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A G-FAMILY OF QUANDLES AND HANDLEBODY-KNOTS

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Abstract. We introduce the notion of a G-family of quandles which is an algebraic system whose axioms are motivated by handlebody-knot theory, and use it to construct invariants for handlebody-knots. Our invariant can detect the chiralities of some handlebody-knots including unknown ones.

1. Introduction

A quandle [11], [15] is an algebraic system whose axioms are motivated by knot theory. Carter, Jelsovsky, Kamada, Langford and Saito [1] defined the quandle homology theory and quandle cocycle invariants for links and surface-links. The quandle chain complex in [1] is a subcomplex of the rack chain complex in [4]. The quandle cocycle invariant extracts information from quandle colorings by a quandle cocycle, and are used to detect the chirality of links in [3], [18].

In this paper, we introduce the notion of a G-family of quandles which is an algebraic system whose axioms are motivated by handlebody-knot theory, and use it to construct invariants for handlebody-knots. A *handlebody-knot* is a handlebody embedded in the 3-sphere. A handlebody-knot can be represented by its trivalent spine, and the first author, in [6], gave a list of local moves connecting diagrams of spatial trivalent graphs which represent equivalent handlebody-knots. The axioms of a G-family of quandles are derived from the local moves.

A G-family of quandles gives us not only invariants for handlebody-knots but also a way to handle a number of quandles at once. We see that a G-family of quandles is indeed a family of quandles associated with a group G. Any

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quandle is contained in some G-family of quandles as we see in Proposition 2.3. We introduce a homology theory for G-families of quandles. A cocycle of a G-family of quandles gives a family of cocycles of quandles. Thus it is efficient to find cocycles of a G-family of quandles, and indeed Nosaka [17] gave some cocycles together with a method to construct a cocycle of a G-family of quandles induced by a G-invariant group cocycle.

A G-family of quandles induces a quandle which contains all quandles forming the G-family of quandles as subquandles. This quandle, which we call the associated quandle, has a suitable structure to define colorings of a diagram of a handlebody-knot. Putting weights on colorings with a cocycle of a G-family of quandles, we define a quandle cocycle invariant for handlebody-knots. In [7], the first and second authors defined quandle colorings and quandle cocycle invariants for handlebody-links by introducing the notion of an A-flow for an abelian group A. Quandle cocycle invariants we define in this paper are nonabelian versions of the invariants. A usual knot can be regarded as a genus one handlebody-knot by taking its regular neighborhood, and some knot invariants have been modified and generalized to construct invariants for handlebody-knots. In [10], the third and fourth authors defined symmetric quandle colorings and symmetric quandle cocycle invariants for handlebody-links by generalizing symmetric quandle cocycle invariants of classical knots given in [12], [13].

A table of genus two handlebody-knots with up to 6 crossings is given in [8], and the handlebody-knots $0_1, \dots, 6_{16}$ in the table were proved to be mutually distinct by using the fundamental groups of their complements, quandle cocycle invariants in [7] and some topological arguments in [9], [14]. Our quandle cocycle invariant can distinguish the handlebody-knots 6_{14} and 6_{15} whose complements have isomorphic fundamental groups, and detect the chiralities of the handlebody-knots 5_2, 5_3, 6_5, 6_9, 6_{11}, 6_{12}, 6_{13}, 6_{14}, 6_{15}. In particular, the chiralities of 5_3, 6_5, 6_{11} and 6_{12} were not known.

This paper is organized as follows. In Section 2, we give the definition of a G-family of quandles together with some examples. In Section 3, we describe colorings with a G-family of quandles for handlebody-links. We define the homology for a G-family of quandles in Section 4 and define several invariants for handlebody-links including quandle cocycle invariants in Section 5. In Section 6, we calculate quandle cocycle invariants for handlebody-knots with up to 6 crossings and show the chirality for some of the handlebody-knots. In Section 7, we prove that our invariants can be regarded as a generalization of the invariants defined in [7].

2. A G-family of quandles

A *quandle* [11], [15] is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x * x = x$.
- For any $x \in X$, the map $S_x : X \to X$ defined by $S_x(y) = y * x$ is a bijection.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

A *rack* is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the second and third axioms. When we specify the binary operation $*$ of a quandle (resp. rack) X, we denote the quandle (resp. rack) by the pair $(X, *)$. An *Alexander quandle* $(M, *)$ is a Λ-module M with the binary operation defined by $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A *conjugation quandle* $(G, *)$ is a group G with the binary operation defined by $x * y = y^{-1}xy$.

Let G be a group with identity element e. A *G-family of quandles* is a non-empty set X with a family of binary operations $*^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms.

- For any $x \in X$ and any $g \in G$, $x *^g x = x$.
- For any $x, y \in X$ and any $g, h \in G$,

$$x *^{gh} y = \left(x *^g y\right) *^h y \quad \text{and} \quad x *^e y = x.$$

- For any $x, y, z \in X$ and any $g, h \in G$,

$$\left(x *^g y\right) *^h z = \left(x *^h z\right) *^{h^{-1}gh} \left(y *^h z\right).$$

When we specify the family of binary operations $*^g : X \times X \to X$ $(g \in G)$ of a G-family of quandles, we denote the G-family of quandles by the pair $(X, \{*^g\}_{g \in G})$.

PROPOSITION 2.1. *Let G be a group. Let $(X, \{*^g\}_{g \in G})$ be a G-family of quandles.*

(1) *For each $g \in G$, the pair $(X, *^g)$ is a quandle.*
(2) *We define a binary operation $* : (X \times G) \times (X \times G) \to X \times G$ by*

$$(x, g) * (y, h) = \left(x *^h y, h^{-1}gh\right).$$

Then $(X \times G, *)$ is a quandle.

We call the quandle $(X \times G, *)$ in Proposition 2.1 the *associated quandle* of X. We note that the involution $f : X \times G \to X \times G$ defined by $f((x, g)) = (x, g^{-1})$ is a good involution of the associated quandle $X \times G$, where we refer the reader to [12] for the definition of a good involution of a quandle. Before proving this proposition, we introduce a notion of a Q-family of quandles. Let $(Q, \triangleleft)$ be a quandle. A *Q-family of quandles* is a non-empty set X with a family of binary operations $*^a : X \times X \to X$ $(a \in Q)$ satisfying the following axioms.

- For any $x \in X$ and any $a \in Q$, $x *^a x = x$.
- For any $x \in X$ and any $a \in Q$, the map $S_{x,a} : X \to X$ defined by $S_{x,a}(y) = y *^a x$ is a bijection.
- For any $x, y, z \in X$ and any $a, b \in Q$, $(x *^a y) *^b z = (x *^b z) *^{a \triangleleft b} (y *^b z)$.

Let Q be a rack. A *Q-family of racks* is a non-empty set X with a family of binary operations $*^a : X \times X \to X$ $(a \in Q)$ satisfying the second and third axioms.

LEMMA 2.2. *Let $(Q, \triangleleft)$ be a quandle (resp. rack). Let $(X, \{*^a\}_{a\in Q})$ be a Q-family of quandles (resp. racks). We define a binary operation $* : (X \times Q) \times (X \times Q) \to X \times Q$ by*

$$(x,a) * (y,b) = \big(x *^b y, a \triangleleft b\big).$$

Then $(X \times Q, *)$ is a quandle (resp. rack).

