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CONDITION R AND HOLOMORPHIC MAPPINGS OF DOMAINS WITH GENERIC CORNERS

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ABSTRACT. A piecewise smooth domain is said to have generic corners if the corners are generic CR manifolds. It is shown that a biholomorphic mapping from a piecewise smooth pseudoconvex domain with generic corners in complex Euclidean space that satisfies Condition R to another domain extends as a smooth diffeomorphism of the respective closures if and only if the target domain is also piecewise smooth with generic corners and satisfies Condition R. Further it is shown that a proper map from a domain with generic corners satisfying Condition R to a product domain of the same dimension extends continuously to the closure of the source domain in such a way that the extension is smooth on the smooth part of the boundary. In particular, the existence of such a proper mapping forces the smooth part of the boundary of the source to be Levi degenerate.

1. Introduction

The question of continuous or smooth extension to the boundary of holomorphic maps is of central importance in complex analysis. One significance of such extension lies in the fact that it reduces the difficult problem of classification of domains in $\mathbb{C}^n$, $n \geq 2$ up to biholomorphism, or the more general problem of deciding the existence of a proper map between two given domains, to the problem of study of CR invariants of the boundary hypersurfaces. After the fundamental result in this direction of Fefferman [18] giving smooth extension up to the boundary of a biholomorphic map between strictly pseudoconvex domains, there were obtained far reaching generalizations to proper

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maps between smoothly bounded pseudoconvex domains (e.g., [9], [4], [5], [6], [8], [16]). In these investigations, the hypothesis on the source domain D of the proper map is that it satisfies *Condition R*: the *Bergman projection*, the orthogonal projection from the Hilbert space $L^2(D)$ of square integrable functions to the closed subspace $\mathcal{H}(D)$ of holomorphic square integrable functions, maps a function smooth up to the boundary to a holomorphic function smooth up to the boundary.

In this note we consider a class of piecewise smooth domains to which the techniques of Bell–Catlin–Diederich–Fornaess–Ligocka et al. mentioned above extend in a natural way. By definition, a *piecewise smooth domain* is an intersection of finitely many smoothly bounded domains in which all possible boundary intersections are transverse. The class of domains we will be considering are the *domains with generic corners* defined below. Such domains have been considered by various authors (see [2], [19], [25]). In [25], Webster considered holomorphic mappings defined on domains with *real analytic* generic corners, and a reflection principle for such corners was developed. These ideas were subsequently developed by Forstnerič (see [19]). A crucial estimate of Bell for holomorphic functions on smooth domains was generalized by Barrett to this class of domains (see [2], and Lemma 2.1 below). In a previous article ([13]) we considered the extension of proper mappings of equidimensional products of smoothly bounded domains. These products are examples of domains with generic corners, and here we generalize some of the results of [13] to the wider class. We now formally define these domains:

Definition 1.1. Let Ω be a bounded domain in $\mathbb{C}^n$ that may be written as an intersection $\bigcap_{j=1}^N \Omega_j$ of smooth domains such that

(i) all intersections of the boundaries $b\Omega_j$ are transverse.
(ii) for each subset $S \subset \{1, \ldots, N\}$ the intersection $B_S = \bigcap_{j \in S} b\Omega_j$, if nonempty, is a CR manifold of CR-dimension $n - |S|$.

We call such a domain a *domain with generic corners.*

Our first result is the following theorem.

Theorem 1.2. *Suppose that $D \subset \mathbb{C}^n$ is a pseudoconvex domain with generic corners which satisfies Condition R. If $G \subset \mathbb{C}^n$ is a domain and $f : D \to G$ is a biholomorphic map, then the following are equivalent*:

(1) *f extends as a $\mathcal{C}^\infty$-smooth diffeomorphism from $\overline{D}$ to $\overline{G}$.*
(2) *G is a domain with generic corners and satisfies Condition R.*

Therefore the property of a domain that it satisfies Condition R and has generic corners is invariant under holomorphic maps smooth up to the boundary. As a result, the classification of domains in this class is reduced to the study of the boundaries. In Section 3, we consider some examples of domains satisfying the hypotheses of Theorem 1.2. These are also the hypotheses on the source domain D in Theorem 1.3 below. In a future work, we will consider further examples of this class of domains.

For a domain Ω with generic corners, let $b\Omega^{\text{sng}} \subset b\Omega$ consist of all those points that lie on the intersection of two or more boundaries $b\Omega_j$ and set $b\Omega^{\text{reg}} = b\Omega \setminus b\Omega^{\text{sng}}$.

THEOREM 1.3. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain with generic corners and $G = G_1 \times G_2 \times \cdots \times G_k \subset \mathbb{C}^n$ a product domain where each $G_j \subset \mathbb{C}^{\mu_j}$ is smoothly bounded and $\mu_1 + \mu_2 + \cdots + \mu_k = n$. Assume that D satisfies Condition R and let $f : D \to G$ be a proper holomorphic mapping. Then f admits a continuous extension to $\overline{D}$ in such a way that the extension is C^∞ smooth on bD^{reg}.*

It is possible to prove continuous extension of holomorphic maps between piecewise smooth domains under hypotheses different from those used in Theorems 1.2 and 1.3. Piecewise smooth pseudoconvex domains that admit plurisubharmonic peak points on their boundaries were considered by Berteloot ([10]) and Hölder continuity at the boundary for proper holomorphic mappings between such domains was established. A similar result that relied on estimates for the Carathéodory metric on strictly pseudoconvex piecewise smooth domains was proved by Range ([23]).

One interesting question that Theorem 1.3 leaves unresolved is whether we can conclude from the hypotheses if the source D itself has a product structure, that is, if there is a biholomorphic map $F : D \to F(D)$ onto a product domain $F(D) \subset \mathbb{C}^n$, where F extends to a diffeomorphism from $\overline{D}$ to $\overline{F(D)}$. It would be interesting to know if this indeed is the case.

2. Bell operator

Let Ω be a domain with generic corners in $\mathbb{C}^n$ and let N and Ω_j have the same meaning as in Definition 1.1. Suppose that r_j (where $j = 1, \ldots, N$) is a defining function of the domain Ω_j, that is, r_j is a smooth function on $\mathbb{C}^n$ such that $\Omega_j = \{r_j < 0\}$ and dr_j is non-zero at each point of $b\Omega_j$. Then the conditions (i) and (ii) in Definition 1.1 may be rephrased as follows: for each point p such that

$$r_{j_1}(p) = r_{j_2}(p) = \cdots = r_{j_k}(p) = 0$$

we have

$$dr_{j_1}(p) \wedge dr_{j_2}(p) \wedge \cdots \wedge dr_{j_k}(p) \neq 0$$

and also

$$\overline{\partial} r_{j_1}(p) \wedge \overline{\partial} r_{j_2}(p) \wedge \cdots \wedge \overline{\partial} r_{j_k}(p) \neq 0. \tag{2.1}$$

LEMMA 2.1 (cf. Barrett [1], [2]). *Let $s = (s_1, \ldots, s_N)$ be a tuple of non-negative integers. There is a linear differential operator Φ^s with smooth coefficients defined on $\overline{\Omega}$ such that for all $f \in \mathcal{C}^\infty(\overline{\Omega})$,*

(i) $P\Phi^s f = Pf$ *and*

(ii) $|\Phi^s f(z)| \le C\|f\|_{\mathcal{C}^{|s|}} d(z)^s$, *where* $|s| = \sum_{j=1}^N s_j$ *and*

$$d(z)^s = d_1(z)^{s_1} \cdots d_N(z)^{s_N},$$

where $d_j(z)$ *is the distance from the point* z *to* $b\Omega_j$.

