

## RIGIDITY OF DERIVATIONS IN THE PLANE AND IN METRIC MEASURE SPACES

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ABSTRACT. Following the work of Weaver, we study generalized differential operators, called (metric) derivations, and their linear algebraic properties. In particular, for  $k = 1, 2$  we show that measures on  $\mathbb{R}^k$  that induce rank- $k$  modules of derivations must be absolutely continuous to Lebesgue measure. An analogous result holds true for measures concentrated on  $k$ -rectifiable sets with respect to  $k$ -dimensional Hausdorff measure.

These rigidity results also apply to the metric space setting and specifically, to spaces that support a doubling measure and a  $p$ -Poincaré inequality. Using our results for the Euclidean plane, we prove the 2-dimensional case of a conjecture of Cheeger, which concerns the non-degeneracy of Lipschitz images of such spaces.

### 1. Introduction

In this work, we consider Weaver's theory of (metric) derivations [Wea00], which are generalizations of differential operators on Riemannian manifolds. For metric spaces equipped with a Borel regular measure, derivations are linear operators from the class of bounded Lipschitz functions to the class of essentially bounded functions with respect to certain weak topologies; see Lemma 2.3 and Definition 2.5.

On  $\mathbb{R}^n$  equipped with the standard metric, Rademacher's theorem states that every Lipschitz function is almost everywhere (a.e.) differentiable with respect to the Lebesgue measure. Put one way, the validity of Rademacher's theorem is encapsulated in the structure of a metric space, if there exists a nonzero derivation with respect to a fixed measure on that space.

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**1.1. Rigidity of measures and derivations.** The framework of [Wea00] includes many examples, such as Riemannian manifolds, the self-similar fractal spaces of Laakso [Laa00], and infinite-dimensional spaces such as Banach manifolds and abstract Wiener spaces. In each example, there are natural choices for the metric and measure, but one may inquire as to how flexible these choices can be made.

QUESTION 1.1. On a given metric space, which measures induce nontrivial derivations? Of those, how many can we expect?

For a fixed space, the set of derivations admits a natural module structure, so the notions of *linear independence* and *basis* are well defined for derivations. We therefore determine “how many” derivations exist on a space in terms of the *rank* of the module.

To clarify, Question 1.1 is not simply a matter of the Hausdorff dimensions of the relevant spaces, but of subtler issues of geometry as well. Given a line in  $\mathbb{R}^n$ , for instance, 1-dimensional Hausdorff measure induces a rank-1 module of derivations [Wea00, Theorem 38]. On the other hand, the standard Sierpiński carpet in  $\mathbb{R}^2$  equipped with its natural Hausdorff measure (of dimension  $\log_3 8$ ) does not admit *any* nonzero derivations [Wea00, Theorem 41].

In this paper, we will focus on the case of Euclidean spaces. The following result indicates that, for  $k = 1, 2$ , there are few choices of Radon measures on  $(\mathbb{R}^k, |\cdot|)$  that induce rank- $k$  modules of derivations.

THEOREM 1.2. *Let  $k \in \{1, 2\}$ . If  $\mu$  is a Radon measure on  $(\mathbb{R}^k, |\cdot|)$  that induces a rank- $k$  module of derivations, then it is absolutely continuous to Lebesgue measure. Moreover, derivations with respect to  $\mu$  are linear combinations of the differential operators  $\{\partial/\partial x_i\}_{i=1}^k$  with scalars in  $L^\infty(\mathbb{R}^k, \mu)$ .*

The class of Lipschitz functions on a space clearly depends on the choice of metric on that space. So in terms of derivations, Theorem 1.2 can be viewed as a *rigidity* result for measures on  $\mathbb{R}^k$  that obey a Rademacher-type theorem.

Regarding the  $k = 2$  case, the proof uses a result of Alberti, Csörnyei, and Preiss [ACP05] about the structure of Lebesgue null sets in  $\mathbb{R}^2$ . Roughly speaking, it asserts that every Lebesgue null set in  $\mathbb{R}^2$  (that is, a subset of zero Lebesgue measure) splits into a horizontal part and a vertical part. So given a measure  $\mu$  that is concentrated on such a set, we also show that each part admits a generalized “tangent” vector field, but whose components satisfy a linear dependence relation for all derivations with respect to  $\mu$ ; see Lemma 4.11.

**1.2. Applications to metric spaces.** Though formulated for Euclidean spaces, the results in Section 1.1 are also relevant to the general setting of

*metric measure spaces*—that is, metric spaces equipped with Borel regular measures.

To obtain a reasonable setting for analysis, we restrict our focus to spaces that support doubling measures. Recall that a Borel measure  $\mu$  on  $(X, d)$  is called *doubling* if there exists  $\kappa \geq 1$  so that

$$(1.1) \quad 0 < \mu(B(x, 2r)) \leq \kappa \mu(B(x, r)) < \infty$$

holds for all  $x \in X$  and all  $r > 0$ . Spaces supporting such measures are particular cases of *spaces of homogeneous type* [CW77]; in particular they have finite Hausdorff dimension and admit generalized dyadic-cube decompositions [Chr90]. Intuitively, the doubling condition (1.1) ensures that the space  $X$  has good scaling properties, from which we obtain a rich theory of “zeroth order” calculus—that is, good analogues of Riesz potentials, the Lebesgue differentiation theorem, and other elements of harmonic analysis.

For a theory of first-order calculus, however, we also require the spaces to support a generalized Poincaré inequality. Indeed, on  $\mathbb{R}^n$  the inequality takes the form

$$\int_{B(x,r)} |f - f_{B(x,r)}| dx \leq C(n, p) r \left( \int_{B(x,r)} |\nabla f|^p dx \right)^{1/p}$$

for all  $p \geq 1$  and all Lipschitz  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Here mean values are denoted by

$$f_A := \int_A f dx := \frac{1}{|A|} \int_A f dx.$$

So at small scales, the inequality asserts that the slopes  $(f - f_{B(x,r)})/r$  are comparable to the usual gradients  $|\nabla f|$  in an averaged sense.

There is an analogous formulation of the Poincaré inequality for metric spaces supporting doubling measures. In this setting, upper gradients replace the usual gradients, but there remains the same consequence that Lipschitz functions have good infinitesimal behavior [HK98], [Sha00]. Indeed, Cheeger [Che99] has shown that Lipschitz functions on such spaces are also a.e. differentiable; see Theorem 5.7. As a result, these spaces admit generalized differentiable structures, and Keith has extended the result for a more general class of spaces [Kei04a].

In particular, Theorem 1.2 gives rise to *geometric rigidity* theorems in cases when the metric space embeds isometrically into a Euclidean space. For example, it implies an affirmative answer to a conjecture of Cheeger [Che99, Conjecture 4.63] when the generalized differentiable structure is 2-dimensional—that is, the  $N \leq 2$  case of Theorem 5.7. The statement of the conjecture is technical, but combined with [Che99, Theorem 14.2] it implies the following fact.

**THEOREM 1.3.** *Let  $(X, d)$  be a complete metric space that supports a doubling measure  $\mu$  and a  $p$ -Poincaré inequality. If the corresponding measurable differentiable structure is (at most) 2-dimensional and if there is an isometric embedding  $\iota: X \rightarrow \mathbb{R}^N$ , for some  $N \in \mathbb{N}$ , then the image  $\iota(X^m)$  of each coordinate chart  $X^m$  is an  $n(m)$ -rectifiable set.*

Recently Bate has proven a more general version of the same result [Bat12, Corollary 6.10]. Though his methods are independent of ours, his argument also relies on the results of Alberti, Csörnyei, and Preiss about Lebesgue null sets in  $\mathbb{R}^2$  [ACP05].

In the context of geometric measure theory, it is a fact that every  $n$ -rectifiable set in  $\mathbb{R}^N$  agrees with a countable union of  $n$ -dimensional,  $C^1$ -smooth submanifolds, up to a set of zero  $n$ -dimensional Hausdorff measure [Mat95, Theorem 15.21]. Theorem 1.3 therefore asserts that 2-dimensional spaces supporting Euclidean metrics and nontrivial derivations must also have locally Euclidean geometry (up to negligible subsets).

As a special case, Theorem 1.2 implies additional rigidity for measures in the plane. The case of  $\mathbb{R}$  was proven by Björn, Buckley, and Keith [BBK06].

**THEOREM 1.4.** *Let  $\mu$  be a doubling measure on  $\mathbb{R}^2$ . If  $(\mathbb{R}^2, |\cdot|, \mu)$  supports a  $p$ -Poincaré inequality, then  $\mu$  is absolutely continuous to Lebesgue measure.*

The hypothesis of a Poincaré inequality is necessary. Namely, there exist doubling measures on  $\mathbb{R}^n$  that give positive mass to Lebesgue null sets; for examples, see [KW95], [Wu98], and [GKS10].

We note that Keith has proven [Che99, Conjecture 4.63] for 1-dimensional differentiable structures, and that our methods are independent of his. His proof relies on a fact about sets of non-differentiability of Lipschitz functions on  $\mathbb{R}$  [PT95]. Alberti, Csörnyei, and Preiss have recently announced an analogous fact in  $\mathbb{R}^2$  [ACP05, Theorem 7.5] and from this, Keith's techniques will also prove the 2-dimensional case of Cheeger's conjecture; see also the proof in [Bat12].

**1.3. Plan of the paper.** Section 2 begins by introducing terminology and recalling basic facts about Lipschitz functions. It also contains the basics of Weaver's theory, clarifies the equivalence of definitions from [Wea00] and [Hei07], and gives new facts about derivations.

The case of derivations on 1-dimensional sets in  $\mathbb{R}^n$  is treated in Section 3; this includes the setting of 1-rectifiable sets. In Section 4, we discuss the structure of Lebesgue null sets in  $\mathbb{R}^2$  and the rigidity of measures that induce rank-2 modules of derivations. Section 5 begins with basic facts about spaces admitting a Poincaré inequality and concludes with a proof of the

2-dimensional case of Cheeger’s conjecture; we also explore the relationship between several open problems.

## 2. Preliminaries

**2.1. Notation and preliminaries.** The identity map on a set  $S$  is denoted by  $\text{id}_S$ . For real-valued functions  $f$  and  $g$ , we denote their pointwise minimum and maximum as  $f \wedge g$  and  $f \vee g$ , respectively.

For a measure  $\mu$  on a set  $X$  and a  $\mu$ -measurable subset  $A$  of  $X$ , the restriction measure  $\mu \lfloor A$  is defined as

$$(\mu \lfloor A)(E) := \mu(A \cap E)$$

for all  $\mu$ -measurable subsets  $E$  in  $X$ . If  $\mu = \mu \lfloor A$ , then we say that  $\mu$  is *concentrated* on  $A$ . A collection  $\{X_i\}_{i=1}^\infty$  of  $\mu$ -measurable subsets of  $X$  is a  *$\mu$ -measurable decomposition* of  $X$  if  $\mu$  is concentrated on  $\bigcup_{i=1}^\infty X_i$  and if  $\mu(X_i \cap X_j) = 0$  holds whenever  $i \neq j$ .

Given  $p \in [1, \infty]$  and a measure  $\mu$  on a set  $X$ , the standard norm on the Banach space  $L^p(X, \mu)$  is denoted by  $\|\cdot\|_{\mu,p}$ . We will write  $\|f\|_\infty$  for the supremum norm of a function  $f$ , whenever it exists. As indicated before, given a function  $u \in L^1_{\text{loc}}(X, \mu)$  and a subset  $A \subset X$  with  $0 < \mu(A) < \infty$ , its mean value is

$$u_A := \int_A u \, d\mu = \frac{1}{\mu(A)} \int_A u \, d\mu.$$

On a metric space  $X$ , a measure  $\mu$  is Radon if it is Borel regular and if balls have positive finite  $\mu$ -measure. We will denote  $\alpha$ -dimensional Hausdorff measure on a metric space  $X$  by  $\mathcal{H}^\alpha_X$ . For  $X = \mathbb{R}^n$ , we write  $\mathcal{H}^\alpha = \mathcal{H}^\alpha_X$  and  $m_n$  for the Lebesgue measure.

The standard basis of vectors on  $\mathbb{R}^n$  is denoted by  $\{e_1, e_2, \dots, e_n\}$ . If  $V$  is a linear subspace of  $\mathbb{R}^n$ , then  $\text{proj}_V : \mathbb{R}^n \rightarrow V$  is the orthogonal projection map onto  $V$ . For  $j = 1, 2, \dots, n$ , the standard partial differential operators on  $\mathbb{R}^n$  are denoted by  $\partial_j := \partial/\partial x_j$ . The class of smooth functions on  $\mathbb{R}^n$  with compact support is denoted by  $C^\infty_0(\mathbb{R}^n)$ .

**2.2. Lipschitz functions.** Let  $(X, \rho_X)$  and  $(Y, \rho_Y)$  be metric spaces. Recall that a function  $f : X \rightarrow Y$  is Lipschitz if

$$(2.1) \quad L(f) := \sup \left\{ \frac{\rho_Y(f(x), f(y))}{\rho_X(x, y)} : x, y \in X, x \neq y \right\} < \infty$$

and we refer to  $L(f)$  as the *Lipschitz constant* of  $f$ .

We write  $\text{Lip}(X; Y)$  for the space of Lipschitz maps from  $X$  to  $Y$  and  $\text{Lip}_b(X; Y)$  for the subspace of bounded Lipschitz maps in  $\text{Lip}(X; Y)$ . For  $Y = \mathbb{R}$ , we write

$$\text{Lip}(X) := \text{Lip}(X; \mathbb{R}) \quad \text{and} \quad \text{Lip}_b(X) := \text{Lip}_b(X; \mathbb{R}).$$

We now recall some basic properties of Lipschitz maps. Their proofs are elementary and we omit them.

LEMMA 2.1. *Let  $X, Y,$  and  $Z$  be metric spaces.*

- (1) *If  $f \in \text{Lip}(X; Y)$  and  $g \in \text{Lip}(Y; Z)$ , then  $g \circ f \in \text{Lip}(X; Z)$ .*
- (2)  *$\text{Lip}(X)$  is a vector space, and  $\text{Lip}_b(X)$  is an algebra over  $\mathbb{R}$ .*
- (3) *If  $f$  and  $g$  are functions in  $\text{Lip}(X)$ , then so are  $f \vee g$  and  $f \wedge g$ .*
- (4) *Let  $A$  be a closed subset of  $X$ . If  $f \in \text{Lip}(A)$ , then there exists  $F \in \text{Lip}(X)$  so that  $F|_A = f$  and  $L(F) = L(f)$ .*

Part (4) of Lemma 2.1 is known as the *McShane–Whitney extension* of a Lipschitz function [McS34]. For  $f \in \text{Lip}(A)$ , an explicit formula is

$$F(x) := \inf\{f(a) + L(f)d(x, a) : a \in A\}.$$

Combining Parts (3) and (4), we obtain an analogous fact for  $\text{Lip}_b(X)$ .

LEMMA 2.2. *Let  $A$  be a closed subset of  $X$ . If  $f \in \text{Lip}_b(A)$ , then there exists  $f^A \in \text{Lip}_b(X)$  so that  $f^A|_A = f$ ,  $L(f^A) = L(f)$ , and  $\|f^A\|_\infty = \|f\|_\infty$ .*

We refer to  $f^A$  as the *bounded McShane extension* of  $f$ . Note that  $\text{Lip}_b(X)$  is a Banach space with respect to the norm

(2.2) 
$$\|f\|_{\text{Lip}} := \|f\|_\infty \vee L(f).$$

For a proof, see [Wea99, Proposition 1.6.2(a)]. In fact,  $\text{Lip}_b(X)$  is a *dual* Banach space [AE56]; see also [Wea96, Propositions 2 and 8].

LEMMA 2.3 (Weaver, 1996). *Let  $X$  be a metric space.*

- (1)  *$\text{Lip}_b(X)$  is a dual Banach space with respect to the norm  $\|\cdot\|_{\text{Lip}}$ .*
- (2) *If  $\{f_\alpha\}_{\alpha \in I}$  is a bounded net in  $\text{Lip}_b(X)$ , then  $f_\alpha$  converges weak-\* to  $f$  in  $\text{Lip}_b(X)$  if and only if  $f_\alpha$  converges pointwise to  $f$ .*

The next lemma follows from the first lemma and from several explicit constructions in [Wea99, Sections 1.7 and 2.2].

LEMMA 2.4. *If  $X$  is a separable metric space, then the pre-dual of  $\text{Lip}_b(X)$  is a separable Banach space.*

*Proof.* Clearly,  $\rho_2 := \rho \wedge 2$  is a metric on  $X$  and the metric space  $X_2 := (X, \rho_2)$  is separable because  $(X, \rho)$  is separable. By [Wea99, Proposition 1.7.1], we have the isometric isomorphism  $\text{Lip}_b(X) \cong \text{Lip}_b(X_2)$ .

Let  $X_2^+$  be the set of all points in  $X_2$  as well as one additional point  $e$ , so  $X_2^+ = X \cup \{e\}$ . We may extend  $\rho_2$  to a metric  $\rho_2^+$  on  $X_2^+$  by the rule

$$\rho_2^+(x, e) := \begin{cases} \text{diam}(X_2), & x \neq e, \\ 0, & x = e. \end{cases}$$

Now consider the space of functions given by

$$\text{Lip}_0(X_2^+) := \{f \in \text{Lip}(X_2^+) : f(e) = 0\}.$$

By [Wea99, Proposition 1.6.2(b)],  $\text{Lip}_0(X_2^+)$  is a Banach space with respect to the norm  $f \mapsto L(f)$ . Moreover, by [Wea99, Theorem 1.7.2] we also have the isometric isomorphism  $\text{Lip}_0(X_2^+) \cong \text{Lip}_b(X_2)$ .

