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SOLUTION TO BIHARMONIC EQUATION WITH VANISHING POTENTIAL

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Abstract. We establish a result on the existence of nontrivial solution for the following class of biharmonic elliptic equation

$$\text{(P)}\qquad \begin{cases} \Delta^2 u + V(x)u = K(x)f(u) & \text{in } R^N, \\ u \neq 0, & \text{in } R^N, u \in \mathcal{D}^{2,2}(R^N), \end{cases}$$

where $\Delta^2 u = \Delta(\Delta u)$, V and K are nonnegative potentials. K vanishes at infinity and f has a subcritical growth at infinity. The technique used here is the variational approach.

1. Introduction

Consider the following biharmonic elliptic equation in R^N:

$$\text{(P)}\qquad \begin{cases} \Delta^2 u + V(x)u = K(x)f(u) & \text{in } R^N, \\ u \neq 0, & \text{in } R^N; u \in \mathcal{D}^{2,2}(R^N), \end{cases}$$

where $\Delta^2 u = \Delta(\Delta u)$, $N \geq 5$, $V, K : R^N \to R$ are continuous potentials. K vanishes at infinity and $f : R \to R$ is a continuous function with subcritical growth at infinity. Here $\mathcal{D}^{2,2}(R^N)$ is the completion of the $\mathcal{C}_0^\infty(R^N)$ with the norm $|u| = (\int_{R^N} |\Delta u|^2 \, dx)^{\frac{1}{2}}$.

We assume that V and K satisfies the conditions I and II below:

I.

$$(K_1)\qquad V(x), K(x) > 0 \quad \text{in } R^N \quad \text{and} \quad K \in L^\infty(R^N) \cap L^1(R^N).$$

II. One of the following conditions occurs:

$$(K_2)\qquad K/V \in L^\infty(R^N),$$

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or

$$\text{there exists } \alpha \in (2, 2_*), \quad \text{with } 2_* = \frac{4N}{N-4}, \text{such that}$$

$$(K_3) \qquad \lim_{|x|\to\infty} \frac{K(x)}{V(x)^{\frac{2_*-\alpha}{2_*-2}}} = 0.$$

We denote $(V,K) \in \mathcal{K}$ to express that V and K satisfy I and II.

As an example let V be a positive constant and K given by

$$K(x) = \begin{cases} e^{-1}, & \text{if } |x| \le 1, \\ e^{-|x|}, & \text{if } |x| > 1. \end{cases}$$

It is easily seen that $(V,K) \in \mathcal{K}$.

We assume that the function f satisfies the following conditions:

III.

$$(f_1) \qquad \limsup_{s\to 0} \frac{f(s)}{s^{2_*-1}} = 0.$$

IV. f has a subcritical growth at infinity, namely,

$$(f_2) \qquad \limsup_{s\to\infty} \frac{f(s)}{s^{2_*-1}} = 0.$$

V. $f(s) = 0$, for $s \le 0$, $s^{-1}f(s)$ a nondecreasing function in $(0,\infty)$ and its primitive F is superquadratic at infinity, that is,

$$(f_3) \qquad \limsup_{s\to\infty} \frac{F(s)}{s^2} = \infty.$$

We recall that the condition (f_3) is weaker than the Ambrosetti–Rabinowitz condition, namely, there exists $\theta > 2$ such that

$$(\mathrm{AR}) \qquad 0 < \theta F(s) \le s f(s), \quad \text{for all } s \in R.$$

For instance, the function $f(s) = s(1+\ln s^2)$ satisfies (f_3) and does not satisfy the (AR) condition. The (AR) condition is very important to ensure that the Euler–Lagrange functional associated to the problem (P) has a mountain pass geometry and also to guarantee that the corresponding Palais–Smale sequence is bounded. Since the condition (AR) is very restrictive, many researchers have tried to drop it. For more information on this subject, we refer to [18] and references therein.

Also, related to the function f we emphasize that the conditions (f_1), (f_2) and (f_3) imply the there exists $c > 0$ such that

$$(1.1) \qquad 0 \le f(s) \le c|s|^{2_*-1}, \quad \text{for all } s \in R.$$

Equations with the biharmonic operator in bounded domains arise in the study of traveling waves in suspension bridges and in the study of the static deflection of an elastic plate in a fluid (see [23] and references therein). For the problem (P), with $K \equiv 1$ and V constant, in a bounded domain, we cite for example, [11].

Let us briefly recount some results involving the biharmonic operator on unbounded regions. It is well known by now that the nonlinear Schrödinger equation with an additional term containing higher-order derivatives is closely related the self-focusing of whistler waves in plasmas in the final stage. In isotropic media, it has the form

$$i\frac{\partial \Psi}{\partial z} + \frac{1}{2}S\Delta\Psi + \lambda\Delta^2\Psi + \mu|\Psi|^2\Psi = 0,$$

where the term with Δ^2 describes the contribution of the higher-order dispersion (see [13]). We recall that fourth order nonlinear Schrödinger equations have been introduced by Karpman, in [14], and Karpman and Shagalov, in [15], to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity (see [20]).

