [2], we introduce a product (in the sense of hypergroups [12]) in spaces of discrete analytic functions. We use our present transforms for computing this product. In the last section, as a corollary to Theorem 7.3, we establish a significant statistical property for our one-parameter family of kernels K_λ, as well as for the corresponding spectral transforms: we show for each kernel within our family of kernels K_λ there is a probability space; hence, an associated indexed family of probability spaces $(\Omega, \mathcal{F}, P^\lambda)$. We prove that as the value of the parameter λ vary, the corresponding measures P^λ are mutually singular. For readers not familiar

with reproducing kernels, we suggest [1], [3], [15], [16], [17]. For a sample of references to combinatorial probability, see, for example, [13], [14], [19].

Motivated by our previous work on discrete analytic functions [2], we study a reproducing kernel space of functions from $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ into $\mathbb{R}$ associated to the binomial coefficients.

We study the reproducing kernel Hilbert space (RKHS) $\mathcal{H}(K)$ generated by the kernel $K(x,y)$ consisting of the numbers in the Pascal triangle, that is, for every (x,y) in $\mathbb{Z}_+ \times \mathbb{Z}_+$, set $K(x,y) = \binom{x+y}{x}$. Using the Chu–Vandermonde formula, one sees that $K(x,y)$ is positive definite in the sense of kernel theory, hence the RKHS. We study a number of analytic features of $\mathcal{H}(K)$: We find families of orthonormal bases in $\mathcal{H}(K)$, and we prove associated transform theorems. Moreover, we compute these transforms explicitly, making use of infinite square matrices of upper and lower triangular form. We further show that our transforms facilitate analysis on $\mathcal{H}(K)$. We further sketch a one-parameter deformation family with application.

In the remaining part of this section, we give results characterizing vectors in their respective reproducing kernel Hilbert spaces; see Theorem 1.1.

DEFINITIONS. Let for $x, n \in \mathbb{Z}_+$ such that $n \leq x$

$$(1.1) \qquad \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

the binomial coefficient. We set

$$(1.2) \qquad e_n(x) = \begin{cases} 0, & \text{if } x < n, \\ \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}, & \text{if } x \geq n, \end{cases}$$

and

$$(1.3) \qquad K(x,y) = \sum_{n=0}^{\infty} e_n(x)e_n(y) = \binom{x+y}{x}, \quad x, y \in \mathbb{Z}_+.$$

The sum (1.3) is always finite and well defined, and defines a positive definite function on $\mathbb{Z}_+$.

Using the Chu–Vandermonde formula (see, for instance, [18, formula (10), p. 217])

$$(1.4) \qquad \binom{x+y}{t} = \sum_{k=0}^{t} \binom{x}{k}\binom{y}{t-k}$$

(with $t, x, y \in \mathbb{N}_0$ and $t \leq y$) one can rewrite $K(x,y)$ as

$$(1.5) \qquad K(x,y) = \sum_{n=0}^{x \wedge y} \binom{x}{n}\binom{y}{n},$$

or, equivalently,

$$(1.6) \qquad K(x,y) = \sum_{n=0}^{x \wedge y} \frac{x^{[n]} y^{[n]}}{(n!)^2},$$

where we have set

$$(1.7) \qquad x^{[n]} = x(x-1) \cdots (x-n+1).$$

Writing the kernel $K(x,y)$ as an infinite matrix K, we note that we have

$$(1.8) \qquad K = LL^*,$$

where L denote the following infinite lower triangular matrix consisting of the binomial coefficients

$$(1.9) \quad L = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdot & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdot & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdot & 0 & \cdots \\
1 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots & \cdot & 0 & \cdots \\
1 & 4 & 6 & 4 & 1 & 0 & 0 & \cdots & \cdot & 0 & \cdots \\
\vdots & \vdots & & & & & & & & & \\
1 & n & \binom{n}{2} & \cdots & \binom{n}{2} & n & 1 & 0 & 0 & \cdots & 0 & \cdots \\
1 & n+1 & \binom{n+1}{2} & \cdots & & \binom{n+1}{2} & n+1 & 1 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & 1 & & \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & &
\end{pmatrix}.$$

We denote by $\mathcal{H}(K)$ the reproducing kernel of real valued functions with domain $\mathbb{Z}_+$ and with reproducing kernel $K(x,y)$. This latter space consists of all functions of the form

$$(1.10) \qquad g(x) = \sum_{n=0}^{\infty} a_n e_n(x),$$

where the $a_n \in \mathbb{R}$, and with norm

$$(1.11) \qquad \|g\|_{\mathcal{H}(K)}^2 = \sum_{n=0}^{\infty} a_n^2.$$

See [17, Corollary 4, p. 169].

It follows from (1.8) that we have:

THEOREM 1.1. *Let $f \in \mathbb{R}^{\mathbb{Z}_+}$. Then,*

$$(1.12) \qquad \sum_{x \in \mathbb{Z}_+} K(\cdot, x) f(x) \in \mathcal{H}(K) \quad \Longleftrightarrow \quad L^* f \in \ell^2(\mathbb{Z}_+).$$

Organization. The paper consists of seven sections besides the Introduction, and its outline is as follows: In Section 2, we review some results on binomial coefficients. Sections 3–4 contain the main results of the paper, on the above mentioned transforms. The first transform, which we call the *binomial Fourier transform*, and its inverse, are studied in Section 3. A natural isomorphism between $\mathcal{H}(K)$ and the Hardy space $\mathbf{H}^2(\mathbb{D})$ is defined in that section. The following two sections, Section 4 is devoted to the second transform. In Section 5, we study some links with the theory of discrete analytic functions. In Section 6, we considered the case of q-binomial coefficients. We present some applications in the last section.

2. Some formulas for binomial coefficients

In this section, we establish some duality relations for the binomial functions in (1.2), beginning with Lemma 2.1. This lemma in turn will be used in our results from Sections 3 through 4 dealing with the two transforms. The transforms throw light on the RKHS of the binomial functions, but they also imply new formulas for these functions, linear relations (Theorem 3.3), and their use in the study of discrete analytic functions (Section 5).

We make the following summation convention. In computations below, we will be using summations over index-values in finite or infinite subsets of $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$; in some cases, multiple summations inside a single computation. These summations will then always entail ranges over summation indices $x, y, k, n, \ldots$ which are limited by choice of segments in $\mathbb{Z}_+$; and the respective summations will be specified by the segment-endpoints.

LEMMA 2.1. *Let $m, n \in \mathbb{Z}_+$ be such that $m \leq n$. Then,*

$$(2.1) \qquad \sum_{j=m}^{n} (-1)^{m+j} \binom{n}{j} \binom{j}{m} = \delta_{m,n}.$$

Proof. The result is clear when $m = n$. Let us assume now $m < n$. We first note that

$$(2.2) \qquad \binom{n}{j} \binom{j}{m} = \frac{n!}{(n-j)!j!} \frac{j!}{(j-m)!m!}$$
$$= \frac{n!}{(n-j)!(j-m)!m!}$$
$$= \frac{n!}{(n-m)!m!} \frac{(n-m)!}{(n-j)!(j-m)!}$$
$$= \binom{n}{m} \binom{n-m}{j-m}.$$

Hence,

$$(2.3) \qquad \sum_{j=m}^{n} (-1)^{m+j} \binom{n}{j} \binom{j}{m} = \sum_{j=m}^{n} (-1)^{m+j} \binom{n}{m} \binom{n-m}{j-m}$$

$$= \binom{n}{m} \sum_{j=m}^{n} (-1)^{m+j} \binom{n-m}{j-m}$$

and setting $\ell = j - m$,

$$= \binom{n}{m} \sum_{\ell=0}^{n-m} (-1)^{\ell} \binom{n-m}{\ell}$$

$$= \binom{n}{m} (1-1)^{n-m}$$

$$= 0. \qquad \square$$

The following result is a corollary of Lemma 2.1, and plays an important role in the study of the $\blacktriangle$-transform. See Theorem 3.6 below.