Proof. Thanks to (2.1), near each $p \in \mathbb{C}^n$ we can find N vector fields $T_1^{(p)}, \dots, T_N^{(p)}$ of type $(0,1)$ such that $T_j^{(p)} r_k \equiv \delta_{jk}$ in a neighborhood of p when $r_j(p) = r_k(p) = 0$. By a partition of unity argument, we obtain vector fields T_j, $j = 1, \dots, N$ on $\mathbb{C}^n$ of type $(0,1)$ such that $T_j r_k \equiv \delta_{jk}$ on a neighborhood U_{jk} of $b\Omega_j \cap b\Omega_k$. (Note that if $j = k$, this means that $T_j r_j \equiv 1$ near $b\Omega_j$.)

For a subset $S \subset \{1, \dots, N\}$, let

$$U_S = \bigcap_{j,k \in S} U_{jk} \Big\backslash \bigcup_{\ell \notin S} b\Omega_\ell.$$

Then the family $\{U_S\}$, as S runs over all possible subsets of $\{1, 2, \dots, N\}$ including the empty set, is an open cover of $\mathbb{C}^n$. Let $\{\chi_S\}$ be a partition of unity subordinate to this cover. Let

$$\langle f, g \rangle = \int_\Omega f \overline{g} \, dV$$

denote the standard inner product on $L^2(\Omega)$, where dV denotes Lebesgue measure on $\mathbb{C}^n$. Let T_j^* denote the formal adjoint of the operator T_j with respect to this inner product structure. Integration by parts shows that $T_j^* = -\overline{(T_j + \operatorname{div} T_j)}$, and is therefore also a first order operator with smooth coefficients. For $f \in \mathcal{C}^\infty(\overline{\Omega})$, we define the operator Φ^s by

$$\Phi^s f = \sum_{S \subset \{1, \dots, N\}} \left(\prod_{j \in S} \frac{r_j^{s_j}}{s_j!} \right) \left(\prod_{j \in S} \left(T_j^*\right)^{s_j} \right) (\chi_S f).$$

Then Φ^s is a linear differential operator of order $|s|$.

Note that for each $S \subset \{1, \dots, N\}$, the smooth function χ_S vanishes to infinite order along the set $\bigcup_{\ell \notin S} b\Omega_\ell$, and therefore, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_{2n})$, if D^α is the partial derivative operator

$$D^\alpha = \prod_{j=1}^n \left(\frac{\partial}{\partial x_j} \right)^{\alpha_{2j-1}} \left(\frac{\partial}{\partial y_j} \right)^{\alpha_{2j}},$$

and σ_j is a non-negative integer for each $j \notin S$, we have an elementary estimate

$$\left| D^\alpha \chi_S(z) \right| \le C_{\sigma,\alpha} \prod_{j \notin S} \left(d_j(z) \right)^{\sigma_j},$$

where $C_{\sigma,\alpha}$ is a constant independent of z. Therefore, we have

$$\left|\prod_{j\in S}\left(T_j^*\right)^{s_j}\left(\chi_S f(z)\right)\right| \le C_S\|f\|_{\mathcal{C}^{|s|}}\prod_{\ell\notin S}\left(d_\ell(z)\right)^{s_\ell}.$$

Since r_j is comparable to d_j for each j, we have that

$$\left|\Phi^s f(z)\right| \le C\|f\|_{\mathcal{C}^{|s|}}\sum_{S\subset\{1,\dots,N\}}\left(\prod_{j\in S}\left(d_j(z)\right)^{s_j}\cdot\prod_{\ell\notin S}\left(d_\ell(z)\right)^{s_\ell}\right),$$

which proves part (ii) of the lemma.

To prove part (i), it suffices to show that for $h\in L^2(\Omega)$ and $f\in\mathcal{C}^\infty(\overline{\Omega})$ we have

$$\left\langle h, P\Phi^s f\right\rangle = \langle h, Pf\rangle.$$

Since P is the (self-adjoint) orthogonal projection from $L^2(\Omega)$ onto $\mathcal{H}(\Omega) = \mathcal{O}(\Omega)\cap L^2(\Omega)$, this is equivalent to

$$\left\langle g, \Phi^s f\right\rangle = \langle g, f\rangle,$$

where $g\in\mathcal{H}(\Omega)$. Now we have,

$$\begin{aligned}\left\langle g,\Phi^s f\right\rangle &= \sum_{S\subset\{1,\dots,N\}}\left\langle g, \left(\prod_{j\in S}\frac{r_j^{s_j}}{s_j!}\right)\left(\prod_{j\in S}\left(T_j^*\right)^{s_j}\right)(\chi_S f)\right\rangle \\ &= \sum_{S\subset\{1,\dots,N\}}\left\langle \prod_{j\in S}\frac{r_j^{s_j}}{s_j!}g, \left(\prod_{j\in S}\left(T_j^*\right)^{s_j}\right)(\chi_S f)\right\rangle.\end{aligned}\tag{2.2}$$

For $\varepsilon>0$, let $\langle\cdot,\cdot\rangle_{\Omega_\varepsilon}$ denote the standard L^2-inner product on the domain

$$\Omega_\varepsilon = \left\{z\in\mathbb{C}^n : r_j(z) < -\varepsilon, 1\le j\le N\right\}.$$

Fix $S\subset\{1,\dots,N\}$, and first suppose that $S\neq\emptyset$. Denote by j_0 the smallest element of S. Then we have, integrating by parts:

$$\begin{aligned}&\left\langle \prod_{j\in S}\frac{r_j^{s_j}}{s_j!}g, \left(\prod_{j\in S}\left(T_j^*\right)^{s_j}\right)(\chi_S f)\right\rangle_{\Omega_\varepsilon} \\ &\quad= \left\langle T_{j_0}\left(\prod_{j\in S}\frac{r_j^{s_j}}{s_j!}g\right), \left(\left(T_{j_0}^*\right)^{s_0-1}\cdot\prod_{j\in S\setminus\{j_0\}}\left(T_j^*\right)^{s_j}\right)(\chi_S f)\right\rangle_{\Omega_\varepsilon} \\ &\qquad -\sum_{k=1}^N\int_{b\Omega_k^\varepsilon}\left(\prod_{j\in S}\frac{r_j^{s_j}}{s_j!}g\right) \\ &\qquad \cdot\overline{\left(\left(T_{j_0}^*\right)^{s_0-1}\cdot\prod_{j\in S\setminus\{j_0\}}\left(T_j^*\right)^{s_j}\right)(\chi_S f)}\cdot\frac{T_{j_0}r_k}{|dr_k|}\,dS,\end{aligned}\tag{2.3}$$

where $b\Omega_k^\varepsilon = \{r_k = -\varepsilon\}$. In the boundary term, only the summand corresponding to $k = j_0$ is non-zero, since by construction, $T_{j_0}r_k = \delta_{kj_0}$ in U_S. Using the Cauchy–Schwarz inequality, and the fact that $f \in \mathcal{C}^\infty(\overline{\Omega})$, the square of the absolute-value of the boundary term may be estimated to be less than or equal to the quantity

