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ON THE EXCEPTIONAL SET IN A CONDITIONAL THEOREM OF LITTLEWOOD

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ABSTRACT. In 1952, Littlewood stated a conjecture about the average growth of spherical derivatives of polynomials, and showed that it would imply that for entire function of finite order, "most" preimages of almost all points are concentrated in a small subset of the plane. In 1988, Lewis and Wu proved Littlewood's conjecture. Using techniques from complex dynamics, we construct entire functions of finite order with a bounded set of singular values for which the set of exceptional preimages is infinite, with logarithmically growing cardinality.

1. Introduction and main result

For a meromorphic function f, let $f^{\#}(z) = \frac{2|f'(z)|}{1+|f(z)|^2}$ denote the spherical derivative of f, and let $\mathrm{Sing}(f^{-1})$ denote the set of singular values, that is, the set of all critical and asymptotic values of f. Let $\mathcal{S}$ be the class of entire functions with finitely many singular values, and $\mathcal{B}$ the class of entire functions with a bounded set of finite singular values. For Borel sets $A \subseteq \mathbb{C}$, we write $\#A$ for the cardinality of a set A, and $|A|$ for its (two-dimensional) Lebesgue measure. If $A, B \subset \mathbb{C}$ are Borel sets with $|B| \in (0,\infty)$, we define $\mathrm{dens}(A,B) = \frac{|A\cap B|}{|B|}$ as the density of A in B. We denote the unit disk by $\mathbb{D}$, and the disk of radius r centered at 0 by $\mathbb{D}_r$.

If R is a rational function of degree n, an application of the Cauchy–Schwarz inequality yields

$$
\begin{aligned}
(1) \qquad \iint_{\mathbb{D}} R^{\#}(z)\,dx\,dy &\le \left(\iint_{\mathbb{D}} R^{\#}(z)^2\,dx\,dy\right)^{1/2}\left(\iint_{\mathbb{D}} dx\,dy\right)^{1/2} \\
&\le (4\pi n)^{1/2}\pi^{1/2} = 2\pi\sqrt{n},
\end{aligned}
$$

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since R covers the sphere n times, and the area of the sphere is 4π. For general rational functions this is asymptotically best possible, but Littlewood conjectured that for polynomials this estimate could be improved. More precisely, he conjectured the following, which was later proved by Lewis and Wu, building on earlier partial results by Eremenko and Sodin in [ES86] and [ES87].

THEOREM 1 (Lewis, Wu [LW88]). *There exist absolute constants C and $\alpha > 0$ with*

$$\iint_{\mathbb{D}} P^{\#}(z)\,dx\,dy \leq C n^{1/2-\alpha}, \tag{2}$$

for all n and all polynomials P of degree n.

In fact, Lewis and Wu originally showed that one can choose $\alpha = 2^{-264}$, and Eremenko proved in [Ere91] that one cannot choose α arbitrarily close to $1/2$. In [Ere02], a connection between the supremum of the values of α for which the theorem holds and the universal integral means spectrum was conjectured. This connection is stated as a corollary of the main result in [BS05], but the proof of the corollary crucially uses an unpublished result by Binder and Jones about "strong fractal approximation". Assuming this result, both upper and lower estimates on the supremum of α-values in the theorem can be improved considerably, see [HS05], [Bel08], and [BS10].

Littlewood showed that his conjecture would have a curious implication for the value distribution of entire functions of finite order. Roughly speaking, most values are taken in a very small subset of the plane. Since the conjecture is now a theorem, Littlewood's conditional theorem becomes a corollary.

COROLLARY 2 (Littlewood [Lit52]). *Let f be an entire function of finite order $\rho \in (0,\infty)$, and let $\beta \in (0,\alpha)$, where α is the constant of Theorem 1. Then there exists a constant C_1 and an open set $S \subset \mathbb{C}$ with* $\operatorname{dens}(S,\mathbb{D}_r) \leq C_1 r^{-2\rho\beta}$ *for all $r > 0$, such that for almost all $w \in \mathbb{C}$ and all $\varepsilon > 0$, there exists a constant C_2 such that the set $E_w = f^{-1}(w) \setminus S$ satisfies $\#(E_w \cap \mathbb{D}_r) \leq C_2 r^{\rho-(\alpha-\beta)\rho+\varepsilon}$ for all $r > 1$.*

We call E_w the set of *exceptional preimages*. Since we expect to have roughly r^ρ preimages in $\mathbb{D}_r$ for a function of order ρ and a typical point w, the estimate on the cardinality of $E_w \cap \mathbb{D}_r$ shows that most preimages of typical points lie in S, whose Lebesgue density in $\mathbb{D}_r$ is decreasing with a power of the radius r. Obviously, meromorphic functions of finite order do not have this property, as shown by the Weierstrass $\wp$-function.

