Illinois Journal of Mathematics
Volume 58, Number 1, Spring 2014, Pages 207–218
S 0019-2082

THE a-POINTS OF THE SELBERG ZETA-FUNCTION ARE UNIFORMLY DISTRIBUTED MODULO ONE

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ABSTRACT. Let $Z(s)$ be the Selberg zeta-function associated with a compact Riemann surface. We prove that the imaginary parts of the nontrivial a-points of $Z(s)$ are uniformly distributed modulo one. We also consider the question whether the eigenvalues of the corresponding Laplacian are uniformly distributed modulo one.

1. Introduction

Let $s = \sigma + it$ be a complex variable and X a compact Riemann surface of genus $g \geq 2$. The surface X can be regarded as a quotient $\Gamma\backslash H$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a strictly hyperbolic Fuchsian group and H is the upper half-plane of $\mathbb{C}$. Then the Selberg zeta-function associated with $X = \Gamma\backslash H$ is defined by (see Hejhal [9, Section 2.4, Definition 4.1])

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} \bigl(1 - N(P_0)^{-s-k}\bigr). \tag{1}$$

Here $\{P_0\}$ is the primitive element of Γ and $N(P_0) = \alpha^2$ if the eigenvalues of P_0 are α and α^{-1} with $|\alpha| > 1$. Equation (1) defines the Selberg zeta-function in the half-plane $\sigma > 1$. The function $Z(s)$ can be extended to an entire function of order 2 (Hejhal [9, Section 2.4, Theorem 4.25]), with so-called trivial zeros at $1, 0, -1, -2, \ldots$ and nontrivial zeros on the critical line $\sigma = 1/2$ with at most finitely many exceptions of zeros on the real segment $0 < s < 1$ (Hejhal [9, Section 2.4, Theorem 4.11] and Randol [13]). All the nontrivial zeros $s_j = 1/2 \pm it_j$ correspond to eigenvalues

$$0 < \lambda_j = s_j(1 - s_j) = 1/4 + t_j^2 \tag{2}$$

Received January 30, 2014; received in final form February 28, 2014.
2010 *Mathematics Subject Classification.* 11M36.

of the hyperbolic Laplacian Δ on $X = \Gamma\backslash H$ (Hejhal [9, Section 2.4, Theorem 4.11]. Moreover, the Selberg zeta-function satisfies the following functional equation (Hejhal [9, Section 2.4, Theorem 4.12])

$$Z(s) = X(s)Z(1-s), \tag{3}$$

where

$$X(s) = \exp\left(4\pi(g-1)\int_0^{s-1/2} v\tan(\pi v)\,dv\right).$$

Let a be a complex number. Solutions of $Z(s) = a$ are called a-points of $Z(s)$. From definition (1) and the functional equation (3), it follows that there are positive constants $A = A(a)$ and $\tau = \tau(a)$ such that $Z(s) \neq a$ for $\sigma \geq A$ and

$$Z(s) \neq a \quad \text{for } \sigma \leq 1-A \quad \text{and} \quad |t| \geq \tau$$

(see [7]). An a-point is called *nontrivial* if it lies in the strip $1-A < \sigma < A$; nontrivial a-points are denoted by $\rho_a = \beta_a + i\gamma_a$. Any a-point inside in the region $\sigma < 1-A$ and $|t| < \tau$ is called a *trivial*. Denote by $N_a(T)$ the number of nontrivial a-points (counted with multiplicities) of $Z(s)$ in the region $\tau < t \leq T$. In [7] it was proved that, for $a \neq 1$,

$$N_a(T) = (g-1)T^2 + o(T) \tag{4}$$

and, for $a = 1$,

$$N_1(T) = (g-1)T^2 - \frac{T}{2\pi}\log N(P_{00}) + o(T),$$

where $N(P_{00}) = \min_{P_0}\{N(P_0)\}$. If $a = 0$, then formula (4) is known to hold with a better error term $O(T/\log T)$ (Hejhal [9, Section 2.8, Theorem 8.19]).