Now we turn out our attention for the biharmonic Schrödinger equation with potentials, in unbounded domains. To begin with, we cite the work [19] where it was considered $V \equiv 0$, nonnegative K and radial potential vanishing at infinity. In [19], the authors obtained the existence of positive radial solutions. Alves, do Ó and Miyagaki, in [3], took the potential V nonegative and the nonlinearity with two nonnegative potentials. Assuming periodicity of the potentials, the authors obtained existence of solutions. In [3] it was also considered small perturbations of the potentials and existence of solutions was obtained. The same authors, in [2], used a potential V that changes sign with some points of singularities. Chabrowski and do Ó, in [9], assumed that K is a continuous bounded potential varying in sign and V is a nonpositive potential. In [9] it was obtained existence of two solutions. Gazzola and Grunau, in [12], considered $K \equiv 1$ and $V \equiv 0$ to investigate existence, uniqueness, asymptotic behaviour and futher qualitative properties of radial solutions. Wang and Shen, in [22], assumed the potential $V \equiv 0$ and the nonlinearity with a nonnegative potential in the subcritical growth and a nonnegative potential that vanishes at infinity in the critical growth. Under a improved Hardy–Rellich's inequality, in [22] the authors studied the existence of multiple and sign-changing solutions by the minimax method and linking theorem. Carrião, Demarque and Miyagaki, in [8], considered $K \equiv 1$ and V radial and vanishing at infinity to get existence of radial solutions results. Finally, Pimenta and Soares, in [21], studied a phenomena of concentration for the problem (P) with $K \equiv 1$ and V satisfying the following proprety: there exists a bounded domain $\Omega \subset R^N$ such that

$$0 < V(x_0) = V_0 = \inf_{R^N} V < \inf_{\partial\Omega} V.$$

As the discussion above shows, the types of potential affect the existence and features of solutions. In the present work, we consider the problem (P) under more general hypothesis on the potentials V and K, besides nonlinearity in a wider class. In fact, our result extends that of Demarque and Miyagaki, in [10] for nonradial potentials V and K. It also extends the result of Alves and do Ó, in [1], for more general potentials and nonlinearity. In addition, our result extends in part, or it complements the result of Alves and Souto, in [5], for the biharmonic operator.

In this work, we use a technique analogous to that used by Alves and Souto, in [5]. In fact, in order to obtain the mountain pass geometry, in addition to a specific condition about the primitive F, we used subcriticals growth conditions of type Hardy–Sobolev on f. We also imposed some convenient conditions on V and K to get an inequality of the Hardy type and, with this, we got a strong convergence in the whole space. In fact, we assume the conditions (K_2) and (K_3) to get the compact embedding of $E \subset \mathcal{D}^{2,2}(R^N)$ in $L^q_K(R^N)$ with $2 < q < 2_*$. The spaces E and $L^q_K(R^N)$ will be defined below. With this tool, we could overcome the lack of compactness in the Sobolev's embedding in the whole space, which is one of the great difficulties for this type of problem.

Before stating the main result, it should be noted that here we use the notion of ground state solution of the problem (P) as defined in [4], that is, a function $u \in E$ verifying $J'(u) = 0$ and $J(u) = c_*$. J is the Euler–Lagrange functional associated to (P) and c_* is the mountain pass minimax level, both to be defined later.

Theorem 1.1. *Suppose* $(V, K) \in \mathcal{K}$, (f_1), (f_2) *and* (f_3). *Then the problem* (P) *has a ground state solution.*

Hereafter, c is a positive constant which can change value in a sequence of inequalities. We denote $B_r = B_r(0)$ the ball in R^N centered in the origin with radius r. $o_n(1)$ denotes a term that tends to zero as $n \to \infty$. The weak $(\rightharpoonup)$ and strong $(\to)$ convergences are always taken as $n \to \infty$. The weighted L^p spaces are denoted by $L^p_Q(A) = \{u : A \to R : \int_A Q(x)|u|^p \, dx < \infty\}$, for $1 \le p \le \infty$ and measurable set $A \subset R^N$.

2. Preliminary results

Aiming to solve the problem (P) with the variational method, we consider its Euler–Lagrange functional:

$$J(u) = \frac{1}{2} \int_{R^N} \left(|\Delta u|^2 + V(x)|u|^2\right) dx - \int_{R^N} K(x) F(u) \, dx,$$

defined on space E, given by

$$E = E\left(R^N\right) = \left\{ u \in \mathcal{D}^{2,2}\left(R^N\right) : \int_{R^N} V|u|^2 \, dx < \infty \right\},$$

with the norm

$$\|u\| = \left(\int_{R^N} \left(|\Delta u|^2 + V(x)|u|^2 \right) dx \right)^{\frac{1}{2}}.$$

Here $F(s) = \int_0^s f(t)\, dt$. From the assumptions on f, it follows that J is C^1 with Gâteaux derivative given by

$$J'(u)v = \int_{R^N} \left(\Delta u \Delta v + V(x)uv \right) dx - \int_{R^N} K(x) f(u) v\, dx, \quad v \in E.$$

LEMMA 2.1. *If the conditions* (K_1), (f_1) *and* (f_3) *are satisfied then the functional* J *satisfies the mountain pass geometry, namely,*

1. *There are* $r, \rho > 0$ *such that* $J(u) \geq \rho$, *for* $\|u\| = r$.
2. *There exists* $e \in E$ *such that* $\|e\| \geq r$ *and* $J(e) \leq 0$.