THEOREM 2.2. *Let $K(\ell, m)$ be defined by* (1.5). *Then it holds that*:

$$(2.4) \qquad \sum_{\ell=0}^{n} (-1)^{n+\ell} \binom{n}{\ell} K(\ell, m) = \begin{cases} \binom{m}{n}, & \text{if } m \geq n, \\ 0, & \text{if } m < n. \end{cases}$$

Proof. By the definition of K, we have

$$\sum_{\ell=0}^{n} (-1)^{n+\ell} \binom{n}{\ell} K(\ell, m) = \sum_{\ell=0}^{n} \sum_{j=0}^{m \wedge \ell} (-1)^{n+\ell} \binom{n}{\ell} \binom{m}{j} \binom{\ell}{j}$$

$$= \sum_{j=0}^{n \wedge m} \binom{m}{j} \left(\sum_{\ell=j}^{n} (-1)^{n+\ell} \binom{n}{\ell} \binom{\ell}{j} \right)$$

and, applying Lemma 2.1

$$= \sum_{j=0}^{n \wedge m} \binom{m}{j} \delta_{j,n}$$

$$= \begin{cases} \binom{m}{n}, & \text{if } m \geq n, \\ 0, & \text{if } m < n. \end{cases} \qquad \square$$

LEMMA 2.3. *Let $K(x, y)$ be defined by* (1.5). *Then it holds that*:

$$(2.5) \qquad K(x, x) = \sum_{n=0}^{x} \binom{x}{n}^2 = \binom{2x}{x} = e_x(2x),$$

$$(2.6) \qquad \left| K(x,y) \right| \leq K(x,x)^{1/2} K(y,y)^{1/2} = \left(\binom{2x}{x} \binom{2y}{y} \right)^{1/2}.$$

Proof. The first claim follows from setting $x = y$ in (1.5). The second claim follows from the Cauchy–Schwarz inequality, since K is positive definite. $\square$

3. The binomial Fourier transform

In this section, we introduce the first family of transforms. We then use them in order to characterize functions on $\mathbb{Z}_+$ belonging to $\mathcal{H}(K)$, their properties, both analytic and algebraic. For example in Theorem 3.3, we give a natural isomorphism between $\mathcal{H}(K)$ and the Hardy space of the disk. In Theorem 3.14, we show that the orthonornal vectors e_n in $\mathcal{H}(K)$ from (1.2) generate an algebra, or rather a hypergroup. See [12].

The definition of the $\blacktriangle$-transform does not need the hypothesis that the sequence is in $\ell_2(\mathbb{Z}_+)$, and is as follows:

DEFINITION 3.1. Let f be a function from $\mathbb{Z}_+$ into $\mathbb{R}$. The binomial Fourier transform of f is defined by

$$(3.1) \qquad f^{\blacktriangle}(x) = \sum_{j=0}^{x} \binom{x}{j} f(j), \quad x = 0, 1, 2, \ldots.$$

Thus,

$$(3.2) \quad f^{\blacktriangle}(x) = f(0) + x f(1) + \frac{x(x-1)}{2} f(2) + \frac{x(x-1)(x-2)}{6} f(3) + \cdots$$
$$= \sum_{n=0}^{\infty} f(n) e_n(x).$$

The sums in (3.1) are finite, and $f^{\blacktriangle}$ exists for every function from $\mathbb{Z}_+$ into $\mathbb{R}$. We note that (3.1) can be written as

$$(3.3) \qquad f^{\blacktriangle} = Lf = L(1)f,$$

where $L(\lambda)$ is defined in Section 7 below by (7.3).

The image of the function $f(j) \equiv 0$ is the zero function. The image of $f(j) = (-1)^j$ is the function

$$(3.4) \qquad f^{\blacktriangle}(x) = \delta_{0,x}$$

while the image of $f(j) = 1$ is $f^{\blacktriangle}(x) = 2^x$.

Another example of interest is:

EXAMPLE 3.2. Let $f_a(j) = a^{*j}$, where $a \in \mathbb{C}$. Then,

$$(3.5) \qquad f^{\blacktriangle}(x) = \sum_{n=0}^{\infty} a^{*n} e_n(x)$$

is given by

$$(3.6) \qquad f_a^{\blacktriangle}(x) = \sum_{j=0}^{x} \binom{x}{j} a^{*j} = (1 + a^*)^x, \quad x = 0, 1, 2, \ldots.$$

Let now $\mathbf{H}^2(\mathbb{D})$ denote the Hardy space of complex-valued functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in the open unit disk and such that

$$(3.7) \qquad \|f\|_2^2 := \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

We define a map ψ from the complexified of $\mathcal{H}(K)$ into $\mathbf{H}^2(\mathbb{D})$ by

$$(3.8) \qquad \psi(e_n) = z^n.$$

THEOREM 3.3. *The map ψ is unitary from the complexified of $\mathcal{H}(K)$ onto the Hardy space $\mathbf{H}^2(\mathbb{D})$. Moreover, it associates to the function $f_a : x \mapsto (1 + a^*)^x, x \in \mathbb{Z}_+$ the function $z \mapsto \frac{1}{1 - za^*}, z \in \mathbb{D}$. In particular, for a, b in the open unit disk $\mathbb{D}$*

$$(3.9) \qquad \langle f_a^{\blacktriangle}, f_b^{\blacktriangle} \rangle_{\mathcal{H}(K)} = \frac{1}{1 - a^*b}.$$

Proof. The map is unitary since it maps an orthonormal basis onto an orthonormal basis. The claim on the image of the sequence f_a follows from (3.6) and (3.5). Finally, (3.9) follows from the unitarity of ψ, the formula

$$(3.10) \qquad \frac{1}{1 - za^*} = \sum_{n=0}^{\infty} z^n a^{*n}$$

and the definition of the inner product of $\mathbf{H}^2(\mathbb{D})$. $\qquad \square$

The next proposition is used to prove that the map $^{\blacktriangle}$ is unitary from $\ell^2(\mathbb{Z}_+)$ onto $\mathcal{H}(K)$. See Theorem 3.6.

