Illinois Journal of Mathematics
Volume 56, Number 3, Fall 2012, Pages 905–930
S 0019-2082

ON THE $(1,p)$-POINCARÉ INEQUALITY

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ABSTRACT. We show that s-John domains satisfy the $(1,p)$-Poincaré inequality for all finite $p > p_0$. We prove that the lower bound p_0 is sharp. We formulate a conjecture concerning (q,p)-Poincaré inequalities in s-John domains, $1 \leq q \leq p$.

1. Introduction

A bounded domain G in $\mathbb{R}^n$, $n \geq 2$, is said to be a (q,p)-Poincaré domain if there exists a finite constant c such that inequality,

$$\left(\int_G \left|u(x) - u_G\right|^q dx\right)^{\frac{1}{q}} \leq c\left(\int_G \left|\nabla u(x)\right|^p dx\right)^{\frac{1}{p}}, \tag{1.1}$$

holds for all $u \in W^{1,p}(G)$; here $1 \leq p < \infty$, $1 \leq q < \infty$, and u_G is the integral average of u. Poincaré inequalities are useful in analysis, especially in the theory of partial differential equations. They have been widely studied in the case $q \geq p$, see, for example, the book of Maz'ya and Poborchi [16]. Poincaré inequalities, (1.1), in the case $1 \leq q \leq p$ have been considered on general domains, for example, in [15, Section 6.4], see also [8]. Maz'ya [15], Theorem 6.4.3/2 on p. 344, gives a characterization for domains which support (1.1) when $q < p$. The characterisation is given in terms of capacity.

We also study the case $1 \leq q \leq p$. Clearly, by Hölder's inequality, if a given domain is a (p,p)-Poincaré domain, then it is a (q,p)-Poincaré domain for every $1 \leq q \leq p$. The benefit is that the inequality with $q < p$ can be satisfied by more irregular domains than the inequality with $q = p$. We provide a sharp quantitative version of this statement for s-John domains. They form a large class of irregular domains including the widely used 1-John domains

Received October 14, 2011; received in final form January 28, 2013.

A. V. Vähäkangas was supported by the Academy of Finland, grants 1118634 and 1134757.

2010 *Mathematics Subject Classification.* Primary 46E35. Secondary 26D10.

and domains that satisfy the quasihyperbolic boundary condition. Our result is given in terms of the upper Minkowski dimension, $\dim_{\mathcal{M}}$, which has been previously used with Poincaré inequalities on domains, for example, in [2], [4].

Let us turn to a detailed discussion of the objectives and results of the present paper. Throughout the paper, we will assume that $n \geq 2$.

The following notation will be convenient to us:

$$\mathbf{C}(q,p,s,\lambda,n) := \frac{(p-q)(\lambda-n)}{pq} + \frac{(s-1)(n-1)}{p}.$$

Smith and Stegenga proved in [17, Theorem 10] that an s-John domain G in $\mathbb{R}^n$ is a (p,p)-Poincaré domain if $1 \leq p < \infty$ and $\mathbf{C}(p,p,s,n,n) < 1$ i.e. if

$$p > (s-1)(n-1).$$

For another proof of this fact, see [7, Corollary 6]. If $\mathbf{C}(p,p,s,n,n) = 1$ and $1 \leq p < \infty$, then we know in some special cases that G is a (p,p)-Poincaré domain. This is true, for instance, in case of rooms and passages -type domains, [9, Remark 5.9] and [5, Example 6.1.1], and s-cups [16, Section 5.1]. We exclude here the discussion about the case $q > p$, for that in s-John domains we refer to [7], [11].

Let us formulate a conjecture.

Conjecture 1.1. *The following statements hold under the assumption that* $1 \leq q \leq p < \infty$, $s > 1$, *and* $\lambda \in [n-1, n)$.

First, let G be an s-John domain in $\mathbb{R}^n$ such that $\dim_{\mathcal{M}}(\partial G) \leq \lambda$. *Then G is a (q,p)-Poincaré domain if either* (1) *or* (2) *holds*:

(1) $\mathbf{C}(q,p,s,\lambda,n) \leq 1$ *and* $1 \leq q = p < \infty$;
(2) $\mathbf{C}(q,p,s,\lambda,n) < 1$ *and* $1 \leq q < p < \infty$.