Clearly  $X_2^+$  is bounded, so by [Wea99, Theorem 2.2.2] the pre-dual of  $\text{Lip}_0(X_2^+)$  is isometrically isomorphic to the Arens–Eells space  $\text{AE}(X_2^+)$ . Since  $X_2^+$  is a bounded, separable metric space, it follows by construction [Wea99, Definition 2.2.1] that  $\text{AE}(X_2^+)$  is a separable Banach space. Thus, the desired result follows from the isometric isomorphism  $\text{Lip}_b(X) \cong [\text{AE}(X_2^+)]^*$ .  $\square$

**2.3. Derivations and basic properties.** Here and in what follows, triples of the form  $(X, \rho, \mu)$  will denote a metric space  $(X, \rho)$  equipped with a Borel measure  $\mu$ .

DEFINITION 2.5 (Weaver). A *derivation*  $\delta : \text{Lip}_b(X) \rightarrow L^\infty(X, \mu)$  is a linear map that satisfies

- (1) the Leibniz Rule:  $\delta(fg) = f\delta g + g\delta f$  holds for all  $f, g \in \text{Lip}_b(X)$ ;
- (2) Weak-\* continuity on bounded sets: if  $\{f_i\}_{i \in I}$  is a bounded net in  $\text{Lip}_b(X)$  so that  $f_i \xrightarrow{*} f$ , then  $\delta f_i \xrightarrow{*} \delta f$  in  $L^\infty(X, \mu)$ .

The set of derivations on  $(X, \rho, \mu)$  is denoted by  $\Upsilon(X, \mu)$ .

REMARK 2.6. The Leibniz rule implies that  $\delta(1) = 2\delta(1)$ , so  $\delta c = 0$  holds for every constant  $c \in \mathbb{R}$ .

REMARK 2.7. Property (2) in Definition 2.5 is better known as *bounded weak-\* continuity*, which refers to continuity with respect to the *bounded weak-\* topology* on the space of linear operators between Banach spaces. For more about this topology, see [DS88, Theorem V.5.3].

We will refer to Property (2) as the *continuity property of derivations*, or simply as *continuity*. For separable metric spaces, this property reduces to familiar modes of continuity from functional analysis. In particular, Definition 2.5 agrees with that in [Hei07, p. 68].

LEMMA 2.8. *Let  $(X, \rho)$  be a separable metric space, let  $\mu$  be a measure on  $X$ , and let  $\delta : \text{Lip}_b(X) \rightarrow L^\infty(X, \mu)$  be a linear operator.*

- (1) *If  $\delta \in \Upsilon(X, \mu)$ , then  $\delta$  is a bounded operator.*
- (2)  *$\delta \in \Upsilon(X, \mu)$  holds if and only if  $\delta$  satisfies the Leibniz rule and is weak-\* continuous with respect to sequences in  $\text{Lip}_b(X)$ .*

*Proof.* Suppose there is a  $\delta \in \Upsilon(X, \mu)$  with the property that, for each  $n \in \mathbb{N}$ , there exists  $f_n \in \text{Lip}_b(X)$  so that  $\|f_n\|_{\text{Lip}} \leq 1$  and  $\|\delta f_n\|_{\infty, \mu} \geq n$ .

By Lemma 2.4,  $\text{Lip}_b(X)$  is the dual of a separable Banach space, so by the Banach–Alaoglu theorem [Rud91, Theorem 3.17], it follows that there is a weak-\* convergent subsequence  $\{f_{n_m}\}_{m=1}^\infty$ . The sequence  $\{\delta f_{n_m}\}_{m=1}^\infty$  is

weak- $*$  convergent in  $L^\infty(X, \mu)$ , by the continuity property of derivations, and therefore bounded. On the other hand, we have, by hypothesis,

$$\|\delta f_{n_m}\|_{\infty, \mu} \geq n_m \rightarrow \infty$$

as  $m \rightarrow \infty$ . This is a contradiction, which gives Part (1).

Since  $\text{Lip}_b(X)$  has a separable pre-dual, by [Rud91, Theorem 3.16] the weak- $*$  topology on  $\bar{B}(0, R) \subset \text{Lip}_b(X)$  is metrizable. As a result, weak- $*$  convergence on bounded sets in  $\text{Lip}_b(X)$  agrees with weak- $*$  convergence with respect to sequences; this gives Part (2).  $\square$

By Lemma 2.8, each derivation in  $\Upsilon(X, \mu)$  has a well-defined operator norm whenever  $X$  is separable. We will denote this norm by

$$(2.3) \quad \|\delta\|_{\text{op}} := \sup\{\|\delta f\|_{\infty, \mu} : f \in \text{Lip}_b(X), \|f\|_{\text{Lip}} \leq 1\}.$$

We now give two examples of metric measure spaces and their derivations. They are [Wea00, Section 5B] and [Wea00, Corollary 35], respectively.

EXAMPLE 2.9. For  $i = 1, 2, \dots, n$ , each  $\partial_i$  lies in  $\Upsilon(\mathbb{R}^n, m_n)$ . The continuity follows from an integration by parts argument.

EXAMPLE 2.10. If  $\mu$  is any measure on  $\mathbb{R}$  that is concentrated on the “middle-thirds” Cantor set, then  $\Upsilon(\mathbb{R}, \mu) = 0$ . This fact follows from Lemma 3.2, but the original proof in [Wea00] relies on the *total disconnectedness* and self-similarity of the Cantor set.

As stated in the Introduction, the set  $\Upsilon(X, \mu)$  is a module over the ring  $L^\infty(X, \mu)$ . Indeed, for  $\delta \in \Upsilon(X, \mu)$  and  $\lambda \in L^\infty(X, \mu)$ , we define  $\lambda\delta \in \Upsilon(X, \mu)$  by the rule

$$(\lambda\delta)f(x) := \lambda(x)\delta f(x).$$

DEFINITION 2.11. A set  $\{\delta_i\}_{i=1}^M$  generates  $\Upsilon(X, \mu)$  if, for all  $\delta \in \Upsilon(X, \mu)$ , there are scalars  $\{c_i\}_{i=1}^M$  in  $L^\infty(X, \mu)$  so that  $\delta = \sum_{i=1}^M c_i \delta_i$ .

A set  $\{\eta_i\}_{i=1}^N$  is linearly independent in  $\Upsilon(X, \mu)$  if whenever there are scalars  $\{\lambda_i\}_{i=1}^N$  in  $L^\infty(X, \mu)$  so that  $\sum_i \lambda_i \eta_i = 0$ , then each  $\lambda_i$  is zero. The rank of the module  $\Upsilon(X, \mu)$  is the largest cardinality of any linearly independent set in  $\Upsilon(X, \mu)$ .

The next lemma follows directly from Definition 2.11, so we omit the proof.

LEMMA 2.12. Let  $N \in \mathbb{N}$  and let  $A$  be a  $\mu$ -measurable subset of  $X$  with  $\mu(A) > 0$ . If  $\{\delta_i\}_{i=1}^N$  is a linearly independent set in  $\Upsilon(X, \mu)$ , then  $\{\chi_A \delta_i\}_{i=1}^N$  is also a linearly independent set in  $\Upsilon(X, \mu)$ .

EXAMPLE 2.13 ([Wea00, Theorem 37]). For  $X = \mathbb{R}^n$ ,  $\{\partial_i\}_{i=1}^n$  is a linearly independent set that generates  $\Upsilon(\mathbb{R}^n, m_n)$ . Moreover, as  $L^\infty(\mathbb{R}^n, m_n)$ -modules,

$$\Upsilon(\mathbb{R}^n, m_n) \cong \bigoplus_{i=1}^n L^\infty(\mathbb{R}^n, m_n).$$



More generally, let  $X$  be a compact Riemannian manifold and let  $\mu$  be the volume element. Then  $\Upsilon(X, \mu)$  is isomorphic to the  $L^\infty(X, \mu)$ -module of bounded measurable sections of the tangent bundle  $TX$ .

Derivations are also known as *measurable vector fields* [Wea00], [Hei07]. In the remainder of the section, we investigate properties of derivations which are similar to those of vector fields on smooth manifolds.

**2.4. Locality and applications.** On a smooth manifold  $M$ , vector fields are *local* objects; that is, their action on a function  $\varphi \in C^\infty(M)$  near a point  $x \in M$  depends only on the behavior of  $\varphi$  near  $x$ . The next theorem shows that derivations enjoy a similar property, called the *locality property*. It is a special case of [Wea00, Theorem 29]; see also [Hei07, Theorem 13.3].

**THEOREM 2.14** (Weaver, 2000). *Let  $\mu$  be a Radon measure on  $X$ . If  $A$  is a  $\mu$ -measurable subset of  $X$ , then we have the  $L^\infty(X, \mu)$ -module isomorphism*

$$\Upsilon(A, \mu) \cong \{\chi_A \delta : \delta \in \Upsilon(X, \mu)\}.$$

By definition, each  $\delta \in \Upsilon(X, \mu)$  acts only on bounded Lipschitz functions. In the case of Radon measures  $\mu$ , however, the domain of definition of  $\delta$  extends to include all Lipschitz functions.

**THEOREM 2.15.** *Let  $\mu$  be a Radon measure on  $X$ . For each  $\delta \in \Upsilon(X, \mu)$ , there is a linear map  $\bar{\delta} : \text{Lip}_{\text{loc}}(X) \rightarrow L^\infty_{\text{loc}}(X, \mu)$  with the following properties:*

- (1) *for all  $f \in \text{Lip}_b(X)$ , we have  $\bar{\delta}f = \delta f$ ,*
- (2) *for all  $f \in \text{Lip}(X)$  and all balls  $B$  in  $X$ , we have*

$$\chi_B \bar{\delta}f = \chi_B \delta((f|B)^B),$$

- (3) *the Leibniz rule holds for  $\bar{\delta}$ ;*
- (4) *if  $X$  is separable, then for all  $f \in \text{Lip}(X)$ , we have*

$$\|\bar{\delta}f\|_{\infty, \mu} \leq \|\delta\|_{\text{op}} L(f).$$

To reiterate,  $(f|B)^B$  refers to the bounded McShane extension of  $f|B$ . Theorem 2.15 will follow from the next lemma and a locality argument.

**LEMMA 2.16.** *Let  $X$  be a separable metric space, let  $\mu$  be a Radon measure on  $X$ , and take a  $\mu$ -measurable decomposition  $\{X_i\}_{i=1}^\infty$  of  $X$ . Suppose that for each  $i \in \mathbb{N}$ , there exists  $\delta_i \in \Upsilon(X_i, \mu)$  so that  $\|\delta_i\|_{\text{op}} \leq 1$ . Then the linear operator  $\delta : \text{Lip}_b(X) \rightarrow L^\infty(X, \mu)$ , given by*

$$\delta f := \sum_{i=1}^\infty \chi_{X_i} \delta_i(f|X_i)$$

*determines a derivation in  $\Upsilon(X, \mu)$ , with  $\|\delta\|_{\text{op}} \leq 1$ .*

*Proof.* For each  $f \in \text{Lip}_b(X)$ , we have

$$\mu(\{x : |\delta f(x)| > 1\}) \leq \sum_{i=1}^{\infty} \mu(\{x \in X_i : |\delta_i(f|X_i)(x)| > 1\}) = 0.$$

Therefore  $\delta f$  is well defined in  $L^\infty(X, \mu)$  with  $\|\delta f\|_{\mu, \infty} \leq 1$ . The map  $\delta$  is clearly linear and satisfies the Leibniz rule. To check continuity, let  $\{f_\alpha\}_{\alpha \in I}$  be a net in  $\text{Lip}_b(X)$  so that  $f_\alpha \rightarrow 0$  and  $\sup_\alpha \|f_\alpha\|_{\text{Lip}} \leq L$  holds, for some  $L \in [0, \infty)$ .

Let  $\varepsilon > 0$  be arbitrary. For each  $h \in L^1(X, \mu)$ , there is an  $N \in \mathbb{N}$  so that

$$\sum_{i=N+1}^{\infty} \int_{X_i} |h| d\mu \leq \frac{\varepsilon}{2L\|\delta\|_{\text{op}}}.$$

Moreover, for each  $i = 1, 2, \dots, N$ , we have  $h|X_i \in L^1(X_i, \mu)$ , so the bound

$$\left| \int_{X_i} h \delta f_\alpha d\mu \right| = \left| \int_{X_i} h \delta_i(f|X_i) d\mu \right| < \frac{\varepsilon}{2N}$$

follows from the continuity of  $\delta_i$ . We then compute

$$\begin{aligned} \left| \int_X h \delta f_\alpha d\mu \right| &\leq \left| \sum_{i=1}^N \int_{X_i} h \delta f_\alpha d\mu \right| + \sum_{i=N+1}^{\infty} \|\delta f_\alpha\|_{\mu, \infty} \int_{X_i} |h| d\mu \\ &\leq N \frac{\varepsilon}{2N} + L \|\delta\|_{\text{op}} \frac{\varepsilon}{2L\|\delta\|_{\text{op}}} = \varepsilon. \end{aligned}$$

As a result, we have  $\delta f_\alpha \xrightarrow{*} 0$  in  $L^\infty(X, \mu)$ , which proves the lemma. □

*Proof of Theorem 2.15.* Without loss of generality, take a  $\mu$ -measurable decomposition  $\{X_n\}_{n=1}^\infty$  of  $X$  where each  $X_n$  is a bounded set. (For example, fix a base point  $a \in X$  and put  $X_n := B(a, n) \setminus B(a, n-1)$  for  $n \in \mathbb{N}$ .) Put

$$\bar{\delta}f := \sum_{n=1}^{\infty} \chi_{X_n} \delta((f|X_n)^{X_n}).$$

Indeed,  $\bar{\delta}f$  is well defined because  $f|X_n \in \text{Lip}_b(X_n)$  holds, for all  $n \in \mathbb{N}$ . Clearly  $\bar{\delta}$  is linear. By Theorem 2.14, we have

$$\chi_{X_n} \delta((f|X_n)^{X_n}) = \chi_{X_n} \delta f$$

for all  $f \in \text{Lip}_b(X)$  and all  $n \in \mathbb{N}$ , so Property (1) follows. Similarly, for each  $n \in \mathbb{N}$  and each ball  $B$  in  $X$ , Property (2) follows from

$$\chi_{B \cap X_n} \delta_{X_n} \delta((f|X_n)^{X_n}) = \chi_{B \cap X_n} \delta((f|B)^B).$$

By a similar argument, Property (3) is a consequence of Property (2), the locality property, and the Leibniz rule for  $\delta$ . Now suppose that  $X$  is separable.

Letting  $\{x_n\}_{n=1}^\infty$  be a countable dense subset of  $X$ , put  $X_0 = \emptyset$  and for each  $n \in \mathbb{N}$ , put

$$X_n := B(x_n, 1/2) \setminus \left( \bigcup_{k=0}^{n-1} X_k \right) \quad \text{and} \quad f_n := f - \inf_{X_n} f.$$

Since each set  $X_n$  has diameter at most 1, we obtain

$$\|f_n\|_\infty = \left| \sup_{X_n} f - \inf_{X_n} f \right| \leq L(f) \operatorname{diam}(X_n) = L(f).$$

Invoking Lemma 2.8 and the estimate above, we now compute

$$\begin{aligned} \|\delta_{X_n}(f|X_n)\|_{\mu, \infty} &= \|\delta((f|X_n)^{X_n})\|_{\mu, \infty} = \|\delta((f_n|X_n)^{X_n})\|_{\mu, \infty} \\ &\leq \|\delta\|_{\text{op}} \|f_n|X_n\|_{\text{Lip}} \leq \|\delta\|_{\text{op}} L(f). \end{aligned}$$

This gives Property (4) and proves the theorem. □

**2.5. Pushforward derivations.** Recall that for smooth manifolds  $M$  and  $N$  with respective tangent bundles  $TM$  and  $TN$ , every smooth bijective map from  $M$  to  $N$  induces a pushforward operator on vector fields. Indeed, for each smooth vector field  $v : M \rightarrow TM$  we obtain a new vector field  $f_\#v : N \rightarrow TN$  from the rule

$$f_\#v(x) := Df(f^{-1}(x))v.$$

A similar procedure holds for derivations, by means of pushforward measures. Recall that on a measure space  $(X, \mu)$ , a set  $Y$ , and a map  $T : X \rightarrow Y$ , one defines the *pushforward measure*  $T_\#\mu$  on  $Y$  by the rule

$$T_\#\mu(A) := \mu(T^{-1}(A)).$$

It is well known that if  $\mu$  is a Borel measure and  $T$  a Borel map, then  $T_\#\mu$  is a Borel measure and we have the “change of variables” formula

$$(2.4) \quad \int_Y \varphi d(T_\#\mu) = \int_X \varphi \circ T d\mu$$

whenever  $\varphi : Y \rightarrow \mathbb{R}$  is a Borel function; see [Mat95, Theorem 1.19].