$$C \int_{b\Omega_{j_0}^{\varepsilon}} \left(\prod_{j\in S} \frac{r_j^{s_j}}{s_j!} \right)^2 |g|^2 \, dS \le C' \varepsilon^2 \int_{b\Omega_{j_0}^{\varepsilon}} |g|^2 \, dS,$$

with C and C' independent of ε. Since $g \in L^2(\Omega)$, we can find a sequence $\varepsilon_i \to 0$ such that $\int_{b\Omega_{j_0}^{\varepsilon_i}} |g|^2 \, dS = o(\varepsilon_i^{-1})$. Taking a limit as $\varepsilon_i \to 0$ in (2.3), we have

$$\begin{aligned}
&\left\langle \prod_{j\in S} \frac{r_j^{s_j}}{s_j!} g, \left(\prod_{j\in S} (T_j^*)^{s_j} \right)(\chi_S f) \right\rangle \\
&\quad = \lim_{i\to\infty} \left\langle \prod_{j\in S} \frac{r_j^{s_j}}{s_j!} g, \left(\prod_{j\in S} (T_j^*)^{s_j} \right)(\chi_S f) \right\rangle_{\Omega_{\varepsilon_i}} \\
&\quad = \left\langle T_{j_0} \left(\prod_{j\in S} \frac{r_j^{s_j}}{s_j!} g \right), \left((T_{j_0}^*)^{s_0 - 1} \cdot \prod_{j \in S\setminus\{j_0\}} (T_j^*)^{s_j} \right)(\chi_S f) \right\rangle \\
&\quad = \left\langle \frac{r_{j_0}^{s_{j_0}-1}}{(s_{j_0}-1)!} \cdot \prod_{j\in S\setminus\{j_0\}} \frac{r_j^{s_j}}{s_j!} g, \left((T_{j_0}^*)^{s_0-1} \cdot \prod_{j\in S\setminus\{j_0\}} (T_j^*)^{s_j} \right)(\chi_S f) \right\rangle,
\end{aligned}$$

where in the last line we have used the facts that $T_{j_0} g = 0$, and $T_{j_0} r_j = \delta_{j j_0}$. Repeating the above process $s_0 - 1$ times more, we conclude that the above expression is equal to

$$\left\langle \prod_{j\in S\setminus\{j_0\}} \frac{r_j^{s_j}}{s_j!} g, \left(\prod_{j\in S\setminus\{j_0\}} (T_j^*)^{s_j} \right)(\chi_S f) \right\rangle,$$

and applying the same process to the smallest element of $S \setminus \{j_0\}$ and continuing till we are left with the empty set of indices, we conclude that

$$\left\langle \prod_{j\in S} \frac{r_j^{s_j}}{s_j!} g, \left(\prod_{j\in S} (T_j^*)^{s_j} \right)(\chi_S f) \right\rangle = \langle g, \chi_S f \rangle.$$

We note that the term corresponding to $S = \emptyset$ in (2.2) is simply $\langle g, \chi_\emptyset f \rangle$, and therefore we can rewrite (2.2) as:

$$\begin{aligned}
\langle g, \Phi^s f \rangle &= \sum_{S\subset\{1,\ldots,N\}} \langle g, \chi_S f \rangle \\
&= \left\langle g, \left(\sum_{S\subset\{1,\ldots,N\}} \chi_S \right) f \right\rangle \\
&= \langle g, f \rangle,
\end{aligned}$$

since $\{\chi_S\}$ is a partition of unity. This proves the result. $\square$

Let K_Ω denote the Bergman kernel of a domain Ω. If $P : L^2(\Omega) \to \mathcal{H}(\Omega) \subset L^2(\Omega)$ denotes the Bergman projection on Ω, K_Ω is characterized by the property that for each $f \in L^2(\Omega)$ we have

$$Pf(w) = \int_\Omega K_\Omega(w, z) f(z)\, dV(z),$$

where dV is Lebesgue measure. It is well known that $K_\Omega(w, z)$ is holomorphic in w, antiholomorphic in z and satisfies the Hermitian symmetry $K_\Omega(w, z) = \overline{K_\Omega(z, w)}$. Lemma 2.1 leads to the following characterization of Condition R on a domain with generic corners.

Proposition 2.2 (cf. [7], [1]). *A domain Ω with generic corners satisfies Condition R if and only if for each multi-index α, there are constants C and m depending only on the domain Ω such that*

$$\left| \frac{\partial^\alpha}{\partial w^\alpha} K_\Omega(w, z) \right| \le C \operatorname{dist}(z, b\Omega)^{-m} \tag{2.4}$$

for all $(w, z) \in \Omega \times \Omega$.

Proof. The method of proof given in [7] may be applied with appropriate minor modification. The crucial point here is the existence of the operator Φ^s. □

3. Some examples

We now consider examples of domains D in $\mathbb{C}^n$ for which the hypotheses of Theorem 1.2 hold, that is, D has generic corners, is pseudoconvex and satisfies Condition R. Note that if D satisfies the hypotheses of Theorem 1.2, it follows from Theorem 1.2 that so does $F(D)$, where $F : D \to F(D)$ is a biholomorphic map extending smoothly to $\overline{D}$. If $n = 1$, the only domains with generic corners are the smoothly bounded ones. For $n \ge 2$, there do exist domains with generic corners in $\mathbb{C}^n$. However, many interesting piecewise smooth domains do not have generic corners, e.g., the intersection of two balls in $\mathbb{C}^2$ (see [3]).

For smoothly bounded domains, Condition R is a consequence of global regularity estimates on the $\overline{\partial}$-Neumann operator (see [14], [24] for details). Indeed it suffices to know that the $\overline{\partial}$-Neumann operator is compact on the space $L^2_{0,1}(D)$ of square integrable $(0, 1)$-forms. However, as [17] already shows, this strategy is unlikely to succeed with general piecewise smooth domains. The question of establishing Condition R on such domains therefore merits deeper study. However there are a few cases where Condition R can be established on a domain with generic corners by elementary means.

3.1. Products. We first show that the hypotheses of Theorem 1.2 propagate to products.

PROPOSITION 3.1. *For $j = 1, \dots, k$, let $D_j \Subset \mathbb{C}^{n_j}$ be a domain with generic corners which satisfies Condition R. Let $n = \sum_{j=1}^{k} n_j$, and let D be the domain in $\mathbb{C}^n$ given as $D = D_1 \times D_2 \times \cdots \times D_k$. Then D has generic corners and satisfies Condition R.*

In [12], [11], the following was proved: if $D_1 \subset \mathbb{C}^{n_1}, D_2 \subset \mathbb{C}^{n_2}$ are bounded pseudoconvex domains (no assumption of generic corners on the boundary) such that each of them satisfies Condition R, then so does their product. Here on the other hand there is no assumption of pseudoconvexity.

Note also that combining this proposition and Theorem 1.2 we recapture the famous observation of Poincaré: the ball and bidisc in $\mathbb{C}^2$ are not biholomorphically equivalent.

Proof of Proposition 3.1. By an induction argument, it is sufficient to prove this for $k = 2$. Assume that as in Definition 1.1 we are given representations $D_1 = \bigcap_{j=1}^{N} G_j$ and $D_2 = \bigcap_{\ell=1}^{M} H_k$. We then have

$$\begin{aligned} D_1 \times D_2 &= \left(D_2 \times \mathbb{C}^{n_2}\right) \cap \left(\mathbb{C}^{n_1} \times D_2\right) \\ &= \left(\bigcap_{j=1}^{N} \left(G_j \times \mathbb{C}^{n_2}\right)\right) \cap \left(\bigcap_{\ell=1}^{M} \left(\mathbb{C}^{n_1} \times H_\ell\right)\right), \end{aligned}$$

which is a representation of $D_1 \times D_2$ as an intersection of smoothly bounded domains. Since D_1, D_2 have generic corners, it is easy to verify that the corners of the product are CR manifolds of the right CR dimension.