Conversely, if neither (1) *nor* (2) *holds, there is an s-John domain G in $\mathbb{R}^n$ such that* $\dim_{\mathcal{M}}(\partial G) = \lambda$ *and G is not a (q,p)-Poincaré domain.*

Our main contribution is a verification of Conjecture 1.1 in the case of $1 = q < p$ and $\lambda < n$. This case is special, and the general case seems to be more difficult.

The following negative result of ours covers the converse statement in Conjecture 1.1. It is restricted to the case $\lambda < n$.

Theorem 1.2. *Let* $s > 1$ *and* $\lambda \in [n-1, n)$. *There is an s-John domain G_s in $\mathbb{R}^n$ such that* $\dim_{\mathcal{M}}(\partial G_s) = \lambda$ *and G_s is not a (q,p)-Poincaré domain if either* (1) *or* (2) *holds*:

(1) $\mathbf{C}(q,p,s,\lambda,n) > 1$ *and* $1 \leq q = p < \infty$;
(2) $\mathbf{C}(q,p,s,\lambda,n) \geq 1$ *and* $1 \leq q < p < \infty$.

This theorem is based on a novel counterexample: Proposition 5.1, Theorem 5.6, and Theorem 5.7. Suppose $s > 1$, $1 \leq q \leq p < \infty$, and

$$\mathbf{C}(q,p,s,n,n) = \mathbf{C}(1,p,s,n,n) > 1.$$

By Theorem 1.2 with parameter λ sufficiently close to n, we obtain an s-John domain G_s in $\mathbb{R}^n$ such that G_s is not a $(1,p)$-Poincaré domain. In particular, it is not a (q,p)-Poincaré domain.

The first statement in Conjecture 1.1 is partially covered by the following positive result of ours. The proof can be found in Section 4.

THEOREM 1.3. *Let $s>1$, $1<p<\infty$, and $\lambda\in[n-1,n]$. Let G be an s-John domain in $\mathbb{R}^n$ such that $\dim_{\mathcal{M}}(\partial G)\le\lambda$. If $\mathbf{C}(1,p,s,\lambda,n)<1$, i.e., if*

$$p>\frac{s(n-1)-\lambda+1}{n-\lambda+1}, \tag{1.2}$$

then G is a $(1,p)$-Poincaré domain.

Conjecture 1.1 is true in the case of $1=q<p<\infty$. This follows by combining Theorem 1.2 and Theorem 1.3.

Structure of the paper. We formulate and prove a decomposition theorem for a (q,p)-Poincaré inequality, $1\le q<p<\infty$, Theorem 3.1 which we use when we prove Theorem 1.3. We formulate and prove several lemmata in Section 4 in order to obtain sharp upper bounds for the requirements in Theorem 3.1. In order to show the sharpness of our result, we introduce the s-version of a 1-John domain, Definition 5.2, using the concept of an s-apartment. Given a 1-John domain and its Whitney decomposition the rough idea is to place an s-apartment into each Whitney cube. The upper Minkowski dimension of the boundary of a 1-John domain is inherited by the s-version, Proposition 5.4, and the s-version is an s-John domain, Proposition 5.5. With the s-version of an explicitly constructed 1-John domain, we are able to prove Theorem 1.2.

2. Notation

Let D and G be bounded domains in $\mathbb{R}^n$, $n\ge 2$, and let $1\le q\le p<\infty$. An open n-dimensional ball centered at x and with radius $r>0$ is denoted by $B^n(x,r)$. We let Q be a cube in $\mathbb{R}^n$, whose sides are parallel to the coordinate axes with x_Q the center and $\ell(Q)$ the side-length. By tQ, $t>0$, we mean the cube that is centered at the same point x_Q but whose side-length is $t\ell(Q)$. The Lebesgue measure of a measurable set E in $\mathbb{R}^n$ is written as $|E|$.