LEMMA 2.17. *Let  $X, Y$  be metric spaces, let  $\mu$  be a Radon measure on  $X$ , and let  $\pi \in \text{Lip}(X; Y)$ . For each  $\delta \in \Upsilon(X, \mu)$ , there is a unique  $\pi_\#\delta \in \Upsilon(Y, \pi_\#\mu)$  so that*

$$(2.5) \quad \int_Y h(\pi_\#\delta) f d(\pi_\#\mu) = \int_X (h \circ \pi) \delta(f \circ \pi) d\mu$$

*holds for all  $h \in L^1(Y, \pi_\#\mu)$  and all  $f \in \text{Lip}_b(Y)$ . If  $X$  is separable, then*

$$(2.6) \quad \|\pi_\#\delta\|_{\text{op}} \leq (1 \vee L(\pi)) \|\delta\|_{\text{op}}.$$

We refer to  $\pi_\#\delta$  as the *pushforward (derivation)* of  $\delta$  with respect to  $\pi$ .

*Proof of Lemma 2.17.* Put  $\nu := \pi_{\#}\mu$ . For  $h \in L^1(Y, \nu)$ , formula (2.4) gives  $h \circ \pi \in L^1(X, \mu)$ , with  $\|h\|_{\nu,1} = \|h \circ \pi\|_{\mu,1}$ . For each  $f \in \text{Lip}_b(Y)$ , define a map  $\lambda_{f,\pi} : L^1(Y, \nu) \rightarrow \mathbb{R}$  by

$$\lambda_{f,\pi}(h) := \int_X (h \circ \pi)\delta(f \circ \pi) d\mu.$$

Clearly  $\lambda_{f,\pi}$  is linear and bounded, so there is a unique function  $(\pi_{\#}\delta)f \in L^\infty(Y, \nu)$  which satisfies, for all  $h \in L^1(Y, \nu)$ , the identity

$$\int_Y h(\pi_{\#}\delta)f d\nu = \int_X (h \circ \pi)\delta(f \circ \pi) d\mu.$$

As constructed, the map  $\pi_{\#}\delta : f \mapsto (\pi_{\#}\delta)f$  satisfies formula (2.5). Moreover, it is linear because  $\delta$  is linear; the same is true of the Leibniz rule.

To show that  $\pi_{\#}\delta$  is continuous, suppose  $\{f_\alpha\}_{\alpha \in I}$  is a net in  $\text{Lip}_b(Y)$  that converges pointwise to 0 and so that  $C := \sup_\alpha \|f_\alpha\|_{\text{Lip}} < \infty$ . Clearly  $f_\alpha \circ \pi$  converges pointwise to 0, and from the estimates

$$(2.7) \quad \begin{cases} \|f_\alpha \circ \pi\|_\infty \leq \|f_\alpha\|_\infty \leq C, \\ L(f_\alpha \circ \pi) \leq L(f_\alpha)L(\pi) \leq CL(\pi) \end{cases}$$

the net  $\{f_\alpha \circ \pi\}_{\alpha \in I}$  is bounded in  $\text{Lip}_b(X)$ . By Lemma 2.3 and the continuity of  $\delta$ , we obtain  $\delta(f_\alpha \circ \pi) \xrightarrow{*} 0$  in  $L^\infty(X, \mu)$ . Since  $h \in L^1(Y, \nu)$  implies  $h \circ \pi \in L^1(X, \mu)$ , it follows that  $(\pi_{\#}\delta)f_\alpha \xrightarrow{*} 0$  in  $L^\infty(Y, \nu)$ .

Let  $f \in \text{Lip}_b(Y)$ . If  $X$  is separable, then by the estimates (2.7), we obtain

$$\begin{aligned} \|(\pi_{\#}\delta)f\|_{\mu,\infty} &= \|\lambda_{f,\pi}\|_{\text{op}} \leq \|h\|_{\nu,1} \|\delta(f \circ \pi)\|_{\mu,\infty} \\ &\leq \|h\|_{\nu,1} \|\delta\|_{\text{op}} \|f \circ \pi\|_{\text{Lip}} \\ &\leq \|h\|_{\nu,1} \|\delta\|_{\text{op}} C(1 \vee L(\pi)) \|f\|_{\text{Lip}}. \end{aligned}$$

This implies inequality (2.6). Lastly, suppose that  $\delta' \in \Upsilon(Y, \nu)$  also satisfies formula (2.5). By linearity, we have, for all  $h \in L^1(Y, \nu)$  and all  $f \in \text{Lip}_b(Y)$ ,

$$\int_Y h(\pi_{\#}\delta - \delta')f d\nu = 0.$$

This means that  $\pi_{\#}\delta = \delta'$ , which gives the desired uniqueness. □

For  $\pi \in \text{Lip}(X; Y)$ , note that  $\Upsilon(X, \mu)$  is an  $L^\infty(Y, \pi_{\#}\mu)$ -module. Indeed, for  $\lambda \in L^\infty(Y, \pi_{\#}\mu)$ ,  $f \in \text{Lip}_b(X)$ , and  $\delta \in \Upsilon(X, \mu)$ , the action is given by

$$(2.8) \quad (\lambda\delta)f := (\lambda \circ \pi)\delta f.$$

Recall that an embedding  $\pi : X \rightarrow Y$  is *bi-Lipschitz* if  $\pi$  and  $\pi^{-1}$  are both Lipschitz maps; it is  $\lambda$ -*bi-Lipschitz* if  $L(\pi) \leq \lambda$  and  $L(\pi^{-1}) \leq \lambda$ .

So if  $\pi$  is bi-Lipschitz, then the proof of Lemma 2.17 (with  $\pi^{-1}$  for  $\pi$ ) also shows that  $\Upsilon(Y, \pi_{\#}\mu)$  is an  $L^\infty(X, \mu)$ -module. Under an appropriate choice of measures, we now obtain a “functorial” property of pushforward derivations with respect to bi-Lipschitz embeddings.

**COROLLARY 2.18.** *Let  $(X, \rho_X, \mu)$  and  $(Y, \rho_Y, \nu)$  be metric measure spaces, with  $\mu$  a Borel measure, and let  $\pi : X \hookrightarrow Y$  be a bi-Lipschitz embedding. If  $\nu$  and  $\pi\#\mu$  are mutually absolutely continuous, then  $\Upsilon(X, \mu)$  and  $\Upsilon(Y, \nu)$  are isomorphic as  $L^\infty(X, \mu)$ -modules.*

**2.6. The Chain Rule.** On Euclidean spaces, derivations exhibit behavior similar to that of the differential operators  $\{\partial_i\}_{i=1}^n$ . For instance, they satisfy a weak form of the Chain Rule from differential calculus. To formulate this fact, recall that by Theorem 2.15, each  $\delta x_j$  is a well-defined function in  $L^\infty(\mathbb{R}^n, \mu)$  for  $j = 1, 2, \dots, n$ .

**LEMMA 2.19.** *Let  $\mu$  be a Radon measure on  $\mathbb{R}^n$ . For each  $f \in \text{Lip}(\mathbb{R}^n)$ , there exist functions  $\{g_f^i\}_{i=1}^n \subset L^\infty(\mathbb{R}^n, \mu)$  so that*

$$(2.9) \quad \|g_f^i\|_{\mu, \infty} \leq L(f),$$

$$(2.10) \quad \delta f = \sum_{i=1}^n g_f^i \delta x_i.$$

*If  $f$  is smooth, then  $g_f^i = \partial_i f$  for  $i = 1, 2, \dots, n$ .*

We refer to Lemma 2.19 as the Chain Rule for derivations. Its proof uses a classical fact about approximation of smooth functions [CH53, Theorem II.4.3].

**LEMMA 2.20.** *Let  $\varphi \in C^\infty(\mathbb{R}^n)$ . For each compact subset  $K$  of  $\mathbb{R}^n$ , there is a sequence of polynomials  $\{P_m\}_{m=1}^\infty$  so that on  $K$ , we have the uniform convergence  $P_m \rightarrow \varphi$  and  $\partial_i P_m \rightarrow \partial_i \varphi$ , for  $1 \leq i \leq n$ .*

*Proof of Lemma 2.19.* Since  $\mathbb{R}^n$  is a countable union of closed cubes  $\{Q_k\}_{k=1}^\infty$ , it suffices to show formula (2.10) for  $\chi_{Q_k} \delta$  in place of  $\delta$ . By the locality property, we therefore assume that  $\delta \in \Upsilon(Q_k, \mu)$ .

We now argue by cases. Formula (2.10) clearly holds when  $f = x_j$ , for each  $j = 1, 2, \dots, n$ , and where  $g_f^i$  is the Kronecker symbol  $\varepsilon_{ij}$ . If  $f$  is a polynomial, then for each  $a \in \mathbb{N}$ , the Leibniz rule implies a “power rule”

$$\delta(x_j^a) = ax_j^{a-1} \delta x_j$$

which further implies formula (2.10), with  $g_f^i := \partial_i f$ .

We next assume that  $f$  is a smooth Lipschitz function on  $\mathbb{R}^n$ . By Lemma 2.20, there is a sequence of polynomials  $\{P_m\}_{m=1}^\infty$  which converges uniformly to  $f$  on  $K$  and where  $\{\nabla P_m\}_{m=1}^\infty$  converges uniformly to  $\nabla f$ . This implies that  $P_m \xrightarrow{*} f$  in  $\text{Lip}_b(Q_k)$ , and by continuity of  $\delta$ , we obtain  $\delta P_m \xrightarrow{*} \delta f$  in  $L^\infty(Q_k, \mu)$ . On the other hand, the convergence  $\nabla P_m \rightarrow \nabla f$  is uniform, hence weak-\*. It follows that  $\partial_i P_m \delta x_i \xrightarrow{*} \partial_i f \delta x_i$  in  $L^\infty(Q_k, \mu)$  and by uniqueness of limits, we obtain formula (2.10), where again  $g_f^i := \partial_i f$ .

For the general case, let  $\varepsilon > 0$  be arbitrary, let  $\eta_\varepsilon$  be a smooth, symmetric mollifier, and consider convolutions  $f_\varepsilon := f * \eta_\varepsilon$ . It is a fact [EG92, Theorem

4.2.1.1] that if  $f$  is continuous, then  $f_\varepsilon$  converges locally uniformly to  $f$ . Moreover, the bound  $L(f_\varepsilon) \leq L(f)$  follows from the computation

$$\begin{aligned}
|f_\varepsilon(x) - f_\varepsilon(y)| &\leq \int_{\mathbb{R}^n} \eta_\varepsilon(z) |f(x-z) - f(y-z)| dz \\
&\leq \int_{\mathbb{R}^n} \eta_\varepsilon(z) L(f) |(x-z) - (y-z)| dz \leq L(f) |x-y|.
\end{aligned}$$

This implies that  $f_\varepsilon \xrightarrow{*} f$  in  $\text{Lip}_b(Q_k)$  and from the continuity of  $\delta$ , we also have  $\delta f_\varepsilon \xrightarrow{*} \delta f$  in  $L^\infty(Q_k, \mu)$ .

However, note that formula (2.10) holds for each  $f_\varepsilon$ , where  $g_f^i = \partial_i f_\varepsilon$ , and note that  $\{\partial_i f_{1/a}\}_{a=1}^\infty$  is a bounded sequence in  $L^\infty(\mathbb{R}^n, \mu)$ , for each  $i$ . It follows from the Banach–Alaoglu theorem that there are weak-\* convergent subsequences  $\{\partial_i f_{1/a_b}\}_{b=1}^\infty$  with weak-\* limits  $g_f^i$ .

By uniqueness of limits, formula (2.10) holds for  $f$  with these choices of  $g_f^i$ . Since the norm on  $L^\infty(\mathbb{R}^n, \mu)$  is lower semi-continuous (with respect to the weak-\* topology), it follows that

$$\|g_f^i\|_{\mu, \infty} \leq \liminf_{b \rightarrow \infty} \|\partial_i f_{1/a_b}\|_{\mu, \infty} \leq L(f)$$

which further implies (2.9). □

The next corollary is a criterion for detecting nonzero derivations on  $\mathbb{R}^n$ . It follows directly from Lemma 2.19, so we omit the proof.

**COROLLARY 2.21.** *Let  $\mu$  be a Radon measure on  $\mathbb{R}^n$  and let  $\delta \in \Upsilon(\mathbb{R}^n, \mu)$ . If  $\delta x_j = 0$  holds for each  $j = 1, 2, \dots, n$ , then  $\delta = 0$ .*

### 3. Derivations on 1-dimensional sets

Adapting the terminology in [Fal86], a subset of  $\mathbb{R}^n$  is called a  $k$ -set if it is  $\mathcal{H}^k$ -measurable and has  $\sigma$ -finite  $\mathcal{H}^k$ -measure. In this section, we will focus on the following fact about measures concentrated on 1-sets and their derivations.

**THEOREM 3.1.** *Let  $\mu$  be a Radon measure on  $\mathbb{R}^n$ . If  $\mu$  is concentrated on a 1-set, then the module  $\Upsilon(\mathbb{R}^n, \mu)$  has rank at most 1.*

The proof uses facts from geometric measure theory, which are discussed in Section 3.2. We begin with a special case.

**3.1. The case of  $\mathbb{R}$ .** Using the Borel regularity of Lebesgue measure, we prove Theorem 1.2 for  $k = 1$ , which is a characterization of  $\Upsilon(\mathbb{R}, \mu)$ . To this end, recall that every Radon measure  $\mu$  on  $\mathbb{R}$  admits a decomposition  $\mu = \mu_{AC} + \mu_S$ , where  $\mu_{AC}$  is absolutely continuous to  $m_1$  and  $\mu_S$  is singular to  $m_1$  [Fol99, Theorem 3.8].

LEMMA 3.2. *Let  $\mu$  be a Radon measure on  $\mathbb{R}$ . If  $\mu_S$  is concentrated on a Lebesgue null set  $E$ , then for all  $\delta \in \Upsilon(\mathbb{R}, \mu)$  and all  $f \in \text{Lip}_b(\mathbb{R})$ ,*

$$(3.1) \quad \delta f(x) = \begin{cases} \delta(\text{id}_{\mathbb{R}})(x)f'(x), & \text{for } x \in \mathbb{R} \setminus E, \\ 0, & \text{for } x \in E, \end{cases}$$

where  $f'$  is the classical derivative of  $f$ . Moreover, as  $L^\infty(\mathbb{R}, \mu)$ -modules,

$$\Upsilon(\mathbb{R}, \mu) \cong L^\infty(\mathbb{R}, \mu_{AC}).$$

*Proof.* Let  $\delta \in \Upsilon(\mathbb{R}, \mu)$  and  $f \in \text{Lip}_b(\mathbb{R})$  be arbitrary. For subsets of  $\mathbb{R} \setminus E$  we have  $\mu = \mu_{AC}$ , so by Rademacher’s theorem,  $f$  is differentiable  $\mu$ -a.e. on  $\mathbb{R} \setminus E$ . The Chain Rule for derivations then implies that

$$\delta f(x) = \delta(\text{id}_{\mathbb{R}})(x)f'(x)$$

for  $\mu_{AC}$ -a.e.  $x \in \mathbb{R}$  and hence for  $\mu$ -a.e.  $x \in \mathbb{R} \setminus E$ .

To show  $\chi_E \delta f = 0$ , assume by locality (Theorem 2.14) that  $E$  is bounded. In particular, let  $E \subset [0, 1]$ , so  $\chi_E \delta \in \Upsilon([0, 1], \mu)$ .

Since  $m_1(E) = 0$ , for each  $j \in \mathbb{N}$  there is a open set  $O_j$  so that  $E \subset O_j$  and  $m_1(O_j) < 2^{-j}$ . We next define functions  $\varphi_j : [0, 1] \rightarrow \mathbb{R}$  by the formula

$$\varphi_j(x) := \int_0^x (1 - \chi_{O_j}) dm_1.$$

Clearly  $\|\varphi_j\|_{\text{Lip}} \leq 1$  holds, for each  $j$ . Estimating further, we see that

$$0 \leq x - \varphi_j(x) = \int_0^x \chi_{O_j} dm_1 \leq m_1(O_j) \leq 2^{-j},$$

and hence  $\{\varphi_j\}_{j=1}^\infty$  converges pointwise to the identity on  $\mathbb{R}$ . By Lemma 2.3, this is equivalent to weak- $*$  convergence in  $\text{Lip}_b([0, 1])$ , and by continuity, we obtain  $\delta \varphi_j \xrightarrow{*} \delta(\text{id}_{\mathbb{R}})$  in  $L^\infty([0, 1], \mu)$ .

However, if  $O'_j$  is a connected component of  $O_j$ , then by construction,  $\varphi_j|_{O'_j}$  is constant for each  $j$ . The locality property implies that  $\delta \varphi_j(x) = 0$  holds for  $\mu$ -a.e.  $x \in O'_j \cap [0, 1]$ , for each  $j$ , and hence  $\chi_E \delta \varphi_j = 0$ . By continuity we obtain  $\chi_E \delta(\text{id}_{\mathbb{R}}) = 0$ , and by the Chain Rule we further obtain  $\chi_E \delta f = 0$ . This proves formula (3.1).