PROPOSITION 3.4. *Let e_n defined by (1.2), and let*

$$(3.11) \qquad \widetilde{e_n}(m) = (-1)^{n+m} e_n(m), \quad and \quad \delta_n(m) = \delta_{n,m}.$$

Then

$$(3.12) \qquad \delta_n^{\blacktriangle} = e_n \quad and \quad (\widetilde{e_n})^{\blacktriangle} = \delta_n.$$

Proof. The first claim follows from the definition of e_n. As for the second claim, we have:

$$(\widetilde{e_n})^{\blacktriangle}(m) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{n+j} e_n(j)$$

$$= \begin{cases} 0, & \text{if } m < n \text{ (since } e_n(j) = 0 \text{ for } j < n) \\ \sum_{j=n}^{m} \binom{m}{j} \binom{j}{n} (-1)^{n+j}, & \text{if } m \geq n. \end{cases}$$

We conclude by using Lemma 2.1 to compute this last sum. $\qquad \square$

It follows from (3.9) that most, if not all, the problems considered in $\mathbf{H}^2(\mathbb{D})$ can be transposed in a natural way in the space $\mathcal{H}(K)$.

We now define the function $\varepsilon_n(j) = e_j(n)$. Thus, $(\varepsilon_n(j))$ can be identified with the finitely nonzero sequence of $\ell^2(\mathbb{Z}_+)$ whose nonzero terms are the first $n+1$ terms given by $\binom{n}{j}, j = 0, \ldots, n$.

LEMMA 3.5. *Let $n, m \in \mathbb{Z}_+$. Then,*

$$(3.13) \qquad \left(\varepsilon_n^{\blacktriangle}\right)(x) = K(x, n),$$

$$(3.14) \qquad \langle \varepsilon_n, \varepsilon_m \rangle_{\ell^2(\mathbb{Z}_+)} = \langle K(\cdot, n), K(\cdot, m) \rangle_{\mathcal{H}(K)}.$$

Proof. By definition of the transform,

$$(3.15) \qquad \left(\varepsilon_n^{\blacktriangle}\right)(x) = \sum_{j=0}^{x} \binom{x}{j} \varepsilon_n(j) = \sum_{j=0}^{x \wedge n} \binom{x}{j} \binom{n}{j} = K(x, n),$$

and so (3.13) holds. To prove (3.14), it suffices to notice that

$$(3.16) \qquad \langle \varepsilon_n, \varepsilon_m \rangle_{\ell^2(\mathbb{Z}_+)} = \sum_{j=0}^{n \wedge m} \binom{n}{j} \binom{m}{j} = K(n, m). \qquad \square$$

THEOREM 3.6. *The map $f \mapsto f^{\blacktriangle}$ defines a unitary mapping from $\ell_2(\mathbb{Z}_+)$ onto $\mathcal{H}(K)$, and its inverse is given by either formulas*

$$(3.17) \qquad g^{\blacktriangledown} = \sum_{n=0}^{\infty} a_n(-1)^n \delta_n \quad \text{for } g = \sum_{n=0}^{\infty} a_n e_n \in \mathcal{H}(K),$$

or

$$(3.18) \qquad g^{\blacktriangledown}(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} g(\ell)(-1)^{\ell-n}.$$

We note that (3.18) can be rewritten as

$$(3.19) \qquad g^{\blacktriangledown} = L(-1)g,$$

where $L(\lambda)$ is defined by (7.3) below.

Proof of Theorem 3.6. We present two proofs. The first is based on Proposition 3.4. Let

$$(3.20) \qquad f = \sum_{n=0}^{N} a_n \delta_n$$

be a real valued function on $\mathbb{Z}_+$ with finite support. Then,

$$(3.21) \qquad f^{\blacktriangle} = \sum_{n=0}^{N} a_n e_n \in \mathcal{H}(K),$$

and

$$(3.22) \qquad \|f^{\blacktriangle}\|^2_{\mathcal{H}(K)} = \|f\|^2_{\ell^2(\mathbb{Z}_+)} = \sum_{n=0}^{N} a_n^2.$$

The result follows by taking limits.

The second proof uses Lemma 3.5. The functions $\varepsilon_0, \varepsilon_1, \ldots$ span a dense set of $\ell_2(\mathbb{Z}_+)$, and therefore (3.14) extends to an isometry. The isometry is onto since the kernels span a dense set of $\mathcal{H}(K)$. Therefore the transform $f^{\blacktriangle}$ is unitary. To prove its inverse is given by (3.18), we note that (2.4) can be rewritten as

$$(3.23) \qquad \left(K_m^{\blacktriangledown}\right)(n) = \varepsilon_m(n), \quad n \in \mathbb{Z}_+. \qquad \square$$

As a corollary of the above, the next results offer an answer to the following question: *Given a function f on $\mathbb{Z}_+$, when does f belong to the reproducing kernel Hilbert space $\mathcal{H}(K)$ with reproducing kernel (1.5)?*

COROLLARY 3.7. *Let $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ be a function, and set*

$$(3.24) \qquad f^{\blacktriangledown}(x) = \sum_{\ell=0}^{x} \binom{x}{\ell} f(\ell)(-1)^{\ell-x}.$$

Then, $f \in \mathcal{H}(K)$ if and only if the transform $x \mapsto f^{\blacktriangledown}(x)$ is in $\ell^2(\mathbb{Z}_+)$.

Proof. In Theorem 3.6, we proved that if $f \in \mathcal{H}(K)$ then its transform $f^{\blacktriangledown}$ belongs to $\ell^2(\mathbb{Z}_+)$. In fact the transform is an isometric isomorphism of $\mathcal{H}(K)$ onto $\ell^2(\mathbb{Z}_+)$.

We turn to the converse: Let f be a fixed function such that $f^{\blacktriangledown} \in \ell^2(\mathbb{Z}_+)$. Recall (see Theorem 2.2) that

$$(3.25) \qquad e_n(x) = \sum_{j=0}^{n} (-1)^{j+n} \binom{n}{j} K_j(x).$$

Since the functions e_n form an orthonormal basis of $\mathcal{H}(K)$, we conclude that

$$(3.26) \qquad F := \sum_{n=0}^{\infty} f^{\blacktriangledown}(n) e_n \in \mathcal{H}(K),$$

and moreover,

$$(3.27) \qquad \langle F, e_n \rangle_{\mathcal{H}(K)} = f^{\blacktriangledown}(n), \quad \forall n \in \mathbb{Z}_+.$$

Using (3.25) again, together with the reproducing kernel property for F we conclude that

$$f(x) = F(x), \quad \forall x \in \mathbb{Z}_+,$$

and so $f \in \mathcal{H}(K)$ as claimed. $\qquad \square$

DEFINITION 3.8. For $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ we define the following Hurwitz transformation:

$$(3.28) \qquad f \mapsto H(f)(z) = \sum_{x=0}^{\infty} \frac{f(x)}{x!} z^x = e^z \sum_{n=0}^{\infty} \frac{f^{\blacktriangledown}(n)}{n!} z^n,$$

where $z \in \mathbb{C}$ is a generating function variable.

That the two expressions for $H(f)$ coincide can be checked as follows. For $n = 0, 1, \ldots$ we have the convolution expression:

$$(3.29) \qquad \frac{f^{\blacktriangledown}(n) z^n}{n!} = \sum_{x=0}^{n} \frac{f(x) z^x}{x!} \cdot \frac{(-1)^{n-x} z^{n-x}}{(n-x)!}.$$

The result follows from the theorem on the coefficients of a product of power series.

We have the following corollary to Corollary 3.7.