It is known that almost all nontrivial a-points are arbitrary close to the critical line $\sigma = 1/2$. More precisely, let $N_a^-(\delta, T)$ and $N_a^+(\delta, T)$ denote the number of nontrivial a-points of $Z(s)$ lying in the corresponding regions $\sigma < 1/2 - \delta$, $1 < t \leq T$, respectively $\sigma > 1/2 + \delta$, $1 < t \leq T$. Furthermore, define

$$N_a^0(\delta, T) = N_a(T) - \big(N_a^-(\delta, T) + N_a^+(\delta, T)\big).$$

Then, for $\delta = (\log\log T)^2/\log T$ we have ([7, Theorem 3])

$$N_a^-(\delta, T) + N_a^+(\delta, T) \ll \frac{T^2}{\log\log T} \tag{5}$$

and

$$N_a^0(\delta, T) = (g-1)T^2 + O\left(\frac{T^2}{\log\log T}\right). \tag{6}$$

In [6] the connection between the distribution of a-points and the growth of $Z(s)$ was considered. The value distribution of the Selberg zeta-function associated to the modular group in light of the universality theorem was investigated in [2].

Here we shall prove:

THEOREM 1. *Let $a \in \mathbb{C}$. The imaginary parts of nontrivial a-points of the Selberg zeta-function $Z(s)$ are uniformly distributed modulo one.*

For the Riemann zeta-function, it was Rademacher [12] who proved under the assumption of the truth of the Riemann hypothesis that the imaginary parts of the nontrivial zeros are uniformly distributed modulo one; Elliott [3] and (independently) Hlawka [10] gave unconditional proofs of this result. Further extensions and generalizations can be found in the articles [1], [4], and [5]; the analogue of Theorem 1 has been proved in [16].

The proof of Theorem 1 relies on the following proposition.

PROPOSITION 2. *Let x be a fixed positive real number not equal to 1. Then, as $T \to \infty$,*

$$\sum_{0<\gamma\leq T} x^{\rho} = O(T).$$

Furthermore, we consider the eigenvalues λ_j of the hyperbolic Laplacian Δ on X.

THEOREM 3. *Let $x = e^{2\pi n}$, $n \in \mathbb{Z}$. The following two statements are equivalent:*

(1) *the eigenvalues λ_j are uniformly distributed modulo one;*
(2) *the following bounds are valid*

$$\int_1^T x^{2t/T+it^2} \frac{Z'}{Z}\left(\frac{1}{2}+\frac{1}{T}-it\right) dt = o\big(T^2\big) \quad \textit{for } n > 0$$

and

$$\int_1^T x^{-2t/T-it^2} \frac{Z'}{Z}\left(\frac{1}{2}+\frac{1}{T}+it\right) dt = o\big(T^2\big) \quad \textit{for } n < 0.$$

In the next section, we state lemmas. Theorems 1, 3, and Proposition 2 are proved in Section 3.

2. Preliminaries

In the proof of Theorem 1, we will use Weyl's criterion.

LEMMA 4 (Weyl's criterion). *A sequence of real numbers y_n is uniformly distributed modulo one if, and only if, for each integer $\ell \neq 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} e^{2\pi i \ell y_j} = 0.$$

For the proof, see Weyl [18], [19].

Lemma 5. *If $f(s)$ is analytic and $f(s_0) \neq 0$ with*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M$$

in $\{s : |s - s_0| \leq r\}$ with $M > 1$, then

$$\left|\frac{f'(s)}{f(s)} - \sum_{\rho} \frac{1}{s-\rho}\right| < C\frac{M}{r}$$

for $|s - s_0| \leq \frac{r}{4}$, where C is some constant and ρ runs through the zeros of $f(s)$ such that $|\rho - s_0| \leq \frac{r}{2}$.

For the proof, see Titchmarsh [17, Section 3.9].

Lemma 5 is applied in the proof of the next lemma.