Consider maps  $S : \Upsilon(\mathbb{R}, \mu) \rightarrow L^\infty(\mathbb{R}, \mu_{AC})$  and  $T : L^\infty(\mathbb{R}, \mu_{AC}) \rightarrow \Upsilon(\mathbb{R}, \mu)$  given by  $S(\delta) := \delta(\text{id}_{\mathbb{R}})$  and  $T(\lambda) := \lambda(d/dx)$ . Clearly,  $S$  and  $T$  are homomorphisms of  $L^\infty(\mathbb{R}, \mu)$ -modules. Using (3.1) and the previous estimates,

$$(T \circ S)(\delta) = T(\chi_{\mathbb{R} \setminus E} \delta(\text{id}_{\mathbb{R}})) = \chi_{\mathbb{R} \setminus E} \delta(\text{id}_{\mathbb{R}})(d/dx) = \delta$$

and hence  $S \circ T = \text{id}_{\Upsilon(\mathbb{R}, \mu)}$ . A similar computation gives  $S \circ T = \text{id}_{L^\infty(\mathbb{R}, \mu)}$ .  $\square$

**3.2. The general case.** We now introduce two types of sets in  $\mathbb{R}^n$ .

DEFINITION 3.3. Let  $k \in \mathbb{N}$ . A subset  $E$  in  $\mathbb{R}^n$  is  $k$ -rectifiable if, for some  $\lambda \in (1, \infty)$  it admits a  $\mathcal{H}^k$ -measurable decomposition of the form

$$(3.2) \quad E = N \cup \bigcup_{i=1}^{\infty} f_i(A_i),$$

where  $\mathcal{H}^k(N) = 0$  and where, for each  $i \in \mathbb{N}$ ,  $A_i$  is a compact subset of  $\mathbb{R}^k$  with  $m_k(A_i) > 0$  and  $f_i : A_i \rightarrow \mathbb{R}^n$  is a  $\lambda$ -bi-Lipschitz embedding.

A subset  $F$  in  $\mathbb{R}^n$  is purely  $k$ -unrectifiable if  $\mathcal{H}^k(E \cap F) = 0$  holds for all  $k$ -rectifiable sets  $E$  in  $\mathbb{R}^n$ .

REMARK 3.4. This definition of  $k$ -rectifiability differs substantially from the standard one; see [Mat95, Definition 15.3] or [Fed69, Definition 3.2.14(1)]. However, by [Fed69, Lemma 3.2.18] these definitions are equivalent.

Indeed, each  $k$ -set is a union of sets of the above types [Mat95, Theorem 15.6].

LEMMA 3.5. Let  $n \in \mathbb{N}$  and let  $k$  be an integer in  $[0, n]$ . If  $A$  is a  $k$ -set in  $\mathbb{R}^n$ , then there is a  $\mathcal{H}^k$ -measurable decomposition  $A = E \cup F$ , where  $E$  is  $k$ -rectifiable and  $F$  is purely  $k$ -unrectifiable.

To prove Theorem 3.1, we will use an alternative characterization of purely  $k$ -unrectifiable subsets in  $\mathbb{R}^n$  [Mat95, Theorem 18.1]. Below,  $\mathcal{G}(n; k)$  denotes the space of  $k$ -dimensional subspaces of  $\mathbb{R}^n$  and “almost everywhere” refers to the Haar measure on  $\mathcal{G}(n; k)$ . When  $k = 1$ , this measure is equivalent to (normalized) surface measure on the half-sphere  $\{x \in \mathbb{S}^{n-1} : x_n > 0\}$ .

THEOREM 3.6 (Besicovitch–Federer). For  $0 \leq k \leq n$ , let  $F$  be a  $k$ -set in  $\mathbb{R}^n$ . Then  $F$  is purely  $k$ -unrectifiable if and only if for a.e.  $V \in \mathcal{G}(n; k)$ , the image  $\text{proj}_V(F)$  has  $\mathcal{H}^k$ -measure zero.

In the remainder of the section, we assume  $k = 1$ . The proof of Theorem 3.1 is split into two cases.

LEMMA 3.7. Let  $\mu$  be a Radon measure on  $\mathbb{R}^n$  that is concentrated on a purely 1-unrectifiable set of Hausdorff dimension (at most) one. Then  $\Upsilon(\mathbb{R}^n, \mu) = 0$ .

*Proof.* By Theorem 3.6,  $F$  satisfies  $\mathcal{H}^1(\text{proj}_V(F)) = 0$  for a.e.  $V \in \mathcal{G}(n; 1)$ . In particular, there exist subspaces  $\{V_i\}_{i=1}^n \subset \mathcal{G}(n; 1)$  whose union spans  $\mathbb{R}^n$  and so that  $\mathcal{H}^1(\text{proj}_{V_i}(F)) = 0$  holds, for each  $1 \leq i \leq n$ .

Put  $p^i := \text{proj}_{V_i}$ . Since  $\mathcal{H}^1(p^i(F)) = 0$ , we observe that  $p^i_{\#}\mu$  is singular to  $\mathcal{H}^1 \llcorner V_i$ . By identifying  $V_i$  with  $\mathbb{R}$ , we further observe that  $\Upsilon(V_i, p^i_{\#}\mu) = 0$ .

We claim that  $\delta p^i = 0$  holds for all  $\delta \in \Upsilon(\mathbb{R}^n, \mu)$ ; if not, the set

$$F_i := \{x \in F : \delta p^i(x) \neq 0\}$$



has positive  $\mu$ -measure. For each bounded domain  $\Omega$  in  $\mathbb{R}^n$ , (2.5) implies

$$0 < \int_{\mathbb{R}^n} \chi_{\Omega \cap F_i} \delta p^i d\mu = \int_{V_i} \chi_{p^i(\Omega \cap F_i)} p_{\#}^i \delta(\text{id}_{\mathbb{R}}) d(p_{\#}^i \mu).$$

However, the rightmost term is zero because  $p_{\#}^i \delta \in \Upsilon(V_i, p_{\#}^i \mu)$ ; therefore  $p_{\#}^i \delta$  is zero. Since  $\Omega$  was arbitrary, the claim follows.

Lastly, the linear functions  $\{p^i\}_{i=1}^n$  generate the coordinates  $\{x_i\}_{i=1}^n$ . This implies that  $\delta x_i = 0$  holds  $\mu$ -a.e. for each  $i$ , so by the Chain Rule for derivations, we conclude that  $\delta = 0$ . □

In the case of 1-rectifiable sets, the next lemma extends a result of Weaver [Wea00, Theorem 38] to arbitrary Radon measures on  $\mathbb{R}^n$ .

LEMMA 3.8. *Suppose that  $\mu$  is a Radon measure on  $\mathbb{R}^n$  that is concentrated on a 1-rectifiable set  $E$ . If  $\Upsilon(\mathbb{R}^n, \mu)$  is nontrivial, then  $\Upsilon(\mathbb{R}^n, \mu)$  has rank-1.*

One can further show that the generator of  $\Upsilon(\mathbb{R}^n, \mu)$  is given by  $f \mapsto \chi_E D_{ap} f$ , where  $D_{ap} f$  is the *approximate derivative* of the restriction  $f|_E$ . For more about approximate derivatives, see [Fed69, Section 3.1.22].

*Proof of Lemma 3.8.* Let  $E$  be a 1-rectifiable set on which  $\mu$  is concentrated. As a first case, assume that  $E = f(A)$ , where  $A \subset \mathbb{R}$  satisfies  $m_1(A) > 0$  and where  $f : A \rightarrow \mathbb{R}^n$  is a bi-Lipschitz embedding. By Lemma 2.17, for each  $\delta \in \Upsilon(\mathbb{R}^n, \mu)$  there is a unique element  $f_{\#}^{-1} \delta$  in  $\Upsilon(A, \nu)$ , where  $\nu := f_{\#}^{-1} \mu$ .

If  $\nu_{AC} = 0$ , then  $f_{\#}^{-1} \delta = 0$  follows from Lemma 3.2. Let  $h \in L^1(\mathbb{R}^n, \mu)$  and  $\varphi \in \text{Lip}_b(\mathbb{R}^n)$  be arbitrary. By formula (2.5) and the previous identity,

$$\begin{aligned} \int_{\mathbb{R}^n} h \delta \varphi d\mu &= \int_A (h \circ f^{-1}) f_{\#}^{-1} \delta(\varphi \circ f^{-1}) d\nu \\ &= \int_A (h \circ f^{-1}) \lambda \left( \chi_{A'} \frac{d}{dx} \right) (\varphi \circ f^{-1}) d\nu \\ &= \int_{\mathbb{R}^n} h(\lambda \circ f) f_{\#} \left( \chi_{A'} \frac{d}{dx} \right) \varphi d\mu. \end{aligned}$$

so  $f_{\#}(\chi_{A'} d/dx)$  generates  $\Upsilon(\mathbb{R}^n, \mu)$ .

For the general case, let  $E = \bigcup_{i=1}^{\infty} f_i(A_i)$ , where each  $A_i$  is compact and each  $f_i : A_i \rightarrow \mathbb{R}^n$  is 2-bi-Lipschitz. Indeed, if  $N$  is an  $\mathcal{H}^1$ -null set in  $\mathbb{R}^n$ , then  $N$  is purely 1-unrectifiable and by Lemma 3.7,  $\Upsilon(N, \mu) = 0$ .

Put  $E_i := f_i(A_i)$  and  $\mu_i := \mu|_{E_i}$ . By the previous case, the derivation

$$\delta_i := (f_i)_{\#}(\chi_{A'_i} d/dx)$$

generates  $\Upsilon(E_i, \mu)$ , where each  $A'_i$  is a subset of  $A_i$  on which  $(f_i^{-1})_{\#} \mu_i$  is concentrated. Moreover, from estimate (2.6), we have

$$\|\delta_i\|_{\text{op}} \leq (1 \vee L(f_i)) \|\chi_{A'_i} d/dx\|_{\text{op}} \leq 2.$$

By Lemma 2.16, the map  $\delta_0 := \sum_{i=1}^\infty \chi_{E_i} \delta_i$  is a well-defined element of  $\Upsilon(\mathbb{R}^n, \mu)$ . For each  $i \in \mathbb{N}$  and each  $\delta \in \Upsilon(\mathbb{R}^n, \mu)$ , put

$$\lambda_i := ((f_i^{-1})_{\#} \delta)(\text{id}_{\mathbb{R}}) \quad \text{and} \quad \lambda := \sum_{i=1}^\infty \chi_{E_i} \lambda_i.$$

By an analogous argument as above, we obtain  $\delta = \lambda \delta_0$ . In addition, for each  $i \in \mathbb{N}$ , the set  $E_i$  is bounded and hence

$$\begin{aligned} \|\chi_{E_i} \lambda_i\|_{\mu, \infty} &\leq \|((f_i^{-1})_{\#} \delta)\|_{\text{op}} \|\text{id}_{E_i}\|_{\text{Lip}} \\ &\leq (1 \vee L(f_i^{-1})) \|\delta\|_{\text{op}} (1 \vee \text{diam}(E_i)) \leq 2\|\delta\|_{\text{op}}. \end{aligned}$$

By Part (1) of Lemma 2.8,  $\delta$  is a bounded operator, so  $\lambda \in L^\infty(\mathbb{R}^n, \mu)$ . □

*Proof of Theorem 3.1.* Let  $A$  be a 1-set on which  $\mu$  is concentrated. By Lemma 3.5, we have the  $\mathcal{H}^1$ -decomposition  $A = E \cup F$ , where  $E$  is 1-rectifiable and  $F$  is purely 1-unrectifiable.

If  $\mu(F) > 0$ , then by the locality property and by Lemma 3.7, the set  $\{\chi_F \delta_1, \chi_F \delta_2\}$  is linearly dependent in  $\Upsilon(\mathbb{R}^n, \mu)$ . It follows from Lemma 2.12 that  $\{\delta_1, \delta_2\}$  is also linearly dependent in  $\Upsilon(\mathbb{R}^n, \mu)$ .

If instead  $\mu(F) = 0$ , then  $\mu$  is concentrated on  $E$  and hence  $\mu = \mu|_E$ . Let  $\delta_0$  be the generator of  $\Upsilon(\mathbb{R}^n, \mu|_E)$ . For each  $i = 1, 2$  there is a nonzero function  $\lambda_i \in L^\infty(\mathbb{R}^n, \mu)$  so that  $\chi_E \delta_i = \delta_i = \lambda_i \delta_0$ . We now put

$$\Lambda_1(x) := \chi_{\text{spt}(\lambda_1)} \left[ 1 \wedge \frac{\lambda_2(x)}{\lambda_1(x)} \right] \quad \text{and} \quad \Lambda_2(x) := \chi_{\text{spt}(\lambda_2)} \left[ 1 \wedge \frac{\lambda_1(x)}{\lambda_2(x)} \right].$$

By construction,  $\Lambda_1 \delta_1 - \Lambda_2 \delta_2 = 0$ . Neither  $\Lambda_1$  nor  $\Lambda_2$  is zero, otherwise one of  $\lambda_1$  and  $\lambda_2$  is zero, which is a contradiction. □

### 4. Derivations on 2-dimensional sets

Let  $\mu$  be a Radon measure on  $\mathbb{R}$ . As a consequence of Theorem 3.2, if  $\mu$  is singular to Lebesgue measure, then  $\Upsilon(\mathbb{R}, \mu)$  has rank 0. A similar statement holds true for Radon measures on  $\mathbb{R}^2$ .

**THEOREM 4.1.** *Let  $\mu$  be a Radon measure on  $\mathbb{R}^2$ . If  $\mu$  is singular to Lebesgue measure, then the module  $\Upsilon(\mathbb{R}^2, \mu)$  has rank 1.*

Recall that the proof of Theorem 3.2 consists of selecting open covers for a Lebesgue null set (on which  $\mu$  is concentrated). From these covers, one constructs a sequence of uniformly Lipschitz functions on  $\mathbb{R}$  that converges to the identity.

The proof of Theorem 4.1 follows similar ideas. However, in order to construct analogous functions, we will use recent results of Alberti, Csörnyei, and Preiss about the structure of Lebesgue null sets [ACP05]. This provides covers of such sets with a suitable geometry. In what follows, we refer to Lebesgue

null sets simply as null sets, Lebesgue singular measures as singular measures, and so on.

**4.1. Null sets in  $\mathbb{R}^2$ .** We begin with a few definitions from [ACP05].

DEFINITION 4.2. An  $x_1$ -curve in  $\mathbb{R}^2$  is a graph of the form

$$\gamma^1(f) := \{(t, f(t)) : t \in \mathbb{R}\},$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is 1-Lipschitz. We call  $f$  the (*Lipschitz*) parametrization of  $\gamma = \gamma^1(f)$ . For  $\delta > 0$ , an  $x_1$ -stripe of thickness  $\delta$  is a set of the form

$$\mathcal{N}^1(g; \delta) := \{(t, y) : |y - g(t)| \leq \delta/2\},$$

where  $g : \mathbb{R} \rightarrow \mathbb{R}$  is also 1-Lipschitz. An  $x_2$ -curve and a  $x_2$ -stripe (of thickness  $\delta$ ) are similarly defined.

We now state a covering theorem for null sets in  $\mathbb{R}^2$  [ACP05, Theorem 2]; for an earlier version of this result, see [Mat97]. The case of compact null sets follows from the proof in [ACP05, pp. 4–5].

THEOREM 4.3 (Alberti–Csörnyei–Preiss, 2005). *Let  $E$  be a null set in  $\mathbb{R}^2$ . Then there is a decomposition  $E = E^1 \cup E^2$ , where each set  $E^i$  satisfies the following property: for each  $\varepsilon > 0$ , there are  $x_i$ -stripes  $\{\mathcal{N}^i(f_j^i; \delta_j^i)\}_{j=1}^\infty$  so that their union covers  $E^i$  and so that  $\sum_{j=1}^\infty \delta_j^i < \varepsilon$ .*

*If  $E$  is compact, then for each  $\varepsilon > 0$ , there exist  $N \in \mathbb{N}$  and  $\delta > 0$  so that each  $E^i$  can be covered by  $N$  many  $x_i$ -stripes  $\mathcal{N}^i(f_j^i; \delta)$ , with  $N\delta < \varepsilon$ , and so that each  $f_j^i$  is piecewise-linear with finitely many points of non-differentiability.*

REMARK 4.4. Strictly speaking, the argument in [ACP05] only shows that shows that for each  $\varepsilon > 0$ , the null set  $E$  can be covered by unions of  $x_1$ - and  $x_2$ -stripes  $\{\mathcal{N}_i^{1,\varepsilon}\}_{i=1}^\infty$  and  $\{\mathcal{N}_j^{2,\varepsilon}\}_{j=1}^\infty$ , respectively, with the desired properties. However, one easily obtains the subsets  $E^1$  and  $E^2$  by putting

$$E^1 := \bigcap_{k=1}^\infty \bigcup_{i=1}^\infty \mathcal{N}_i^{1,1/k} \quad \text{and} \quad E^2 := E \setminus E^1.$$

The next theorem will be a crucial step in the proof of Theorem 4.1.

THEOREM 4.5. *Let  $E$  be a compact null set in  $\mathbb{R}^2$ . In addition to the properties given in Theorem 4.3, for  $i = 1, 2$  and for each  $\varepsilon > 0$  the covering  $x_i$ -stripes for  $E^i$  can be chosen to have pairwise-disjoint interiors.*

To prove the theorem, we first require a lemma. It guarantees that  $x_i$ -curves associated to the covering  $x_i$ -stripes can be chosen without transversal crossings. (The basic idea is to take pointwise maxima among the collection of  $x_i$ -curves, and iterate. See Figure 1.)