Denote by K_j the Bergman kernel of D_j. The derivatives of K_j satisfy the estimate (2.4), since D_j satisfies Condition R. Thanks to [21, Theorem 6.1.11] the Bergman kernel K of the product $D_1 \times D_2$ can be represented $K_1 \otimes K_2$, that is, for $z = (z_1, z_2) \in D_1 \times D_2$ and $w = (w_1, w_2) \in D_1 \times D_2$, we have

$$K(w, z) = K_1(w_1, z_1) \cdot K_2(w_2, z_2).$$

Then it follows that the derivatives of K satisfy the estimate (2.4), and it follows that D satisfies Condition R. $\square$

3.2. Domains with circular symmetry. Recall that a domain $D \subset \mathbb{C}^n$ is said to be *circular* if it is invariant under the natural action of the circle group, that is, if for each $z \in D$ and each real number θ, we have that $e^{i\theta} z \in D$. Clearly, the boundary bD of D has the same circular symmetry. Further, if D has piecewise smooth boundary, it is clear that every stratum is invariant under the circle group. Further, we call a domain *complete circular* if for each $z \in D$, and for each complex number λ in the closed unit disc (i.e., if $|\lambda| \leq 1$), we have $\lambda z \in D$.

For a piecewise smooth domain D, by a *face* we mean a connected component of bD^{reg}. If D is represented as the intersection $\bigcap_{j=1}^{N} D_j$, where each D_j is smoothly bounded and all intersections of the boundaries are transverse, then it is clear that each face is a connected component of $bD_j \cap \overline{D}$ for some j. The following result, extending a classical argument of Boas and Bell, gives simple examples of domains with generic corners satisfying Condition R:

Proposition 3.2 (cf. [7, Theorem 2′]). *Let $D \subset \mathbb{C}^n$ be a bounded complete circular domain with generic corners such that for each $\zeta \in bD$, the radial line from the origin to ζ meets each face of bD that passes through ζ transversely. Then D satisfies Condition R.*

We begin by noting a symmetry property of the Bergman kernel of a circular domain.

Lemma 3.3. *Let K denote the Bergman kernel of a circular domain D, where $0 \in D$. Then if λ is a complex number and $z, w \in D$ are such that the points $\lambda w, \overline{\lambda} z$ are in D, then we have*

$$K(\lambda w, z) = K(w, \overline{\lambda} z).$$

Proof. We claim that there is an orthonormal basis $\{\eta_j\}_{j=1}^{\infty}$ of the Bergman space $\mathcal{H}(D)$ whose elements are homogeneous polynomials. Indeed, it is well-known that any holomorphic function on the circular domain D can be expanded in a series of the form $f(z) = \sum_{k=1}^{\infty} P_k(z)$, where each P_k is a homogenous polynomial and the series converges uniformly on compact subsets of D (see, e.g., [22]). Choosing a basis of the space of homogeneous polynomials of degree d, and taking the union as d ranges over the non-negative integers, we obtain a family of homogeneous polynomials whose span is dense in $L^2(\Omega)$. Further, if P and Q are homogeneous polynomials of degrees p and q respectively, they are orthogonal in $L^2(D)$ if $p \neq q$. Indeed, if θ is a real number such that $e^{i(p-q)\theta} \neq 1$, we have using the change of variables formula and the fact that the unitary transformation $z \mapsto e^{i\theta} z$ has real Jacobian determinant identically equal to 1,

$$\begin{aligned} \int_D P(z)\overline{Q(z)}\, dV(z) &= \int_D P\big(e^{i\theta} w\big)\overline{Q\big(e^{i\theta} w\big)}\, dV(w) \\ &= e^{i(p-q)\theta} \int_D P(z)\overline{Q(z)}\, dV(z). \end{aligned}$$

Consequently, if the Gram–Schmidt process is applied to the spanning family of homogeneous polynomials, it yields the orthonormal sequence $\{\eta_j\}$, and the Bergman kernel is then represented as $K(w, z) = \sum_{j=1}^{\infty} \eta_j(w)\overline{\eta_j(z)}$. Recalling that each η_j is homogenous of some degree, the result follows. $\square$

We also note the two useful properties of the Bergman kernel: the Cauchy–Schwarz inequality

$$\big|K(w,z)\big| \le \big(K(w,w)K(z,z)\big)^{\frac{1}{2}}, \tag{3.1}$$

and the fact that on any bounded domain D

$$K(z,z) \le (\mathrm{const})d(z)^{-n-1}, \tag{3.2}$$

obtained by comparing the Bergman kernel of D with that of a ball centered at z and radius $d(z)$.

From now on, let D be complete circular. Then for a point $w \in D$, we can define the *radial boundary distance* $\rho(w)$ in the following way. Let w^* be the unique point on the boundary bD which is collinear with 0 and w. We define $\rho(w) = |w^* - w|$. We also denote $d(w) = \mathrm{dist}(w, bD)$, and call this the *standard boundary distance.* We will be interested in domains in which there is a constant $C > 1$ such that

$$\rho(w) \le Cd(w). \tag{3.3}$$

Since we always have $d(w) \le \rho(w)$ we will say that on such domains the radial and standard boundary distances are comparable. We first note that this property holds on the domains considered in Proposition 3.2.

Lemma 3.4. *Let D be a piecewise smooth complete circular domain such that for each $\zeta \in bD$, the radial line from 0 to ζ meets each face of bD which passes through ζ transversally. Then the standard and the radial distance are comparable on D.*

Proof. Let bD be smooth, and fix a tubular neighborhood U of bD in D. For a point z in U denote by $\hat{z}$ the unique point on bD closest to z, and by z^* the point where the radial line from 0 to z meets bD. From the transversality of the line zz^* to bD, it follows that the angle between zz^* and $z\hat{z}$ is bounded away from $\frac{\pi}{2}$, and the result follows in this case.

Assuming now that there are at least two faces, it is sufficient to prove (3.3) for z in some neighborhood U of bD in D. Let U be the union of U_j, where each U_j is a tubular neighborhood of bD_j, where the domain D is represented as an intersection $\bigcap_{j=1}^N D_j$. Let $\rho_j(z)$ represent the radial distance from z to the boundary bD_j, and $d_j(z) = \mathrm{dist}(z, bD_j)$. Then if $z \in U_j$, we have $\rho_j(z) \le Cd_j(z)$, where C may be taken independent of j. But $\rho(z) \le \rho_j(z) \le Cd_j(z)$ for each z in U_j. If a point z in U belongs to more than one U_j, it follows that we must have $\rho(z) \le C \min d_j(z) = Cd(z)$ where the minimum is taken over all j such that the point z belongs to U_j. The result is proved. ☐

The proof is now completed by the following lemma, and an appeal to Proposition 2.2.

LEMMA 3.5. *Let D be a bounded complete circular domain in $\mathbb{C}^n$. If the standard and radial boundary distances on D are comparable* (*i.e.*, (3.3) *holds*), *then D satisfies the estimate* (2.4).