Lemma 6. *Let $a \in \mathbb{C}$. Let $B, b \geq 1/2$ be such that $Z(s) \neq a$ for $\sigma < -b$ and $\sigma > B - 1$. If T is such that $Z(\sigma + iT) \neq a$ for $1 - b \leq \sigma \leq B$, then*

$$\int_{1-b}^{B} \left|\frac{Z'(\sigma+iT)}{Z(\sigma+iT)-a}\right| d\sigma \ll T.$$

Proof. In Lemma 5, we choose $s_0 = B + iT$ and $r = 4(B - (1 - b))$. We can take $M = cT$ with some $c > 0$ (see Randol [15, Lemma 2] or Garunkštis [7, comment above Theorem 5]). Then Lemma 5 gives

$$\frac{Z'(s)}{Z(s)-a} = \sum_{|\rho_a - s_0| \leq \frac{r}{2}} \frac{1}{s-\rho_a} + O(T), \tag{7}$$

for $|s - s_0| \leq \frac{r}{4}$. Thus,

$$\begin{aligned}
&\int_{1-b}^{B} \left|\frac{Z'(\sigma+iT)}{Z(\sigma+iT)-a}\right| d\sigma \\
&\quad \leq \int_{1-b}^{B} \sum_{|\rho_a - s_0| \leq \frac{r}{2}} \left|\frac{1}{\sigma+iT-\rho_a}\right| d\sigma + O(T) \\
&\quad = \sum_{|\rho_a - s_0| \leq \frac{r}{2}} \int_{1-b}^{B} \frac{1}{\sqrt{(\sigma-\beta_a)^2 + (T-\gamma_a)^2}} d\sigma + O(T) \\
&\quad = \sum_{|\rho_a - s_0| \leq \frac{r}{2}} \Big(\log\big(B - \beta_a + \sqrt{(T-\gamma_a)^2 + (B-\beta_a)^2}\big) \\
&\qquad - \log\big(1 - b - \beta_a + \sqrt{(T-\gamma_a)^2 + (1-b-\beta_a)^2}\big)\Big) + O(T) \\
&\quad \ll T
\end{aligned}$$

since the disc $|\rho_a - s_0| \leq \frac{r}{2}$ contains $O(T)$ many a-points. □

In the following lemma, we express the Selberg zeta-function by a general Dirichlet series.

LEMMA 7. *There is an unbounded sequence* $1 < x_2 < x_3 \cdots$ *of real numbers and real numbers* a_n, $n = 2, 3, \ldots$, *such that*

$$Z(s) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{x_n^s}, \tag{8}$$

where the Dirichlet series converges absolutely for $\sigma > 1$.

Proof. Multiplying the Euler product, we obtain a formal Dirichlet series

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} \left(1 - N(P_0)^{-s-k}\right) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{x_n^s}.$$

In view of the properties of Dirichlet series (Hardy and Riesz [8, Section 2.2, Theorem 1]), it is enough to prove that the series (8) converges absolutely at $s = \sigma > 1$. For any positive x, we have that

$$1 + \sum_{x_n \le x} \frac{|a_n|}{x_n^{\sigma}} \le \prod_{\{P_0\}} \prod_{k=0}^{\infty} \left(1 + N(P_0)^{-\sigma-k}\right).$$

In the last formula, the product converges for $\sigma > 1$ since (Hejhal [9, Section 1.2, Proposition 2.5])

$$\sum_{\substack{\{P_0\} \\ N(P_0) \le x}} 1 = O(x).$$

This proves the lemma. □

The next lemma is essentially due to Landau [11] and deals with general Dirichlet series. Let $1 = x_1 < x_2 < \cdots$ be an unbounded sequence X of real numbers and define

$$S = \{x_{k_1} x_{k_2}, \ldots, x_{k_m} : m \in \mathbb{N}, k_1 \in \mathbb{N}, \ldots, k_m \in \mathbb{N}\}$$

as the set of all possible products of elements of the sequence X. Let $1 = y_1 < y_2 < \cdots$ be an ordered sequence of all different numbers of S.