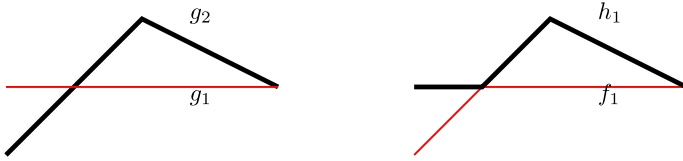


FIGURE 1. Uncrossing  $x_1$ -curves, for  $n = 2$ .

LEMMA 4.6. *Let  $i = 1, 2$ . For each collection of  $x_i$ -curves  $\{\alpha_j\}_{j=1}^N$ , there is a collection of  $x_i$ -curves  $\{\beta_j\}_{j=1}^N$ , with  $\beta_j := \gamma^i(f_j)$ , so that*

$$(4.1) \quad \alpha_1 \cup \dots \cup \alpha_N = \beta_1 \cup \dots \cup \beta_N$$

and so that, for all  $t \in \mathbb{R}$  and all  $1 < j \leq N$ , we have

$$(4.2) \quad f_{j-1}(t) \leq f_j(t).$$

If the curves  $\{\alpha_j\}_{j=1}^N$  are piecewise-linear, then so are the curves  $\{\beta_j\}_{j=1}^N$ .

*Proof.* By symmetry, we assume that  $i = 1$ . We argue by induction, and for  $N = 1$ , the lemma trivially holds with  $\beta_1 = \alpha_1$ .

Fix  $n \in \mathbb{N}$  and let  $\{\alpha_j\}_{j=1}^{n+1}$  be any collection of  $x_1$ -curves. By the induction hypothesis, for  $\{\alpha_j\}_{j=1}^n$  there are  $x_1$ -curves  $\{b_j\}_{j=1}^n$  which satisfy (4.1) and (4.2). For  $j = 1, 2, \dots, n$ , let  $g_j : \mathbb{R} \rightarrow \mathbb{R}$  be the parametrization of  $b_j$ , so  $g_j \leq g_{j+1}$ , and let  $g_{n+1}$  be the parametrization of  $\alpha_{n+1}$ . We now define

$$h_j := \begin{cases} g_1 \vee g_{n+1}, & j = 1, \\ g_j \vee h_{j-1}, & 1 < j \leq n, \end{cases} \quad f_j := \begin{cases} g_1 \wedge g_{n+1}, & j = 1, \\ g_j \wedge h_{j-1}, & 1 < j \leq n, \\ h_n, & j = n + 1. \end{cases}$$

By construction, for each  $x \in \mathbb{R}$  and each  $k$ , there is a unique index  $j$  so that  $g_k(x) = f_j(x)$ . Putting  $\beta_j := \gamma^1(f_j)$ , we see that equation (4.1) holds for the collections of curves  $\{\alpha_j\}_{j=1}^{n+1}$  and  $\{\beta_j\}_{j=1}^{n+1}$ .

For each  $j$ , we have  $g_j \leq g_{j+1}$  by hypothesis and  $h_j \leq h_{j+1}$  by construction, so

$$f_j = g_j \wedge h_{j-1} \leq g_j \leq g_{j+1}, \\ f_j = g_j \wedge h_{j-1} \leq g_j \vee h_{j-1} = h_j.$$

By definition of  $f_{j+1}$ , it follows that inequality (4.2) holds for all  $j$ . □

For Theorem 4.5, the basic idea is that if  $x_i$ -stripes overlap, then by uncrossing the corresponding  $x_i$ -curves, the top stripe can then be “pushed” off the bottom one. See Figure 2.

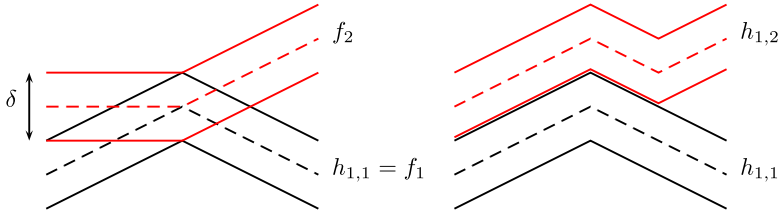


FIGURE 2. Choosing stripes with pairwise-disjoint interiors.

*Proof of Theorem 4.5.* Let  $\varepsilon > 0$  be given. By Theorem 4.3, for each set  $E^i$  there is a  $\delta > 0$  and there are  $x_i$ -stripes  $\{\mathcal{N}^i(g_j^i; \delta)\}_{j=1}^N$  so that their union covers  $E^i$ , so that each  $g_j^i$  is piecewise-linear, and so that  $N\delta < \varepsilon$ . The argument is symmetric, so we assume that  $i = 1$ . We also write  $g_j := g_j^i$ .

By Lemma 4.6, there are 1-Lipschitz functions  $\{f_j\}_{j=1}^N$  so that the  $x_1$ -curves  $\{\gamma^1(g_j)\}_{j=1}^N$  and  $\{\gamma^1(f_j)\}_{j=1}^N$  satisfy equations (4.1) and (4.2). Put

$$h_{1,j} := \begin{cases} f_1, & j = 1, \\ f_j \vee (f_1 + \delta), & j > 1. \end{cases}$$

By construction, for  $j > 1$  none of the  $\mathcal{N}^1(h_{1,j}; \delta)$  meets the interior of the stripe  $\mathcal{N}^1(h_{1,1}; \delta)$ . It also remains that  $h_{1,j} \leq h_{1,j+1}$ . We now claim that

$$(4.3) \quad \bigcup_{j=1}^N \mathcal{N}^1(f_j; \delta) \subset \bigcup_{j=1}^N \mathcal{N}^1(h_{1,j}; \delta).$$

Fix  $(t, y) \in \mathcal{N}^1(f_j; \delta)$ . If  $h_{1,j}(t) = f_j(t)$ , it follows by construction that  $(t, y) \in \bigcup_{j=1}^N \mathcal{N}^1(h_{1,j}; \delta)$ . If instead  $h_{1,j}(t) = f_1(t) + \delta$ , the point  $(t, y)$  satisfies

$$(4.4) \quad \delta/2 < |y - f_1(t)|,$$

$$(4.5) \quad f_j(t) < f_1(t) + \delta = h_{1,j}(t),$$

where again,  $j > 1$ . From inequality (4.5), we obtain

$$y - (f_1(t) + \delta) \leq y - f_j(t) \leq \delta/2.$$

Since  $j > 1$  and  $(t, y) \in \mathcal{N}^1(f_j; \delta)$ , we may further assume by inequality (4.2) that  $y - f_1(t) > \delta/2$ . This in turn gives the estimate

$$-\delta/2 = \delta/2 - \delta < y - f_1(t) - \delta$$

from which we obtain  $(t, y) \in \mathcal{N}^1(h_{1,j}; \delta)$ . This gives the set inclusion (4.3).

We now iterate the argument. For  $k = 1, 2, \dots, N$ , put

$$h_{k,j} := \begin{cases} h_{k-1,j}, & j \leq k, \\ f_k \vee (h_{k,k} + \delta), & j > k. \end{cases}$$

Arguing similarly, we see that inclusion (4.3) holds with  $h_{k,j}$  in place of  $h_{1,j}$  and that, for  $k \leq j \leq N$ , none of the stripes  $\mathcal{N}^1(h_{k,j}; \delta)$  meets the interiors of the previous  $k$  many  $x_1$ -stripes. Thus,  $\{\mathcal{N}^1(h_{j,j}; \delta)\}_{j=1}^N$  is the desired collection of  $x_1$ -stripes for  $E^1$ .  $\square$

Before returning to derivations, we recall a fact [ACP05, Remark 3(ii)] about the geometry of  $E^1$  and  $E^2$ . For completeness, we prove it below.

LEMMA 4.7. *Let  $E$  be a null set in  $\mathbb{R}^2$  and let  $L \in (0, 1)$ . For  $\{i, k\} = \{1, 2\}$ , if  $E^i$  is the subset from Theorem 4.3 and if  $g : \mathbb{R} \rightarrow \mathbb{R}$  is  $L$ -Lipschitz, then*

$$\mathcal{H}^1(E^i \cap \gamma^k(g)) = 0.$$

*Proof.* By Theorem 4.3, for each  $\varepsilon > 0$  there are  $x_i$ -stripes  $\mathcal{N}_j^i := \mathcal{N}^i(f_j^i; \delta_j^i)$ ,  $j \in \mathbb{N}$  so that  $E^i \subset \bigcup_j \mathcal{N}_j^i$  and so that  $\sum_i \delta_j^i < \varepsilon$ . Clearly, the same union of  $x_i$ -stripes also covers the subset  $E^i \cap \gamma^k(g)$ .

For each  $j \in \mathbb{N}$ , let  $p_j$  be the point in  $\gamma^k(g) \cap \mathcal{N}_j^i$  with least  $x_k$ -coordinate. Note that  $\gamma^k(g) \cap \mathcal{N}_j^i$  can be covered by the set  $C(p_j) \cap \mathcal{N}_j^i$ , where  $C(p_j)$  is a one-sided cone with vertex  $p_j$ , direction  $\vec{e}_k$ , and opening angle  $2 \tan^{-1}(1/L)$ . In particular,  $C(p_j) \cap \mathcal{N}_j^i$  has diameter at most  $C\delta_j^i$ , where  $C$  is a positive constant depending only on  $L$ .

In this way we cover  $E^i \cap \gamma^k(g)$  with open sets  $\{O_j\}_{j=1}^\infty$ , each of diameter at most  $2C\delta_j^i$  and hence at most  $2C\varepsilon$ . We now estimate:

$$\mathcal{H}^1(E^i \cap \gamma^k(g)) \leq \limsup_{\varepsilon \rightarrow 0} \sum_{j=1}^\infty \text{diam}(O_j) \leq \limsup_{\varepsilon \rightarrow 0} \sum_{j=1}^\infty 2C\delta_j^i < 2C\varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, the lemma follows.  $\square$

**4.2. Approximating the coordinate functions.** In this section, we prove Theorems 4.1 and 1.2. The proof of Theorem 4.1 consists of two main steps, each of which is a separate lemma below.

LEMMA 4.8. *Let  $E$  be a compact null set in  $\mathbb{R}^2$  and let  $E^1$  and  $E^2$  be as in Theorem 4.5. For  $i \in \{1, 2\}$ , there exist Lipschitz functions  $\{\varphi_{i,j}\}_{j=1}^\infty$  on  $\mathbb{R}^2$  so that*

- (1)  $\varphi_{i,j}$  converges pointwise to  $x_2$ ;
- (2) each  $\varphi_{i,j}$  is 3-Lipschitz and piecewise linear;
- (3) for each  $p \in E^i$  there is a closed neighborhood  $K$  containing  $p$  so that  $\varphi_{i,j}|_K$  depends only on the variable  $x_i$ .

To simplify the proof, we divide it into cases of increasing geometric complexity.

*Proof of Lemma 4.8.* By Theorems 4.3 and 4.5 we have  $E = E^1 \cup E^2$ , where each  $E^i$  has the following properties: given  $j \in \mathbb{N}$ , there are numbers  $N \in \mathbb{N}$  and  $\delta > 0$  and  $x_i$ -stripes  $\{\mathcal{N}^i(f_{l,j}; \delta_j)\}_{l=1}^N$  so that

- (1')  $E^i \subset \bigcup_{l=1}^{N_j} \mathcal{N}^i(f_{l,j}; \delta)$  and  $N\delta < 2^{-j}$ ;
- (2') for  $l \neq l'$ , the interiors of  $\mathcal{N}^i(f_{l,j}; \delta)$  and  $\mathcal{N}^i(f_{l',j}; \delta)$  are disjoint;
- (3') each  $f_{l,j}$  is a piecewise-linear function, with finitely many points of non-differentiability.

For simplicity, let  $i = 1$  and  $E \subset [0, 1]^2$ . For each  $l$ , put  $\mathcal{N}_l^1 := \mathcal{N}^1(f_{l,j}; \delta)$ . To emphasize the dependence on  $j$ , put  $\mathcal{M}_j := \mathbb{R}^2 \setminus \bigcup_l \mathcal{N}_l^1$ . Now put

$$\varphi_{1,j}(p) := \int_{\{p_1\} \times [0, p_2]} \chi_{\mathcal{M}_j} d\mathcal{H}^1,$$

where  $p = (p_1, p_2) \in \mathbb{R}^2$ . Property (1') then implies that

$$0 \leq p_2 - \varphi_{1,j}(p_1) \leq \sum_{j=1}^N \int_{\{p_1\} \times \mathbb{R}} \chi_{\mathcal{N}_l^1} d\mathcal{H}^1 = N\delta < 2^{-j}$$

from which we obtain Property (1).

CLAIM 4.9. *The sequence  $\{\varphi_{1,j}\}_{j=1}^\infty$  is uniformly 3-Lipschitz.*

Let  $p = (p_1, p_2)$  and  $q = (q_1, q_2)$  be points in  $\mathbb{R}^2$ . We argue by cases.

Case A:  $p$  and  $q$  lie on the same vertical line. By construction,  $\varphi_{1,j}$  is 1-Lipschitz in the variable  $x_2$ . The claim then follows from

$$(4.6) \quad |\varphi_{1,j}(p) - \varphi_{1,j}(q)| \leq |p_2 - q_2| \leq |p - q|.$$

Case B:  $p$  and  $q$  lie on the same stripe  $\mathcal{N}_l^1$ . Since  $\varphi_{1,j}$  is constant on each of the vertical segments  $\mathcal{N}_l^1 \cap (\{p_1\} \times \mathbb{R})$  and  $\mathcal{N}_l^1 \cap (\{q_1\} \times \mathbb{R})$ , the corresponding (lower) endpoints  $p' = (p_1, f_{l,j}(p_1) - \delta/2)$  and  $q' = (q_1, f_{l,j}(q_1) - \delta/2)$  satisfy

$$(4.7) \quad \varphi_{1,j}(p) = \varphi_{1,j}(p') \quad \text{and} \quad \varphi_{1,j}(q) = \varphi_{1,j}(q').$$

By Property (2'), the interiors of  $\{\mathcal{N}_l^1\}_{l=1}^N$  are pairwise disjoint. A ray with initial point  $p'$  and direction  $-\vec{e}_2$  goes through  $l - 1$  stripes of thickness  $\delta$ , so

$$(4.8) \quad \varphi_{1,j}(p') = f_{l,j}(p_1) - (l - 1)\delta \quad \text{and} \quad \varphi_{1,j}(q') = f_{l,j}(q_1) - (l - 1)\delta.$$

Since  $L(f_{l,j}) \leq 1$ , the claim follows from equations (4.7) and (4.8):

$$(4.9) \quad \begin{cases} |\varphi_{1,j}(p) - \varphi_{1,j}(q)| = |\varphi_{1,j}(p') - \varphi_{1,j}(q')| \\ \qquad \qquad \qquad = |f_{l,j}(p_1) - f_{l,j}(q_1)| \leq |p_1 - q_1| \leq |p - q|. \end{cases}$$

Case C:  $p, q \notin \mathcal{M}_j$ , and both points lie between the same pair of stripes. The argument is similar to Case B. If  $p$  and  $q$  lie between  $\mathcal{N}_{l+1}^1$  and  $\mathcal{N}_l^1$ , for some  $l$ , then put  $p'' := (p_1, f_{l,j}(p_1) + \delta/2)$  and  $q'' := (q_1, f_{l,j}(q_1) + \delta/2)$ . From the observation

$$\varphi_{1,j}(p) = p''_2 - l\delta \quad \text{and} \quad \varphi_{1,j}(q) = q''_2 - l\delta,$$

we obtain estimates (4.8) and (4.9) as before.

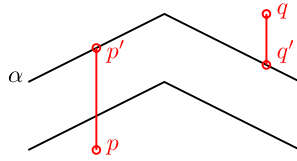


FIGURE 3. A possible configuration for  $p, q, p',$  and  $q'$ .

Case D:  $p$  and  $q$  are arbitrary. Suppose that  $p$  and  $q$  are separated by a boundary curve  $\alpha$  of some stripe  $\mathcal{N}_l^1$ . See Figure 3. Without loss of generality, let

$$\alpha = \{(x_1, x_2) : x_2 = f_{l,j}(x_1) + \delta/2\}$$

and moreover, assume  $p$  lies below  $\alpha$  and  $q$  lies above  $\alpha$ :

$$\begin{cases} p'_2 := f_{l,j}(p_1) + \delta/2 \geq p_2, \\ q'_2 := f_{l,j}(q_1) + \delta/2 \leq q_2. \end{cases}$$

Note that  $p' := (p_1, p'_2)$  and  $q' := (q_1, q'_2)$  lie on the same vertical lines as  $p$  and  $q$ , respectively, and both  $p'$  and  $q'$  lie on  $\alpha$ .

Using the Triangle Inequality and inequalities (4.6) and (4.9),

$$(4.10) \quad \left\{ \begin{aligned} |\varphi_{1,j}(p) - \varphi_{1,j}(q)| &\leq |\varphi_{1,j}(p) - \varphi_{1,j}(p')| + |\varphi_{1,j}(p') - \varphi_{1,j}(q')| \\ &\quad + |\varphi_{1,j}(q') - \varphi_{1,j}(q)| \\ &\leq |p_2 - p'_2| + |f_{l,j}(p_1) - f_{l,j}(q_1)| + |q_2 - q'_2| \\ &\leq |p_2 - p'_2| + |p_1 - q_1| + |q_2 - q'_2|. \end{aligned} \right.$$

CLAIM 4.10. For all choices of  $p_2 \leq p'_2$  and  $q'_2 \leq q_2$ , we have

$$|p_2 - p'_2| + |q'_2 - q_2| \leq |p_2 - q_2| + |p'_2 - q'_2|.$$

The argument is combinatorial, so we further proceed by sub-cases. Consider intervals  $I_p := [p'_2, p_2]$  and  $I_q := [q_2, q'_2]$ .