Proof. Without loss of generality we can assume that the diameter of D is less than or equal to one, since (2.4) holds on a domain if and only if it holds on any dilation. We proceed as in [7]. We fix once for all $z \in D$. We consider two cases. First, assume that $w \in D$ is such that $|w| > \frac{1}{2}d(0)$.

We choose a number $0 < \delta < \frac{d(0)}{4}$ such that

$$\frac{4\delta}{d(0) - 4\delta} < \frac{1}{2}\rho(z).$$

The number δ exists since $x \mapsto \frac{4x}{d(0)-4x}$ is increasing on $[0, \frac{d(0)}{4})$. Let $\lambda = (1 - \frac{2\delta}{|w|})^{-1}$. For future, use we note that

$$\begin{aligned} \rho(\lambda z) &= \rho(z) - (\lambda - 1)|z| \\ &= \rho(z) - \frac{2\delta}{|w| - 2\delta}|z| \\ &\geq \rho(z) - \frac{2\delta}{\frac{\delta(0)}{2} - 2\delta} \\ &> \frac{1}{2}\rho(z), \end{aligned} \tag{3.4}$$

where the last line follows from the choice of δ. By Lemma 3.3,

$$\begin{aligned} K(w, z) &= K\big(\lambda^{-1}w, \lambda z\big) \\ &= K(t, \lambda z), \end{aligned} \tag{3.5}$$

where $t = (1 - \frac{2\delta}{|w|})w$, and therefore we have that $\rho(t) \geq 2\delta$. Noting that we are considering such $w \in D$ as $|w| > \frac{d(0)}{2}$, we see that $|\frac{\partial^\alpha}{\partial w^\alpha}(\frac{1}{|w|})|$, and therefore $|\frac{\partial^\alpha}{\partial w^\alpha}(\frac{w_j}{|w|})|$ are bounded (the latter for each j). It follows that for any multi-index α with $|\alpha| \geq 2$, we have that $|\frac{\partial^\alpha}{\partial w^\alpha} t_j| \leq C\delta$, and that $|\frac{\partial^\alpha}{\partial w^\alpha}\lambda| \leq C\delta$, where here and in the sequel the constant C depends on α but is independent of z (and therefore δ) and w (with $|w| > \frac{d(0)}{2}$), but C may have different values at different occurrences. Using the alternative representation of $K(w, z)$ in (3.5), and the repeated use of the chain and the product rule (i.e., the Faa di Bruno formula), one may compute an expression for $\frac{\partial^\alpha}{\partial w^\alpha}K(w, z)$, in terms of the t and z-derivatives of $K(t, z)$ and the w derivatives of t and λ. It follows that

$$\left|\frac{\partial^\alpha}{\partial w^\alpha}K(w, z)\right| \leq C\delta \cdot \sum_{\beta+\gamma\leq\alpha}\left|\frac{\partial^\beta}{\partial t^\beta}\frac{\partial^\gamma}{\partial \overline{z}^\gamma}K(t, \lambda z)\right|, \tag{3.6}$$

since higher powers of δ may be absorbed into δ itself (since $\delta < \frac{1}{4}$). Now, thanks to the comparability of the standard and radial distances to the boundary, we see that there is a polydisc of polyradius $C(\delta, \delta, \dots, \delta)$ with center at $t = (1 - \frac{2\delta}{|w|})w$ and located within $\{\zeta \in D : \rho(\zeta) > \delta\}$. Recalling that K is holomorphic in the first and antiholomorphic in the second argument, and applying the Cauchy estimates in both arguments to this polydisc we conclude:

$$\begin{aligned}
\left|\frac{\partial^\beta}{\partial t^\beta}\frac{\partial^\gamma}{\partial \overline{z}^\gamma} K(t, \lambda z)\right| &\le \frac{C}{\delta^{|\beta|+|\gamma|}} \sup_{\rho(t)>\delta} \sqrt{K(t,t)}\sqrt{K(\lambda z, \lambda z)} && \text{using (3.1)}\\
&\le \frac{C}{\delta^{|\beta|+|\gamma|}} \delta^{-\frac{n+1}{2}} \rho(\lambda z)^{-\frac{n+1}{2}} && \text{using (3.2)}\\
&\le \frac{C}{\delta^{|\beta|+|\gamma|}} \delta^{-\frac{n+1}{2}} \rho(z)^{-\frac{n+1}{2}} && \text{using (3.4)}\\
&\le \frac{C}{d(z)^{|\beta|+|\gamma|+n+1}}.
\end{aligned}$$

Combining this with (3.5), we conclude that

$$\left|\frac{\partial^\alpha}{\partial w^\alpha} K(w, z)\right| \le \frac{C}{d(z)^{|\alpha|+n}},$$

for w such that $|w| > \frac{d(0)}{2}$.

We now consider a $w \in D$ such that $|w| \le \frac{d(0)}{2}$. Then there is an η independent of w such that $\rho(w) > \eta$. By the comparability of ρ and d, we conclude that there is an $\varepsilon > 0$ such that the polydisc centered at w and of radius ε is contained in the set $\{\zeta \in D : d(\zeta) > \varepsilon\}$. (Note that ε depends only on $d(0)$ and the constant C in (3.3).) Applying the Cauchy estimates to this polydisc we see that

$$\begin{aligned}
\left|\frac{\partial^\alpha}{\partial w^\alpha} K(w, z)\right| &\le \frac{C}{\varepsilon^{|\alpha|}} \sup_{d(w)>\varepsilon} \left|K(w,z)\right|\\
&\le C\sqrt{K(z,z)} \sup_{d(w)>\varepsilon} \sqrt{\left|K(w,w)\right|}\\
&\le C d(z)^{-\frac{(n+1)}{2}} \varepsilon^{-\frac{(n+1)}{2}},
\end{aligned}$$

therefore the estimate (2.4) is established and the result is proved. □

4. Hopf lemma on domains with generic corners

Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain and $\phi : D \to [-\infty, 0)$ a plurisubharmonic exhaustion function. The Hopf lemma asserts that $|\phi(z)|$ decays to zero near the boundary bD at least at the rate of $\operatorname{dist}(z, bD)$, that is,

$$\left|\phi(z)\right| \gtrsim \operatorname{dist}(z, bD) \tag{4.1}$$

for all $z \in D$. For a given proper holomorphic mapping $f : D \to G$, this estimate plays a useful role in controlling the ratio $\operatorname{dist}(f(z), bG)/\operatorname{dist}(z, bD)^{\eta}$ for some $\eta > 0$. Thus, we are interested in obtaining (4.1) on non-smooth domains as well. For piecewise smooth domains, this was done in [10], [23] by showing that each point sufficiently close to the boundary lies in a cone of uniform aperture with vertex on the boundary. In other words, a planar sector of uniform aperture containing a given point near the boundary was shown to exist. On a product domain G, it is evident that a sector whose aperture angle is $\pi/2$, that is, a quadrant, can be fitted at each boundary point. Therefore, the techniques of [10] show that a negative plurisubharmonic exhaustion ϕ on a product domain satisfies

$$\big|\phi(z)\big| \gtrsim \operatorname{dist}(z, bG)^2$$

for all $z \in G$. A different approach was used in [13] for product domains wherein a disc that satisfies certain uniform geometric properties was used instead of a sector. Similar ideas can be applied to domains with generic corners as well which yield a better growth estimate.