COROLLARY 3.9. *Let* $f \in \mathcal{H}(K)$, *and consider* $f^{\blacktriangledown} \in \ell^2(\mathbb{Z}_+)$, *see* (3.24). *Then the following hold for the respective Hurwitz-transforms:*

$$(3.30) \qquad H(f)(z) = e^z H\left(f^{\blacktriangledown}\right)(z).$$

In particular, $H(f)$ *is an entire function.*

The following proposition will be used in Corollary 3.9.

PROPOSITION 3.10. *Assume* $(f(x))_{x \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$. *Then the Hurwitz transform* $H(f)$ *is an entire function of* z.

Proof. Indeed, for any $z \in \mathbb{C}$, the sequence

$$\left(\frac{z^x}{x!} \right)_{x \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+).$$

By the Cauchy–Schwarz inequality, we see that the series (3.28) converges then absolutely for all complex numbers z. $\qquad \square$

Proof of Corollary 3.9. By Corollary 3.7, $f \in \mathcal{H}(K)$ if and only if there exists $(a_n)_{n \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$ such that

$$(3.31) \qquad f = \sum_{n=0}^{\infty} a_n e_n,$$

with

$$(3.32) \qquad a_n = f^{\blacktriangledown}(n) = \sum_{x=0}^{n} (-1)^{n+x} \binom{n}{x} f(x).$$

In a way similar to (3.29), we have:

$$(3.33) \qquad \frac{f^{\blacktriangledown}(n)}{n!} = \sum_{x=0}^{n} \frac{f(x)}{x!} \frac{(-1)^{n-x}}{(n-x)!}.$$

By Proposition 3.10, the function $H(f^{\blacktriangledown})$ is entire. Since

$$(3.34) \qquad \frac{f(n)}{n!} = \sum_{x=0}^{n} \frac{f^{\blacktriangledown}(x)}{x!} \frac{1}{(n-x)!},$$

classical results on power series and convolution implies that

$$(3.35) \qquad H(f)(z) = \big(H\big(f^{\blacktriangledown}\big)(z)\big)e^{z},$$

and $H(f)$ is in particular an entire function. $\qquad\square$

PROPOSITION 3.11. *Let* $f : \mathbb{Z}_{+} \longrightarrow \mathbb{R}$, *and assume that* $f \in \mathcal{H}(K)$. *Then, for every* $\varepsilon > 0$ *there exists* $M > 0$ *such that*

$$(3.36) \qquad \big|f(n)\big| \le Mn!\varepsilon^{n}, \quad n \in \mathbb{Z}_{+}.$$

Proof. From the preceding proposition, we know that $H(f)$ is an entire function. Thus for every $r > 0$ there exists an $M > 0$ such that

$$r^{n}\frac{|f(n)|}{n!} \le M, \quad n \in \mathbb{Z}_{+}.$$

The result follows with $r = 1/\varepsilon$. $\qquad\square$

We note that (3.36) is not sufficient to guarantee that $f \in \mathcal{H}(K)$. Indeed, the series

$$\sum_{n=0}^{\infty} \frac{f^{\blacktriangledown}(n)}{n!} z^{n}$$

may define an entire function even when $(f^{\blacktriangledown}(n))_{n \in \mathbb{Z}_{+}} \notin \ell^{2}(\mathbb{Z}_{+})$. For example, consider the function g_{a} given by:

$$(3.37) \qquad g_{a}(y) = \frac{a^{y}}{y!}, \quad y \in \mathbb{Z}_{+}.$$

Then

$$g_{a}^{\blacktriangledown}(n) = (-1)^{n} \sum_{x=0}^{n} \binom{n}{x} \frac{(-a)^{x}}{x!}$$

$$= (-1)^{n} \sum_{x=0}^{n} (-a)^{x} \frac{n(n-1)\cdots(n-x+1)}{(x!)^{2}},$$

and so $(g_{a}^{\blacktriangledown}(n))_{n \in \mathbb{Z}_{+}} \notin \ell^{2}(\mathbb{Z}_{+})$. On the other hand,

$$H\big(g_{a}^{\blacktriangledown}\big)(z) = e^{-z}\big(H(g_{a})(z)\big) = e^{-z} \sum_{n=0}^{\infty} \frac{(az)^{n}}{(n!)^{2}}$$

is entire. But $g_{a} \notin \mathcal{H}(K)$ since $(g_{a}^{\blacktriangledown}(n))_{n \in \mathbb{Z}_{+}} \notin \ell^{2}(\mathbb{Z}_{+})$.

COROLLARY 3.12. *Let $f : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ be a function, and assume that $f \in \mathcal{H}(K)$. Then there exists $M < \infty$ such that*

$$(3.38) \qquad \left| f(x) \right|^2 \leq M \binom{2x}{x}, \quad \forall x \in \mathbb{Z}_+.$$

(*In other words, condition* (3.38) *is necessary for f to be in $\mathcal{H}(K)$.*)

Proof. Using the Cauchy–Schwarz inequality we get

$$\left| f(x) \right|^2 \leq \|f\|^2_{\mathcal{H}(K)} \cdot K(x,x)$$
$$= \|f\|^2_{\mathcal{H}(K)} \cdot \binom{2x}{x}.$$

See (2.5) for the latter. $\square$

REMARK 3.13. Let $a \in \mathbb{R} \setminus \{0\}$, and let g_a be given by (3.37), that is

$$(3.39) \qquad g_a(x) = \frac{a^x}{x!}, \quad x \in \mathbb{Z}_+,$$

with as usual $0! = 1$. The condition (3.38) is satisfied, but $g_a^{\blacktriangledown}$ is not in $\ell_2(\mathcal{Z}_+)$. We conclude that condition (3.38) is not sufficient and $g_a \notin \mathcal{H}(K)$.

We now define an isometric isomorphism between the reproducing kernel Hilbert space $\mathcal{H}(K)$ and the reproducing kernel Hilbert space $\mathcal{H}(K_2)$ of entire functions associated to the positive definite function

$$(3.40) \qquad K_2(\zeta, z) = \sum_{n=0}^{\infty} \frac{\zeta^n z^{*n}}{(n!)^2}, \quad \zeta, z \in \mathbb{C}.$$

An orthonormal basis of the space $\mathcal{H}(K_2)$ is given by the functions

$$(3.41) \qquad \zeta \mapsto \frac{\zeta^n}{n!}, \quad n = 0, 1, \ldots.$$

For f a real- or complex-valued function defined on $\mathbb{Z}_+$ we define the map

$$(3.42) \qquad \left(\mathscr{H}(f) \right)(\zeta) = e^{-\zeta} \left(H(f) \right)(\zeta).$$

THEOREM 3.14. *The map $\mathscr{H}$ is unitary from $\mathcal{H}(K)$ onto $\mathcal{H}(K_2)$, i.e., is isometric and onto.*

Proof. Corollary 3.7 implies that $\mathscr{H}$ maps $\mathcal{H}(K)$ into $\mathcal{H}(K_2)$. Indeed f belongs to $\mathcal{H}(K)$ if and only if the transform $f^{\blacktriangledown}$ belongs to $\ell^2(\mathbb{Z}_+)$. But the function

$$(3.43) \qquad \left(\mathscr{H}(f) \right)(\zeta) = e^{-\zeta} \sum_{n=0}^{\infty} f(x) \frac{\zeta^x}{x!} = \sum_{n=0}^{\infty} f^{\blacktriangledown}(n) \frac{\zeta^n}{n!}$$

belongs to $\mathcal{H}(K_2)$ since the functions (3.41) form an orthonormal basis of $\mathcal{H}(K_2)$ and $f^{\blacktriangledown} \in \ell^2(\mathbb{Z}_+)$.