LEMMA 8. *For* $n \in \mathbb{N}$ *let* a_n *and* b_n *be complex numbers such that the general Dirichlet series* $A(s) = \sum_n a_n x_n^{-s}$ *and* $B(s) = \sum_n b_n x_n^{-s}$ *converge absolutely in the right half-plane* $\sigma > \sigma_0$. *If* $b_1 \neq 0$, *then there exist a real number* $\sigma_1 \ge \sigma_0$ *and complex numbers* c_n, $n = 1, 2, \ldots$, *such that*

$$\frac{A(s)}{B(s)} = \sum_{n=1}^{\infty} \frac{c_n}{y_n^s}$$

and the series converges absolutely for $\sigma > \sigma_1$.

Proof. Without loss of generality, we assume that $b_1 = 1$. Then there exists $\sigma_1 \geq \sigma_0$ such that $|B(s) - 1| < 1$, for $\sigma > \sigma_1$, and the series of $B(s) - 1$ converges absolutely. Thus, there exist complex numbers d_n such that

$$\frac{1}{B(s)} = \sum_{n=0}^{\infty} (-1)^n \big(B(s) - 1\big)^n = \sum_{n=1}^{\infty} \frac{d_n}{y_n^s},$$

where the last series converges absolutely for $\sigma > \sigma_1$. Now the lemma follows in view of the absolute convergence of the series for $A(s)$ and $B(s)^{-1}$. □

The following lemma describes the asymptotic behavior of the factor $X(s)$ from the functional equation (3).

LEMMA 9. *For* $t \geq 1$,

$$X(s) = \exp\bigg(2\pi i(g-1)\Big(s - \frac{1}{2}\Big)^2 + \frac{\pi i(g-1)}{6} + O\Big(\frac{t}{e^{2\pi t}}\Big) + O\Big(\frac{(\sigma - 1/2)^2}{e^{2\pi t}}\Big) + O\Big(\frac{(\sigma - 1/2)t}{e^{2\pi t}}\Big)\bigg) \quad (t \to \infty)$$

uniformly in σ.

Proof. This is Lemma 1 in [7]. □

3. Proofs

Proof of Proposition 2. First, we may assume $a \neq 1$. Let B be a sufficiently large fixed number, such that $B \geq A$, where A is defined in Introduction. Then the strip $1 - B \leq \sigma \leq B$ contains all the nontrivial a-points and a finite number of trivial a-points.

Next, let T be such that there are no a-points on the line $t = T$. Using the residue theorem and the fact that the logarithmic derivative of $Z(s) - a$ has simple poles at each a-point ρ_a with residue equal to the order of ρ_a, we get

$$\sum_{0<\gamma_a \leq T} x^{\rho_a} = \frac{1}{2\pi i} \int_{\square} x^s \frac{Z'(s)}{Z(s) - a}\, ds + O(1);$$

here □ denotes the counterclockwise oriented rectangular contour with vertices $B + i$, $B + iT$, $1 - B + iT$, $1 - B + i$. If the line $t = 1$ contains a-points, we slightly alter the lower edge of the rectangular contour □.

In order to evaluate the integral, we write

$$\int_{\square} x^s \frac{Z'(s)}{Z(s) - a}\, ds = \bigg\{\int_{B+i}^{B+iT} + \int_{B+iT}^{1-B+iT} + \int_{1-B+iT}^{1-B+i} + \int_{1-B+i}^{B+i}\bigg\} x^s \frac{Z'(s)}{Z(s) - a}\, ds = \sum_{j=1}^{4} I_j.$$

We shall evaluate each I_j individually.