Proposition 4.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain with generic corners. Let $\phi : \Omega \to [-\infty, 0)$ be a plurisubharmonic exhaustion. Then*

$$\big|\phi(z)\big| \gtrsim \operatorname{dist}(z, bD)$$

for all $z \in \Omega$.

We first recall some geometric conditions on an analytic disc from [13] that are sufficient to prove (4.1). Let $D \subset \mathbb{C}^n$ be a bounded domain and take a tubular neighborhood U of bD. The domain $U \cap D$ whose boundary consists of two disjoint components, namely bD and $B = bU \cap D$ will be relevant to us. Suppose that there is a constant $\theta = \theta(D) \in (0, 2\pi)$ and points $\kappa(z) \in B, \zeta(z) \in bD$ (both possibly non-unique) for every $z \in U \cap D$ such that the following hold:

(i) The points $\zeta(z), z, \kappa(z)$ are collinear and z lies between $\zeta(z)$ and $\kappa(z)$.
(ii) $\zeta(z)$ is the nearest point to z on bD which means that $|\zeta(z) - z| = \operatorname{dist}(z, bD)$.
(iii) The affine analytic disc $\alpha_z : \Delta(0,1) \to \mathbb{C}^n$ given by

$$\alpha_z(\lambda) = \kappa(z) + \lambda\big(\zeta(z) - \kappa(z)\big)$$

lies in D.
(iv) There exists a neighborhood of $\partial\Omega$ in $\mathbb{C}^n$, say V which is compactly contained in U such that the portion of the boundary of $\alpha_z(\Delta(0,1))$, that is, $\alpha_z(b\Delta(0,1))$ that lies in $D \setminus V$ subtends an angle of at least $\theta = \theta(D) > 0$ at the centre $\kappa(z)$. Note that $\alpha_z(0) = \kappa(z)$ and $\alpha_z(1) = \zeta(z)$.

In short, these properties allow the existence of an analytic disc passing through a given point p near bD and also containing p^*, a nearest point to p

on bD, whose centre is at a uniform distance away from bD and such that a uniform piece of its boundary is also uniformly away from bD. We say that it is possible to *roll an analytic disc* in D if these properties hold. Theorem 4.4 in [13] shows that the Hopf lemma holds on a domain if it is possible to roll an analytic disc in it.

Proof of Proposition 4.1. It suffices to show that it is possible to roll an analytic disc in a domain Ω with generic corners as in Definition 1.1. Fix a point $p \in b\Omega$ and let $S \subset \{1, 2, \dots, N\}$ be such that

$$p \in B_S = \bigcap_{j \in S} b\Omega_j.$$

Without loss of generality we may assume that $S = \{1, 2, \dots, k\}$ where $k \leq N$. Then

$$r_1(p) = r_2(p) = \cdots = r_k(p) = 0$$

and (2.1) holds. Thanks to this transversality condition, we may choose coordinates in a neighborhood U around $p = 0$ so that the defining functions become

$$r_j(z) = 2 \operatorname{Re} z_j + \phi_j(z),$$

where $\phi_j \in \mathcal{C}^\infty(U)$ and $d\phi_j(0) = 0$ for all $1 \leq j \leq k$. The smoothness of each r_j implies that for a given point $z \in U$ there is a unique point z_j^* on $\{r_j = 0\} = b\Omega_j \cap U$ such that

$$\tau_j = \operatorname{dist}(z, b\Omega_j \cap U) = \left| z - z_j^* \right|$$

for all $1 \leq j \leq k$. The analytic disc

$$\zeta \mapsto z + \zeta \tau_j \left(\overline{\partial} r_j \left(z_j^* \right) \right)$$

for $|\zeta| < 1$ is centered at z and is contained in $\{r_j < 0\} = \Omega_j \cap U$. Thus, through a given point $z \in \Omega \cap U$ there are k analytic discs which approximately point in the direction of the coordinate axes $z_1, z_2, \dots, z_k$. This observation will allow us to choose the right direction for the disc $\alpha_z(\lambda)$ as in (iii) above. Let $C > 0$ be such that

$$C^{-1} \left| r_j(z) \right| \leq \operatorname{dist}(z, b\Omega_j \cap U) \leq C \left| r_j(z) \right| \tag{4.2}$$

for all $z \in \Omega \cap U$ and $1 \leq j \leq k$. Furthermore, since $d\phi_j(0) = 0$ we may also assume that $|d\phi_j(z)| \leq 1/2C$ for all $z \in \Omega \cap U$. For $\varepsilon > 0$, let

$$\Omega_\varepsilon = \left\{ z \in U : r_j(z) < -\varepsilon, 1 \leq j \leq k \right\}.$$

Pick $z \in U \cap \Omega$ and note that the nearest point to it (which is possibly non-unique) on $b\Omega \cap U$ lies on one or possibly more of the boundaries $b\Omega_j \cap U = \{r_j = 0\}$. For the sake of definiteness, assume that it lies on $b\Omega_1 \cap U = \{r_1(z) = 0\}$ and denote it by $\zeta(z)$. Extend the real inner normal l to the smooth real hypersurface $\{r_1(z) = 0\}$ at $\zeta(z)$ till it intersects $b\Omega_\varepsilon \cap U$. Denote this point of intersection by $\kappa(z)$. Note that $|\kappa(z)-\zeta(z)| = \operatorname{dist}(\kappa(z), \{r_1(z)=0\}\cap U) \geq \varepsilon/C$ by (4.2). The affine analytic disc

$$\alpha_z(\lambda) = \kappa(z) + \lambda\big(\zeta(z) - \kappa(z)\big)$$

defined for $|\zeta|<1$ is evidently contained in $\{r_1<0\}\cap U$. For $1<j\leq k$ observe that

$$\big|r_j\big(\alpha_z(\lambda)\big) - r_j\big(\kappa(z)\big)\big| = \big|\lambda\big(\zeta(z)-\kappa(z)\big)\cdot dr_j(\tilde{z})\big|$$

for some $\tilde{z}=\tilde{z}(\lambda)\in\Omega\cap U$. Again (4.2) shows that $|\kappa(z)-\zeta(z)|\leq C\varepsilon$ and by construction we have $|dr_j(\tilde{z})|\leq 1/2C$. Combining these estimates shows that

$$r_j\big(\alpha_z(\lambda)\big) \leq r_j\big(\kappa(z)\big) + \varepsilon/2 \leq -\varepsilon + \varepsilon/2 = -\varepsilon/2.$$

Thus, the analytic disc $\alpha_z(\lambda)$ is contained in $\{r_1<0\}\cap U$ and stays at a uniform distance from the other hypersurfaces $\{r_j=0\}\cap U$ where $1<j\leq k$. Let $V=\{z\in U : |r_j(z)|<\varepsilon/2, 1\leq j\leq k\}$—this is a neighborhood of $b\Omega\cap U$ of uniform width $\varepsilon/2$. The smoothness of r_1 shows that there is a uniform portion of $b\alpha_z(\lambda)$ that lies in $(\Omega\cap U)\setminus\{r_1>-\varepsilon/2\}$. The arguments given above show that the closure of $\alpha_z(\lambda)$ lies in $(\Omega\cap U)\setminus\{r_j>-\varepsilon/2, 1<j\leq k\}$ and hence a uniform portion of $b\alpha_z(\lambda)$ lies in $(\Omega\cap U)\setminus V$. These estimates are uniform for all $z\in\Omega\cap U$ and hence for all z near $b\Omega$ by compactness. Hence, it is possible to roll an analytic disc in Ω. $\square$

5. Proper maps of domains with generic corners

5.1. Distortion estimate on domains with generic corners. We now generalize some well-known properties of proper maps of smoothly bounded pseudoconvex domains to domains with generic corners. In these results, D and G are pseudoconvex domains with generic corners, and $f: D\to G$ is a proper holomorphic mapping. Let $Z=\{f(z): \det f'(z)=0\}\subset G$ be the set of critical values of f. Then Z is a codimension one subvariety in G, and on $G\setminus Z$, we can define locally well-defined holomorphic branches $F_1, F_2, \ldots, F_m$ of f^{-1}. The following consequence of the Hopf lemma is well-known in the case of smoothly bounded domains.