Proof. The first axiom of a quandle follows from the equalities

$$(x,a) * (x,a) = \big(x *^a x, a \triangleleft a\big) = (x,a).$$

For any $(x,a),(y,b) \in X \times Q$, there is a unique $(z,c) \in X \times Q$ such that $x = z *^b y$ and $a = c \triangleleft b$. By the equalities $(x,a) = (z *^b y, c \triangleleft b) = (z,c) * (y,b)$, we have the second axiom of a quandle. The third axiom of a quandle follows from

$$\begin{aligned}\big((x,a) * (y,b)\big) * (z,c) &= \big(\big(x *^b y\big) *^c z, (a \triangleleft b) \triangleleft c\big)\\ &= \big(\big(x *^c z\big) *^{b \triangleleft c} \big(y *^c z\big), (a \triangleleft c) \triangleleft (b \triangleleft c)\big)\\ &= \big((x,a) * (z,c)\big) * \big((y,b) * (z,c)\big).\end{aligned}$$

□

Conversely, we can prove the following. Let $\triangleleft$ be a binary operation on a non-empty set Q. Let $*^a$ be a binary operation on a non-empty set X for $a \in Q$. We define a binary operation $* : (X \times Q) \times (X \times Q) \to X \times Q$ by

$$(x,a) * (y,b) = \big(x *^b y, a \triangleleft b\big).$$

If $(X \times Q, *)$ is a quandle (resp. rack), then $(Q, \triangleleft)$ is a quandle (resp. rack) and $(X, \{*^a\}_{a\in Q})$ is a Q-family of quandles (resp. racks).

Proof of Proposition 2.1. (1) The first and third axioms of a quandle are easily checked. The second axiom of a quandle follows from the equalities

$$\big(x *^g y\big) *^{g^{-1}} y = \big(x *^{g^{-1}} y\big) *^g y = x.$$

Then $(X, *^g)$ is a quandle.

(2) Let $(G, \triangleleft)$ be the conjugation quandle. By Lemma 2.2, $(X \times G, *)$ is a quandle. □

The following proposition gives us many examples for a G-family of quandles.

PROPOSITION 2.3. (1) *Let $(X, *)$ be a quandle. Let $S_x : X \to X$ be the bijection defined by $S_x(y) = y * x$. Let m be a positive integer such that $S_x^m = \mathrm{id}_X$ for any $x \in X$ if such an integer exists. We define the binary operation $*^i : X \times X \to X$ by $x *^i y = S_y^i(x)$. Then X is a $\mathbb{Z}$-family of quandles and a $\mathbb{Z}_m$-family of quandles, where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.*

(2) *Let R be a ring, and G a group with identity element e. Let X be a right $R[G]$-module, where $R[G]$ is the group ring of G over R. We define the binary operation $*^g : X \times X \to X$ by $x *^g y = xg + y(e - g)$. Then X is a G-family of quandles.*

Proof. (1) We verify the axioms of a G-family of quandles.

$$x *^0 y = S_y^0(x) = \mathrm{id}_X(x) = x,$$
$$x *^i x = S_x^i(x) = x,$$
$$\bigl(x *^i y\bigr) *^j y = S_y^j\bigl(S_y^i(x)\bigr) = S_y^{i+j}(x) = x *^{i+j} y.$$

For the last axiom of a G-family of quandles, we can prove

$$\bigl(x *^j z\bigr) *^i \bigl(y *^j z\bigr) = \bigl(x *^i y\bigr) *^j z$$

by induction.

(2) We verify the axioms of a G-family of quandles.

$$\begin{aligned}
&x *^e y \\
&\quad = xe + y(e - e) = x, \\
&x *^g x \\
&\quad = xg + x(e - g) = x, \\
&\bigl(x *^g y\bigr) *^h y \\
&\quad = (xg + y - yg)h + y - yh = x *^{gh} y, \\
&\bigl(x *^h z\bigr) *^{h^{-1}gh} \bigl(y *^h z\bigr) \\
&\quad = (xh + z - zh)h^{-1}gh + (yh + z - zh) - (yh + z - zh)h^{-1}gh \\
&\quad = (xg + y - yg)h + z - zh \\
&\quad = \bigl(x *^g y\bigr) *^h z.
\end{aligned}$$

□

3. Handlebody-links and X-colorings

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *spatial graph* is a finite graph embedded in S^3. Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. When a handlebody-link H is a regular neighborhood of a spatial graph K, we say that K *represents* H, or H *is represented by* K. In this paper, a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link.

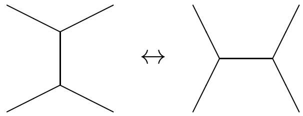

FIGURE 1

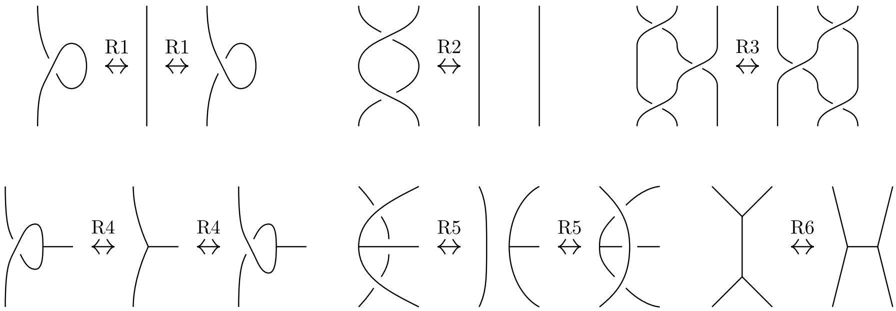

FIGURE 2

An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1, where the replacement is applied in a 3-ball embedded in S^3. Then we have the following theorem.

THEOREM 3.1 ([6]). *For spatial trivalent graphs K_1 and K_2, the following are equivalent.*

- *K_1 and K_2 represent an equivalent handlebody-link.*
- *K_1 and K_2 are related by a finite sequence of IH-moves.*
- *Diagrams of K_1 and K_2 are related by a finite sequence of the moves depicted in Figure 2.*

Let D be a diagram of a handlebody-link H. We set an orientation for each edge in D. Then D is a diagram of an oriented spatial trivalent graph K. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on the diagram. We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. For an arc α incident to a vertex ω, we define $\varepsilon(\alpha;\omega) \in \{1, -1\}$ by

$$\varepsilon(\alpha;\omega) = \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \omega, \\ -1 & \text{otherwise.} \end{cases}$$

Let X be a G-family of quandles, and Q the associated quandle of X. Let p_X (resp. p_G) be the projection from Q to X (resp. G). An *X-coloring* of D is a map $C : \mathcal{A}(D) \to Q$ satisfying the following conditions at each crossing χ and each vertex ω of D (see Figure 3).

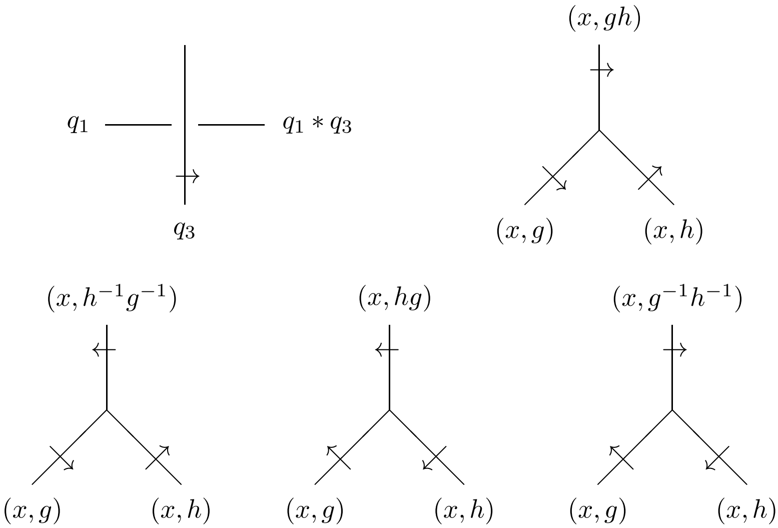

FIGURE 3

- Let χ_1, χ_2 and χ_3 be respectively, the under-arcs and the over-arc at a crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2. Then
$$C(\chi_2) = C(\chi_1) * C(\chi_3).$$
- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex ω arranged clockwise around ω. Then
$$(p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),$$
$$(p_G \circ C)(\omega_1)^{\varepsilon(\omega_1;\omega)}(p_G \circ C)(\omega_2)^{\varepsilon(\omega_2;\omega)}(p_G \circ C)(\omega_3)^{\varepsilon(\omega_3;\omega)} = e.$$

We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D. We call $C(\alpha)$ the *color* of α. For two diagrams D and E which locally differ, we denote by $\mathcal{A}(D,E)$ the set of arcs that D and E share.