In view of Lemmas 7 and 8, we may suppose that the logarithmic derivative of $Z(s)-a$ has an absolutely convergent Dirichlet series expansion for $\sigma > B$, namely

$$\frac{Z'(s)}{Z(s)-a} = \sum_{n=2}^{\infty} \frac{c_n}{y_n^s}.$$

Now we interchange summation and integration on the right-hand side of the rectangle, which gives

$$\begin{aligned} I_1 &= \sum_{n=2}^{\infty} c_n \int_{B+i}^{B+iT} \left(\frac{x}{y_n}\right)^s ds = \sum_{n=2}^{\infty} c_n i \int_1^T \exp\bigl((B+it)\log(x/y_n)\bigr)\, dt \\ &= \sum_{n=2}^{\infty} c_n i \exp\bigl(B\log(x/y_n)\bigr) \int_1^T \exp\bigl(it\log(x/y_n)\bigr)\, dt. \end{aligned}$$

By

$$\begin{aligned} &\int_1^T \exp\bigl(it\log(x/y_n)\bigr)\, dt \\ &\quad = \begin{cases} T-1 & \text{if } x = y_n, \\ (\exp(iT\log(x/y_n)) - \exp(i\log(x/y_n)))/(i\log(x/y_n)) & \text{otherwise,} \end{cases} \end{aligned}$$

we obtain

$$I_1 = ic(x)T + O(1).$$

Here $c(x)$ equals the Dirichlet coefficient c_n if $x = y_n$ and 0 otherwise.

Next, we estimate the integrals along the horizontal segments. Clearly, $I_4 = O(1)$. In view of Lemma 6, the contribution of the upper horizontal segment gives

$$I_2 = \int_{1-B}^{B} x^{\sigma+it'} \frac{Z'(\sigma+iT)}{Z(\sigma+iT)-a}\, d\sigma \ll x^{\sigma} \int_{1-B}^{B} \left| \frac{Z'(\sigma+iT)}{Z(\sigma+iT)-a} \right| d\sigma \ll T.$$

It remains to estimate the integral along the left-hand side:

$$I_3 = O(1) - \int_{1-B+it_0}^{1-B+iT} x^s \frac{Z'(s)}{Z(s)-a}\, ds. \tag{9}$$

In view of the expression of $Z(s)$ by a Dirichlet series (Lemma 7), we may assume $|Z(1-\sigma-it)| \geq 1/2$ for $\sigma \leq 1-B$ and all t; it follows from Lemma 9 above that

$$Z(1-B+it) \gg \exp(t),$$

as $t \to \infty$. Hence there exists t_0 such that the absolute value of $Z(1-B+it)$ is greater than $2|a|$ for $t > t_0$ and we obtain the following expansion into a geometric series:

$$\frac{Z(s)}{Z(s)-a} = \frac{Z'}{Z}(s)\frac{1}{1-a/Z(s)} = \frac{Z'}{Z}(s)\left(1 + \sum_{k=1}^{\infty}\left(\frac{a}{Z(s)}\right)^k\right).$$

Then, in view of the bound $Z'/Z(1-B+it) \ll t$, for $t \to \infty$ (see Randol [14, Lemma 2]), we get

$$\int_{1-B+it_0}^{1-B+iT} x^s \frac{Z'}{Z}(s) \sum_{k=1}^{\infty} \left(\frac{a}{Z(s)}\right)^k ds \ll x^{1-B} T^2 \sum_{k=1}^{\infty} \left(\frac{1}{\exp(T)}\right)^k \ll 1.$$

By Hejhal [9, Chapter 2, Proposition 4.2] we have

$$\frac{Z'}{Z}(s) = \sum_{\{P_0\}} \sum_{k=1}^{\infty} \frac{\log(N(P_0))(1-N(P_0)^{-k})^{-1}}{N(P_0)^{ks}}, \tag{10}$$

where the series converges absolutely in the half-plane $\sigma > 1$.