Subcase D1:  $I_p$  and  $I_q$  are disjoint. Relative to  $p_2 \leq q_2$  or  $q_2 \leq p_2$ , the union  $[p_2, p'_2] \cup [q'_2, q_2]$  lies in either  $[p_2, q_2]$  or  $[q'_2, p'_2]$ , so the claim follows from

$$|p_2 - p'_2| + |q'_2 - q_2| \leq |p_2 - q_2| \vee |p'_2 - q'_2| \leq |p_2 - q_2| + |p'_2 - q'_2|.$$

Subcase D2:  $I_p \subset I_q$ . Under this set inclusion, we have the identities

$$I_q = [p'_2, q'_2] \cup [p_2, q_2], \quad I_p = [p'_2, q'_2] \cap [p_2, q_2]$$

from which we obtain the claim, also as an identity:

$$\begin{aligned} |p_2 - p'_2| + |q'_2 - q_2| &= m_1(I_p) + m_1(I_q) \\ &= m_1([p'_2, q'_2] \cup [p_2, q_2]) + m_1([p'_2, q'_2] \cap [p_2, q_2]) \\ &= |p_2 - q_2| + |p'_2 - q'_2|. \end{aligned}$$

By symmetry, the claim also holds for  $I_q \subset I_p$ .



*Subcase D3:*  $I_p \not\subset I_q$ ,  $I_p \not\subset I_p$ , and  $I_p \cap I_q \neq \emptyset$ . Of the intervals  $[q_2, p_2]$  and  $[p'_2, q'_2]$ , one is  $I_p \cup I_q$  and the other is  $I_p \cap I_q$ . We then compute

$$\begin{aligned} |p_2 - p'_2| + |q'_2 - q_2| &= m_1(I_p) + m_1(I_q) \\ &= m_1(I_p \cup I_q) + m_1(I_p \cap I_q) = |p_2 - q_2| + |q'_2 - p'_2|. \end{aligned}$$

This proves Claim 4.10. Using this and (4.10), Claim 4.9 follows from

$$\begin{aligned} |\varphi_{1,j}(p) - \varphi_{1,j}(q)| &\leq |p_2 - p'_2| + |p_1 - q_1| + |q_2 - q'_2| \\ &\leq |p_2 - q_2| + |p_1 - q_1| + |p'_2 - q'_2| \\ &\leq |p_2 - q_2| + 2|p_1 - q_1| \leq 3|p - q|. \end{aligned}$$

Let  $O$  be any connected component of  $\mathcal{M}_j$ . From the argument in Case C, the restriction  $\varphi_{1,j}|_O$  is a translate of  $x_2$ , so  $\varphi_{i,j}|_O$  is piecewise linear. On the other hand, by Property (3'), each  $f_{i,j}$  is piecewise-linear, so by Case B, the restriction of  $\varphi_{i,j}$  to any  $x_1$ -stripe  $\mathcal{N}_j^1$  is also piecewise linear. Both types of sets  $O$  and  $\mathcal{N}_j^1$  partition  $\mathbb{R}^2$ , so  $\varphi_{1,j}$  must be piecewise-linear on all of  $\mathbb{R}^2$ . This gives Property (2).

Lastly, recall from Case B that for all stripes  $\mathcal{N}_l^1$ , we have

$$(4.11) \quad \varphi_{1,j}(p) = f_{l,j}(p_1) + (l - 1)\delta$$

for all  $p = (p_1, p_2) \in \mathcal{N}_l^1$ . This gives Property (3): for all  $p \in E^1$ , there is a closed neighborhood  $K$  containing  $p$  so that  $\varphi_{1,j}|_K$  depends only on  $x_1$ .  $\square$

**4.3. Linearly independent derivations on  $\mathbb{R}^2$ .** Using the approximating sequence from Lemma 4.8, we proceed to a linear dependence relation for derivations (with respect to singular measures).

LEMMA 4.11. *Let  $\mu$  be a singular Radon measure on  $\mathbb{R}^2$ , and let  $E$  be a subset on which  $\mu$  is concentrated. There exist  $F^1, F^2 \subset \mathbb{R}^2$  and  $g_1, g_2 \in L^\infty(\mathbb{R}^2, \mu)$  so that  $E = F^1 \cup F^2$  and so that, for all  $\delta \in \Upsilon(\mathbb{R}^2, \mu)$ ,*

$$(4.12) \quad \begin{cases} \delta x_2 = g_1 \delta x_1, & \mu\text{-a.e. on } F_1, \\ \delta x_1 = g_2 \delta x_2, & \mu\text{-a.e. on } F_2. \end{cases}$$

*Proof.* We proceed by cases.

Case 1: assume that  $E$  is compact, so by Lemma 4.8 there exist piecewise-linear, 3-Lipschitz functions  $\{\varphi_{1,j}\}_{j=1}^\infty$  and  $\{\varphi_{2,j}\}_{j=1}^\infty$  that satisfy Properties (1) and (2) of Lemma 4.8. Assume again that  $i = 1$  and put  $\varphi_j := \varphi_{1,j}$ .

From the proof of Lemma 4.8, each  $\varphi_j$  is formed from a covering of  $E^1$  by  $x_1$ -stripes  $\{\mathcal{N}^1(f_l; \delta)\}_{l=1}^N$  with  $N\delta < 2^{-j}$ . Moreover, the set of non-differentiability of  $\varphi_j$  consists of two parts:

- (1) a finite union of  $x_1$ -stripe boundaries, written  $\Gamma := \bigcup_l \partial \mathcal{N}^1(f_l; \delta)$ ;
- (2) a finite union of vertical line segments, written  $\ell := \bigcup_k \ell_k$ , where each segment  $\ell_l$  projects to a point of non-differentiability of  $f_l$ .

Let  $\delta_1, \delta_2 \in \Upsilon(\mathbb{R}^2, \mu)$  be arbitrary and write  $\mathcal{N}_l^1 := \mathcal{N}^1(f_l; \delta)$ . Several reductions follow, which we state below as claims.

CLAIM 4.12. *Lemma 4.11 is true for  $\mu(\mathbb{R}^2 \setminus \ell) = 0$ .*

We may assume that  $\mu(\ell) > 0$ , so  $\chi_\ell \neq 0$ . Since  $\mathcal{H}^1(E^1 \cap \ell_k) = 0$  holds for each  $k$ , it follows from Lemma 4.7 that  $E^1 \cap \ell$  is purely 1-unrectifiable. Moreover, by Theorem 3.7, we have  $\chi_\ell \delta = 0$ , for all  $\delta \in \Upsilon(\mathbb{R}^2, \mu)$ , from which we obtain  $\chi_\ell(\delta_1 + \delta_2) = 0$ . This proves the claim.

CLAIM 4.13. *Lemma 4.11 is true for  $\mu(\mathbb{R}^2 \setminus \Gamma) = 0$ .*

For each  $j \in \mathbb{N}$ , let  $S_l := \partial \mathcal{N}^1(f_l; \delta)$ , so  $\varphi_j$  is non-differentiable on  $S_l$ . In particular, every such  $S_l$  is a Lipschitz curve, hence 1-rectifiable, so by Theorem 3.8 the module  $\Upsilon(S_l, \mu)$  has rank-1. This means that the set  $\{\chi_{S_l} \delta_i\}_{i=1}^2$  is linearly dependent, and by Lemma 2.12, so is  $\{\chi_\Gamma \delta_i\}_{i=1}^2$ . The claim follows.

Without loss of generality, assume that  $E^1 = E^1 \setminus (\ell \cup \Gamma)$ . By Theorem 4.5,  $f_l$  is piecewise-linear and  $f'_l(\mathbb{R} \setminus \text{proj}_{\mathbb{R} \times 0}(\ell))$  is a finite set in  $\mathbb{R}$ . We may then cover  $E^1$  by a finite union of sets of the form

$$\mathcal{N}_l^1(\xi) := \{p \in \mathcal{N}_l^1 : f'_l(p_1) = \xi\}.$$

Note that the restriction of  $\varphi_{1,j}$  to the interior of  $\mathcal{N}_l^1(\xi)$  is linear and hence smooth. From formulas (2.10) and (4.11), we then obtain the  $\mu$ -a.e. identity

$$\chi_{\mathcal{N}_l^1(\xi)} \delta \varphi_{1,j} = \chi_{\mathcal{N}_l^1(\xi)} f'_l \delta x_1,$$

for all  $l$  and all  $\xi$ . Putting  $G_j^1 := \sum_{l=1}^N \chi_{\mathcal{N}_l^1} f'_l$ , we further obtain

$$\delta \varphi_{1,j} = G_j^1 \delta x_1$$

$\mu$ -a.e. on  $E^1$ . Clearly  $\|G_j^1\|_{\infty, \mu} \leq 1$  holds for each  $j$ , and by the Banach–Alaoglu theorem there is a weak- $*$  convergent subsequence  $\{G_{j_m}^1\}_{m=1}^\infty$  in  $L^\infty(\mathbb{R}^2, \mu)$ ; let  $G^1$  be the weak- $*$  limit. For all  $\delta \in \Upsilon(\mathbb{R}^2, \mu)$ , we also have

$$G_{j_m}^1 \delta x_1 \xrightarrow{*} G^1 \delta x_1 \quad \text{in } L^\infty(\mathbb{R}^2, \mu).$$

However, by Property (1) of Lemma 4.8, we have  $\varphi_{1,j} \xrightarrow{*} x_2$  in  $\text{Lip}_b(\mathbb{R}^2)$  and by continuity of  $\delta$ , we obtain  $\delta \varphi_{1,j} \xrightarrow{*} \delta x_2$  in  $L^\infty(\mathbb{R}^2, \mu)$  for all  $\delta$ . Putting  $F^1 := E^1$  and  $g_1 := G^1$ , formula (4.12) follows from uniqueness of limits.

Case 2: For non-compact  $E$ , consider subsets in  $\mathbb{R}^2$  of the form

$$Q_{ab} := [a, a + 1) \times [b, b + 1), \quad a, b \in \mathbb{Z}.$$

Indeed, each  $Q = Q_{ab}$  is bounded and therefore has finite  $\mu$ -measure. From the Borel regularity of  $\mu$ , there are sequences of compact sets  $\{K_c\}_{c=1}^\infty$  so that

$$\lim_{c \rightarrow \infty} \mu((E \cap Q) \setminus K_c) = 0.$$

Since  $K_c$  is compact, there exist subsets  $F_c^1, F_c^2$  in  $\mathbb{R}^2$  so that

$$K_c = F_c^1 \cup F_c^2$$

and there exist functions  $G_c^1 \in L^\infty(\mathbb{R}^2, \mu)$  so that the identity

$$(4.13) \quad \chi_{K_c} \delta x_2 = \chi_{K_c} G_c^1 \delta x_1$$

holds  $\mu$ -a.e. on  $F_c^1$ , for all  $\delta \in \Upsilon(\mathbb{R}^2, \mu)$ . We now put

$$F^i := \bigcup_{abc} F_c^i, \quad \text{for } i = 1, 2,$$

$$E_c := \begin{cases} K_1, & c = 1, \\ K_c \setminus K_{c-1}, & c \geq 2, \end{cases}$$

$$g_1 := \sum_c \chi_{E_c} G_c^1.$$

Clearly,  $E = \bigcup_c E_c$  and  $E = F^1 \cup F^2$ . From (4.13) and from the definitions of  $E_c$  and  $g_1$ , we also obtain (4.12) for  $i = 1$ . The lemma follows.  $\square$

*Proof of Theorem 4.1.* If  $\mu$  is a singular Radon measure on  $\mathbb{R}^2$ , then let  $E$  be a null set on which  $\mu$  is concentrated. Let  $\delta_1, \delta_2 \in \Upsilon(\mathbb{R}^2, \mu)$  be arbitrary.

By Theorem 4.11, there are subsets  $F^1$  and  $F^2$  so that  $E = F^1 \cup F^2$  and there are functions  $g_1, g_2 \in L^\infty(\mathbb{R}^2, \mu)$  so that the system of equations (4.12) holds  $\mu$ -a.e. for  $\delta_1$  and for  $\delta_2$ . Now consider

$$(4.14) \quad \begin{cases} \lambda_1 := \chi_{F^1} \delta_2 x_1 + \chi_{F^2} \delta_2 x_2, \\ \lambda_2 := \chi_{F^1} \delta_1 x_1 + \chi_{F^2} \delta_1 x_2. \end{cases}$$

We first observe that, for  $\mu$ -a.e.  $p \in F^1$ , we have the identities

$$\begin{aligned} \chi_{F^1}(\lambda_1 \delta_1 - \lambda_2 \delta_2)x_1 &= \chi_{F^1}(\delta_2 x_1 \delta_1 x_1 - \delta_1 x_1 \delta_2 x_1) = 0, \\ \chi_{F^1}(\lambda_1 \delta_1 - \lambda_2 \delta_2)x_2 &= \chi_{F^1}(\delta_2 x_1 \delta_1 x_2 - \delta_1 x_1 \delta_2 x_2) \\ &= \chi_{F^1}(\delta_2 x_1 g_1 \delta_1 x_1 - \delta_1 x_1 g_1 \delta_2 x_1) = 0. \end{aligned}$$

Arguing similarly for  $F^2$ , we see that  $\lambda_1 \delta_1 - \lambda_2 \delta_2$  annihilates both  $x_1$  and  $x_2$ . By Lemma 2.19, it follows that  $\lambda_1 \delta_1 - \lambda_2 \delta_2 = 0$ .

Now suppose that both  $\lambda_1$  and  $\lambda_2$  are zero. By Equations (4.12) and (4.14), the four functions  $\delta_1 x_1, \delta_1 x_2, \delta_2 x_1,$  and  $\delta_2 x_2$  would all be zero, which implies that  $\delta_1 = \delta_2 = 0$ . This is a contradiction, so either  $\lambda_1 \neq 0$  or  $\lambda_2 \neq 0$ , and therefore the set  $\{\delta_1, \delta_2\}$  is linearly dependent in  $\Upsilon(\mathbb{R}^2, \mu)$ .  $\square$

We now prove the rigidity theorem for derivations.

*Proof of Theorem 1.2.* We argue by contradiction. For  $k = 2$ , let  $\mu$  be a Radon measure on  $\mathbb{R}^2$ . If  $\mu_S \neq 0$ , then let  $A$  be a null set on which  $\mu_S$  is concentrated. For any two derivations  $\delta_1, \delta_2$  in  $\Upsilon(\mathbb{R}^2, \mu)$ , the restrictions  $\chi_A \delta_1, \chi_A \delta_2$  lie in  $\Upsilon(\mathbb{R}^2, \mu_S)$ , by Theorem 2.14, and by Theorem 4.1, there exist functions  $\lambda_1, \lambda_2 \in L^\infty(\mathbb{R}^2, \nu_S)$ , not both zero, so that

$$\lambda_1(\chi_A \delta_1) + \lambda_2(\chi_A \delta_2) = 0.$$

So from the choice of scalars  $\Lambda_i = \chi_A \lambda_i, i = 1, 2$ , we see that  $\{\delta_1, \delta_2\}$  is a linearly dependent set in  $\Upsilon(\mathbb{R}^2, \mu)$ .

A similar argument holds for  $k = 1$ , with Lemma 3.2 in place of Theorem 4.1.  $\square$

In the previous section, we studied Radon measures concentrated on 1-sets. From Lemma 3.2 we deduced Theorem 3.1, which asserts that the rank of such modules of derivations is at most one. As an application of Theorem 4.1, we now deduce the following result about derivations on 2-sets in  $\mathbb{R}^n$ .

PROPOSITION 4.14. *Let  $\mu$  be a Radon measure on  $\mathbb{R}^n$ .*

- (1) *If  $\mu$  is concentrated on a 2-set  $A$ , then  $\Upsilon(\mathbb{R}^n, \mu)$  has rank at most 2.*
- (2) *If  $A$  contains a purely 2-unrectifiable subset of positive  $\mathcal{H}^2$ -measure, then  $\Upsilon(\mathbb{R}^n, \mu)$  has rank at most 1.*

*Sketch of Proof.* By Lemma 3.5, we have  $A = E \cup F$ , where  $E$  is 2-rectifiable and  $F$  is purely 2-unrectifiable. It is easy to see that derivations restricted to  $E$  are pushforwards of derivations on  $\mathbb{R}^2$ ; by Theorem 1.2, the rank of  $\Upsilon(E, \mu)$  is therefore at most 2.