To finish the proof, we show that the map $\mathscr{H}$ sends an orthonormal basis of $\mathcal{H}(K)$ onto an orthonormal basis of $\mathcal{H}(K_2)$. We claim that

$$(3.44) \qquad \left(\mathscr{H}(e_n)\right)(\zeta) = \frac{\zeta^n}{n!}, \quad n = 0, 1, \ldots.$$

Indeed

$$(3.45) \qquad \left(\mathscr{H}(e_n)\right)(\zeta) = e^{-\zeta} \sum_{x=n}^{\infty} e_n(x) \frac{\zeta^x}{x!}$$

$$= e^{-\zeta} \frac{\zeta^n}{n!} \sum_{x=n}^{\infty} \frac{\zeta^{x-n}}{(x-n)!}$$

$$= \frac{\zeta^n}{n!}$$

concluding the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem establishes a co-product for the binomial functions e_n in (1.2). It follows from our analysis that the functions $\{e_n | n \in \mathbb{N}_0\}$ generate a hypergroup HG; in the sense of Lasser et al., see [12]. Here we refer to our formula (3.47) below for the co-product in HG. And in (3.48) we give an explicit formula for the coefficients defining the co-product.

THEOREM 3.15.

(1) *It holds that*

$$(3.46) \qquad e_n^{\blacktriangledown}(x) = \delta_{n,x}.$$

(2) *We have*

$$(3.47) \qquad e_n \cdot e_m = \sum_{k=m\vee n}^{m+n} (e_m \cdot e_n)^{\blacktriangledown}(k) e_k,$$

where

$$(3.48) \qquad (e_m \cdot e_n)^{\blacktriangledown}(k) = (-1)^{n+k} \binom{k}{n} \left(\sum_{\ell=0}^{k-n} \binom{k-n}{\ell} \binom{\ell+n}{m} \right).$$

For example, we have

$$(3.49) \qquad e_1 \cdot e_n = n e_n + (n+1) e_{n+1}.$$

Proof of Theorem 3.15. The first claim is a mere rewriting of the second equality in (3.12). To prove (3.48), we assume $m \le n$ (so that $m \vee n = n$) and

write

$$(e_m \cdot e_n)^{\blacktriangledown}(k) = \sum_{y=n}^{k} (-1)^{y+k} \binom{k}{y}\binom{y}{n}\binom{y}{m}$$

$$= \sum_{y=n}^{k} \binom{k}{n}\binom{k-n}{y-n}\binom{y}{m}$$

$$= \binom{k}{n}(-1)^{n+k}\left(\sum_{\ell=0}^{k-n}(-1)^{\ell}\binom{k-n}{\ell}\binom{\ell+n}{m}\right). \qquad \square$$

4. The second transform

We show that the two classes of transforms in our study are two sides of a harmonic analysis for the reproducing kernel Hilbert space $\mathcal{H}(K)$, see Theorem 1.1 and Theorem 4.5 below. In the sense of computations, a main distinction between the two is that the first class of transforms involves only finite summations, while the second infinite; see, for example, Definition 4.3, and Proposition 4.6. In the terminology of the infinite square matrices in Sections 1 and 7, the distinction reflects the difference in the algebra of infinite lower triangular matrices, vs upper triangular ones. In Theorem 4.8, we establish explicit inversion formulas. Caution, while the range of the respective isometric transforms are Hilbert spaces, they differ from one to the other.

We note that the function

$$(4.1) \qquad (-1)^{y+z}K(y,z) = (-1)^{y}(-1)^{z}K(y,z)$$

is still positive on $\mathbb{Z}_+$, and recall that we denote by $\ell_0(\mathbb{Z}_+)$ the vector space of sequences from $\mathbb{Z}_+$ into $\mathbb{R}$, with compact support. We endow $\ell_0(\mathbb{Z}_+)$ with the bilinear form

$$(4.2) \qquad \langle f, g \rangle_K = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}(-1)^{y+z}f(x)g(y)K(x,y).$$

PROPOSITION 4.1. *The space $\ell_0(\mathbb{Z}_+)$ endowed with the bilinear form* (4.2) *is a pre-Hilbert space.*

Proof. Bilinearity of $\langle \cdot, \cdot \rangle_K$ is clear from the definition, while the positivity property

$$(4.3) \qquad \langle f, f \rangle_K \geq 0, \quad \forall f \in \ell_0(\mathbb{Z}_+)$$

follows from the fact that the function (4.1) is positive definite on $\mathbb{Z}_+$. We now show that the form $\langle \cdot, \cdot \rangle_K$ is nondegenerate. Let $g \in \ell_0(\mathbb{Z}_+)$ be such that

$$(4.4) \qquad \langle f, g \rangle_K = 0, \quad \forall f \in \ell_0(\mathbb{Z}_+).$$

In particular,

$$(4.5) \qquad \langle g, g \rangle_K = 0.$$

This cannot be since all the submatrices built from the kernel K from different points are strictly positive (see Theorem 1.5). $\qquad\square$

We denote by $\ell^2_K(\mathbb{Z}_+)$ the closure of $\ell_0(\mathbb{Z}_+)$ with respect to $\langle \cdot, \cdot \rangle_K$.

PROPOSITION 4.2. *The map*

$$(4.6) \qquad f \mapsto \sum_{y=0}^{\infty} K(x,y)(-1)^y f(y),$$

first defined for $f \in \ell_0(\mathbb{Z}_+)$, extends to a unitary map from $\ell^2_K(\mathbb{Z}_+)$ onto $\mathcal{H}(K)$.

Proof. Let $f \in \ell_0(\mathbb{Z}_+)$. We have (recall that all the sums are finite since f has finite support)

$$(4.7) \qquad \langle f, f \rangle_K = \sum_{n,m=0}^{\infty} (-1)^n (-1)^m f(n) f(m) K(n,m)$$

$$= \left\langle \sum_{n=0}^{\infty} K_n (-1)^n f(n), \sum_{m=0}^{\infty} K_m (-1)^m f(m) \right\rangle_{\mathcal{H}(K)}.$$

The claim then follows from density and from the continuity of the inner product. $\qquad\square$

DEFINITION 4.3. Let $f \in \ell_0(\mathbb{Z}_+)$. We set

$$(4.8) \qquad f^{\Delta}(x) = \sum_{y=x}^{\infty} (-1)^{x+y} \binom{y}{x} f(y).$$

We note that (4.8) can be rewritten as

$$(4.9) \qquad f^{\Delta} = M(-1)f,$$

where $M(\lambda)$ is defined by (7.12).

Explicitly, the series is written as

$$(4.10) \qquad f^{\Delta}(x) = (-1)^x \left(f(x) - (x+1)f(x+1) + \frac{(x+2)(x+1)}{2} f(x+2) \right.$$

$$\left. - \frac{(x+3)(x+2)(x+1)}{6} f(x+3) + \cdots \right).$$

Note that $f^{\Delta} \in \ell_0(\mathbb{Z}_+)$, and more precisely, if $f(x) = 0$ for $x \geq N$ then $f^{\Delta}(x) = 0$ for $x \geq N$.