Recall that $x \neq 1$. By the functional equation (Lemma 9) and (10), for the second part of the integral in (9) we get

$$\begin{aligned}
&-\int_{1-B+it_0}^{1-B+iT} x^s \frac{Z'(s)}{Z(s)} ds \\
&= \int_{1-B+it_0}^{1-B+iT} x^s \left(\frac{Z'}{Z}(1-s) - \frac{X'}{X}(s)\right) ds \\
&= -ix^{1-B} \sum_{P_0} \sum_{k=1}^{\infty} \frac{\log(N(P_0))(1-N(P_0)^{-k})^{-1}}{N(P_0)^{kB}} \int_{t_0}^{T} \big(xN(P_0)^k\big)^{it} dt \\
&\quad + ix^{1-B} \int_{t_0}^{T} x^{it} \big(-4\pi(g-1)t + O(1)\big) dt. \\
&\ll T.
\end{aligned}$$

Thus, $I_3 \ll T$.

So far we have been considering the case $a \neq 1$. Now we consider the case $a = 1$. In the expression of $Z(s)$ by a Dirichlet series (Lemma 7), we can suppose that $a_2 \neq 0$. Let us define the function:

$$\ell(s) = x_2^s \big(Z(s) - 1\big) = 1 + \sum_{n=3}^{\infty} \frac{a_n}{a_2} \left(\frac{x_2}{x_n}\right)^s.$$

Then the logarithmic derivative of ℓ is given by

$$\frac{\ell'}{\ell}(s) = \log x_2 + \frac{Z'(s)}{Z(s) - 1}.$$

Applying contour integration and the above reasoning to this function proves Proposition 2. □

Proof of Theorem 1. Our argument goes along the lines of the proof of Theorem 1 in [16]. We use the property that non-trivial a-values are clustered around the critical line. By formulas (5) and (6), we have

$$\sum_{1<\gamma_a\le T}\left|\beta_a-\frac{1}{2}\right| = \left\{\sum_{1<\gamma_a\le T, |\beta_a-1/2|>\delta} + \sum_{1<\gamma_a\le T, |\beta_a-1/2|\le\delta}\right\}\left|\beta_a-\frac{1}{2}\right| \ll \frac{T^2}{\log\log T} + \frac{T^2(\log\log T)^2}{\log T}.$$

Since the function $Z(s)$ has only a bounded number of nontrivial a-points satisfying $0<t\le 1$, we get

$$\sum_{0<\gamma_a\le T}\left|\beta_a-\frac{1}{2}\right| \ll \frac{T^2}{\log\log T}.$$

Since, for any real number y,

$$\left|\exp(y)-1\right| = \left|\int_0^y \exp(t)\,dt\right| \le |y|\max\{1,\exp(y)\},$$

we find

$$\left|x^{1/2+i\gamma_a} - x^{\beta_a+i\gamma_a}\right| = x^{\beta_a}\left|\exp\left(\left(\frac{1}{2}-\beta_a\right)\log x\right)-1\right| \le \left|\beta_a-\frac{1}{2}\right| |\log x| \max\{x^{\beta_a}, x^{1/2}\}.$$

Furthermore,

$$\frac{1}{N_a(T)}\sum_{0<\gamma_a\le T}\left|x^{1/2+i\gamma_a} - x^{\beta_a+i\gamma_a}\right| \le \frac{X}{N_a(T)}\sum_{0<\gamma_a\le T}\left|\beta_a-\frac{1}{2}\right|,$$

where $X=\max\{x^B,1\}|\log x|$. Hence,

$$\frac{1}{N_a(T)}\sum_{0<\gamma_a\le T}\left(x^{1/2+i\gamma_a} - x^{\beta_a+i\gamma_a}\right) \ll \frac{X}{\log\log T}.$$

By Theorem 2,

$$\sum_{0<\gamma_a\le T} x^{\beta_a+i\gamma_a} \ll T.$$

Therefore, as $T\to\infty$,

$$\frac{1}{N_a(T)}\sum_{0<\gamma_a\le T} x^{1/2+i\gamma_a} \ll \frac{1}{\log\log T}.$$

Now let $x=z^m$ with some positive $z\ne 1$ and $m\in\mathbb{N}$. It follows from the latter formula that

$$\lim_{T\to\infty}\frac{1}{N_a(T)}\sum_{0<\gamma_a\le T}\exp(im\gamma_a\log z) = 0.$$