We say that D is a *(q,p)-Poincaré domain* if there is a finite positive constant $\kappa_{q,p}(D)$ such that

$$\left(\fint_D |u-u_D|^q\,dy\right)^{\frac{1}{q}}\le\kappa_{q,p}(D)\left(\fint_D|\nabla u|^p\,dy\right)^{\frac{1}{p}} \tag{2.1}$$

for all $u\in W^{1,p}(D)$; where

$$u_D:=\fint_D u(x)\,dx=\frac{1}{|D|}\int_D u(x)\,dx$$

is the integral average of function u over D, and the constant $\kappa_{q,p}(D)$ depends only on n, p, q and D. By Hölder's inequality D is a (q,p)-Poincaré domain whenever D is a (p,p)-Poincaré domain and furthermore $\kappa_{q,p}(D) \le \kappa_{p,p}(D)$. The inequality (2.1) is often written in the form

$$\left(\int_D |u-u_D|^q\,dy\right)^{\frac{1}{q}} \le \kappa_{q,p}(D)|D|^{\frac{1}{q}-\frac{1}{p}}\left(\int_D |\nabla u|^p\,dy\right)^{\frac{1}{p}}$$

and $\kappa_{q,p}(D)|D|^{\frac{1}{q}-\frac{1}{p}}$ is called a (q,p)-Poincaré constant.

REMARK 2.1. We frequently use the well-known fact

$$\kappa_{q,p}(Q) \le \kappa_{p,p}(Q) \le c(n)|Q|^{\frac{1}{n}}$$

for a cube Q, [6, p. 157].

By $\mathcal{W}_D$ we denote a Whitney decomposition of the domain D. This is a family of those closed dyadic cubes Q in the Whitney decomposition of $\mathbb{R}^n \setminus \partial D$ for which $Q \subset D$. However, we modify the standard construction, cf. [18, p. 167], such that $\mathcal{W}_D$ consists of cubes Q for which $\frac{9}{8}\operatorname{diam}(Q) \le 1$ and

$$\kappa_{q,p}\left(\operatorname{int}\frac{9}{8}Q\right) \le c(n)\left|\frac{9}{8}Q\right|^{\frac{1}{n}} \le 1. \tag{2.2}$$

If the domain D is clear from the context we write simply $\mathcal{W}$ for $\mathcal{W}_D$. For every $k \in \mathbb{N}$, we write

$$\mathcal{W}_k := \left\{Q \in \mathcal{W}_D : \ell(Q) = 2^{-k}\right\}$$

and by $\sharp\mathcal{W}_k$ we denote the number of cubes in this family. Note that $\mathcal{W}_D = \bigcup_{k=0}^{\infty} \mathcal{W}_k$.

Let E in R^n be a non-empty bounded set. By $\mathcal{H}^\lambda(E)$ we mean the λ-dimensional Hausdorff measure of E. The Hausdorff dimension of E is written as $\dim_{\mathcal{H}}(E)$. The upper Minkowski dimension of E is

$$\dim_{\mathcal{M}}(E) := \sup\left\{d \ge 0 : \limsup_{r\to 0+} \mathcal{M}_d(E,r) = \infty\right\},$$

where

$$\mathcal{M}_d(E,r) := \frac{|E+B^n(0,r)|}{r^{n-d}} := \frac{|\bigcup_{x\in E} B^n(x,r)|}{r^{n-d}}, \quad r>0,$$

is the d-dimensional Minkowski precontent.

3. Poincaré decomposition

The following Poincaré decomposition is from [8] which, in turn, is based on [9]. A collection $\mathcal{C}(D) = \{D_0, D_1, \ldots, D_k\}$ of bounded domains in $\mathbb{R}^n$ with

$D_k = D$ is said to be a *chain* from D_0 to D whenever $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The length of a chain $\mathcal{C}(D)$ is denoted by $\ell(\mathcal{C}(D)) = k$.

Let Π be a collection of bounded (q,p)-Poincaré domains. Let us fix constants $N \geq 1$ and $c_1 > 0$. We call Π a (q,p)-*Poincaré decomposition* of a domain G, if

(i) $G = \bigcup_{D\in\Pi} D$;
(ii) $\sum_{D\in\Pi} \chi_D(x) \leq N\chi_G(x)$ for all $x \in \mathbb{R}^n$, where χ_G is the characteristic function of G; and
(iii) there is a domain $D_0 \in \Pi$ such that for each $D \in \Pi$ there exists a chain $\mathcal{C}(D) = \{D_0, D_1, \ldots, D_{\ell(C(D))-1}, D\}$ of domains in Π with

$$\max\{|D_i|, |D_{i-1}|\} \leq c_1 |D_i \cap D_{i-1}| \tag{3.1}$$

for $i = 1, \ldots, \ell(\mathcal{C}(D))$.