LEMMA 3.2. *Let X be a G-family of quandles. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the* R1–R6 *moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D,E)$. For $C \in \mathrm{Col}_X(D)$, there is a unique X-coloring $C_{D,E} \in \mathrm{Col}_X(E)$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$.*

Proof. The colors of arcs in $\mathcal{A}(E) - \mathcal{A}(D,E)$ are uniquely determined by those of arcs in $\mathcal{A}(D,E)$, since we have
$$a *^g a = a$$

for the R1, R4 moves, and

$$\left(a *^g b\right) *^{g^{-1}} b = a *^e b = a$$

for the R2 move, and

$$\left(a *^g b\right) *^h c = \left(a *^h c\right) *^{h^{-1}gh} \left(b *^h c\right)$$

for the R3 move, and

$$\left(\left(b *^g a\right) *^h a\right) *^{(gh)^{-1}} a = a *^e b = b$$

for the R5 move, and only the coloring condition for the R6-move. □

Let X be a G-family of quandles, and Q the associated quandle of X. An X*-set* is a non-empty set Y with a family of maps $*^g : Y \times X \to Y$ satisfying the following axioms, where we note that we use the same symbol $*^g$ as the binary operation of the G-family of quandles.

- For any $y \in Y$, $x \in X$, and any $g, h \in G$,

$$y *^{gh} x = \left(y *^g x\right) *^h x \quad \text{and} \quad y *^e x = y.$$

- For any $y \in Y$, $x_1, x_2 \in X$, and any $g, h \in G$,

$$\left(y *^g x_1\right) *^h x_2 = \left(y *^h x_2\right) *^{h^{-1}gh} \left(x_1 *^h x_2\right).$$

Put $y * (x, g) := y *^g x$ for $y \in Y$, $(x, g) \in Q$. Then the second axiom implies that $(y * q_1) * q_2 = (y * q_2) * (q_1 * q_2)$ for $q_1, q_2 \in Q$. Any G-family of quandles $(X, \{*^g\}_{g\in G})$ itself is an X-set with its binary operations. We call it the *primitive* X*-set*. Any singleton set $\{y\}$ is also an X-set with the maps $*^g$ defined by $y *^g x = y$ for $x \in X$ and $g \in G$, which is a trivial X-set.

Let D be a diagram of an oriented spatial trivalent graph. We denote by $\mathcal{R}(D)$ the set of complementary regions of D. Let X be a G-family of quandles, and Y an X-set. Let Q be the associated quandle of X. An X_Y*-coloring* of D is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \to Q \cup Y$ satisfying the following conditions.

- $C(\mathcal{A}(D)) \subset Q$, $C(\mathcal{R}(D)) \subset Y$.
- The restriction $C|_{\mathcal{A}(D)}$ of C on $\mathcal{A}(D)$ is an X-coloring of D.
- For any arc $\alpha \in \mathcal{A}(D)$, we have

$$C(\alpha_1) * C(\alpha) = C(\alpha_2),$$

  where α_1, α_2 are the regions facing the arc α so that the normal orientation of α points from α_1 to α_2 (see Figure 4).

We denote by $\mathrm{Col}_X(D)_Y$ the set of X_Y-colorings of D.

For two diagrams D and E which locally differ, we denote by $\mathcal{R}(D, E)$ the set of regions that D and E share. Since colors of regions are uniquely determined by those of arcs and one region, Lemma 3.2 implies the following lemma.

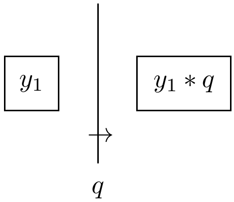

FIGURE 4

LEMMA 3.3. *Let X be a G-family of quandles, Y an X-set. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the* R1–R6 *moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D,E)$. For $C \in \mathrm{Col}_X(D)_Y$, there is a unique X_Y-coloring $C_{D,E} \in \mathrm{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$.*

4. A homology

Let X be a G-family of quandles, and Y an X-set. Let $(Q,*)$ be the associated quandle of X. Let $B_n(X)_Y$ be the free Abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$\big((y,q_1,\dots,q_i)*q,q_{i+1},\dots,q_n\big) := (y*q,q_1*q,\dots,q_i*q,q_{i+1},\dots,q_n)$$

for $y \in Y$ and $q,q_1,\dots,q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \to B_{n-1}(X)_Y$ by

$$\begin{aligned}\partial_n(y,q_1,\dots,q_n) = {} & \sum_{i=1}^{n}(-1)^i(y,q_1,\dots,q_{i-1},q_{i+1},\dots,q_n) \\ & - \sum_{i=1}^{n}(-1)^i\big((y,q_1,\dots,q_{i-1})*q_i,q_{i+1},\dots,q_n\big)\end{aligned}$$

for $n>0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1], [2], [4], [5]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1}\left\{ \big(y,q_1,\dots,q_{i-1},(x,g),(x,h),q_{i+2},\dots,q_n\big) \;\middle|\; \begin{array}{l} y \in Y, x \in X, g,h \in G, \\ q_1,\dots,q_n \in Q \end{array} \right\}$$

and

$$\bigcup_{i=1}^{n}\left\{ \begin{array}{l} (y,q_1,\dots,q_{i-1},(x,gh),q_{i+1},\dots,q_n) \\ -(y,q_1,\dots,q_{i-1},(x,g),q_{i+1},\dots,q_n) \\ -((y,q_1,\dots,q_{i-1})*(x,g),(x,h),q_{i+1},\dots,q_n) \end{array} \;\middle|\; \begin{array}{l} y \in Y, x \in X, \\ g,h \in G, \\ q_1,\dots,q_n \in Q \end{array} \right\}.$$

We remark that

$$\big(y, q_1, \dots, q_{i-1}, (x, e), q_{i+1}, \dots, q_n\big)$$

and

$$\begin{aligned}&\big(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n\big)\\ &\quad + \big((y, q_1, \dots, q_{i-1}) * (x, g), \big(x, g^{-1}\big), q_{i+1}, \dots, q_n\big)\end{aligned}$$

belong to $D_n(X)_Y$.