Proof.

Step 1. Combining (K_1), the growth condition of f, given by inequality (1.1) and the inequality

$$\left(\int_{R^N} |u|^{2_*}\, dx \right)^{\frac{1}{2_*}} \leq S \left(\int_{R^N} |\Delta u|^2\, dx \right)^{\frac{1}{2}}, \tag{2.1}$$

(see [11, Theorem 2.1]), we have

$$\int_{R^N} K(x) F(u)\, dx \leq c \left(S \int_{R^N} |\Delta u|^2\, dx \right)^{\frac{2_*}{2}} \leq c\|u\|^{2_*}. \tag{2.2}$$

Hence,

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{R^N} K(x) F(u)\, dx \geq \frac{1}{2}\|u\|^2 - c\|u\|^{2_*}.$$

Thus, we have $J(u) \geq \rho := \frac{1}{2} r^2 - c r^{2_*} > 0$, for $\|u\| = r$, small enough.

Step 2. By (f_3) it follows that, for all $M > 0$, there exists $s_M > 0$, such that

$$F(s) \geq M s^2, \quad \text{for all } |s| > s_M.$$

By setting $C_M = \sup_{|s| \leq s_M} F(s)$ we have $0 < C_M < \infty$. Thus, we have

$$F(s) \geq M s^2 - C_M, \quad \text{for all } s \in R. \tag{2.3}$$

Fixed $u \in E$ and using (2.3) we obtain

$$\begin{aligned} J(tu) &= \frac{1}{2}\|tu\|^2 - \int_{R^N} K(x) F(tu)\, dx \\ &\leq t^2 \left[\frac{\|u\|^2}{2} - M \int_{R^N} K(x) u^2\, dx \right] + c. \end{aligned}$$

Taking M big enough, $J(tu) \to -\infty$ as $t \to \infty$, which finishes the proof. $\square$

Since J satisfies mountain pass geometry we conclude, by Mountain Pass theorem ([7, Theorem 2.2]), that there exists a Palais–Smale sequence, ((PS) sequence for short), $(u_n) \subset E$ for J, that is, (u_n) satisfies $J(u_n) \to c_*$ and $J'(u_n) \to 0$, where c_*, is the mountain pass level, given by

$$c_* = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J\big(\gamma(t)\big),$$

with $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0 \text{ and } J(\gamma(1)) \leq 0\}$.

PROPOSITION 2.2. *Assume that* $(V,K) \in \mathcal{K}$. *Then*

1. *If* (K_2) *holds,* E *is compactly embedded in* $L^q_K(R^N)$, *for all* $q \in (2,2^*)$.
2. *If* (K_3) *holds,* E *is compactly embedded in* $L^\alpha_K(R^N)$, *with* α *as in the hypothesis* (K_3).

Proof.

Part 1. We assume (K_2) and $v_n \rightharpoonup v$ to show that $v_n \to v$ in $L^q_K(R^N)$, for $q \in (2,2^*)$.

Given $\varepsilon > 0$, from $2 < q$, there exists $s_0 > 0$ such that $|s|^q \leq \varepsilon|s|^2$, in $|s| \leq s_0$. Since $K/V \in L^\infty(R^N)$, there exists $c > 0$ such that

$$K(x)|s|^q \leq \varepsilon c V(x)|s|^2, \quad \text{for } |s| \leq s_0 \text{ and } x \in R^N.$$

Given $\varepsilon > 0$, from $K \in L^\infty(R^N)$ and by $q < 2_*$, we can take $s_1 > 0$ such that

$$K(x)|s|^q \leq \varepsilon c|s|^{2_*}, \quad \text{for } |s| \geq s_1 \text{ and } x \in R^N.$$

By the continuity of the functions involved, we see that there exists $c > 0$ such that

$$K(x)|s|^q \leq cK(x)\chi_{[s_0,s_1]}\big(|s|\big)|s|^{2_*}, \quad \text{for } s_0 \leq |s| \leq s_1 \text{ and } x \in R^N.$$

Here $\chi_{[s_0,s_1]}$ is the characteristic function in the interval $[s_0,s_1]$.