For the purely 2-unrectifiable part, by Theorem 3.6 the image of  $F$  under a generic projection is a null set in  $\mathbb{R}^2$ . This produces linear dependence relations between derivations as in Lemma 4.11. Arguing similarly as in the proof of Lemma 3.7, these linear relations can be “pulled back” to  $\mathbb{R}^n$ . By choosing scalars  $\lambda_i \in L^\infty(\mathbb{R}^n, \mu)$  similarly to those in the proof of Theorem 4.1, we conclude that  $\Upsilon(F, \mu)$  must have rank at most 1.  $\square$

## 5. Derivations on spaces supporting a Poincaré inequality

We now turn to the class of metric measure spaces which admit a Poincaré inequality in a suitably weak sense. These were first considered in the work of Heinonen and Koskela in their study of quasiconformal mappings on metric spaces [HK98], and it is known that such spaces possess good geometric properties, such as quasi-convexity [DS90]. As stated before in the Introduction, Cheeger has also proven an analogue of the Rademacher theorem on such spaces [Che99].

In what follows, we discuss facts about Sobolev spaces on metric measure spaces that support a  $p$ -Poincaré inequality and then construct derivations on such spaces with respect to the underlying measure. As an application, we also prove the 2-dimensional case of Cheeger’s conjecture about the structure of such measures.

**5.1. Calculus on metric spaces.** As before,  $(X, \rho, \mu)$  denotes a metric measure space. Here and in the remainder of the section we assume that  $\mu$  is *doubling*, as defined in Equation (1.1).

REMARK 5.1. If  $X$  admits a doubling measure then the metric on  $X$  is also *doubling*, that is: there exists  $N \in \mathbb{N}$  so that every ball  $B$  in  $X$  can be covered by  $N$  balls of half the radius of  $B$ .

By iterating the doubling property above, we see that every ball  $B$  in  $X$  is a separable metric space. It follows from Part (2) of Lemma 2.8 that a linear operator  $\delta : \text{Lip}_b(B) \rightarrow L^\infty(B, \mu)$  is weak-\* continuous on bounded sets if and only if it is sequentially weak-\* continuous.

Following [HK98], we now introduce the notion of an upper gradient.

DEFINITION 5.2. Let  $u : X \rightarrow \mathbb{R}$  be Borel. A Borel function  $g : X \rightarrow [0, \infty]$  is an *upper gradient* for  $u$  if the inequality

$$(5.1) \quad |u(y) - u(x)| \leq \int_a^b g(\gamma(t)) dt$$

holds for all rectifiable curves  $\gamma : [a, b] \rightarrow X$  which are parametrized by arc-length and which satisfy  $x = \gamma(a)$  and  $y = \gamma(b)$ .

DEFINITION 5.3. We say that  $(X, \rho, \mu)$  supports a *p-Poincaré inequality* if there exist  $\Lambda \geq 1$  and  $C > 0$  so that for all balls  $B$  in  $X$  and all  $u \in L^1_{\text{loc}}(X, \mu)$ , we have

$$(5.2) \quad \int_B |u - u_B| d\mu \leq C \text{diam}(B) \left( \int_{\Lambda B} g^p d\mu \right)^{1/p},$$

whenever  $g$  is an upper gradient of  $u$ . As a shorthand, we call  $(X, \rho, \mu)$  a *p-PI space* if  $\mu$  is doubling and if  $(X, \rho, \mu)$  admits a *p-Poincaré inequality*.

Following [Che99, Section 2], for  $u \in L^p(X, \mu)$  we now define

$$(5.3) \quad \|u\|_{1,p} := \|u\|_{\mu,p} + \inf_{\{g_i\}} \liminf_{i \rightarrow \infty} \|g_i\|_{\mu,p},$$

where the infimum is taken over all sequences  $\{u_i\}_{i=1}^\infty$  in  $L^p(X, \mu)$  so that  $u_i \rightarrow u$  in  $L^p$ -norm and so that  $g_i$  is an upper gradient for  $u_i$ , for each  $i \in \mathbb{N}$ .

DEFINITION 5.4. The Sobolev space  $H^{1,p}(X, \mu)$  is the subspace of functions  $u \in L^p(X, \mu)$  for which  $\|u\|_{1,p} < \infty$ .

Indeed,  $\|\cdot\|_{1,p}$  is a norm on  $H^{1,p}(X, \mu)$ , but more is true; the next theorem summarizes [Che99, Theorems 2.7, 2.10, 2.18, 4.48].

THEOREM 5.5 (Cheeger, 1999). *The space  $(H^{1,p}(X, \mu), \|\cdot\|_{1,p})$  is a Banach space. Moreover, it enjoys the following properties:*

- (1) *if  $p > 1$  and if  $(X, \rho, \mu)$  is a  $p$ -PI space, then  $H^{1,p}(X, \mu)$  is reflexive;*
- (2) *for each  $f \in H^{1,p}(X, \mu)$ , there exists  $g_f \in L^p(X, \mu)$  so that*

$$\|f\|_{1,p} = \|f\|_{\mu,p} + \|g_f\|_{\mu,p}.$$

*If  $g$  is an upper gradient of  $f$ , then  $g_f \leq g$  holds  $\mu$ -a.e. on  $X$ .*

We call  $g_f$  the *minimal (generalized) upper gradient of  $f$* .

REMARK 5.6. Shanmugalingam has defined *Newtonian spaces*  $N^{1,p}(X, \mu)$  that are isometrically equivalent to the spaces  $H^{1,p}(X, \mu)$ , for  $p \in (1, \infty)$  [Sha00, Theorem 4.10]. Her approach uses the notion of *weak upper gradients*, and the spaces  $N^{1,p}(X, \mu)$  are norm completions of functions in  $L^p(X, \mu)$  which admit weak upper gradients in  $L^p(X, \mu)$ . Moreover, for  $p \in (1, \infty)$  we have

$$W^{1,p}(\mathbb{R}^n) \cong H^{1,p}(\mathbb{R}^n, m_n) \cong N^{1,p}(\mathbb{R}^n, m_n)$$

For further details, see [Sha00], [Hei01, Chapters 5–6], and [Hei07].

For  $f \in \text{Lip}(X)$ , the constant  $L(f)$  is always an upper gradient for  $f$  but rarely the minimal generalized upper gradient. We instead consider the upper and lower pointwise Lipschitz constants of  $f$ , defined as

$$(5.4) \quad \begin{cases} \text{Lip}[f](x) := \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{\rho(x,y)}, \\ \text{lip}[f](x) := \liminf_{r \rightarrow 0} \sup_{\rho(x,y) \leq r} \frac{|f(x) - f(y)|}{r}, \end{cases}$$

respectively. It is clear from formula (5.4) that, for all  $x \in X$ ,

$$(5.5) \quad \text{lip}[f](x) \leq \text{Lip}[f](x) \leq L(f).$$

Semmes has shown that  $\text{Lip}[f]$  and  $\text{lip}[f]$  are upper gradients of  $f$  [Sem95, Lemma 1.20]. Moreover, for  $p$ -PI spaces  $X$ , we have the  $\mu$ -a.e. identities [Che99, Theorem 6.1]

$$(5.6) \quad g_f(x) = \text{Lip}[f](x) = \text{lip}[f](x).$$

**5.2. Derivations from differentiability.** We now state a Rademacher-type theorem for  $p$ -PI spaces. To fix notation, for  $f : X \rightarrow \mathbb{R}^k$  and  $a \in \mathbb{R}^k$ , we write  $a \bullet f := \sum_i a_i f_i$  for their (pointwise) inner product.

THEOREM 5.7 (Cheeger, 1999). *Let  $(X, \rho, \mu)$  be a  $p$ -PI space. There exists  $N \in \mathbb{N}$  and a  $\mu$ -measurable decomposition  $\{X^n\}_{n=1}^\infty$  with the following properties: for each  $n \in \mathbb{N}$ , there exist  $k = k(n) \in \mathbb{N}$ ,  $1 \leq k \leq N$  and  $\xi^n \in \text{Lip}(X; \mathbb{R}^k)$  so that*

(1) *There exists  $K = K(n) > 0$  so that for all  $x \in X^n$ ,*

$$(5.7) \quad K \leq \inf \{ \text{Lip}[a \bullet \xi^n](x) : a \in \mathbb{R}^k, |a| = 1 \}.$$

(2) *For each  $f \in \text{Lip}(B)$ , there is a unique map  $D^n f : X \rightarrow \mathbb{R}^k$ , with components in  $L^\infty(X^n, \mu)$ , so that for  $\mu$ -a.e.  $x \in X^n$ ,*

$$(5.8) \quad \limsup_{y \rightarrow x} \left| \frac{f(y) - f(x) - D^n f(x) \bullet (\xi^n(y) - \xi^n(x))}{\rho(x,y)} \right| = 0.$$

Put  $\xi^n := (\xi_1^n, \dots, \xi_k^n)$ . To mimic the terminology of manifolds, we refer to  $\xi_i^n$  as (*Cheeger*) *coordinates* on  $X^n$ , to  $(\xi^n, X^n)$  as (*Cheeger*) *coordinate charts* on  $X$ , and to  $D^n f$  as the (*Cheeger*) *differential* of  $f$  on  $X^n$ .

REMARK 5.8. Inequality (5.7) is a tacit consequence of the proof of [Che99, Theorem 4.38] and is used to show  $\xi_i^n \in L^\infty(X^n, \mu)$ . In fact, the measurable decomposition is chosen so that it is valid on each  $X^n$ .

Equation (5.8) is a reformulation of Part (iii) of [Che99, Theorem 4.38]. In the notation of [Che99],

$$D^\alpha f(z) = (b_1^\alpha(z; f), \dots, b_k^\alpha(z; f)).$$

On  $\mathbb{R}^n$ , the coordinate  $x_i$  is precisely the Lipschitz function whose gradient is the vector  $e_i$ . The next corollary is an analogue of this fact for  $p$ -PI spaces, and it follows directly from the uniqueness of Cheeger differentials.

COROLLARY 5.9. *Assuming the hypotheses of Theorem 5.7, let  $n \in \mathbb{N}$  and let  $1 \leq i \leq k(n)$ . Then  $D^n \xi_i^n(x) = e_i$  holds for  $\mu$ -a.e.  $x \in X^n$ .*

For a  $p$ -PI space  $(X, \rho, \mu)$ , Cheeger and Weaver have shown that  $\Upsilon(X, \mu)$  is nontrivial [Wea00, Theorem 43]. However, their argument is non-constructive, so we will prove a quantitative form of their theorem below.

THEOREM 5.10. *Let  $(X, \rho, \mu)$  be a  $p$ -PI space. For  $f \in \text{Lip}(X)$  and  $n \in \mathbb{N}$ , let  $D^n f : X^n \rightarrow \mathbb{R}^k$  be as in Theorem 5.7. For  $1 \leq i \leq k$ , the linear operator  $\delta_i^n : \text{Lip}(X^n) \rightarrow L^\infty(X^n, \mu)$  given by*

$$(5.9) \quad \delta_i^n f := D^n f \bullet e_i$$

*is a derivation in  $\Upsilon(X^n, \mu)$ .*

To prove Theorem 5.10, we require two lemmas. The first is similar to the  $L^\infty$ -regularity argument in [Che99, p. 457].

LEMMA 5.11. *Let  $(X, \rho, \mu)$  be a  $p$ -PI space. For each  $n \in \mathbb{N}$ , there exists  $C = C(n) > 0$  so that for all  $f \in \text{Lip}(X)$  and  $\mu$ -a.e.  $x \in X^n$ , we have*

$$|\delta_f^n(x)| \leq C \text{Lip}[f](x).$$

*Proof.* Fix  $x \in X^n$  and put  $a_0 = D^n f(x)/|D^n f(x)|$ . By Part (2) of Theorem 5.7, we have

$$\begin{aligned} \text{Lip}[a_0 \bullet \xi^n](x) &= \frac{1}{|D^n f(x)|} \limsup_{y \rightarrow x} \frac{|D^n f(x) \bullet (\xi^n(y) - \xi^n(x))|}{\rho(x, y)} \\ &= \frac{1}{|D^n f(x)|} \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{\rho(x, y)} = \frac{\text{Lip}[f](x)}{|D^n f(x)|}. \end{aligned}$$

By Theorem 5.7, there exists  $K = K(n) > 0$  so that for  $\mu$ -a.e.  $x \in X^n$ , inequality (5.7) holds for all  $|a| = 1$ . In particular, the vector  $a_0$  has norm 1, so from the above identity, we obtain the lemma with  $C = 1/K$ . □

LEMMA 5.12. *Let  $p > 1$ , let  $n \in \mathbb{N}$ , and let  $\{f_a\}_{a=1}^\infty$  be a sequence in  $\text{Lip}_b(X^n)$  so that  $f_a \xrightarrow{*} 0$ . If  $B$  is a ball in  $X$  so that  $\mu(B \cap X^n) > 0$ , then  $f_a|_B \rightarrow 0$  in  $H^{1,p}(B \cap X^n, \mu)$ .*

*Proof.* Let  $B$  be a ball in  $X$  so that  $\mu(B \cap X^n) > 0$ . In what follows, we write

$$B^n := B \cap X^n$$

and for each  $a \in \mathbb{N}$ , we write  $f_a = f_a|_B$ .

Let  $\{f_{a_b}\}_{b=1}^\infty$  be any subsequence of  $\{f_a\}_{a=1}^\infty$ . Note that  $\{f_{a_b}\}_{b=1}^\infty$  is a bounded set in  $H^{1,p}(B^n, \mu)$  because by Equations (5.5) and (5.6), we have

$$\begin{aligned} \|f\|_{\mu,p} &:= \left[ \int_{B^n} |f|^p d\mu \right]^{1/p} \leq \|f\|_\infty (\text{diam}(B))^{1/p}, \\ \|g_f\|_{\mu,p} &:= \left[ \int_{B^n} |g_f|^p d\mu \right]^{1/p} \leq \left[ \int_{B^n} L(f)^p d\mu \right]^{1/p} = L(f) (\text{diam}(B))^{1/p} \end{aligned}$$

for each  $f \in H^{1,p}(X, \mu)$ . Therefore, for  $C' := C(\text{diam}(B))^{1/p}$ , we obtain

$$\|f_{a_b}\|_{H^{1,p}(B^n, \mu)} \leq C'.$$

By Theorem 5.5 and weak compactness, there is a further subsequence  $h_c := f_{a_{b_c}}$ ,  $c \in \mathbb{N}$ , and a function  $h \in H^{1,p}(B^n, \mu)$  so that  $h_c \rightharpoonup h$  in  $H^{1,p}(B^n, \mu)$ . We now invoke Mazur’s lemma, so there is a sequence of (finite) convex combinations  $\tilde{h}_c := \sum_\alpha \lambda_{c\alpha} h_\alpha$  which converge in norm to  $h$  in  $H^{1,p}(B^n, \mu)$ . In particular,  $\tilde{h}_c$  converges in norm to  $h$  in  $L^p(B^n, \mu)$ , so there is a further subsequence  $\{\tilde{h}_{c_d}\}_{d=1}^\infty$  that converges  $\mu$ -a.e. to  $h$  on  $B^n$ .

By hypothesis,  $f_a \xrightarrow{*} 0$  in  $\text{Lip}_b(X^n)$ . Since  $\|f_a\|_{\text{Lip}} \leq C$ , it follows from Lemma 2.3 that  $f_a$  converges pointwise to 0, and therefore  $h_c$  also converges pointwise to 0. A sharper form of Mazur’s lemma<sup>1</sup> also assures that  $\tilde{h}_c$  converges pointwise to 0, and therefore  $\tilde{h}_{c_d}$  also converges pointwise to 0. This shows that  $h = 0$   $\mu$ -a.e. on  $B^n$ , so every subsequence of  $\{f_a\}_{a=1}^\infty$  has a further subsequence which converges weakly to 0 in  $H^{1,p}(B^n, \mu)$ . It follows that  $f_a \rightharpoonup 0$  in  $H^{1,p}(B, \mu)$ .  $\square$

*Proof of Theorem 5.10.* Let  $n, k \in \mathbb{N}$  be as given in Theorem 5.7, and let  $\delta_i^n : \text{Lip}_b(X^n) \rightarrow L^\infty(X^n, \mu)$  be the map from formula (5.9).

By the uniqueness of Cheeger differentials, the map  $f \mapsto D^n f$  is linear, so each  $\delta_i^n$  is linear. It is known that  $D^n$  satisfies the Leibniz rule [Che99, Equation 4.43], and by a similar argument as above,  $\delta_i^n$  also satisfies the Leibniz rule. It remains to show that  $\delta_i^n$  is continuous. By Lemma 2.8 and Remark 5.1, it suffices to check weak- $*$  convergent sequences in  $\text{Lip}_b(X^n)$ .

To this end, let  $\{f_a\}_{a=1}^\infty \subset \text{Lip}_b(X^n)$  satisfy  $f_a \xrightarrow{*} 0$  and  $\sup_\alpha \|f_a\|_{\text{Lip}} \leq C$ , for some  $C \in (0, \infty)$ . Fix  $p \in (1, \infty)$ , and let  $q = p/(p - 1)$ . As a shorthand, we suppress the notation  $d\mu$  below.

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<sup>1</sup> This fact follows from applying the usual form of Mazur’s lemma to each of the sequences  $\{f_a\}_{a=\alpha}^\infty$ , for  $\alpha \in \mathbb{N}$ , and then taking an appropriate “diagonal” subsequence.