LEMMA 4.1. *For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.*

Proof. It is sufficient to show the equalities

$$\begin{aligned}&\partial_n\big(y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n\big) = 0,\\ &\partial_n\big(y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_n\big)\\ &\quad = \partial_n\big(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n\big)\\ &\qquad + \partial_n\big((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_n\big)\end{aligned}$$

in $B_{n-1}(X)_Y/D_{n-1}(X)_Y$. We verify the first equality in the quotient group.

$$\begin{aligned}&\partial_n\big(y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n\big)\\ &\quad = (-1)^i\big(y, q_1, \dots, q_{i-1}, (x, h), q_{i+2}, \dots, q_n\big)\\ &\qquad + (-1)^{i+1}\big(y, q_1, \dots, q_{i-1}, (x, g), q_{i+2}, \dots, q_n\big)\\ &\qquad - (-1)^i\big((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+2}, \dots, q_n\big)\\ &\qquad - (-1)^{i+1}\big(\big(y, q_1, \dots, q_{i-1}, (x, g)\big) * (x, h), q_{i+2}, \dots, q_n\big)\\ &\quad = (-1)^i\big(y, q_1, \dots, q_{i-1}, (x, h), q_{i+2}, \dots, q_n\big)\\ &\qquad + (-1)^{i+1}\big(y, q_1, \dots, q_{i-1}, (x, gh), q_{i+2}, \dots, q_n\big)\\ &\qquad - (-1)^{i+1}\big((y, q_1, \dots, q_{i-1}) * (x, h), \big(x, h^{-1}gh\big), q_{i+2}, \dots, q_n\big)\\ &\quad = 0,\end{aligned}$$

where the first equality follows from

$$\big(\big(y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_{j-1}\big) * q_j, q_{j+1}, \dots, q_n\big) = 0.$$

We verify the second equality in the quotient group.

$$\begin{aligned}&\partial_n\big(y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_n\big)\\ &\quad = \sum_{j<i}(-1)^j\big(y, q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n\big)\\ &\qquad + \sum_{j<i}(-1)^j\big((y, q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_n\big)\\ &\qquad + (-1)^i(y, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j>i}(-1)^j\big(y,q_1,\dots,q_{i-1},(x,g),q_{i+1},\dots,q_{j-1},q_{j+1},\dots,q_n\big)\\
&+\sum_{j>i}(-1)^j\big((y,q_1,\dots,q_{i-1})*(x,g),(x,h),q_{i+1},\dots,q_{j-1},q_{j+1},\dots,q_n\big)\\
&-\sum_{j<i}(-1)^j\big((y,q_1,\dots,q_{j-1})*q_j,q_{j+1},\dots,q_{i-1},(x,g),q_{i+1},\dots,q_n\big)\\
&-\sum_{j<i}(-1)^j\big(\big((y,q_1,\dots,q_{j-1})*q_j,q_{j+1},\dots,q_{i-1}\big)\\
&*(x,g),(x,h),q_{i+1},\dots,q_n\big)\\
&-(-1)^i\big((y,q_1,\dots,q_{i-1})*(x,gh),q_{i+1},\dots,q_n\big)\\
&-\sum_{j>i}(-1)^j\big(\big(y,q_1,\dots,q_{i-1},(x,gh),q_{i+1},\dots,q_{j-1}\big)*q_j,q_{j+1},\dots,q_n\big)\\
=\,&\partial_n\big(y,q_1,\dots,q_{i-1},(x,g),q_{i+1},\dots,q_n\big)\\
&+\partial_n\big((y,q_1,\dots,q_{i-1})*(x,g),(x,h),q_{i+1},\dots,q_n\big),
\end{aligned}
$$

where the last equality follows from

$$
\begin{aligned}
&\big((y,q_1,\dots,q_{i-1})*(x,gh),q_{i+1},\dots,q_n\big)\\
&\quad=\big(\big((y,q_1,\dots,q_{i-1})*(x,g)\big)*(x,h),q_{i+1},\dots,q_n\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(\big(y,q_1,\dots,q_{i-1},(x,gh),q_{i+1},\dots,q_{j-1}\big)*q_j,q_{j+1},\dots,q_n\big)\\
&\quad=\big(\big(y,q_1,\dots,q_{i-1},(x,g),q_{i+1},\dots,q_{j-1}\big)*q_j,q_{j+1},\dots,q_n\big)\\
&\qquad+\big(\big((y,q_1,\dots,q_{i-1})*(x,g),(x,h),q_{i+1},\dots,q_{j-1}\big)*q_j,q_{j+1},\dots,q_n\big).
\end{aligned}
$$

Then $\partial_n(D_n(X)_Y)\subset D_{n-1}(X)_Y$. □

We put $C_n(X)_Y=B_n(X)_Y/D_n(X)_Y$. Then $C_*(X)_Y=(C_n(X)_Y,\partial_n)$ is a chain complex. For an Abelian group A, we define the cochain complex $C^*(X;A)_Y=\mathrm{Hom}(C_*(X)_Y,A)$. We denote by $H_n(X)_Y$ the nth homology group of $C_*(X)_Y$.

5. Cocycle invariants

Let X be a G-family of quandles, and Y an X-set. Let D be a diagram of an oriented spatial trivalent graph. For an X_Y-coloring $C\in\mathrm{Col}_X(D)_Y$, we define the weight $w(\chi;C)\in C_2(X)_Y$ at a crossing χ of D as follows. Let χ_1,χ_2 and χ_3 be respectively, the under-arcs and the over-arc at a crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2. Let R_χ be the region facing χ_1 and χ_3 such that the normal orientations χ_1 and χ_3 point from R_χ to the opposite regions with respect to χ_1 and χ_3, respectively. Then we define

$$w(\chi;C)=\varepsilon(\chi)\big(C(R_\chi),C(\chi_1),C(\chi_3)\big),$$

where $\varepsilon(\chi)\in\{1,-1\}$ is the sign of a crossing χ. We define a chain $W(D;C)\in C_2(X)_Y$ by

$$W(D;C)=\sum_{\chi} w(\chi;C),$$

where χ runs over all crossings of D.

LEMMA 5.1. *The chain $W(D;C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles θ,θ' of $C^*(X;A)_Y$, we have $\theta(W(D;C))=\theta'(W(D;C))$.*

Proof. It is sufficient to show that $W(D;C)$ is a 2-cycle of $C_2(X)_Y$. We denote by $\mathcal{SA}(D)$ the set of curves obtained from D by removing (small neighborhoods of) crossings and vertices. We call a curve in $\mathcal{SA}(D)$ a *semi-arc* of D. We note that a semi-arc is obtained by dividing an over-arc at all crossings. We denote by $\mathcal{SA}(D;\xi)$ the set of semi-arcs incident to ξ, where ξ is a crossing or a vertex of D.

We define the orientation and the color of a semi-arc by those of the arc including the semi-arc. For a semi-arc α, there is a unique region R_α facing α such that the orientation of α points from the region R_α to the opposite region with respect to α. For a semi-arc α incident to a crossing or a vertex χ, we define

$$\varepsilon(\alpha;\chi):=\begin{cases}1 & \text{if the orientation of } \alpha \text{ points to } \chi,\\ -1 & \text{otherwise.}\end{cases}$$

Let χ_1,χ_2 be the semi-arcs incident to a crossing χ such that they originate from the under-arcs at χ and that the normal orientation of the over-arc points from χ_1 to χ_2. Let χ_3,χ_4 be the semi-arcs incident to a crossing χ such that they originate from the over-arc at χ and that the normal orientation of the under-arcs points from χ_3 to χ_4 (see Figure 5). Then we have

$$\begin{aligned}\partial_2\big(w(\chi;C)\big)= &-\varepsilon(\chi)\big(C(R_{\chi_1}),C(\chi_3)\big)+\varepsilon(\chi)\big(C(R_{\chi_1}),C(\chi_1)\big)\\ &+\varepsilon(\chi)\big(C(R_{\chi_1})*C(\chi_1),C(\chi_3)\big)\\ &-\varepsilon(\chi)\big(C(R_{\chi_1})*C(\chi_3),C(\chi_1)*C(\chi_3)\big)\\ =&\sum_{\alpha\in\mathcal{SA}(D;\chi)}\varepsilon(\alpha;\chi)\big(C(R_\alpha),C(\alpha)\big).\end{aligned}$$