Thus, fixed $q \in (2,2_*)$ and given $\varepsilon > 0$, there are $c > 0$ and $0 < s_0 < s_1$ such that, for all $s \in R$ and $x \in R^N$, we have

$$K(x)|s|^q \leq \varepsilon c\big(V(x)|s|^2 + |s|^{2_*}\big) + cK(x)\chi_{[s_0,s_1]}\big(|s|\big)|s|^{2_*}. \tag{2.4}$$

For a given $r > 0$ and $u \in E$, the equation (2.4) give us

$$\begin{aligned}
&\int_{B_r^c} K(x)|u|^q\,dx \\
&\quad\leq \varepsilon c\bigg(\int_{B_r^c} V(x)|u|^2\,dx + \int_{B_r^c} |u|^{2_*}\,dx\bigg) \\
&\qquad + c\int_{B_r^c} K(x)\chi_{[s_0,s_1]}\big(|u|\big)|u|^{2_*}\,dx \\
&\quad\leq \varepsilon cQ(u) + c\int_{A\cap B_r^c} K(x)\,dx,
\end{aligned} \tag{2.5}$$

being

$$Q(u) = \int_{R^N} V(x)|u|^2\,dx + \int_{R^N} |u|^{2_*}\,dx \tag{2.6}$$

and $A = \{x \in R^N : s_0 \le |u(x)| \le s_1\}$.

Since $v_n \rightharpoonup v$ in E, the sequence (v_n) is bounded in E. Then, for all n we have $\int_{R^N} V(x)|v_n|^2\,dx \le c$, and, using (2.1) we get $\int_{R^N} |v_n|^{2_*}\,dx \le c$, for all n, so that

$$Q(v_n) \le c, \quad \text{for all } n. \tag{2.7}$$

Hereafter, we will refer to inequality (2.7) as "the boundedness of $(Q(v_n))$".

From (K_1), we can take $r_1 > 0$ such that

$$\int_{A_n \cap B^c_{r_1}} K(x)\,dx \le \varepsilon. \tag{2.8}$$

From the inequalities (2.5), (2.7) and (2.8), we have

$$\int_{B^c_{r_1}} K(x)|v_n|^q\,dx \le \varepsilon, \quad \text{for all } n. \tag{2.9}$$

Considering the particular case in which $v_n = v$, for all n, we see that (2.9) allow us to choose a $r_2 \ge r_1$ such that

$$\int_{B^c_{r_2}} K(x)|v|^q\,dx \le \varepsilon, \tag{2.10}$$

which, along with (2.9), give us

$$\int_{B^c_{r_2}} K(x)|v_n|^q\,dx \to \int_{B^c_{r_2}} K(x)|v|^q\,dx. \tag{2.11}$$

Notice that $\mathcal{D}^{2,2}(R^N)$ is the completion of the $\mathcal{C}_0^\infty(R^N)$ with the norm $|u| = (\int_{R^N} |\Delta u|^2\,dx)^{\frac{1}{2}}$. Thus, if $u \in \mathcal{D}^{2,2}(R^N)$, by [16, p. 164] we have $\|D^2 u\|_{2,R^N} = (\int_{R^N} |D^2 u|^2\,dx)^{\frac{1}{2}} < \infty$, so that $D^2 u \in L^2(B_{r_2})$. By using [17, Corollary, p. 8], we have that $D^\alpha u \in L^2(B_{r_2})$, with $|\alpha| = 0, 1$. Then $u \in \mathcal{H}^2(B_{r_2})$. Since K is a continuous function and $q \in (2, 2_*)$, it follows from usual Sobolev embeddings and Dominated Convergence theorem that

$$\int_{B_{r_2}} K(x)|v_n|^q\,dx \to \int_{B_{r_2}} K(x)|v|^q\,dx. \tag{2.12}$$

By (2.11) and (2.12), we get

$$\int_{R^N} K(x)|v_n|^q\,dx \to \int_{R^N} K(x)|v|^q\,dx, \quad \text{for all } q \in (2, 2_*),$$

so that E is compactly embedded in $L^q_K(R^N)$, for all $q \in (2, 2^*)$.

Part 2. We consider (K_3) and $v_n \rightharpoonup v$ to show that $v_n \to v$ in $L^\alpha_K(R^N)$ with α as in the hypothesis (K_3).

Fixed $x \in R^N$, the function of $s > 0$ given by $g(s) = V(x)s^{2-\alpha} + s^{2_*-\alpha}$, has a minimum value given by

$$m_\alpha V(x)^{\frac{2_*-\alpha}{2_*-2}}, \quad \text{being } m_\alpha = \left(\frac{2_*-2}{2_*-\alpha}\right)\left(\frac{\alpha-2}{2_*-2}\right)^{\frac{2-\alpha}{2_*-2}}.$$

Thus, we get

$$m_\alpha V(x)^{\frac{2_*-\alpha}{2_*-2}} \le V(x)|s|^{2-\alpha} + |s|^{2_*-\alpha}, \quad \text{for } s \in R^* \text{ and } x \in R^N. \tag{2.13}$$

By (K_3), for a given $\varepsilon > 0$, there exists $r_1 > 0$, such that

$$K(x) \le \varepsilon m_\alpha V(x)^{\frac{2_*-\alpha}{2_*-2}}, \quad \text{for } |x| \ge r_1. \tag{2.14}$$