For each D in Π, we fix a chain $\mathcal{C}(D)$ satisfying (3.1) and call this the *Poincaré chain* from D_0 to D. For a fixed $A \in \Pi$, we write

$$A(\Pi) := \{D \in \Pi : A \in \mathcal{C}(D)\}.$$

Various chains and/or decompositions are available in the literature, for example [1], [3], [7], [9], [10], [11], [17]. The optimal (q,p)-Poincaré inequalities for rooms and passages-type domains are obtained in [8] by using a Poincaré decomposition arising from the geometry of the underlying domain.

We prove a slight modification of [8, Theorem 2.4] and [9, Theorem 4.4]. For the sake of completeness, we present the proof.

THEOREM 3.1. *Let $1 \leq q < p < \infty$. Let G be a bounded domain in $\mathbb{R}^n$ and let Π be a (q,p)-Poincaré decomposition of G. If $\kappa_{q,p}(D) \leq 1$ for every $D \in \Pi$ and there are positive and finite constants c and $\varkappa$ such that*

$$\sum_{D\in\Pi} \kappa_{q,p}(D)^{\frac{pq}{p-q}-\varkappa} |D| \leq c, \tag{3.2}$$

and for every $A \in \Pi$

$$\sum_{D\in A(\Pi)} \ell(\mathcal{C}(D))^{q-1} |D| \leq c\kappa_{q,p}(A)^{-\varkappa\frac{p-q}{p}} |A|, \tag{3.3}$$

then the domain G is a (q,p)-Poincaré domain.

Proof. Let D_0 be a fixed domain in Π. The Hölder's inequality yields

$$\left(\int_G |u(x) - u_G|^q \, dx\right)^{\frac{1}{q}} \leq 2\left(\int_G |u(x) - u_{D_0}|^q \, dx\right)^{\frac{1}{q}}.$$

By the elementary inequalities

$$|a+b|^q \leq 2^{q-1}(|a|^q + |b|^q), \qquad |a+b|^{\frac{1}{q}} \leq |a|^{\frac{1}{q}} + |b|^{\frac{1}{q}},$$

with $1 \le q < \infty$, we obtain

$$\begin{aligned}(3.4)\qquad \Big(\int_G \big|u(x)-u_{D_0}\big|^q\,dx\Big)^{\frac1q} &\le \Big(\sum_{D\in\Pi}\int_D \big|u(x)-u_{D_0}\big|^q\,dx\Big)^{\frac1q}\\ &\le c\underbrace{\Big(\sum_{D\in\Pi}\int_D \big|u(x)-u_D\big|^q\,dx\Big)^{\frac1q}}_{=:\,\mathcal{I}}\\ &\quad + c\underbrace{\Big(\sum_{D\in\Pi}\int_D |u_D-u_{D_0}|^q\,dx\Big)^{\frac1q}}_{=:\,II}.\end{aligned}$$

The term $\mathcal{I}$ in (3.4) is estimated by the (q,p)-Poincaré inequality in D and Hölder's inequality for sums with $(\frac{p}{q},\frac{p}{p-q})$

$$\begin{aligned}(3.5)\qquad \mathcal{I} &\le \Bigg(\sum_{D\in\Pi}\kappa_{q,p}(D)^q|D|^{1-\frac qp}\Big(\int_D\big|\nabla u(x)\big|^p\,dx\Big)^{\frac qp}\Bigg)^{\frac1q}\\ &\le \Bigg(\sum_{D\in\Pi}\big(\kappa_{q,p}(D)^q|D|^{1-\frac qp}\big)^{\frac{p}{p-q}}\Bigg)^{\frac{p-q}{pq}}\Big(\sum_{D\in\Pi}\int_D\big|\nabla u(x)\big|^p\,dx\Big)^{\frac1p}\\ &\le c\Big(\int_G\big|\nabla u(x)\big|^p\,dx\Big)^{\frac1p},\end{aligned}$$

where in the last inequality we used the estimate

$$\sum_{D\in\Pi}\kappa_{q,p}(D)^{\frac{pq}{p-q}}|D| \le \sum_{D\in\Pi}|D| \le N|G| < \infty,$$

which follows from the properties of the (q,p)-Poincaré decomposition Π and the boundedness of G.