EXAMPLE 4.4. Let g_a be defined by (3.37): $g_a(y) = \frac{a^y}{y!}$, where $a \in \mathbb{R}$. Then,

$$(4.11) \qquad g_a^{\Delta} = e^{-a} g_a.$$

Indeed,

$$
\begin{aligned}
g_a^\Delta(x) &= \sum_{y=x}^\infty (-1)^{x+y} \binom{y}{x} \frac{a^y}{y!} \\
&= \sum_{y=x}^\infty (-1)^{x+y} \frac{a^y}{x!(y-x)!} \\
&= \frac{1}{x!} \sum_{y=x}^\infty (-1)^{x+y} \frac{a^y}{(y-x)!} \\
&= \frac{a^x}{x!} \sum_{u=0}^\infty (-1)^u \frac{a^u}{u!} \\
&= e^{-a} \frac{a^x}{x!},
\end{aligned}
$$

where we have used the change of index $y - x = u$ to go from the third to the fourth line.

We note the following: though g_a does not belong to $\mathcal{H}(K)$ unless $a = 0$, the Δ transform of g_a exists for all real a.

In preparation of Theorem 4.5, we introduce the notation $s(f)$ as:

(4.12) $$ f \mapsto s(f) := \sum_{x \in \mathbb{Z}_+} K_x f(x). $$

THEOREM 4.5. *The map $f \mapsto s(f^\Delta)$ extends to a unitary map from $\mathcal{H}(K)$ onto $\ell^2(\mathbb{Z}_+)$.*

Proof. Let $f \in \ell_0(\mathbb{Z}_+)$. We have (recall that all the sums are finite since f has finite support):

$$
\begin{aligned}
\langle s(f^\Delta), s(f^\Delta) \rangle_{\mathcal{H}(K)} &= \sum_{x_1=0}^\infty \sum_{x_2=0}^\infty s(f^\Delta)(x_1) s(f^\Delta)(x_2) K(x_1, x_2) \\
&= \sum_{x_1=0}^\infty \sum_{x_2=0}^\infty \left(\sum_{y=x_1}^\infty (-1)^{x_1+y} \binom{y}{x_1} f(y) \right) \\
&\quad \times K(x_1, x_2) \left(\sum_{z=x_2}^\infty (-1)^{x_2+z} \binom{z}{x_2} f(z) \right) \\
&= \sum_{x_1=0}^\infty \sum_{x_2=0}^\infty \sum_{y=x_1}^\infty \sum_{z=x_2}^\infty \sum_{j=0}^{x_1 \wedge x_2} (-1)^{x_1+x_2+y+z} f(y) f(z) \\
&\quad \times \binom{y}{x_1} \binom{z}{x_2} \binom{x_1}{j} \binom{x_2}{j}
\end{aligned}
$$

$$= \sum_{y=0}^{\infty}\sum_{z=0}^{\infty} f(y)f(z)\left(\sum_{j=0}^{\infty}\left(\sum_{x_1=j}^{y}(-1)^{x_1+y}\binom{y}{x_1}\binom{x_1}{j}\right)\right.$$
$$\left.\times\left(\sum_{x_2=j}^{z}(-1)^{x_2+z}\binom{z}{x_2}\binom{x_2}{j}\right)\right)$$

and using Lemma 2.1

$$= \sum_{j=0}^{\infty}\sum_{y=0}^{\infty}\sum_{z=0}^{\infty} f(y)f(z)\delta_{j,y}\delta_{j,z}$$
$$= \sum_{j=0}^{\infty} f(j)^2. \qquad \square$$

We set $\mathscr{A}(f) = s(f^\triangle)$. It turns out (see the next proposition), that although $s(f)$ is difficult to compute for general value of f, the operator $\mathscr{A}$ and its adjoint are easier to handle.

PROPOSITION 4.6. *The adjoint $\mathscr{A}^*$ of the operator $\mathscr{A}$ is given by:*

$$(4.13) \qquad \left(\mathscr{A}^*(\varphi)\right)(x) = \sum_{y=x}^{\infty}(-1)^{x+y}\binom{y}{x}\varphi(y), \quad \varphi \in \mathcal{H}(K),$$

and with values in $\ell^2(\mathbb{Z}_+)$.

Proof. We check the result for $\varphi_{K_{y_0}}$. The general case follows then by continuity. Let $f \in \ell^2(\mathbb{Z}_+)$. We have:

$$(4.14) \qquad \langle \mathscr{A}f, \varphi\rangle_{\mathcal{H}(K)} = (\mathscr{A}f)(y_0)$$
$$= \sum_{t=0}^{\infty} f^\triangle(t)K(y_0,t)$$
$$= \sum_{t=0}^{\infty}\sum_{s=t}^{\infty}(-1)^{t+s}f(s)\binom{s}{t}K(y_0,t)$$
$$= \sum_{s=0}^{\infty} f(s)(-1)^s\sum_{t=0}^{s}\binom{s}{t}K(y_0,t),$$

and hence the result. $\qquad \square$

DEFINITION 4.7. Let $g \in \ell_0(\mathbb{Z}_+)$. We set

$$(4.15) \qquad g^\nabla(n) = \sum_{x=n}^{\infty}\binom{x}{n}g(x).$$

THEOREM 4.8. *The map $g \mapsto g^\nabla$ extends to a unitary map from $\ell^2_K(\mathbb{Z}_+)$ onto $\ell^2(\mathbb{Z}_+)$, and its adjoint is the map $f \mapsto f^\triangle$. In particular:*

(4.16)
$$\left(g^\nabla\right)^\triangle = g, \quad \forall g \in \ell^2_K(\mathbb{Z}_+),$$
$$\left(f^\triangle\right)^\nabla = f, \quad \forall f \in \ell^2(\mathbb{Z}_+).$$

Proof. We divide the proof in two steps.

Step 1: We first prove that the two transforms are inverse of each other. Then we show that they are adjoint of each other.

Let $g \in \ell_0(\mathbb{Z}_+)$. We have:

$$
\begin{aligned}
\left(g^\nabla\right)^\triangle(y) &= \sum_{n=0}^{\infty} (-1)^{n+y} \binom{n}{y} \left(g^\triangle(n)\right) \\
&= \sum_{n=y}^{\infty} (-1)^{n+y} \binom{n}{y} \left(\sum_{x=n}^{\infty} \binom{x}{n} g(x)\right) \\
&= \sum_{x=y}^{\infty} \left(\sum_{n=y}^{x} (-1)^{n+y} \binom{n}{y} \binom{x}{n}\right) \\
&= g(y)
\end{aligned}
$$

since (see Lemma 2.1)

(4.17)
$$\sum_{n=y}^{x} (-1)^{n+y} \binom{n}{y} \binom{x}{n} = \delta_{x,y}.$$

Similarly, we have for $f \in \ell^2(\mathbb{Z}_+)$ and $n \geq 0$:

$$
\begin{aligned}
\left(\left(f^\triangle\right)^\nabla\right)(n) &= \sum_{x=n}^{\infty} \binom{x}{n} f^\triangle(x) \\
&= \sum_{x=n}^{\infty} \binom{x}{n} \left(\sum_{m=x}^{\infty} \binom{m}{x} f(m)\right) \\
&= \sum_{m=n}^{\infty} f(n) \left(\sum_{x=n}^{m} (-1)^{x+m} \binom{m}{x} \binom{x}{n}\right) \\
&= f(m)
\end{aligned}
$$

since (see Lemma 2.1)

(4.18)
$$\sum_{x=n}^{m} (-1)^{x+m} \binom{m}{x} \binom{x}{n} = \delta_{m,n}.$$

Step 2: We now prove that the two transforms are adjoint of each other.