Proposition 5.1. *There exists a $\delta\in(0,1)$ such that*

$$\operatorname{dist}(z,bD)^{1/\delta} \lesssim \operatorname{dist}\big(f(z),bG\big) \lesssim \operatorname{dist}(z,bD)^{\delta}$$

for all $z\in D$.

Proof. We begin by noting that if Ω is a pseudoconvex domain with generic corners, then there is a negative strictly plurisubharmonic exhaustion ϱ of Ω which decays to zero at the boundary no faster than a power of the distance to the boundary, that is, for some $0<\eta<1$ and all $z\in\Omega$ we have

$$\big|\varrho(z)\big| \lesssim \operatorname{dist}(z,b\Omega)^{\eta}.$$

This follows directly (even for Lipschitz Ω) from [20]. We can also deduce it from the fact that if as in Definition 1.1, the domain Ω is represented as an intersection $\bigcap_{j=1}^{N}\Omega_j$ of smoothly bounded pseudoconvex domains, then by famous results of Diederich and Fornaess [15], each Ω_j admits a bounded plurisubharmonic exhaustion ϱ_j satisfying $|\varrho_j(z)| \lesssim \operatorname{dist}(z, b\Omega_j)^{\eta_j}$ for some $\eta_j \in (0,1)$ and for each $z \in \Omega_j$. We can simply take $\varrho = \max_{1\le j\le N}\varrho_j$.

Therefore, let ϱ_D and ϱ_G be bounded plurisubharmonic exhaustions on D and G such that for some $\eta, \tau \in (0,1)$ and

$$\left|\varrho_D(z)\right| \lesssim \operatorname{dist}(z, bD)^{\eta}$$

for all $z \in D$, and

$$\left|\varrho_G(w)\right| \lesssim \operatorname{dist}(z, bG)^{\tau}$$

for all $w \in G$. Then $\varrho_G \circ f$ is a negative plurisubharmonic exhaustion on D and satisfies

$$-\varrho_G \circ f(z) = \left|\varrho_G \circ f(z)\right| \gtrsim \operatorname{dist}(z, bD)$$

for all $z \in D$ by the Hopf lemma. Thus we get

$$\operatorname{dist}(z, bD) \lesssim -\varrho_G \circ f(z) \lesssim \operatorname{dist}\big(f(z), bG\big)^{\tau}$$

which is the left-hand side inequality in the proposition.

Recall that $F_1, \dots, F_m$ denote the branches of the inverse mapping f^{-1}, which are locally well-defined on $G \setminus Z$, where Z is the set of critical values of the mapping F. Then

$$\psi = \max\{\varrho_D \circ F_j : 1 \le j \le m\}$$

is a bounded continuous plurisubharmonic function on $G \setminus Z$ which extends to a plurisubharmonic exhaustion on G. Therefore, for each $1 \le j \le m$ and $w \in G$, we have

$$-\varrho_D \circ F_j(w) \ge -\psi(w) = \left|\psi(w)\right| \gtrsim \operatorname{dist}(w, bG),$$

where the last inequality follows from the Hopf lemma. Rewriting this as

$$\left|\varrho_D(z)\right| = -\varrho_D(z) \gtrsim \operatorname{dist}\big(f(z), bG\big)$$

and combining with the rate of decay of ϱ_D near bD we get

$$\operatorname{dist}\big(f(z), bG\big) \lesssim \operatorname{dist}(z, bD)^{\eta}$$

for all $z \in D$ which completes the proof. □

5.2. Smoothness of the Jacobian up to the boundary. We now note that the following lemma, well-known for smoothly bounded domains, continues to hold for domains with generic corners. For a domain Ω in $\mathbb{C}^n$, we denote by $\mathcal{H}^\infty(\Omega)$ the space $\mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ of holomorphic functions on Ω which are smooth up to the boundary of Ω.

LEMMA 5.2. *Suppose that D satisfies Condition R, and let $u = \det(f')$ be the Jacobian determinant of the mapping f. If $h \in \mathcal{H}^\infty(G)$, we have*

$$u \cdot (h \circ f) \in \mathcal{H}^\infty(D).$$

Proof. We adapt the classical proof from [5]. Let ℓ be a given positive integer. We need to show that $u\cdot(h\circ f)\in\mathcal{C}^{\ell}(\overline{D})$. Denote by P and Q the Bergman projections on the domains D and G, respectively. Now, thanks to the classical transformation formula for the Bergman projection, we have for each $g\in L^2(G)$ that

$$P\big(u\cdot(g\circ f)\big)=u\cdot\big(Q(g)\circ f\big).$$

For an N-tuple $s=(s_1,\dots,s_N)$ of positive integers, let Φ^s be the operator on G as constructed in Lemma 2.1, and set $g_s=\Phi^s h$. Then $Qg_s=h$, and we have

$$u\cdot(h\circ f)=P\big(u\cdot(g_s\circ f)\big).$$

Since D satisfies Condition R, it follows that there is an integer k such that P maps $\mathcal{C}^k(\overline{D})$ into $\mathcal{C}^{\ell}(\overline{D})$. Therefore, to prove the result, it suffices to show that there is a tuple s such that $u\cdot(g_s\circ f)\in\mathcal{C}^k(\overline{D})$. It will be sufficient to show that derivatives of order $k+1$ of the function $u\cdot(g_s\circ f)$ on D are all bounded.

Denote the map f in components as $f=(f_1,\dots,f_n)$, where each f_j is complex valued on D. Note that each f_j is bounded. Consequently, we have the Cauchy estimates

$$\big|D^\alpha f_j(z)\big|\lesssim \operatorname{dist}(z,bD)^{-|\alpha|}$$

and

$$\big|D^\alpha u(z)\big|\lesssim \operatorname{dist}(z,bD)^{-|\alpha|-n}.$$

We will take the tuple s be to of the form $s=(\sigma,\dots,\sigma)$, that is, all N entries are equal to the same positive integer σ. If $N\sigma>|\alpha|$, and $w\in G$, by Lemma 2.1 we have an estimate

$$\big|D^\alpha\big(g_s(w)\big)\big|\lesssim \operatorname{dist}(w,bG)^{N\sigma-|\alpha|}.$$