From (2.13) and (2.14), we have

$$K(x) \le \varepsilon\big(V(x)|s|^{2-\alpha} + |s|^{2_*-\alpha}\big), \quad \text{for } s \in R^* \text{ and } |x| \ge r_1. \tag{2.15}$$

Thus, we get

$$K(x)|s|^\alpha \le \varepsilon\big(V(x)|s|^2 + |s|^{2_*}\big), \quad \text{for } s \in R \text{ and } |x| \ge r_1,$$

which, along with (2.6), yields

$$\int_{B_{r_1}^c} K(x)|u|^\alpha\, dx \le \int_{B_{r_1}^c} \varepsilon\big(V(x)|u|^2 + |u|^{2_*}\big)\, dx \le \varepsilon Q(u). \tag{2.16}$$

Taking $u = v_n$ in (2.16) and using the boundedness of $(Q(v_n))$, we have

$$\int_{B_{r_1}^c} K(x)|v_n|^\alpha\, dx \le \varepsilon, \quad \text{for all } n.$$

Since $\alpha \in (2, 2_*)$, from (2.10) we can choose $r_2 \ge r_1 > 0$ such that

$$\int_{B_{r_2}^c} K(x)|v|^\alpha\, dx \le \varepsilon. \tag{2.17}$$

Proceeding as in the previous part, we have

$$\int_{R^N} K(x)|v_n|^\alpha\, dx \to \int_{R^N} K(x)|v|^\alpha\, dx,$$

so that E is compactly embedded in $L_K^\alpha(R^N)$, with α as in the hypothesis (K_3). □

LEMMA 2.3. *Suppose* $(V, K) \in \mathcal{K}$, (f_1) *and* (f_2). *Let* (v_n) *be a sequence such that* $v_n \rightharpoonup v$ *in* E. *Then*

$$\int_{R^N} K(x)G(v_n)\, dx \to \int_{R^N} K(x)G(v)\, dx,$$

for $G(v_n) = F(v_n), G(v_n) = f(v_n)v_n$ *and* $G(v_n) = f(v_n)v$.

Proof.

Part 1. We consider (K_2) and we start with the case $G(v_n) = F(v_n)$.

Given $\varepsilon > 0$, from (f_1) and by $K/V \in L^\infty(R^N)$, we conclude that there exists $s_0 > 0$ such that

$$K(x)F(s) \le \varepsilon c V(x)|s|^2, \quad \text{for } |s| \le s_0 \text{ and } x \in R^N.$$

Given $\varepsilon > 0$, from $K \in L^\infty(R^N)$ and by (f_2), we can take $s_1 > 0$ such that

$$K(x)F(s) \le \varepsilon c|s|^{2^*}, \quad \text{for } |s| \ge s_1 \text{ and } x \in R^N.$$

By the continuity of the functions involved, we see that there exists $c > 0$ such that

$$K(x)F(s) \le cK(x)|s|^q, \quad \text{for } s_0 \le |s| \le s_1 \text{ and } x \in R^N.$$

Thus, fixed $q \in (2, 2_*)$ and given $\varepsilon > 0$, there exists $c > 0$ such that, for all $s \in R$ and $x \in R^N$, we have

$$K(x)F(s) \le \varepsilon c\big(V(x)|s|^2 + |s|^{2_*}\big) + cK(x)|s|^q. \tag{2.18}$$

By the boundedness of $(Q(v_n))$ and from the previous inequality, we get

$$\begin{aligned}\int_{B_r^c} K(x)F(v_n)\,dx &\le \int_{B_r^c} \varepsilon c\big(V(x)|v_n|^2 + |v_n|^{2_*}\big)\,dx + \int_{B_r^c} K(x)|v_n|^q\,dx \\ &\le \varepsilon c Q(v_n) + \int_{B_r^c} K(x)|v_n|^q\,dx \le \varepsilon c + \int_{B_r^c} K(x)|v_n|^q\,dx,\end{aligned} \tag{2.19}$$

for all n and $r > 0$.

From Proposition 2.2, we have

$$\int_{R^N} K(x)|v_n|^q\,dx \to \int_{R^N} K(x)|v|^q\,dx, \quad \text{for all } q \in (2, 2_*), \tag{2.20}$$

so that we can take $r_1 > 0$ for which we have

$$\int_{B_{r_1}^c} K(x)|v_n|^q\,dx \le \varepsilon, \quad \text{for all } n.$$

This, along with (2.19), give us

$$\int_{B_{r_1}^c} K(x)F(v_n)\,dx \le \varepsilon, \quad \text{for all } n. \tag{2.21}$$

Considering the particular case in which $v_n = v$, for all n, it follows from (2.21) that we can choose $r_2 \ge r_1 > 0$ such that

$$\int_{B_{r_2}^c} K(x)F(v)\,dx \le \varepsilon. \tag{2.22}$$

From (2.21) and (2.22), it follows that

$$\int_{B_{r_2}^c} K(x)F(v_n)\,dx \to \int_{B_{r_2}^c} K(x)F(v)\,dx. \tag{2.23}$$