Let $f \in \ell^2(\mathbb{Z}_+)$ and $g \in \ell_K^2(\mathbb{Z}_+)$. We have:

$$(4.19) \qquad \langle g, f^\triangle \rangle_{\ell_K^2(\mathbb{Z}_+)} = \sum_{x,y=0}^{\infty} g(x) f^\triangle(y) K(x,y)$$

$$= \sum_{x,y=0}^{\infty} g(x) \left(\sum_{t=y}^{\infty} (-1)^{t+y} f(t) \binom{t}{y} \right) K(x,y)$$

$$= \sum_{t=0}^{\infty} f(t) \left(\sum_{x=0}^{\infty} g(x) \sum_{y=0}^{t} (-1)^{t+y} \binom{t}{y} \right) K(x,y)$$

$$= \sum_{t=0}^{\infty} f(t) \left(\sum_{x=t}^{\infty} g(x) \binom{x}{t} \right),$$

where we use (2.4)

$$= \langle g^\nabla f \rangle_{\ell^2(\mathbb{Z}_+)}.$$

To conclude, we now prove that the ∇ transform is an isometry. We first prove this on the space $\mathcal{H}_0$ of functions from $\mathbb{Z}_+$ into $\mathbb{R}$ with finite support. For $g \in \ell_0$ we have:

$$(4.20) \qquad \left\| g^\triangle \right\|_{\mathcal{H}(K)}^2 = \sum_{n=0}^{\infty} \left(\sum_{x=n}^{\infty} \binom{x}{n} g(x) \right)^2$$

$$= \sum_{x,y=0}^{\infty} g(x) g(y) \left(\sum_{n=0}^{x \wedge y} \binom{x}{n} \binom{y}{n} \right)$$

$$= \sum_{x,y=0}^{\infty} g(x) g(y) K(x,y)$$

$$= \| g \|_{\ell^2(K)}^2. \qquad \qquad \square$$

5. Motivation from discrete analytic functions

Here we offer a brief outline of the significance of our binomial functions in the study of discrete analytic functions. While our binomial functions are functions of a single discrete variable (here $\mathbb{Z}_+$), we show in [2] that the binomial functions extend uniquely to discrete analytic functions on the $2D$ lattice, so formulas obtained in the earlier part of our paper have direct implications for Hilbert spaces of discrete analytic functions.

Recall first that a function $f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ is said to be discrete analytic if

$$(5.1) \quad \forall (x,y) \in \mathbb{Z}^2, \quad \frac{f(x+1,y+1) - f(x,y)}{1+i} = \frac{f(x+1,y) - f(x,y+1)}{1-i}.$$

See [7], [6].

Given a function $f_0 : \mathbb{Z} \longrightarrow \mathbb{C}$ there are infinitely discrete analytic functions f on $\mathbb{Z}^2$ such that $f(x,0) = f_0(x)$. However, when f_0 is a polynomial, only one of these discrete analytic extensions will be a polynomial in x, y. See [6] (we gave a new proof of this fact in [2]). Thus, there exists a unique discrete analytic polynomial $\zeta_n(x,y)$ determined by $\zeta_n(x,0) \equiv x^{[n]}$, where $x^{[n]}$ has been defined in (1.7).

6. The quantum case

In a number of questions from mathematical physics (see [9], [8], [10] and the papers cited there), it is of interest to consider the questions from Sections 2 through 3 for a deformation family of the binomial functions (1.2). Traditionally the complex deformation parameter is denoted q, and the q-versions of the binomial formulas were first studied in [4]. The purpose of this section is to prove an analogue of the orthonormal basis theorems but adapted to the q-binomials.

The q-deformed binomial coefficients were defined by Carlitz; see [5], [4]. We fix $q \in \mathbb{R} \setminus \{1\}$, and set

$$
\begin{aligned}
[x] &:= \frac{q^x - 1}{q - 1}, \\
[x]_n &:= [x][x-1]\cdots[x-n+1], \\
[n]! &:= [n]_n, \quad \text{and} \quad [0]! = [0] = 1, \\
\begin{bmatrix} x \\ n \end{bmatrix} &:= \frac{[x]_n}{[n]!},
\end{aligned}
\tag{6.1}
$$

where all the expressions depend on the value of q. To lighten the notation, we do not stress the q-dependence. We introduce

$$
e_n(x) = \begin{cases} 0, & \text{if } x < n, \\ \begin{bmatrix} x \\ n \end{bmatrix} = \frac{[x]_n}{[n]!}, & \text{if } x \geq n, \end{cases}
\tag{6.2}
$$

and

$$
K(x,y) = \sum_{n=0}^{\infty} e_n(x) e_n(y), \quad x, y \in \mathbb{Z}_+.
\tag{6.3}
$$

The sum (6.3) is always finite and well defined, and defines a positive definite function on $\mathbb{Z}_+$. It can also be rewritten as:

$$
K(x,y) = \sum_{n=0}^{x \wedge y} \begin{bmatrix} x \\ n \end{bmatrix} \begin{bmatrix} y \\ n \end{bmatrix}.
\tag{6.4}
$$

The functions $e_0, e_1, \ldots$ are easily seen to be linearly independent, and so we have:

THEOREM 6.1. *The functions $e_0, e_1, \ldots$ form an orthonormal basis of $\mathcal{H}(K)$.*

7. A one parameter group of transforms

Let $\lambda \in (0, \infty)$. The function

$$(7.1) \qquad K_\lambda(x, y) = \sum_{n=0}^{x \wedge y} \lambda^{x+y-2n} \binom{x}{n} \binom{y}{n}$$

is positive definite in $\mathbb{Z}_+$, and we denote by $\mathcal{H}(K_\lambda)$ the associated reproducing kernel Hilbert space. The functions

$$(7.2) \qquad e_n^{(\lambda)}(x) = \lambda^{x-n} e_n(x), \quad n = 0, 1, \ldots$$

form an orthonormal basis of the associated reproducing kernel Hilbert space $\mathcal{H}(K_\lambda)$. Note that the case $\lambda = 1$ corresponds to formulas (1.5) and (1.2). Let $L(\lambda)$ denote the Zadeh-transform of the lower triangular infinite matrix L given by (1.9), that is the lower triangular infinite matrix $M(\lambda)$ as follows: For a point $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, let x denote row-index and y denote column-index. The matrix $L(\lambda)$ is defined by:

$$(7.3) \qquad \big(L(\lambda)\big)_{(x,y)} = \begin{cases} \lambda^{x-y} \binom{x}{y}, & \text{if } 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $L(\lambda)$ is given by