Let  $\psi \in L^1(X, \mu)$  be given, and fix  $\varepsilon > 0$  and  $x_0 \in X^n$ . Since  $\int_X |\psi|$  is finite, there exists  $R > 0$  so that

$$(5.10) \quad \int_{X \setminus B(x_0, R)} |\psi| \leq \frac{\varepsilon}{3C'}.$$

Put  $B = B(x_0, R)$ . Since  $\mu(B) < \infty$ ,  $L^q(B, \mu)$  is a dense subset of  $L^1(B, \mu)$ , so there exists  $\varphi \in L^q(B, \mu)$  so that

$$(5.11) \quad \int_B |\psi - \varphi| \leq \frac{\varepsilon}{3C'}.$$

Now consider the linear operator given by

$$T_\varphi(f) := \int_{B \cap X^n} \varphi \delta_i^n f.$$

Put  $C'' := C(n)\|\varphi\|_{\mu, q}$ . From formula (5.6) and Lemma 5.11, we obtain

$$|T_\varphi(f)| \leq \int_B |\varphi| |\delta_i^n f| \leq C(n) \int_B |\varphi| g_f \leq C'' \|f\|_{H^{1,p}(B, \mu)}$$

for all  $f \in \text{Lip}_b(X^n)$ , so  $T_\varphi$  is a bounded linear functional on  $\text{Lip}_b(X^n) \cap H^{1,p}(B, \mu)$ . By the Hahn–Banach theorem, it extends to an element in the dual  $[H^{1,p}(B \cap X^n, \mu)]^*$ , which we also call  $T_\varphi$ .

From our hypothesis we have  $f_a \xrightarrow{*} 0$  in  $\text{Lip}_b(X^n)$ , so by Lemma 5.12, we obtain  $f_a \rightarrow 0$  in  $H^{1,p}(B, \mu)$ . This implies that, for sufficiently large  $a \in \mathbb{N}$ ,

$$(5.12) \quad |T_\varphi(f_a)| = \left| \int_B \varphi \delta_i^n f_a \right| \leq \frac{\varepsilon}{3}.$$

We now combine estimates (5.10) through (5.12) to obtain

$$\begin{aligned} \left| \int_X \psi \delta_i^n f_a \right| &\leq \left| \int_{X \setminus B} \psi \delta_i^n f_a \right| + \left| \int_B (\psi - \varphi) \delta_i^n f_a \right| + \left| \int_X \varphi \delta_i^n f_a \right| \\ &\leq C' \int_{X \setminus B} |\psi| + C' \int_B |\psi - \varphi| + \left| \int_B \varphi \delta_i^n f_a \right| \\ &\leq C' \frac{\varepsilon}{3C'} + C' \frac{\varepsilon}{3C'} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since the above estimates hold for all  $\varepsilon > 0$ , we obtain

$$\int_X \psi \delta_i^n f_a \rightarrow 0.$$

However,  $\psi \in L^1(X, \mu)$  was also arbitrary, so  $\delta_i^n f_a \xrightarrow{*} 0$  in  $L^\infty(X^n, \mu)$ . □

**5.3. Geometric rigidity and Cheeger’s conjecture.** As discussed in the introduction, Theorem 5.7 indicates that  $p$ -PI spaces have good infinitesimal geometry, in the sense of a differentiability property for Lipschitz functions.

Regarding the *global* geometric structure of such spaces, the following result was proven by Cheeger [Che99, Theorem 14.2]. In addition to the hypotheses for differentiability, as in Theorem 5.7, one further requires that the coordinate charts remain nondegenerate in a measure-theoretic way.

**THEOREM 5.13** (Cheeger, 1999). *Let  $(X, d)$  be a complete metric space that supports a doubling measure and a  $p$ -Poincaré inequality, for some  $1 < p < \infty$ . Assume in addition that  $X$  admits an isometric embedding  $\iota : X \rightarrow \mathbb{R}^N$  for some  $N \in \mathbb{N}$ . If  $\mathcal{H}^{k(n)}(X^n) > 0$ , then  $\iota(X^n)$  is  $k(n)$ -rectifiable.*

In light of this theorem, Cheeger has conjectured that the images of coordinate charts are always measurably non-degenerate [Che99, Conjecture 4.63].

**CONJECTURE 5.14** (Cheeger, 1999). *Let  $(X, \rho, \mu)$ ,  $\{X_n\}_{n=1}^\infty$  and  $\xi^n : X \rightarrow \mathbb{R}^{k(n)}$  be as in Theorem 5.7. Then  $\mathcal{H}^{k(n)}(\xi^n(X^n)) > 0$ .*

Since Lipschitz maps do not increase Hausdorff dimension, the validity of Conjecture 5.14 is consistent with the hypothesis of Theorem 5.13.

Several special cases of Conjecture 5.14 are known. Cheeger has proven it under the hypothesis that  $\mu$  is *lower Ahlfors regular* [Che99, Theorem 13.12] with exponent  $k(n)$ ; that is, there exists  $C \geq 1$  so that

$$C^{-1}r^{k(n)} \leq \mu(B(x, r))$$

holds, for all  $x \in X$  and all  $r > 0$ . Keith has also proven the conjecture for the case  $k = 1$ , but without additional hypotheses [Kei04a].

Using results from Section 4, we now prove the conjecture for  $k = 2$  and without additional hypotheses. To begin, we prove a lower bound for the rank of  $\Upsilon(X, \mu)$ .

**LEMMA 5.15.** *Let  $(X, \rho, \mu)$  be a  $p$ -PI space. If  $\{\delta_i^n\}_{i=1}^k$  are the derivations from formula (5.9), then they form a linearly independent set in  $\Upsilon(X^n, \mu)$ .*

*Proof.* We argue by contradiction. Suppose there exist  $\{\lambda_i\}_{i=1}^k$  in  $L^\infty(X^n, \mu)$ , not all zero, so that  $\delta' := \sum_i \lambda_i \delta_i^n$  is zero. As a result,  $\chi_B \delta' g = 0$ , for all  $g \in \text{Lip}(X)$  and for all balls  $B$  which meet  $X^n$ .

In particular, let  $g = \xi_i^n$ . From Corollary 5.9, it follows that  $\delta_i^n \xi_i^n = 1$  and  $\delta_i^n \xi_j^n = 0$  whenever  $i \neq j$ . Computing further,

$$0 = \chi_B \delta' \xi_j^n = \chi_B \sum_{i=1}^k \lambda_i \delta_i^n \xi_j^n = \chi_B \lambda_j.$$

So each  $\lambda_j$  is zero on every ball  $B$ , and  $\lambda_j = 0$  holds  $\mu$ -a.e. on  $X^n$ . □

**LEMMA 5.16.** *Let  $(X, \rho, \mu)$  be a  $p$ -PI space, and let  $\{X_n\}_{n=1}^\infty$  and  $\xi^n : X \rightarrow \mathbb{R}^k$  be as in Theorem 5.7. Then  $\Upsilon(\mathbb{R}^k, \xi_\#(\mu \lfloor X^n))$  has rank at least  $k$ .*

*Proof.* By Lemma 5.15, the measure  $\mu|_{X^n}$  admits a linearly independent set of  $k$  derivations on  $X^n$ . We now claim that for  $1 \leq i \leq k$ , the pushforward derivations  $\delta'_i := \xi^n_{\#} \delta_i^n$  form a linearly independent set in  $\Upsilon(\mathbb{R}^k, \nu_n)$ , where  $\nu_n := \xi^n_{\#}(\mu|_{X^n})$ .

Let  $\{\lambda_i\}_{i=1}^k$  be functions in  $L^\infty(\mathbb{R}^k, \nu_n)$  so that  $\delta' := \sum_i \lambda_i \delta'_i$  is zero. This implies that for all  $f \in \text{Lip}(\mathbb{R}^k)$  and all balls  $B$  in  $\mathbb{R}^k$ , we have  $\chi_B \delta' f = 0$ .

In particular, put  $f = x_j$  and observe that for  $i \neq j$ , we have  $\delta'_i x_j = 0$ . Indeed, it follows from Lemma 2.17, Corollary 5.9, and formula (5.9) that for each  $h \in L^1(\mathbb{R}^k, \nu_n)$  and each ball  $B$  in  $\mathbb{R}^k$ ,

$$\int_B h \delta'_i x_j d\nu_n = \int_{(\xi^n)^{-1}(B)} (h \circ \xi^n) \delta_i^n \xi_j^n d\mu = 0.$$

Next, put  $Z_n := \{\delta'_i x_i = 0\}$  and  $h := \chi_{Z_n}$ . A similar computation gives

$$\begin{aligned} 0 &= \int_B h \delta'_i x_i d\nu_n = \int_{(\xi^n)^{-1}(B)} \chi_{(\xi^n)^{-1}(Z_n)} \delta_i^n \xi_i^n d\mu \\ &= \mu((\xi^n)^{-1}(B \cap Z_n)) = \xi^n_{\#} \mu(B \cap Z_n). \end{aligned}$$

Letting  $B = B(0, m)$ , for  $m \in \mathbb{N}$ , we obtain  $\xi^n_{\#} \mu(Z_n) = 0$  and therefore  $\delta'_i x_i(x) \neq 0$  for  $\xi^n_{\#} \mu$ -a.e.  $x \in \mathbb{R}^k$ . From these observations, we conclude that

$$0 = \delta' x_j = \sum_{i=1}^k \lambda_i \delta'_i x_j = \lambda_j \delta'_j x_j$$

holds, for each  $1 \leq j \leq k$ , and therefore  $\lambda_j = 0$ . This proves the lemma. □

**COROLLARY 5.17.** *If  $k = k(n) = 2$ , then  $\xi^n_{\#}(\mu|_{X^n})$  is a nonzero measure on  $\mathbb{R}^2$  that is absolutely continuous to Lebesgue 2-measure. In particular, Conjecture 5.14 and Theorem 1.3 are true for  $k = 2$ .*

*Proof.* Put  $\nu_n := \xi^n_{\#}(\mu|_{X^n})$ . By Lemma 5.16, the module  $\Upsilon(\mathbb{R}^k, \nu_n)$  has rank at least  $k$ , so  $\nu_n$  must be nonzero. For  $k = 2$ , this implies that  $\nu_n$  is absolutely continuous to  $m_2$ ; supposing otherwise, if  $\nu_n$  had nonzero Lebesgue singular part, then by Theorem 4.1, the rank of  $\Upsilon(\mathbb{R}^2, \nu_n)$  would be at most 1.

By the definition of pushforward measure,  $\nu_n$  is concentrated on the image set  $\xi^n(X^n)$ . Because  $\nu_n$  is nonzero, we see that  $\nu_n(\xi^n(X^n)) > 0$ , and because  $\nu_n$  is absolutely continuous to  $m_2$ , we further obtain  $m_2(\xi^n(X^n)) > 0$ . □

Lastly, we prove the rigidity theorem for doubling measures on  $\mathbb{R}^2$  that support a Poincaré inequality (as formulated in the Introduction). We begin with a lemma.

**LEMMA 5.18.** *Assuming the hypotheses of Theorem 1.4, let  $\{X^n\}_{n=1}^\infty$  be the measurable decomposition of  $X = \mathbb{R}^2$  from Theorem 5.7. Then  $\Upsilon(X^n, \mu)$  has rank at most 2, for all  $n \in \mathbb{N}$ .*

*Proof.* If  $\{\delta_i\}_{i=1}^3$  were linearly independent in  $\Upsilon(X, \mu)$ , then the derivations

$$(5.13) \quad \begin{cases} \delta'_1 := (\delta_1 x_2)\delta_2 - (\delta_2 x_2)\delta_1, \\ \delta'_2 := (\delta_2 x_1)\delta_1 - (\delta_1 x_1)\delta_2 \end{cases}$$

satisfy  $\delta'_i x_j = 0$   $\mu$ -a.e. on  $\mathbb{R}^2$  whenever  $i \neq j$ , as well as the  $\mu$ -a.e. identity

$$\delta'_1 x_1 = \delta'_2 x_2 \neq 0$$

on  $\mathbb{R}^2$ , otherwise the Chain Rule would imply that  $\delta'_i = 0$ , so  $\{\delta_i\}_{i=1}^3$  would be linearly dependent. On the other hand, (5.13) implies that the derivation

$$\begin{aligned} \delta'_3 &:= (\delta'_1 x_1)\delta_3 - (\delta_3 x_1)\delta'_1 - (\delta_3 x_2)\delta'_2 \\ &= (\delta'_1 x_1)\delta_3 - (\delta_3 x_1)[(\delta_1 x_2)\delta_2 - (\delta_2 x_2)\delta_1] - (\delta_3 x_2)[(\delta_2 x_1)\delta_1 - (\delta_1 x_1)\delta_2] \\ &= (\delta'_1 x_1)\delta_3 + \det \begin{bmatrix} \delta_1 x_1 & \delta_1 x_2 \\ \delta_3 x_1 & \delta_3 x_2 \end{bmatrix} \delta_2 + \det \begin{bmatrix} \delta_2 x_2 & \delta_2 x_1 \\ \delta_3 x_2 & \delta_3 x_1 \end{bmatrix} \delta_1 \end{aligned}$$

acts as zero on every polynomial in  $\mathbb{R}^2$  and hence, on every  $f \in \text{Lip}_b(\mathbb{R}^2)$ . This therefore contradicts the linear independence of  $\{\delta_i\}_{i=1}^3$ , since  $\delta'_1 x_1$  must be nonzero in  $L^\infty(\mathbb{R}^2, \mu)$ .  $\square$

*Proof of Theorem 1.4.* Assuming all of the hypotheses, let  $\{X^n\}_{n=1}^\infty$  again denote the measurable decomposition of  $X = \mathbb{R}^2$  from Theorem 5.7. Moreover, a theorem of Keith [Kei04b, Theorem 2.7] states that on each chart  $X^n$ , coordinates can be chosen to be “distance vectors”—that is, there exist points  $\{p_i\}_{i=1}^{k(n)}$  on  $X$  so that

$$\xi^n(x) := (d(x, p_1), \dots, d(x, p_{k(n)}))$$

satisfies the differentiability property (5.8).

Applying Theorem 5.10 and Lemma 5.18, each  $\Upsilon(X^n, \mu)$  must either have rank-1 or rank-2. If  $\Upsilon(X^n, \mu)$  has rank-1, for some  $n \in \mathbb{N}$ , then fix a coordinate function  $\xi^n(x) = |x - p|$  on  $X^n$ .

Let  $x = (x_1, x_2)$  be a point of  $\mu$ -density in  $X^n$ . By rotating the space as necessary, assume that  $x$  and  $p$  do not lie on the same vertical line or the same horizontal line, and let  $\theta$  be the acute angle formed by the line joining  $x$  to  $p$  and the horizontal line containing  $x$ .

Choosing sequences  $\{y^i\}_{i=1}^\infty$  and  $\{z^i\}_{i=1}^\infty$  in  $\mathbb{R}^2$ , both converging to  $x$  and

$$\frac{y_1^i - x_1}{|y^i - x|} \rightarrow 1 \quad \text{and} \quad \frac{z_1^i - x_1}{|z^i - x|} \rightarrow 0, \quad \text{as } i \rightarrow \infty,$$

a trigonometric computation then yields the limit

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{|y^i - p| - |x - p|}{|y^i - x|} &= \lim_{i \rightarrow \infty} \frac{1}{|y^i - x|} \frac{|y^i - p|^2 - |x - p|^2}{|y^i - p| + |x - p|} \\ &= \lim_{i \rightarrow \infty} \frac{|y^i - x|^2 - 2|x - p||y^i - x| \cos \theta}{|y^i - x|(|y^i - p| + |x - p|)} \end{aligned}$$

$$= \lim_{i \rightarrow \infty} \frac{|y^i - x| - 2|x - p| \cos \theta}{|y^i - p| + |x - p|} = -\cos \theta \neq 0$$

and by a similar computation, the limit

$$\lim_{i \rightarrow \infty} \frac{|z^i - p| - |x - p|}{|z^i - x|} = -\sin \theta \neq 0.$$

Applying Theorem 5.7 to the Euclidean coordinate function  $f(x) = x_1$  and using the above limits, it follows that

$$\begin{aligned} D^n f(x) \cos \theta &= \lim_{i \rightarrow \infty} \frac{y_2^i - x_2 - D^n f(z)(|y^i - p| - |x - p|)}{|y^i - x|} \\ &= 0 = \lim_{i \rightarrow \infty} \frac{z_2^i - x_2 - D^n f(z)(|z^i - p| - |x - p|)}{|z^i - x|} \\ &= 1 - D^n f(x) \sin \theta \end{aligned}$$

which is a contradiction. Therefore each  $\Upsilon(X^n, \mu)$  must have rank-2, so the desired conclusion follows from Theorem 1.2.  $\square$

It is interesting to note that Lemma 5.16 holds true for all  $k \in \mathbb{N}$ . We would obtain all cases of Cheeger's measure conjecture if an analogue of Theorem 4.1 were also true for all  $k \in \mathbb{N}$ . Recalling further, Theorem 4.1 relies crucially on Theorem 4.5, which is an adaptation of the covering theorem of Alberti, Csörnyei, and Preiss (Theorem 4.3).

There are other covering theorems for null sets in  $\mathbb{R}^k$ , for all  $k \in \mathbb{N}$  [ACP05, Proposition 8.4]. However, for  $k \geq 3$ , such covers consist of neighborhoods of both 1-dimensional curves and  $(k - 1)$ -dimensional hypersurfaces in  $\mathbb{R}^k$ . It is easy to see that the argument in Section 4 generalizes for the “ $(k - 1)$ -dimensional” part of a  $m_k$ -null set, but not the “1-dimensional” part.

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