We are left to handle the term II in (3.4). Let us connect every domain $D\in\Pi$ to the fixed domain D_0 by a Poincaré chain $\mathcal{C}(D)=(D_0,D_1,\dots,D_{k-1},D)$. By the inequality

$$\Big(\sum_{i=1}^k t_i\Big)^q \le k^{q-1}\sum_{i=1}^k t_i^q,$$

with $1\le q<\infty$, we obtain

$$\begin{aligned}II &\le \Bigg(\sum_{D\in\Pi}\int_D \ell\big(\mathcal{C}(D)\big)^{q-1}\sum_{i=1}^{\ell(\mathcal{C}(D))}|u_{D_i}-u_{D_{i-1}}|^q\,dx\Bigg)^{\frac1q}\\ &= \Bigg(\sum_{D\in\Pi}\int_D \ell\big(\mathcal{C}(D)\big)^{q-1}\sum_{i=1}^{\ell(\mathcal{C}(D))}\fint_{D_i\cap D_{i-1}}|u_{D_i}-u_{D_{i-1}}|^q\,dy\,dx\Bigg)^{\frac1q}\end{aligned}$$

$$\le \Bigg(\sum_{D\in\Pi}|D|\ell\big(\mathcal{C}(D)\big)^{q-1}\sum_{i=1}^{\ell(\mathcal{C}(D))}|D_i\cap D_{i-1}|^{-1}2^{q-1}\Big\{\int_{D_i}\big|u(y)-u_{D_i}\big|^q\,dy + \int_{D_{i-1}}\big|u(y)-u_{D_{i-1}}\big|^q\,dy\Big\}\Bigg)^{\frac{1}{q}}.$$

By the (q,p)-Poincaré inequality and condition (3.1)

$$II \le c\Bigg(\sum_{D\in\Pi}|D|\ell\big(\mathcal{C}(D)\big)^{q-1}\sum_{i=1}^{\ell(\mathcal{C}(D))}|D_i\cap D_{i-1}|^{-1} \cdot\Big\{\kappa_{q,p}(D_i)^q|D_i|^{1-\frac{q}{p}}\Big(\int_{D_i}\big|\nabla u(y)\big|^p\,dy\Big)^{\frac{q}{p}} + \kappa_{q,p}(D_{i-1})^q|D_{i-1}|^{1-\frac{q}{p}}\Big(\int_{D_{i-1}}\big|\nabla u(y)\big|^p\,dy\Big)^{\frac{q}{p}}\Big\}\Bigg)^{\frac{1}{q}}$$

$$\le \underbrace{c\Bigg(\sum_{D\in\Pi}|D|\ell\big(\mathcal{C}(D)\big)^{q-1}\sum_{A\in\mathcal{C}(D)}\kappa_{q,p}(A)^q|A|^{-\frac{q}{p}}\Big(\int_A|\nabla u|^p\,dy\Big)^{\frac{q}{p}}\Bigg)^{\frac{1}{q}}}_{=:\,III}.$$

Rearranging the double sum and using (3.3), we obtain

$$III \le \Bigg(\sum_{A\in\Pi}\sum_{D\in A(\Pi)}\ell\big(\mathcal{C}(D)\big)^{q-1}|D|\kappa_{q,p}(A)^q|A|^{-\frac{q}{p}}\Big(\int_A|\nabla u|^p\,dy\Big)^{\frac{q}{p}}\Bigg)^{\frac{1}{q}}$$

$$\le c\Bigg(\sum_{A\in\Pi}\kappa_{q,p}(A)^{q-\varkappa\frac{p-q}{p}}|A|^{1-\frac{q}{p}}\Big(\int_A|\nabla u|^p\,dy\Big)^{\frac{q}{p}}\Bigg)^{\frac{1}{q}}.$$