The question how large the set of exceptional preimages can be is related to Epstein's "order conjecture", the question whether the order of an entire function $f \in \mathcal{S}$ is invariant under topological equivalence in the sense of Eremenko and Lyubich [EL92]. If the number of exceptional preimages $\#E_w$ is uniformly bounded on every compact set $K \subset \mathbb{C} \setminus \operatorname{Sing}(f^{-1})$, then f has the "area property", that is, $\iint_{f^{-1}(K)} \frac{dx\,dy}{1+|z|^2} < \infty$ for every such compact set K. This in turn implies invariance of the order of f under topological equivalence in the class $\mathcal{S}$. For more background, technical details and a similar construction to the one in this paper of a function $f \in \mathcal{B}$ which does not have the area property see Epstein and Rempe-Gillen [ER13]. The order conjecture has recently been disproved by Bishop [Bis13].

In this note, we show that in the class $\mathcal{B}$ the exceptional set can indeed contain $\geq c \log r$ points, as made precise in the following theorem.

THEOREM 3. *For almost every* $\rho \in (\log 2/\log 3, \infty)$, *there exists a function* $f \in \mathcal{B}$ *of order* ρ *and a set* W *of positive measure such that for any constants* $C, \delta > 0$, *any Borel set* $S \subset \mathbb{C}$ *satisfying* $\operatorname{dens}(S, \mathbb{D}_r) \leq C r^{-\delta}$ *for every* $r > 0$, *and every* $w \in W$, *the set* $E_w = f^{-1}(w) \setminus S$ *satisfies*

$$\liminf_{r\to\infty} \frac{\#(E_w \cap \mathbb{D}_r)}{\log r} \geq \frac{\rho}{\log 2}. \tag{3}$$

Furthermore, for every $\varepsilon > 0$ *there exists a function* $f \in \mathcal{B}$ *of order* $\rho \in (1/2, 1/2 + \varepsilon)$ *and a set* W *of positive measure satisfying* (3) *for every* $w \in W$ *under the same assumptions.*

REMARK. Entire functions in $\mathcal{B}$ always have order $\rho \geq 1/2$ (see [BE95] and [Lan95]). It is possible that our construction may be tweaked to yield the result for all $\rho > 1/2$, but it will never produce examples of order $\rho = 1/2$.

2. Proof

The entire functions we use will be Poincaré functions of quadratic polynomials at repelling fixed points. We obtain the exceptional preimages as preimages of a rotation domain under the Poincaré function. In order to get almost all $\rho > \log 2/\log 3$, we use explicit polynomials with fixed Siegel disks; in order to obtain ρ arbitrarily close to $1/2$, we choose perturbations of the Chebyshev polynomial $T(z) = z^2 - 2$ with periodic Siegel disks. For background on complex dynamics, see [CG93].

Let $P(w) = \lambda w + w^2$ with $\lambda = e^{2\pi i \gamma}$. By a classical result of Siegel [Sie42], for almost all $\gamma \in \mathbb{R}$ the function P can be linearized near 0, that is, there exists an analytic linearizing map $h(z) = z + O(z^2)$ near 0 such that $P(h(z)) = h(\lambda z)$. The power series of h has a finite radius of convergence $R > 0$, and h maps $\mathbb{D}_R$ conformally onto the *Siegel disk* V of P centered at 0. The polynomial P has another finite fixed point at $z_0 = 1 - \lambda$ with multiplier $\mu := P'(z_0) = 2 - \lambda$. Since $|\mu| > 1$, there exists a local linearizing function $f(z) = z_0 + z + O(z^2)$ with $P(f(z)) = f(\mu z)$. In this case, the functional equation allows to extend f to an entire function of order $\rho = \log 2/\log|\mu|$ (see [Val13, §48]), the *Poincaré function* of P at z_0. We now fix such a function f associated to a polynomial P with a Siegel disk $V = h(\mathbb{D}_R)$ centered at 0, as well as the sub-Siegel disk $W := h(\mathbb{D}_{R/2})$.

Now let $C,\delta>0$ be constants, and let $S\subset\mathbb{C}$ be a Borel set satisfying $\operatorname{dens}(S,\mathbb{D}_r)\le Cr^{-\delta}$ for every $r>0$. In the following we use C_k for constants depending only on f and S.

We will show that $f\in\mathcal{B}$ and that the exceptional set $E_w=f^{-1}(w)\setminus S$ satisfies the asymptotic estimate (3) for almost every $w\in W$. Since $|\mu|=|2-\lambda|$ attains almost every value in the interval $(1,3)$, this proves the theorem.

The set of singular values $\operatorname{Sing}(f^{-1})$ equals the post-critical set of the polynomial P, that is, the closure of the forward orbit of the critical point [MBP12, Proposition 4.2]. Since the latter is contained in the Julia set of P, it is a bounded set disjoint from the simply connected Siegel disk V. This implies both that $f\in\mathcal{B}$, and that f maps every component of $f^{-1}(V)$ conformally onto V.

The Koebe distortion theorem implies that there exists an absolute constant M such that

$$\frac{1}{M}\le\frac{\operatorname{dens}(g(A),g(W))}{\operatorname{dens}(A,W)}\le M, \tag{4}$$

for all conformal maps $g:V\to\mathbb{C}$ and all Borel sets $A\subseteq W$ of positive measure.