Since $v_n \rightharpoonup v$, we have $v_n \to v$ in $L^q(B_{r_2})$, for $q \in (2, 2_*)$, and $v_n \to v$ a.e. in B_{r_2} as $n \to \infty$. So that by continuity of F, we have $F(v_n) \to F(v)$ a.e. in B_{r_2} as $n \to \infty$. On the other hand, since (v_n) is bounded in E, from inequality (2.1), we infer that $\sup_n \int_{B_{r_2}} |v_n|^{2_*}\,dx < \infty$. From (f_2) we have $\frac{F(v_n)}{|v_n|^{2_*}} \to 0$ as $v_n \to \infty$. By using the lemma of Strauss [6, Theorem A.I, p. 338] we get

$$\int_{B_{r_2}} K(x)F(v_n)\,dx \to \int_{B_{r_2}} K(x)F(v)\,dx. \tag{2.24}$$

Hence, from equations (2.23) and (2.24), we have

$$\int_{R^N} K(x)F(v_n)\,dx \to \int_{R^N} K(x)F(v)\,dx,$$

which completes the proof.

The other cases are completely analogous. For the case $G(v_n) = f(v_n)v$ it is enough to observe that $v \in L^\infty(R^N)$, since $v \in \mathcal{D}^{2,2}(R^N)$.

Part 2. We assume (K_3) and we start with the case $G(v_n) = F(v_n)$.

Exactly as in the part 2 of Proposition 2.2 we can say that, for a given $\varepsilon > 0$, by (K_3), there exists a $r_1 > 0$, such that

$$K(x) \le \varepsilon\big(V(x)|s|^{2-\alpha} + |s|^{2_*-\alpha}\big), \quad \text{for } s \in R^* \text{ and } |x| \ge r_1. \tag{2.25}$$

Then, for all $s \in R$ and $|x| \ge r_1$, we have

$$K(x)F(s) \le \varepsilon\big(V(x)F(s)|s|^{2-\alpha} + F(s)|s|^{2_*-\alpha}\big). \tag{2.26}$$

From (f_1), for a given $\varepsilon > 0$, there exists $0 < s_0$ such that

$$F(s) \le c|s|^\alpha, \quad \text{for } |s| \le s_0, \tag{2.27}$$

which, along with (2.26), yields

$$K(x)F(s) \le \varepsilon\big(V(x)|s|^2 + |s|^{2^*}\big), \quad \text{for } |s| \le s_0 \text{ and } |x| \ge r_1. \tag{2.28}$$

From (f_2), for a given $\varepsilon > 0$, we can find $s_1 > s_0 > 0$ such that

$$\frac{K(x)F(s)}{(V(x)|s|^2 + |s|^{2^*})} \le \|K\|_\infty \frac{F(s)}{|s|^{2^*}} \le c\varepsilon, \quad \text{for } |s| \ge s_1,$$

so that,

$$K(x)F(s) \le \varepsilon c\big(V(x)|s|^2 + |s|^{2^*}\big), \quad \text{for } |s| \ge s_1 \text{ and } x \in R^N. \tag{2.29}$$

Thus, by (2.28) and (2.29), we have

$$K(x)F(s) \le \varepsilon c\big(V(x)|s|^2 + |s|^{2^*}\big), \quad \text{for } s \in \mathcal{I} \text{ and } |x| \ge r_1, \tag{2.30}$$

being $\mathcal{I} = \{s \in R : |s| \le s_0 \text{ or } |s| \ge s_1\}$.

Using (2.30), the boundedness of $(Q(v_n))$ and $A_n = \{x \in R^N : s_0 \le |v_n(x)| \le s_1\}$ we get

$$\begin{aligned}
(2.31)\qquad & \int_{B^c_{r_1}} K(x)F(v_n)\,dx \\
&\le \int_{B^c_{r_1}\cap A^c_n} K(x)F(v_n)\,dx + \int_{B^c_{r_1}\cap A_n} K(x)F(v_n)\,dx \\
&\le \int_{B^c_{r_1}\cap A^c_n} \varepsilon c\big(V(x)|v_n|^2 + |v_n|^{2_*}\big)\,dx + c\int_{B^c_{r_1}\cap A_n} K(x)\,dx \\
&\le c\varepsilon Q(v_n) + c\int_{B^c_{r_1}\cap A_n} K(x)\,dx \\
&\le c\varepsilon + c\int_{B^c_{r_1}\cap A_n} K(x)\,dx, \quad \text{for all } n.
\end{aligned}$$

From (K_1), we can take $r_2 \ge r_1$ such that

$$\int_{B^c_{r_2}\cap A_n} K(x)\,dx \le \varepsilon, \quad \text{for all } n. \tag{2.32}$$

Then, from (2.31) and (2.32), we get

$$\int_{B^c_{r_2}} K(x)F(v_n)\,dx \le c\varepsilon, \quad \text{for all } n. \tag{2.33}$$

From now on, following the steps as after (2.21), we conclude the proof of part 2.