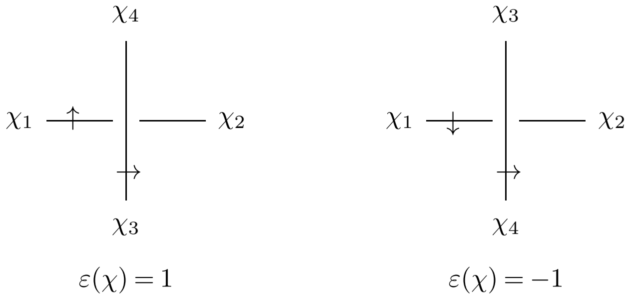

FIGURE 5

Since $\sum_{\alpha \in \mathcal{SA}(D;\omega)} \varepsilon(\alpha;\omega)(C(R_\alpha), C(\alpha))$ is an element of $D_1(X)_Y$ for a vertex ω, we have

$$\begin{aligned}
\partial_2\Bigl(\sum_\chi w(\chi;C)\Bigr) &= \sum_\chi \sum_{\alpha\in\mathcal{SA}(D;\chi)} \varepsilon(\alpha;\chi)\bigl(C(R_\alpha), C(\alpha)\bigr)\\
&= \sum_\chi \sum_{\alpha\in\mathcal{SA}(D;\chi)} \varepsilon(\alpha;\chi)\bigl(C(R_\alpha), C(\alpha)\bigr)\\
&\quad + \sum_\omega \sum_{\alpha\in\mathcal{SA}(D;\omega)} \varepsilon(\alpha;\omega)\bigl(C(R_\alpha), C(\alpha)\bigr)\\
&= \sum_{\alpha\in\mathcal{SA}(D)} \Bigl(\bigl(C(R_\alpha), C(\alpha)\bigr) - \bigl(C(R_\alpha), C(\alpha)\bigr)\Bigr)\\
&= 0
\end{aligned}$$

in $C_1(X)_Y$, where χ and ω, respectively run over all crossings and vertices of D. □

We recall that, for $C \in \mathrm{Col}_X(D)_Y$, there is a unique X_Y-coloring $C_{D,E} \in \mathrm{Col}_X(E)$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{A}(R,E)}$ by Lemma 3.3.

LEMMA 5.2. *Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the* R1–R6 *moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D,E)$. For $C \in \mathrm{Col}_X(D)_Y$ and $C_{D,E} \in \mathrm{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$, we have $[W(D;C)] = [W(E;C_{D,E})]$ in $H_2(X)_Y$.*

Proof. We have the invariance under the R1, R4 and R5 moves, since the difference between $[W(D;C)]$ and $[W(E;C_{D,E})]$ is an element of $D_2(X)_Y$. The invariance under the R2 move follows from the signs of the crossings which appear in the move. We have the invariance under the R3 move, since the difference between $[W(D;C)]$ and $[W(E;C_{D,E})]$ is an image of ∂_3. We have the invariance under the R6 move, since no crossings appear in the move. $\square$

We denote by G_H (resp. G_K) the fundamental group of the exterior of a handlebody-link H (resp. a spatial graph K). When H is represented by K, the groups G_H and G_K are identical. Let D be a diagram of an oriented spatial trivalent graph K. By the definition of an X_Y-coloring C of D, the map $p_G \circ C|_{\mathcal{A}(D)}$ represents a homomorphism from G_K to G, which we denote by $\rho_C \in \mathrm{Hom}(G_K, G)$. For $\rho \in \mathrm{Hom}(G_K, G)$, we define

$$\mathrm{Col}_X(D;\rho)_Y = \{C \in \mathrm{Col}_X(D)_Y \mid \rho_C = \rho\}.$$

For a 2-cocycle θ of $C^*(X;A)_Y$, we define

$$\begin{aligned}
\mathcal{H}(D) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \mathrm{Col}_X(D)_Y\},\\
\Phi_\theta(D) &:= \{\theta(W(D;C)) \in A \mid C \in \mathrm{Col}_X(D)_Y\},\\
\mathcal{H}(D;\rho) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \mathrm{Col}_X(D;\rho)_Y\},\\
\Phi_\theta(D;\rho) &:= \{\theta(W(D;C)) \in A \mid C \in \mathrm{Col}_X(D;\rho)_Y\}
\end{aligned}$$

as multisets.

LEMMA 5.3. *Let D be a diagram of an oriented spatial trivalent graph K. For $\rho, \rho' \in \mathrm{Hom}(G_K, G)$ such that ρ and ρ' are conjugate, we have $\mathcal{H}(D;\rho) = \mathcal{H}(D;\rho')$ and $\Phi_\theta(D;\rho) = \Phi_\theta(D;\rho')$.*

Proof. Let g_0 be an element of G such that $\rho'(x) = g_0^{-1}\rho(x)g_0$ for any $x \in G_K$. Fix $x_0 \in X$. We set $q_0 := (x_0, g_0)$. Let $f : \mathrm{Col}_X(D;\rho)_Y \to \mathrm{Col}_X(D;\rho')_Y$ be the bijection defined by $f(C)(x) = C(x) * q_0$ (see Figure 6).

We prove $[W(D;C)] = [W(D;f(C))] \in H_2(X)_Y$ for $C \in \mathrm{Col}_X(D;\rho)_Y$. We assume that spatial trivalent graphs are drawn in $\mathbb{R}^2 (\subset S^2)$. Let D' be a diagram obtained from D by putting an oriented loop γ in the outermost region R_∞ so that the loop bounds a disk, where the loop is oriented counterclockwise (see Figure 7). Let C' be the X_Y-coloring of D' defined by $C'(\gamma) = q_0$

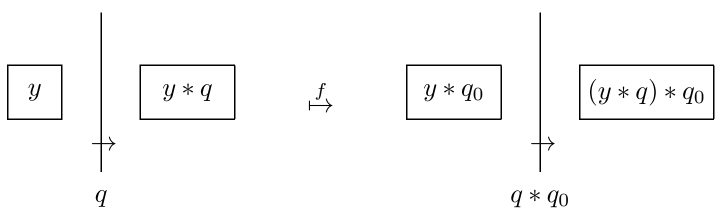

FIGURE 6

D' D''

FIGURE 7

and $C' = C$ on $\mathcal{A}(D,D') \cup \mathcal{R}(D,D')$. Then we note that $C'(R'_\infty) = C(R_\infty) * q_0$ for the region R'_∞ surrounded by the loop γ in D'. We deform the diagram D' by using R2, R3 and R5 moves so that the loop passes over all arcs of D exactly once. Then we denote by D'' and $C'' \in \mathrm{Col}_X(D'')_Y$ the resulting diagram and the corresponding X_Y-coloring of D'', respectively. We obtain the X_Y-coloring $f(C)$ from C'' by removing the loop from D'', which also implies that f is well-defined.

Since no crossings increase or decrease when we add or remove the loop γ, we have

$$\bigl[W(D;C)\bigr] = \bigl[W(D';C')\bigr] = \bigl[W(D'';C'')\bigr] = \bigl[W\bigl(D;f(C)\bigr)\bigr],$$

where the second equality follows from Lemma 5.2. Then we have $\mathcal{H}(D;\rho) = \mathcal{H}(D;\rho')$ and $\Phi_\theta(D;\rho) = \Phi_\theta(D;\rho')$. $\square$

We denote by $\mathrm{Conj}(G_K,G)$ the set of conjugacy classes of homomorphisms from G_K to G. By Lemma 5.3, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for $\rho \in \mathrm{Conj}(G_K,G)$.