Let U_0 be a component of $f^{-1}(V)$, and let $U_k=\mu^kU$. Then $f(U_k)=f(\mu^kU_0)=P^k(V)=V$, so (U_k) is a sequence of components of $f^{-1}(V)$. Let $W_k=f^{-1}(W)\cap U_k$ and $S_k=S\cap W_k$. Since W_0 is a Borel set of measure $|W_0|>0$ contained in some disk $\mathbb{D}_{C_1}$, we get that $W_k=\mu^kW_0$ is a Borel set of measure $|W_k|=\mu^{2k}|W_0|$ contained in $\mathbb{D}_{\mu^kC_1}$, so

$$\operatorname{dens}(S,W_k)=\operatorname{dens}(S_k,W_k)\le C_2\operatorname{dens}(S_k,\mathbb{D}_{\mu^kC_1})\le C_3\mu^{-k\delta},$$

for all k. Applying (4) to the branch of f^{-1} mapping V to U_k yields

$$\operatorname{dens}\bigl(f(S_k),W\bigr)\le C_4|\mu|^{-k\delta}.$$

Setting

$$E:=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}f(S_k),$$

we get

$$\operatorname{dens}(E,W)\le\sum_{k=n}^{\infty}C_4|\mu|^{-k\delta},$$

for every n, so $\operatorname{dens}(E,W)=0$, and hence $|E|=0$. This implies that almost every $w\in W$ satisfies $g^{-1}(w)\cap S_k=\emptyset$ for all but finitely many indices k. Thus for almost every $w\in W$, the exceptional set E_w contains μ^kz_w for some $z_w\neq0$ and all $k\ge0$, and hence

$$\liminf_{k\to\infty}\frac{\#(E_w\cap\mathbb{D}_r)}{\log r}\ge\frac{1}{\log|\mu|}=\frac{\rho}{\log 2}. \tag{5}$$

In order to produce examples of order arbitrarily close to $1/2$, we need to modify the construction slightly. Instead of using explicit polynomials with Siegel fixed points, we use perturbations of the Chebyshev polynomial $T(z) = z^2 - 2$ which have cycles of Siegel disks. Existence of these polynomials is well-known, but for the convenience of the reader we give a sketch of the proof.

The intermediate value theorem shows that there exists a decreasing sequence of real numbers (a_n) with $a_n \to -2$ such that $Q_n(z) = z^2 + a_n$ has a super-attracting periodic point, that is, it satisfies $Q_n^{q_n}(0) = 0$ for some $q_n \geq 1$. Perturbing Q_n and using the implicit function theorem, we get a sequence of polynomials $R_n(z) = z^2 + b_n$ with $-2 < b_n < a_n$ having a parabolic periodic point z_n with multiplier -1, that is, $R_n^{q_n}(z_n) = z_n$ and $(R_n^{q_n})'(z_n) = -1$. (Essentially this is the well-known fact that there are Feigenbaum bifurcations arbitrarily close to the Chebyshev polynomial in the quadratic family.) By the same result of Siegel that we used in the first part of the proof, there are numbers γ arbitrarily close to $1/2$ such that any analytic function $F(z) = e^{2\pi i \gamma} z + O(z^2)$ is linearizable. Since the multiplier is a non-constant analytic function of the parameter near b_n, there exists $c_n \in \mathbb{C}$ with $|c_n - b_n| < \frac{1}{n}$ such that $P_n(z) = z^2 + c_n$ has a periodic Siegel disk of period q_n. In this way, we have constructed a sequence of polynomials $P_n(z) = z^2 + c_n$ with periodic Siegel disks and $c_n \to -2$.

The repelling fixed point $z = 2$ of $T(z) = z^2 - 2$ varies analytically with the parameter, so P_n has a repelling fixed point z_n with $z_n \to 2$ and $\mu_n = P_n'(z_n) \to 4$ for $n \to \infty$. It follows from classical results in complex dynamics that the Julia set of P_n is connected, and this implies $|\mu_n| < 4$ (see [Buf03]). Let f_n denote the Poincaré function of P_n at the fixed point z_n. Since P_n has connected Julia set, we get $f_n \in \mathcal{B}$ with order $\rho_n = \log 2 / \log \mu_n \to 1/2$. It remains to show that f_n satisfies (3), and this follows along the same lines as in the first part of the proof.

Let n be fixed and, and let $U_1, \ldots, U_{q-1}$ be the cycle of Siegel disks of P_n containing periodic points $\zeta_1, \ldots, \zeta_{q-1}$. Let $h_k : \mathbb{D}_{R_k} \to U_k$ be the linearizing map of P^q in U_k, normalized as $h_k(z) = \zeta_k + z + O(z^2)$. Now we let $W = \bigcup_{k=1}^{q-1} h_k(\mathbb{D}_{R_k/2})$. Then $P_n(W) = W$, and W is a finite union of sub-Siegel disks, so we also get (4) for all maps g which are conformal in any U_j, where the constants do not depend on j. Applying this to branches of f_n^{-1} exactly as in the first part of the proof yields the desired estimate (3).

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