$$(7.4) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & \cdots \\ \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & \cdots \\ \lambda^2 & 2\lambda & 1 & 0 & 0 & 0 & 0 & \cdots & & \cdots \\ \lambda^3 & 3\lambda^2 & 3\lambda & 1 & 0 & 0 & 0 & \cdots & & \cdots \\ \lambda^4 & 4\lambda^3 & 6\lambda^2 & 4\lambda & 1 & 0 & 0 & \cdots & & \cdots \\ \vdots & \vdots & & & & & & & & \\ \lambda^n & n\lambda^{n-1} & \lambda^{n-2}\binom{n}{2} & \cdots & n\lambda & 1 & 0 & 0 & \cdots & \cdots \\ \lambda^{n+1} & (n+1)\lambda^n & \lambda^{n-1}\binom{n+1}{2} & \cdots & & (n+1)\lambda & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & 1 & \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Then the matrix product $L(\lambda)L(\mu)$ makes sense for all $\lambda, \mu \in \mathbb{C}$, with

$$(7.5) \qquad \big(L(\lambda)L(\mu)\big)_{(x,y)} = \sum_{k=y}^{x} L(\lambda)_{(x,k)} L(\mu)_{(k,y)}, \quad \lambda, \mu \in \mathbb{C}.$$

LEMMA 7.1. *With the notation above, it holds that*

$$(7.6) \qquad L(\lambda + \mu) = L(\lambda)L(\mu), \quad \lambda, \mu \in \mathbb{C},$$

that is, the matrices in (7.3) define a one-parameter semi-group of infinite matrices.

Proof. We give the details, but note that the computation of (7.5), is essential the same as that given in the proof of Lemma 2.1 above. In other words, when the product in (7.5) is computed, the group law (7.6) follows. More precisely,

$$(7.7) \qquad L(\lambda)L(\mu) = D(\lambda)L(1)D(\lambda)^{-1}D(\mu)L(1)D(\mu)^{-1}$$

$$= D(\lambda)L(1)D\left(\frac{\mu}{\lambda}\right)L(1)D(\mu)^{-1}.$$

But for $y \le x$ we have:

$$\left(L(1)D\left(\frac{\mu}{\lambda}\right)L(1)\right)_{(x,y)} = \sum_{k=y}^{x} \binom{x}{k}\left(\frac{\mu}{\lambda}\right)^k \binom{k}{y}$$

and using $\binom{x}{k}\binom{k}{y} = \binom{x}{y}\binom{x-y}{k-y}$, this equality is equal to

$$= \binom{x}{y}\left(\sum_{k=y}^{x}\left(\frac{\mu}{\lambda}\right)^k \binom{x-y}{k-y}\right)$$

$$= \left(1+\frac{\mu}{\lambda}\right)^{x-y}\left(\frac{\mu}{\lambda}\right)^y \binom{x}{y}.$$

Thus

$$(7.8) \qquad L(1)D\left(\frac{\mu}{\lambda}\right)L(1) = D\left(1+\frac{\mu}{\lambda}\right)L(1)D\left(\frac{\mu}{\mu+\lambda}\right),$$

and this, together with (7.7) leads to the result.   □

COROLLARY 7.2. *The one-parameter group $\lambda \mapsto L(\lambda)$ of infinite matrices in (7.4) has the form*

$$(7.9) \qquad\qquad\qquad L(\lambda) = \exp(\lambda A),$$

where A is the following sub-diagonal banded matrix

$$(7.10) \qquad\qquad\qquad A_{(x,y)} = x\delta_0(x-y-1),$$

see (7.11).

$$(7.11) \qquad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdot & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdot & \cdots \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdot & \cdots \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & \cdots & \cdot & \cdots \\
\vdots & \vdots & & & & & & & & \\
0 & 0 & \cdots & \cdots & n-1 & 0 & 0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & & n & 0 & 0 & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & n+1 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}.
$$

We denote

$$(7.12) \qquad M(\lambda) = L(\lambda)^*.$$

We define

$$(7.13) \qquad \big(T_{-\lambda}\big(e_n^{(\lambda)}\big)\big)(x) = \delta_{n,x}.$$

The proof of the following theorem is easy and will be omitted.

THEOREM 7.3. *The map*

$$(7.14) \qquad T_{-\lambda} : f \mapsto L(\lambda)^* f$$

is an isometry from $\mathcal{H}(K_\lambda)$ *into* $\ell^2(\mathbb{Z}_+)$.

As a corollary, we have:

$$(7.15) \qquad \big(L(\lambda)L(\lambda)^*\big)_{(x,y)} = K_\lambda(x,y) = \sum_{n=0}^{x \wedge y} e_n^{(\lambda)}(x) e_n^{(\lambda)}(y)^*,$$

where $e_n^{(\lambda)}(x) = \lambda^{x-n} e_n(x)$.

Before stating another corollary to this theorem, we mention the following lemma, whose proof is left to the reader.

LEMMA 7.4. *Let* $(X_x(\cdot))$ *be a Gaussian process with index* x, *with mean zero and probability space* $(\Omega, \mathcal{F}, P)$. *Let* f *be a nonvanishing function on the index set. Then, the Gaussian process* $(Y_x(\cdot)) = (f(x)X_x(\cdot))$ *has the same probability space but with covariance* $\mathbb{E}(Y_x Y_y) = f(x)\mathbb{E}(X_x X_y)f(y)$.

COROLLARY 7.5. *Let* $(\Omega, \mathcal{F}, P^\lambda)$ *be probability spaces and associated stochastic processes* $X^{(\lambda)}$ *such that*

$$(7.16) \qquad \mathbb{E}_\lambda\big(X_x^{(\lambda)} X_y^{(\lambda)}\big) = K_\lambda(x,y).$$

Then the probability measures P^λ *are mutually singular, as we vary* $\lambda \in (0,\infty)$.

Proof. By the above lemma, the process $Y_x^{(\lambda)} = \frac{1}{\lambda^x}X_x^{(\lambda)}$ has the same probability space but covariance function

$$(7.17) \qquad \mathbb{E}_\lambda\big(Y_x^{(\lambda)} Y_y^{(\lambda)}\big) = \frac{1}{\lambda^x \lambda^y} K_\lambda(x,y).$$

We write for λ_1 and $\lambda_2 \in (0,\infty)$

$$\frac{1}{\lambda_1^x \lambda_1^y} K_{\lambda_1}(x,y) = \sum_{n=0}^{\infty} \lambda_1^{-2n} e_n(x) e_n(y),$$

$$(7.18) \qquad \frac{1}{\lambda_2^x \lambda_2^y} K_{\lambda_2}(x,y) = \sum_{n=0}^{\infty} \lambda_2^{-2n} e_n(x) e_n(y)$$

$$= \sum_{n=0}^{\infty} \mu_n \lambda_1^{-2n} e_n(x) e_n(y),$$

where we have set $\mu_n = \frac{\lambda_2^{2n}}{\lambda_1^{2n}}$. By [11, Theorems 4.3 and 4.4, p. 27], we see that P^{λ_1} and P^{λ_2} are equivalent if and only if

$$(7.19) \qquad \sum_{n=0}^{\infty} (1 - \mu_n)^2 < \infty,$$

which holds if and only if $\lambda_1 = \lambda_2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

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