LEMMA 5.4. *Let D be a diagram of an oriented spatial trivalent graph K. Let E be a diagram obtained from D by reversing the orientation of an edge e. For $\rho \in \mathrm{Hom}(G_K,G)$, we have $\mathcal{H}(D) = \mathcal{H}(E)$, $\Phi_\theta(D) = \Phi_\theta(E)$, $\mathcal{H}(D;\rho) = \mathcal{H}(E;\rho)$ and $\Phi_\theta(D;\rho) = \Phi_\theta(E;\rho)$.*

Proof. It is sufficient to show that $\mathcal{H}(D;\rho) = \mathcal{H}(E;\rho)$. We define a bijection $f:\ \mathrm{Col}_X(D;\rho)_Y \to \mathrm{Col}_X(E;\rho)_Y$ by $f(C)(\alpha) = (p_X(C(\alpha)), p_G(C(\alpha))^{-1})$ if α is an arc originates from the edge e, $f(C)(\alpha) = C(\alpha)$ otherwise. We remark that $\rho_{f(C)} = \rho_C = \rho$. The map f is well-defined, since $z_1 * (x,g) = z_2$ is equivalent to $z_2 * (x,g^{-1}) = z_1$. Then we have $w(\chi;C) = w(\chi;f(C))$ for every crossing χ, since we have

$$\begin{aligned}\bigl(y,(x_1,g_1),(x_2,g_2)\bigr) &= -\bigl(y*(x_1,g_1),\bigl(x_1,g_1^{-1}\bigr),(x_2,g_2)\bigr)\\ &= -\bigl(y*(x_2,g_2),(x_1,g_1)*(x_2,g_2),\bigl(x_2,g_2^{-1}\bigr)\bigr)\\ &= \bigl(\bigl(y*(x_1,g_1)\bigr)*(x_2,g_2),\bigl(x_1,g_1^{-1}\bigr)*(x_2,g_2),\bigl(x_2,g_2^{-1}\bigr)\bigr)\end{aligned}$$

in $C_2(X)_Y$ (see Figure 8). Then we have $\mathcal{H}(D;\rho) = \mathcal{H}(E;\rho)$. $\square$

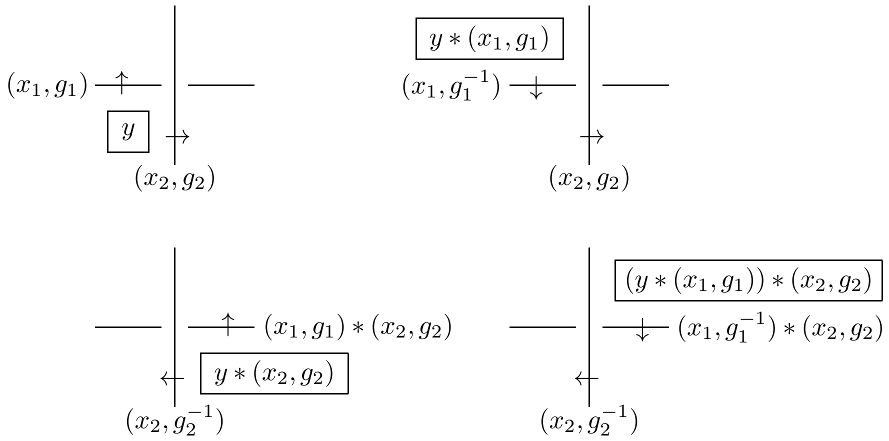

FIGURE 8

By Lemma 5.4, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for a diagram D of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram D of a handlebody-link H, we define

$$\begin{aligned}
\mathcal{H}^{\mathrm{hom}}(D) &:= \big\{\mathcal{H}(D;\rho)\mid \rho\in\mathrm{Hom}(G_H,G)\big\},\\
\Phi_\theta^{\mathrm{hom}}(D) &:= \big\{\Phi_\theta(D;\rho)\mid \rho\in\mathrm{Hom}(G_H,G)\big\},\\
\mathcal{H}^{\mathrm{conj}}(D) &:= \big\{\mathcal{H}(D;\rho)\mid \rho\in\mathrm{Conj}(G_H,G)\big\},\\
\Phi_\theta^{\mathrm{conj}}(D) &:= \big\{\Phi_\theta(D;\rho)\mid \rho\in\mathrm{Conj}(G_H,G)\big\}
\end{aligned}$$

as “multisets of multisets.” We remark that, for X_Y-colorings C and $C_{D,E}$ in Lemma 5.2, we have $\rho_C=\rho_{C_{D,E}}$. Then, by Lemmas 5.1–5.4, we have the following theorem.

THEOREM 5.5. *Let X be a G-family of quandles, Y an X-set. Let θ be a 2-cocycle of $C^*(X;A)_Y$. Let H be a handlebody-link represented by a diagram D. Then the following are invariants of a handlebody-link H.*

$$\mathcal{H}(D),\qquad \Phi_\theta(D),\qquad \mathcal{H}^{\mathrm{hom}}(D),\qquad \Phi_\theta^{\mathrm{hom}}(D),\qquad \mathcal{H}^{\mathrm{conj}}(D),\qquad \Phi_\theta^{\mathrm{conj}}(D).$$

We denote the invariants of H given in Theorem 5.5 by

$$\begin{gathered}
\mathcal{H}(H),\qquad \Phi_\theta(H),\qquad \mathcal{H}^{\mathrm{hom}}(H),\\
\Phi_\theta^{\mathrm{hom}}(H),\qquad \mathcal{H}^{\mathrm{conj}}(H),\qquad \Phi_\theta^{\mathrm{conj}}(H),
\end{gathered}$$

respectively.

Let $\{y\}$ be a trivial X-set. For the trivial 2-cocycle 0 of $C^*(X;A)_{\{y\}}$, we have

$$\Phi_0(H) = \{0 \mid C \in \mathrm{Col}_X(D)_{\{y\}}\},$$
$$\Phi_0^{\mathrm{hom}}(H) = \{\{0 \mid C \in \mathrm{Col}_X(D;\rho)_{\{y\}}\} \mid \rho \in \mathrm{Hom}(G_H, G)\},$$
$$\Phi_0^{\mathrm{conj}}(H) = \{\{0 \mid C \in \mathrm{Col}_X(D;\rho)_{\{y\}}\} \mid \rho \in \mathrm{Conj}(G_H, G)\}.$$

Thus

$$\#\,\mathrm{Col}_X(H) := \#\,\mathrm{Col}_X(D)_{\{y\}},$$
$$\#\,\mathrm{Col}_X^{\mathrm{hom}}(H) := \{\#\,\mathrm{Col}_X(D;\rho)_{\{y\}} \mid \rho \in \mathrm{Hom}(G_H, G)\},$$
$$\#\,\mathrm{Col}_X^{\mathrm{conj}}(H) := \{\#\,\mathrm{Col}_X(D;\rho)_{\{y\}} \mid \rho \in \mathrm{Conj}(G_H, G)\}$$

are invariants of a handlebody-link H represented by a diagram D, where $\#S$ denotes the cardinality of a multiset S. We remark that these invariants do not depend on the choice of the singleton set $\{y\}$.

We denote by H^* the mirror image of a handlebody-link H. Then we have the following theorem.

THEOREM 5.6. *For a handlebody-link H, we have*

$$\mathcal{H}(H^*) = -\mathcal{H}(H), \qquad \Phi_\theta(H^*) = -\Phi_\theta(H),$$
$$\mathcal{H}^{\mathrm{hom}}(H^*) = -\mathcal{H}^{\mathrm{hom}}(H), \qquad \Phi_\theta^{\mathrm{hom}}(H^*) = -\Phi_\theta^{\mathrm{hom}}(H),$$
$$\mathcal{H}^{\mathrm{conj}}(H^*) = -\mathcal{H}^{\mathrm{conj}}(H), \qquad \Phi_\theta^{\mathrm{conj}}(H^*) = -\Phi_\theta^{\mathrm{conj}}(H),$$

where $-S = \{-a \mid a \in S\}$ for a multiset S.