By Hölder's inequality with $(\frac{p}{q},\frac{p}{p-q})$ and by (3.2), this yields

$$III \le c\Bigg(\sum_{A\in\Pi}\big(\kappa_{q,p}(A)^{q-\varkappa\frac{p-q}{p}}|A|^{1-\frac{q}{p}}\big)^{\frac{p}{p-q}}\Bigg)^{\frac{p-q}{pq}}\Bigg(\sum_{A\in\Pi}\int_A\big|\nabla u(y)\big|^p\,dy\Bigg)^{\frac{1}{p}}$$

$$\le c\Big(\int_G|\nabla u|^p\,dy\Big)^{\frac{1}{p}}.$$

This completes the proof. □

REMARK 3.2. Theorem 3.1 is a generalization of [9, Theorem 4.4], where Hurri showed that G is a (p,p)-Poincaré domain if condition (3.3) is replaced by

$$\sum_{D\in A(\Pi)}\ell\big(\mathcal{C}(D)\big)^{p-1}|D| \le c\kappa_{p,p}(A)^{-p}|A| \tag{3.6}$$

and condition (3.2) is omitted. Note that condition (3.3) gives condition (3.6) by a limiting process: If we choose $\varkappa = pq/(p-q)$, then condition (3.2) holds. Condition (3.3) is now

$$\sum_{D\in A(\Pi)} \ell\big(\mathcal{C}(D)\big)^{q-1}|D| \le c\kappa_{q,p}(A)^{-q}|A|,$$

which yields (3.6) as $q\to p$.

Remark 3.3. The two conditions (3.2) and (3.3) were used in the proof of Theorem 3.1 to establish the following estimate:

$$\sum_{A\in\Pi}\bigg(\sum_{D\in A(\Pi)} \ell\big(\mathcal{C}(D)\big)^{q-1}|D|\kappa_{q,p}(A)^{q}|A|^{-\frac{q}{p}}\bigg)^{\frac{p}{p-q}} < \infty. \tag{3.7}$$

An examination of the proof reveals that the two conditions above can be replaced with (3.7) in the formulation of Theorem 3.1. We will use this single condition later to obtain sharp estimates in s-John domains.

4. Proof of Theorem 1.3

First, we need some preparations. The actual proof of Theorem 1.3 is presented at the end of this section.

Let us begin with definition of s-John domains.

Definition 4.1. Let $s\ge 1$. A bounded domain G in $\mathbb{R}^n$, $n\ge 2$, is an *s-John domain* if there exists a point x_0 in G and a constant $c>0$ such that every point x in G can be joined to x_0 by a rectifiable path $\gamma\colon [0,l]\to G$ parametrized by its arc length for which $\gamma(0)=x$, $\gamma(l)=x_0$, $l\le c$, and

$$\operatorname{dist}\big(\gamma(t),\partial G\big) \ge t^s/c \quad \text{for } t\in[0,l].$$

The point x_0 is called an *s-John center* of G.

Observe the following reductions: The case $\lambda=n$ in Theorem 1.3 follows from Theorem 10 in [17]. Hence, we can assume that $\lambda<n$. Choose $\lambda'\in(\lambda,n)$ such that (1.2) is true if λ is replaced by λ'. Then $\dim_{\mathcal{M}}(\partial G)<\lambda'$ and hence we may assume that $\dim_{\mathcal{M}}(\partial G)$ is strictly less than $\lambda\in[n-1,n)$. This assumption is later used with the aid of the following lemma.

Lemma 4.2. *Let K in $\mathbb{R}^n$ be a compact set such that*

$$\dim_{\mathcal{M}}(K)<\lambda, \quad \textit{where } \lambda\in[n-1,n).$$

There is a positive constant c as follows: Assume that $\{B_1,B_2,\dots,B_N\}$ is a family of N disjoint balls in $\mathbb{R}^n$, each of which is centered in K and whose radius is $r\in(0,1]$. Then $N\le cr^{-\lambda}$.