The other cases are completely analogous. For the case $G(v_n) = f(v_n)v$ it is enough to observe that $v \in L^\infty(R^N)$, since $v \in \mathcal{D}^{2,2}(R^N)$. ☐

LEMMA 2.4. *The* (PS) *sequence* (u_n) *given by Lemma* 2.1, *is bounded.*

Proof. From the proof of Lemma 2.1, we can take $t_n \in [0,1]$ such that $J(t_n u_n) = \max_{\beta\ge 0} J(\beta u_n)$.

We claim that the sequence $(J(t_n u_n))$ is bounded from above.

For $t_n = 0$, we have $(J(t_n u_n)) = (J(0))$ and, for $t_n = 1$, $(J(t_n u_n)) = (J(u_n))$. In both cases, we have the boundedness, since $J(u_n) \to c_*$.

Thus, we can assume $t_n \in (0,1)$. Since $J'(t_n u_n)t_n u_n = 0$ we have

$$\begin{aligned}
(2.34)\qquad 2J(t_n u_n) &= 2J(t_n u_n) - J'(t_n u_n)t_n u_n \\
&= \int_{R^N} K(x)\big[-2F(t_n u_n) + f(t_n u_n)t_n u_n\big]\,dx \\
&= \int_{R^N} K(x)H(t_n u_n)\,dx,
\end{aligned}$$

being $H(s) = -2F(s) + sf(s)$, for all $s \in R$.

From (f_3), H is a nondecreasing function in $(0,\infty)$ and we have

$$
\begin{aligned}
(2.35) \qquad 2J(t_n u_n) &= \int_{R^N} K(x)H(t_n u_n)\,dx \le \int_{R^N} K(x)H(u_n)\,dx \\
&\le 2J(u_n) - J'(u_n)u_n = 2J(u_n) + o_n(1),
\end{aligned}
$$

so that $(J(t_n u_n))$ is bounded from above, since $J(u_n)\to c_*$.

We will assume that $\|u_n\|\to\infty$ to reach a contradiction with the equation (2.35). For this purpose consider $w_n = u_n/\|u_n\|$. Since (w_n) is bounded, we conclude that there exists $w\in E$ such that $w_n \rightharpoonup w$ in E.

We claim that $w=0$ in R^N. We will prove it later.

Proceeding with the proof of the lemma, note that for $B>0$ and n big enough we have $B/\|u_n\|\in[0,1]$. Then

$$
\begin{aligned}
J(t_n u_n) &= \max_{\beta\ge 0} J\big(\beta u_n\big) \ge J\big(Bu_n/\|u_n\|\big) = J(Bw_n) \\
&= \frac{1}{2}\|Bw_n\|^2 - \int_{R^N} K(x)F(Bw_n)\,dx \\
&= \frac{B^2}{2} - \int_{R^N} K(x)F(Bw_n)\,dx.
\end{aligned}
$$

Since $w_n \rightharpoonup 0$, by Lemma 2.3, we have

$$
\int_{R^N} K(x)F(Bw_n)\,dx \to \int_{R^N} K(x)F(0)\,dx = 0.
$$

Thus, for all $B>0$, we get

$$
\liminf_{n\to\infty} J(t_n u_n) \ge \liminf_{n\to\infty}\left(\frac{B^2}{2} - \int_{R^N} K(x)F(Bw_n)\,dx\right) = \frac{B^2}{2}.
$$

Then, $(J(t_n u_n))$ is unbounded from above, which contradicts the equation (2.35). Hence, the (PS) sequence (u_n) is bounded. $\square$

Proof of claim. First, we consider the sequence (u_n) bounded in $L^\infty(R^N)$. Then $w_n(x) = u_n(x)/\|u_n\| \le c/\|u_n\| \to 0$, for all $x\in R^N$, since $\|u_n\|\to\infty$. From the compact embedding of E in $L^q_K(R^N)$, we have $w_n(x)\to w(x)$ a.e. in R^N. Thus,

$$
w\equiv 0, \quad \text{in } R^N.
$$

Now, consider that there exists a subsequence, renamed by (u_n), unbounded in $L^\infty(R^N)$ and define $\Omega = \{x\in R^N : u_n(x)\ne 0\} = \{x\in R^N : w_n(x)\ne 0\}$.