Proof. Let D be a diagram of a handlebody-link H. We suppose that D is depicted in an xy-plane $\mathbb{R}^2$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the involution defined by $\varphi(x,y) = (-x,y)$. Let $\widetilde{\varphi} : S^3 \to S^3$ be the involution defined by $\varphi(x,y,z) = (-x,y,z)$ and $\varphi(\infty) = \infty$, where we regard the 3-sphere S^3 as $\mathbb{R}^3 \cup \{\infty\}$. Then $\varphi(D)$ is a diagram of the handlebody-link $H^* = \widetilde{\varphi}(H)$. For $\rho \in \mathrm{Hom}(G_H, G)$ and $C \in \mathrm{Col}_X(D;\rho)_Y$, we have $\widetilde{\varphi}_*(\rho) \in \mathrm{Hom}(G_{H^*}, G)$ and $C \circ \varphi \in \mathrm{Col}_X(\varphi(D); \widetilde{\varphi}_*(\rho))_Y$, where $\widetilde{\varphi}_*$ is the isomorphism induced by $\widetilde{\varphi}$. For each crossing χ of D, $\varepsilon(\chi) = -\varepsilon(\varphi(\chi))$, and hence we have $w(\varphi(\chi), C\circ\varphi) = -w(\chi, C)$. Then $[W(\varphi(D); C\circ\varphi)] = -[W(D;C)]$, which implies the equalities in this theorem. $\square$

6. Applications

In this section, we calculate cocycle invariants defined in the previous section for the handlebody-knots $0_1, \ldots, 6_{16}$ in the table given in [8], by using a 2-cocycle given by Nosaka [17]. This calculation enables us to distinguish some of handlebody-knots from their mirror images, and a pair of handlebody-knots whose complements have isomorphic fundamental groups.

Let $G = SL(2;\mathbb{Z}_3)$ and $X = (\mathbb{Z}_3)^2$. Then X is a G-family of quandles with the proper binary operation as given in Proposition 2.3(2). Let Y be the trivial X-set $\{y\}$. We define a map $\theta : Y \times (X \times G)^2 \to \mathbb{Z}_3$ by

$$\theta\big(y,(x_1,g_1),(x_2,g_2)\big) := \lambda(g_1)\det\big(x_1 - x_2, x_2\big(1 - g_2^{-1}\big)\big),$$

where the Abelianization $\lambda : G \to \mathbb{Z}_3$ is given by

$$\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+d)(b-c)(1-bc).$$

By [17], the map θ is a 2-cocycle of $C^*(X;\mathbb{Z}_3)_Y$. Table 1 lists the invariant $\Phi_\theta^{\mathrm{conj}}(H)$ for the handlebody-knots $0_1,\dots,6_{16}$. We represent the multiplicity of elements of a multiset by using subscripts. For example, $\{\{0_2,1_3\}_1,\{0_3\}_2\}$ represents the multiset $\{\{0,0,1,1,1\},\{0,0,0\},\{0,0,0\}\}$.

From Table 1, we see that our invariant can distinguish the handlebody-knots 6_{14}, 6_{15}, whose complements have the isomorphic fundamental groups. Together with Theorem 5.6, we also see that handlebody-knots 5_2, 5_3, 6_5, 6_9, 6_{11}, 6_{12}, 6_{13}, 6_{14}, 6_{15} are not equivalent to their mirror images. In particular,

TABLE 1

H	$\Phi_\theta^{\mathrm{conj}}(H)$
0_1	$\{\{0_9\}_{76}\}$
4_1	$\{\{0_9\}_{83},\{0_{27}\}_{22},\{0_{81}\}_3\}$
5_1	$\{\{0_9\}_{76}\}$
5_2	$\{\{0_9\}_{95},\{0_{27}\}_6,\{0_{81}\}_1,\{0_9,1_{18}\}_4,\{0_{27},1_{54}\}_2\}$
5_3	$\{\{0_9\}_{102},\{0_{27}\}_4,\{0_{27},2_{54}\}_2\}$
5_4	$\{\{0_9\}_{74},\{0_{81}\}_2\}$
6_1	$\{\{0_9\}_{91},\{0_{27}\}_{16},\{0_{81}\}_1\}$
6_2	$\{\{0_9\}_{106},\{0_{45},1_{18},2_{18}\}_2\}$
6_3	$\{\{0_9\}_{74},\{0_{27}\}_2\}$
6_4	$\{\{0_9\}_{76}\}$
6_5	$\{\{0_9\}_{74},\{0_9,1_{18}\}_2\}$
6_6	$\{\{0_9\}_{72},\{0_{27}\}_4\}$
6_7	$\{\{0_9\}_{85},\{0_{27}\}_{16},\{0_{81}\}_3,\{0_{45},1_{18},2_{18}\}_4\}$
6_8	$\{\{0_9\}_{76}\}$
6_9	$\{\{0_9\}_{91},\{0_{27}\}_6,\{0_{81}\}_1,\{0_9,1_{18}\}_6,\{0_{27},1_{54}\}_2,\{0_{27},2_{54}\}_2\}$
6_{10}	$\{\{0_9\}_{76}\}$
6_{11}	$\{\{0_9\}_{70},\{0_9,1_{18}\}_6\}$
6_{12}	$\{\{0_9\}_{97},\{0_{81}\}_1,\{0_9,1_{18}\}_8,\{0_9,1_{36},2_{36}\}_2\}$
6_{13}	$\{\{0_9\}_{95},\{0_{27}\}_6,\{0_{81}\}_1,\{0_9,2_{18}\}_4,\{0_{27},2_{54}\}_2\}$
6_{14}	$\{\{0_9\}_{119},\{0_{27}\}_6,\{0_{81}\}_{11},\{0_9,1_{18}\}_{12},\{0_{27},1_{54}\}_{24}\}$
6_{15}	$\{\{0_9\}_{119},\{0_{27}\}_6,\{0_{81}\}_{11},\{0_9,2_{18}\}_{12},\{0_{27},1_{54}\}_{24}\}$
6_{16}	$\{\{0_9\}_{44},\{0_{81}\}_{32}\}$

TABLE 2

	Chirality	M	II	LL	IKO	IIJO
0_1	$\bigcirc$					
4_1	$\bigcirc$					
5_1	$\times$			$\checkmark$		
5_2	$\times$		$\checkmark$	$\checkmark$		$\checkmark$
5_3	$\times$					$\checkmark$
5_4	$\times$				$\checkmark$	
6_1	$\times$	$\checkmark$				
6_2	?					
6_3	?					
6_4	$\times$			$\checkmark$		
6_5	$\times$					$\checkmark$
6_6	$\bigcirc$					
6_7	$\bigcirc$					
6_8	?					
6_9	$\times$		$\checkmark$			$\checkmark$
6_{10}	?					
6_{11}	$\times$					$\checkmark$
6_{12}	$\times$					$\checkmark$
6_{13}	$\times$		$\checkmark$	$\checkmark$		$\checkmark$
6_{14}	$\times$				$\checkmark$	$\checkmark$
6_{15}	$\times$				$\checkmark$	$\checkmark$
6_{16}	$\bigcirc$					

the chiralities of 5_3, 6_5, 6_{11} and 6_{12} were not known. Table 2 shows us known facts on the chirality of handlebody-knots in [8] so far. In the column of "chirality", the symbols $\bigcirc$ and $\times$ mean that the handlebody-knot is amphichiral and chiral, respectively, and the symbol ? means that it is not known whether the handlebody-knot is amphichiral or chiral. The symbols $\checkmark$ in the right five columns mean that the handlebody-knots can be proved chiral by using the method introduced in the papers corresponding to the columns. Here, M, II, LL, IKO and IIJO denote the papers [16], [7], [14], [9] and this paper, respectively.