By Weyl's criterion (Lemma 4), the sequence of numbers $\gamma_a \log z/2\pi$ is uniformly distributed modulo 1. This proves Theorem 1. □

Proof of Theorem 3. In view of Weyl's criterion (Lemma 4), the eigenvalues λ_j are uniformly distributed modulo one if, and only if, for any fixed $n \in \mathbb{Z}\setminus\{0\}$,

$$\sum_{0<\lambda_j\le T^2} x^{i\lambda_j} = o\biggl(\sum_{0<\lambda_j\le T^2} 1\biggr),$$

where $x = e^{2\pi n}$. By the relation between eigenvalues and nontrivial zeros (2) and by the formula for the number of nontrivial zeros (4), it follows that

$$\sum_{0<\lambda_j\le T^2+\frac14} 1 = \sum_{0<t_j\le T} 1 = (g-1)T^2 + O\biggl(\frac{T}{\log T}\biggr).$$

First, we consider the case $x > 1$. If T is not an ordinate of a zero, then

$$\begin{aligned}(11)\qquad \sum_{0<\lambda_j<T^2+\frac14} x^{i\lambda_j} &= \sum_{0<t_j<T} x^{\frac{i}{4}+it_j^2} = \sum_{-T<-t_j<0} x^{\frac{i}{4}+it_j^2}\\ &= \frac{1}{2\pi i}\int_{\square} x^{\frac{i}{4}+is^2}\frac{Z'(s+\frac12)}{Z(s+\frac12)}\,ds + O(1)\\ &=: I_1 + I_2 + I_3 + I_4 + O(1),\end{aligned}$$

where the integration is over the counterclockwise oriented rectangular contour $\square$ in the lower half-plane with vertices $1/T-i$, $-1-i$, $-1-iT$, $1/T-iT$.

Clearly, for the integral on the upper horizontal line segment of $\square$ we have $I_1 \ll 1$.

For the integral I_2 over the left vertical line we use the bound $Z'/Z(-1+iT) \ll T$, $T\to\infty$ (Randol [14, Lemma 2]). Then, in view of $x^{is^2} = x^{-2\sigma t + i(\sigma^2-t^2)}$, we deduce $I_2 \ll 1$.

For the integral I_3 over the lower horizontal line, we use once more formula (7) and derive

$$\begin{aligned} I_3 &= \int_{-1-iT}^{1/T-iT} x^{\frac{i}{4}+is^2}\frac{Z'(s+\frac12)}{Z(s+\frac12)}\,ds \ll \int_{-1-iT}^{1/T-iT}\left|\frac{Z'(s+\frac12)}{Z(s+\frac12)}\right| ds\\ &= \sum_{|\rho-s_0|\le\frac{r}{2}}\int_{-1}^{1/T}\frac{1}{\sqrt{\sigma^2+(T-\gamma)^2}}\,d\sigma + O(T)\\ &= \sum_{|\rho-s_0|\le\frac{r}{2}}\biggl(\log\biggl(\frac1T+\sqrt{(T-\gamma)^2+\frac{1}{T^2}}\biggr)\\ &\qquad -\log\bigl(1+\sqrt{(T-\gamma)^2+1}\bigr)\biggr) + O(T)\\ &\ll T\log T.\end{aligned}$$

Further, the integral I_4 can be estimated by

$$I_4 = ix^{i\sigma^2} \int_{-T}^{-1} x^{-2\sigma t - it^2} \frac{Z'}{Z}\left(\frac{1}{2} + \frac{1}{T} + it\right) dt.$$

This proves the assertion of the theorem in the case $n > 0$.

In order to prove the assertion in the case $n \leq 0$, we choose the rectangular contour in the upper half-plane with vertices $-1+i$, $1/T+i$, $1/T+iT$, $-1+iT$ in formula (11) and proceed as in the previous case. This proves Theorem 3. □

Acknowledgment. The first author is supported by grant No. MIP-049/2014 from the Research Council of Lithuania.

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