Since $J(u_n)\to c_*$, we have

$$
\frac{1}{2}\|u_n\|^2 - \int_{R^N} K(x)F(u_n)\,dx = c + o_n(1),
$$

so that

$$
(2.36) \qquad o_n(1) + \frac{1}{2} = \int_{R^N} \frac{K(x)F(u_n)}{\|u_n\|^2}\,dx = \int_{\Omega} \frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2\,dx.
$$

From (f_3), we see that, given $\tau > 0$, there exists $M > 0$ such that $F(s)/s^2 \geq \tau$, for $s \geq M$. Define ψ_n and χ_n, the characteristic functions for $\{u_n \leq M\} = \{x \in \Omega : 0 < u_n(x) \leq M\}$ and $\{u_n > M\} = \{x \in \Omega : u_n(x) > M\}$, respectively. Applying this in the equation (2.36), we have

$$\begin{aligned} o_n(1) + \frac{1}{2} &= \int_\Omega \frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2\,dx \\ &= \int_\Omega \psi_n(x)\frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2\,dx + \int_\Omega \chi_n(x)\frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2\,dx \\ &\geq \int_\Omega \psi_n(x)\frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2\,dx + \tau\int_\Omega \chi_n(x)K(x)|w_n|^2\,dx. \end{aligned} \tag{2.37}$$

Let Ω^- and Ω^+ be the sets limit of $\{u_n \leq M\}$ and $\{u_n > M\}$, respectively. Using the same argument of the first paragraph of this proof, we have $w_n(x) \to 0$ in $\{u_n \leq M\}$, so that

$$w \equiv 0, \quad \text{in } \Omega^-.$$

Moreover, $K(x)F(u_n)/|u_n|^2$ is bounded in $\{u_n \leq M\}$, for all n. In fact, when $u_n(x) \to 0$, we use (f_1) to get the conclusion. When $0 < \varepsilon \leq u_n(x) \leq M$, we use the continuity of F and the fact that $K \in L^\infty(R^N)$, which completes the argument. Thus, from this uniform boundedness with respect to n we conclude that $K(x)F(u_n)/|u_n|^2$ is bounded in Ω^-.

Since in Ω^- we have $w_n(x) \to 0$ and $K(x)F(u_n)/|u_n|^2$ is bounded we conclude that

$$\liminf_{n\to\infty} \frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2 = 0, \quad \text{in } \Omega^-, \tag{2.38}$$

so that

$$\liminf_{n\to\infty} \psi_n(x)\frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2 = 0, \quad \text{in } \Omega. \tag{2.39}$$

Using (2.39) and the Fatou's lemma in (2.37), we get

$$\begin{aligned} \frac{1}{2} &\geq \liminf_{n\to\infty} \int_\Omega \psi_n(x)\frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2\,dx + \liminf_{n\to\infty} \tau\int_\Omega \chi_n(x)K(x)|w_n|^2\,dx \\ &\geq \int_\Omega \liminf_{n\to\infty} \psi_n(x)\frac{K(x)F(u_n)}{|u_n|^2}|w_n|^2\,dx + \tau\int_{\Omega^+} K(x)|w|^2\,dx \\ &\geq \tau\int_{\Omega^+} K(x)|w|^2\,dx. \end{aligned}$$

Thus, we have

$$\frac{1}{2} \geq \tau\int_{\Omega^+} K(x)|w|^2\,dx, \quad \text{for all } \tau > 0,$$

so that

$$\int_{\Omega^+} K(x)|w|^2\,dx = 0.$$

Since $K(x) > 0$ for all $x \in R^N$ we conclude that

$$w \equiv 0, \quad \text{in } \Omega^+,$$

which completes the proof. □

3. Main result: Theorem 1.2

Proof of Theorem 1.2.

Let (u_n) be the (PS) sequence given by Lemma 2.1. By Lemma 2.4, (u_n) is bounded and there exists $u \in E$ such that, up to a subsequence,

$$u_n \rightharpoonup u, \quad \text{in } E.$$

From $J'(u_n)u_n = o_n(1)$, we derive

$$\lim_{n\to\infty} \|u_n\|^2 = \lim_{n\to\infty} \int_{R^N} K(x)f(u_n)u_n\,dx.$$

By Lemma 2.3, we have

$$\int_{R^N} K(x)f(u_n)u_n\,dx \to \int_{R^N} K(x)f(u)u\,dx,$$

so that

$$\|u_n\|^2 \to \int_{R^N} K(x)f(u)u\,dx. \tag{3.1}$$

Since $J'(u_n)u = o_n(1)$, we have

$$\int_{R^N} \Delta u_n \Delta u + V(x)u_n u\,dx - \int_{R^N} K(x)f(u_n)u\,dx = o_n(1). \tag{3.2}$$

Note that $\phi(u_n) = \int_{R^N} \Delta u_n \Delta u + V(x)u_n u\,dx$ defines a continuous linear functional. Then

$$\int_{R^N} \Delta u_n \Delta u + V(x)u_n u\,dx \to \int_{R^N} |\Delta u|^2 + V(x)|u|^2\,dx = \|u\|^2.$$

Using this, Proposition 2.3 and taking the limit in (3.2), we get

$$\|u\|^2 = \int_{R^N} K(x)f(u)u\,dx, \tag{3.3}$$

which, along with the equation (3.1) give us

$$\|u_n\|^2 \to \|u\|^2,$$

so that we have the convergence

$$u_n \to u, \quad \text{in } E.$$

Consequently,

$$J(u) = c_* \quad \text{and} \quad J'(u) = 0.$$

As $c_* > 0$ we have $u \not\equiv 0$, and hence, u is a nontrivial ground state solution for the problem (P).

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