7. A generalization

In this section, we show that our invariant is a generalization of the invariant $\Phi_\theta^{\mathrm{I}}(H)$ defined by the first and second authors in [7]. We refer the reader to [7] for the details of the invariant $\Phi_\theta^{\mathrm{I}}(H)$. We recall the definition of the chain complex for the invariant $\Phi_\theta^{\mathrm{I}}(H)$.

Let X be a $\mathbb{Z}_m$-family of quandles, Y an X-set. Let $B_n^{\mathrm{I}}(X)_Y$ be the free Abelian group generated by the elements of $Y \times X^n$ if $n \geq 0$, and let $B_n^{\mathrm{I}}(X)_Y = 0$ otherwise. We put

$$\big((y, x_1, \dots, x_i) *^j x, x_{i+1}, \dots, x_n\big) := \big(y *^j x, x_1 *^j x, \dots, x_i *^j x, x_{i+1}, \dots, x_n\big)$$

for $y \in Y$, $x, x_1, \dots, x_n \in X$ and $j \in \mathbb{Z}_m$. We define a boundary homomorphism $\partial_n : B_n^{\mathrm{I}}(X)_Y \to B_{n-1}^{\mathrm{I}}(X)_Y$ by

$$\begin{aligned}\partial_n(y, x_1, \dots, x_n) = &\sum_{i=1}^{n} (-1)^i (y, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ &- \sum_{i=1}^{n} (-1)^i \big((y, x_1, \dots, x_{i-1}) *^1 x_i, x_{i+1}, \dots, x_n\big)\end{aligned}$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*^{\mathrm{I}}(X)_Y = (B_n^{\mathrm{I}}(X)_Y, \partial_n)$ is a chain complex. Let $D_n^{\mathrm{I}}(X)_Y$ be the subgroup of $B_n^{\mathrm{I}}(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1} \big\{ (y, x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_n) \mid y \in Y, x, x_1, \dots, x_n \in X \big\}$$

and

$$\bigcup_{i=1}^{n} \left\{ \sum_{j=0}^{m-1} \big((y, x_1, \dots, x_{i-1}) *^j x_i, x_i, \dots, x_n\big) \mid y \in Y, x_1, \dots, x_n \in X \right\}.$$

Then $D_*^{\mathrm{I}}(X)_Y = (D_n^{\mathrm{I}}(X)_Y, \partial_n)$ is a chain complex.

We put $C_n^{\mathrm{I}}(X)_Y = B_n^{\mathrm{I}}(X)_Y / D_n^{\mathrm{I}}(X)_Y$. Then $C_*^{\mathrm{I}}(X)_Y = (C_n^{\mathrm{I}}(X)_Y, \partial_n)$ is a chain complex. For an Abelian group A, we define the cochain complex $C_{\mathrm{I}}^*(X; A)_Y = \mathrm{Hom}(C_*^{\mathrm{I}}(X)_Y, A)$. We denote by $H_n^{\mathrm{I}}(X)_Y$ the nth homology group of $C_*^{\mathrm{I}}(X)_Y$.

Proposition 7.1. *For $n \in \mathbb{Z}$, we have*

$$H_n^{\mathrm{I}}(X)_Y \cong H_n(X)_Y.$$

Proof. The homomorphism $f_n : C_n^{\mathrm{I}}(X)_Y \to C_n(X)_Y$ defined by

$$f_n\big((y, x_1, \dots, x_n)\big) = \big(y, (x_1, 1), \dots, (x_n, 1)\big)$$

is an isomorphism, since the homomorphism $g_n : C_n(X)_Y \to C_n^{\mathrm{I}}(X)_Y$ defined by

$$\begin{aligned}&g_n\big(y, (x_1, s_1), \dots, (x_n, s_n)\big) \\ &\quad = \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \cdots \sum_{i_n=0}^{s_n-1} \Big(\cdots \Big(\big(y *^{i_1} x_1, x_1\big) *^{i_2} x_2, x_2\Big) \cdots *^{i_n} x_n, x_n\Big)\end{aligned}$$

is the inverse map of f_n. It is easy to see that $f = \{f_n\}$ is a chain map from $C_*^{\mathrm{I}}(X)_Y$ to $C_*(X)_Y$. Therefore, $H_n^{\mathrm{I}}(X)_Y \cong H_n(X)_Y$. $\square$

For a 2-cocycle θ of $C^*_{\mathrm{I}}(X;A)_Y$, the composition $\theta \circ g_2$ is a 2-cocycle of $C^*(X;A)_Y$, and we have

$$\Phi^{\mathrm{I}}_{\theta}(H) = \Phi^{\mathrm{hom}}_{\theta \circ g_2}(H),$$

where g_2 is the map defined in Proposition 7.1. Then our invariant is a generalization of the invariant introduced in [7].

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References

[1] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003), 3947–3989. MR 1990571

[2] J. S. Carter, D. Jelsovsky, S. Kamada and M. Saito, *Quandle homology groups, their Betti numbers, and virtual knots*, J. Pure Appl. Algebra **157** (2001), 135–155. MR 1812049

[3] J. S. Carter, S. Kamada and M. Saito, *Geometric interpretations of quandle homology*, J. Knot Theory Ramifications **10** (2001), 345–386. MR 1825963

[4] R. Fenn, C. Rourke and B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures **3** (1995), 321–356. MR 1364012

[5] R. Fenn, C. Rourke and B. Sanderson, *The rack space*, Trans. Amer. Math. Soc. **359** (2007), 701–740. MR 2255194

[6] A. Ishii, *Moves and invariants for knotted handlebodies*, Algebr. Geom. Topol. **8** (2008), 1403–1418. MR 2443248

[7] A. Ishii and M. Iwakiri, *Quandle cocycle invariants for spatial graphs and knotted handlebodies*, Canad. J. Math. **64** (2012), 102–122. MR 2932171

[8] A. Ishii, K. Kishimoto, H. Moriuchi and M. Suzuki, *A table of genus two handlebody-knots up to six crossings*, J. Knot Theory Ramifications **21** (2012), 1250035, 1–9. MR 2890461

[9] A. Ishii, K. Kishimoto and M. Ozawa, *Knotted handle decomposing spheres for handlebody-knots*, to appear in J. Math. Soc. Japan.

[10] Y. Jang and K. Oshiro, *Symmetric quandle colorings for spatial graphs and handlebody-links*, J. Knot Theory Ramifications **21** (2012), no. 4, 1250050, 1–16. MR 2890467

[11] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), 37–65. MR 0638121

[12] S. Kamada, *Quandles with good involutions, their homologies and knot invariants*, Intelligence of low dimensional topology 2006 (J. S. Carter et al., eds.), World Scientific Publishing Co., Hackensack, NJ, 2007, pp. 101–108. MR 2371714

[13] S. Kamada and K. Oshiro, *Homology groups of symmetric quandles and cocycle invariants of links and surface-links*, Trans. Amer. Math. Soc. **362** (2010), 5501–5527. MR 2657689

[14] J. H. Lee and S. Lee, *Inequivalent handlebody-knots with homeomorphic complements*, Algebr. Geom. Topol. **12** (2012), 1059–1079. MR 2928904

[15] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119** (1982), no. 161, 78–88. MR 0672410

[16] M. Motto, *Inequivalent genus 2 handlebodies in S^3 with homeomorphic complement*, Topology Appl. **36** (1970), 283–290. MR 1070707

[17] T. Nosaka, *Quandle cocycles from invariant theory*, Adv. Math. **245** (2013), 423–438. MR 3084434

[18] C. Rourke and B. Sanderson, *There are two 2-twist-spun trefoils*, available at http://msp.warwick.ac.uk/~cpr/.

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