Note that we have

$$D^\alpha(g_s\circ f)(z)=\sum D^\beta g_s\big(f(z)\big)D^{\delta_1}f_{i_1}D^{\delta_2}f_{i_2}\cdots D^{\delta_p}f_{i_p},$$

where the sum extends over all tuples $1\le i_1,\dots,i_p\le n$, and multi-indices β, $\delta_1,\dots,\delta_p$ with $|\beta|\le|\alpha|$ and $|\delta_1|+\dots+|\delta_p|=|\alpha|$. Therefore, we have the estimate

$$\begin{aligned}\big|D^\alpha(g_s\circ f)(z)\big|&\lesssim \operatorname{dist}\big(f(z),bG\big)^{N\sigma-|\alpha|}\cdot\operatorname{dist}(z,bD)^{-(|\alpha|+1)}\\&\lesssim \operatorname{dist}(z,bD)^{\delta(N\sigma-|\alpha|)}\cdot\operatorname{dist}(z,bD)^{-(|\alpha|+1)}\\&\lesssim \operatorname{dist}(z,bD)^{\delta N\sigma-(1+\delta)|\alpha|-1},\end{aligned}$$

where in the second line, the right half of the distortion estimate from Proposition 5.1 has been used. It follows that by taking σ to be sufficiently large, we can make the function $D^\alpha(g_s\circ f)$ vanish to arbitrarily high order on the boundary bD. Using the Cauchy estimates on the derivatives of u, and the Leibniz rule for the derivative of a product it now follows that by taking σ

sufficiently large, we can ensure that for any multi-index α, the derivative $D^\alpha(u\cdot(g_s\circ f))$ is bounded on D. Consequently there is an N-tuple s such that $u\cdot(g_s\circ f)\in\mathcal{C}^k(\overline{D})$, and our result is proved. □

5.3. Smoothness to the boundary of symmetric functions of the inverse branches.

PROPOSITION 5.3. *If G is as in Theorem 1.3, that is, G is a product of smoothly bounded domains, then for $h\in\mathcal{H}^\infty(D)$, an elementary symmetric function of $h\circ F_1, h\circ F_2,\dots,h\circ F_m$ (defined on $G\setminus Z$) extends to a function in $\mathcal{H}^\infty(G)$.*

Proof. For smoothly bounded domains D and G, this is a classical result of Bell (see [5]). It was shown in [13, Proposition 5.3] that the same arguments, with minor modifications work when each of D and G is a product of smoothly bounded domains. We note here that the proof given in [13] actually works in the more general situation when D is merely a domain with generic corners and is not necessarily a product. □

6. Proof of Theorem 1.2

1 ⇒ 2. Since f maps bD diffeomorphically to bG, it follows that G must have piecewise smooth boundary. Since the map f is CR on each of the manifolds constituting bD, it follows that G is a domain with generic corners.

Let $g=f^{-1}$, and let $K_G(z,w)$ and $K_D(Z,W)$ denote the Bergman kernels on the domains G and D respectively. Since by hypothesis, D satisfies Condition R, it follows from Proposition 2.2 that for each multi-index α, there is an m_α such that we have an estimate

$$\left|\left(\frac{\partial}{\partial W}\right)^\alpha K_D(W,Z)\right|\lesssim \operatorname{dist}(Z,bD)^{-m_\alpha},$$

valid for all $(W,Z)\in D\times D$. The kernels K_D and K_G are related by the classical formula ([5] or [21, Proposition 6.1.7])

$$K_G(w,z)=\det g'(w)K_D\big(g(w),g(z)\big)\overline{\det g'(z)}.$$

Therefore

$$\left(\frac{\partial}{\partial w}\right)^\alpha K_G(w,z)=\left(\left(\frac{\partial}{\partial w}\right)^\alpha \det g'(w)K_D\big(g(w),g(z)\big)\right)\cdot\overline{\det g'(z)}.$$

Since g is smooth up to the boundary,

$$\big|\det g'(z)\big|\lesssim 1. \tag{6.1}$$

Again, since g is smooth up to the boundary, by repeated application of the chain rule and the product rule we obtain

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial w}\right)^{\alpha}\big(\det g'(w)K_D\big(g(w),g(z)\big)\big)\right| \\
&\quad\lesssim \sum_{|\beta|\le|\alpha|}\left|\left(\frac{\partial}{\partial W}\right)^{\beta}K_D\big(g(w),g(z)\big)\right| \\
&\quad\lesssim \sum_{|\beta|\le|\alpha|}\operatorname{dist}\big(g(z),bD\big)^{-m_\beta} \\
&\quad\lesssim \operatorname{dist}\big(g(z),bD\big)^{-M} \quad (M \text{ being the largest of the } m_\beta\text{'s}) \\
&\quad\lesssim \operatorname{dist}(z,bG)^{-\frac{M}{\delta}},
\end{aligned}$$

where in the last line we have used Proposition 5.1. Combining this with (6.1), and invoking Proposition 2.2 our result follows.

2 ⇒ 1. Taking $h\equiv 1$ in Lemma 5.2, we see that $u\in\mathcal{C}^\infty(\overline{D})$. Applying the lemma again to the mapping $f^{-1}: G\to D$, we obtain that $\det((f^{-1})')\in\mathcal{C}^\infty(\overline{G})$. But this implies that $u^{-1}\in\mathcal{C}^\infty(\overline{D})$. It therefore follows that for each holomorphic h on G such that $h\in\mathcal{C}^\infty(\overline{G})$, we have that $h\circ f\in\mathcal{C}^\infty(\overline{D})$. Taking h to be the coordinate functions $z\mapsto z_j$ from D to $\mathbb{C}$, we see that each component of f extends smoothly to the boundary, and the result is proved.

7. Proof of Theorem 1.3

The proof of Theorem 1.3 is for most part identical to the first part of the argument for the proof of [13, Theorem 1.1], where it is further assumed that D is also a product domain. We review the main steps of the proof below, noting in each step that the hypothesis of product structure is not really used in the proof of continuous extension to the boundary. (It does become relevant in the latter part of the proof of [13, Theorem 1.1], i.e., Lemma 5.7 onward.) What is important is that D is piecewise smooth, pseudoconvex, satisfies Condition R, and there is a Bell operator on D.

As in Lemma 5.2, let $u=\det(f')$ be the Jacobian determinant of the map $f: D\to G$. We claim that u vanishes to at most finite order at each point of ∂D. For smoothly bounded domains, the proof can be found in [6], [9]. It was shown in [13, Lemma 5.5] that essentially the same argument continues to work for the piecewise smooth domains considered here.

From this, as in [13, Lemma 5.6], it follows that f extends to a continuous map from $\overline{D}$ to $\overline{G}$. The key ingredient here is the weak division result [16, Lemma 10] which states the following: On a smoothly bounded domain $\Omega\subset\mathbb{C}^n$, let $u\in\mathcal{H}^\infty(\Omega)$ be a function that does not vanish to infinite order at any point on $b\Omega$. If h is a bounded holomorphic function on Ω such that $u\cdot h^N\in\mathcal{H}^\infty(\Omega)$ for all $N\ge 1$, then h is continuous on $\overline{\Omega}$. To prove that h

is continuous at $p \in b\Omega$, the only geometric requirement is the existence of a complex line through p that enters Ω near p and which is transverse to $b\Omega$ near p. This condition is clearly satisfied at all points of $bD^{\rm reg}$ while at the generic corners such a complex line may be chosen to be transverse to the tangent cone to $b\Omega$ at such points. Thus the proof of [13, Lemma 5.6] carries over to the case of domains with generic corners. To show that f is smooth at all points of $bD^{\rm reg}$, the finite order vanishing of u at the boundary can be combined with the strong form of the division theorem (which is a local statement) as in [8] or [16]. This completes the proof of Theorem 1.3.

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