Proof. By definition, we have

$$\inf_{a>0}\left\{\sup_{r\in(0,a)}\frac{|K+B^n(0,r)|}{r^{n-\lambda}}\right\}=\limsup_{r\to 0+}\mathcal{M}_\lambda(K,r)<\infty.$$

In particular, there is $a\in(0,1)$ such that

$$\sup_{r\in(0,a)}\frac{|K+B^n(0,r)|}{r^{n-\lambda}}=C<\infty. \tag{4.1}$$

We consider a family $\{B_1,\dots,B_N\}$ of disjoint balls in $\mathbb{R}^n$, each of which is centered in K and whose radius is $r\in(0,1]$. We separate two cases I and II:

Case I. $r\in[a,1]$. In this case, we have

$$\begin{aligned}N&\le c_n\sum_{i=1}^N\frac{|B_i|}{r^n}\le c_na^{-n}\sum_{i=1}^N|B_i|=c_na^{-n}\left|\bigcup_{i=1}^N B_i\right|\\&\le c_na^{-n}\big|K+B^n(0,r)\big|\le c_na^{-n}\big|K+B^n(0,1)\big|=c_1\le c_1r^{-\lambda}.\end{aligned}$$

Case II. $r\in(0,a)$. The estimate (4.1) yields

$$\begin{aligned}N&\le c_nr^{-n}\sum_{i=1}^N|B_i|\le c_nr^{-n}\big|K+B^n(0,r)\big|=c_nr^{-\lambda}\frac{|K+B^n(0,r)|}{r^{n-\lambda}}\\&\le c_nCr^{-\lambda}=c_2r^{-\lambda}.\end{aligned}$$

Combining the Cases I and II the required estimate holds true with a constant $c=\max\{c_1,c_2\}$. □

For the proof of Theorem 1.3, we fix a Whitney decomposition $\mathcal{W}=\mathcal{W}_G$ satisfying (2.2).

We write

$$\frac{9}{8}\mathcal{W}:=\left\{\operatorname{int}\frac{9}{8}Q:Q\in\mathcal{W}\right\}.$$

In order to equip this family with Poincaré chains, we fix $Q_0\in\mathcal{W}$ and state that the s-John center of G is x_{Q_0}. We wish to join Q_0 to every cube R in $\mathcal{W}$. It is convenient first to connect x_R to x_{Q_0} by an s-John path γ_R that joins a sequence of midpoints of intersecting Whitney cubes to each other. Indeed, such a path will yield a Poincaré chain with nice properties. The following construction is essentially from [17, p. 86]. Other constructions are used in [9], [12].

Fix a rectifiable path γ that is parametrized by its arc length and joins the points x_R and x_{Q_0} as in Definition 4.1. Assume that x_{Q_0} lies in one of the cubes intersecting R. Then join x_R to x_{Q_0} by an arc that is contained in $R\cup Q_0$ and whose length is comparable to $\ell(R)$. Otherwise there is $r>0$ such that $\gamma(r)$ lies in the boundary of a cube $P\in\mathcal{W}$ that intersects R and $\gamma(t)$ belongs to a cube that is not intersecting R whenever $t\in(r,\ell(\gamma)]$. Now we connect the midpoint of x_R to the midpoint of x_P by an arc whose length is comparable to $\ell(R)$ and that is contained in $R\cup P$. Then we iterate the steps above but with R replaced by P. This procedure is repeated until we reach x_{Q_0}. Finally, we collect the arcs in the order that they were constructed, and arc length parametrize them by a path γ_R. It is straightforward to verify that

$$t^s \le c\,\mathrm{dist}\big(\gamma_R(t),\partial G\big) \quad \text{if } t\in\big[0,\ell(\gamma_R)\big], \tag{4.2}$$

where $c>0$ depends on the s-John constant of G and n.

We define $P(R)$, $R\in\mathcal{W}$, to be the union of those cubes in $\mathcal{W}$ whose midpoints lie in the trace of γ_R. If $Q\in\mathcal{W}$, we write

$$S(Q) := \bigcup\big\{R\in\mathcal{W} : Q\subset P(R)\big\}.$$

This is the *shadow* of Q. Let $D\in\frac{9}{8}\mathcal{W}$. Then $D=\mathrm{int}\frac{9}{8}Q$ for some $Q\in\mathcal{W}$, and we define $\mathcal{C}(D)$ to be the Poincaré chain

$$\Big\{\mathrm{int}\frac{9}{8}R : R\in\mathcal{W} \text{ and } R\subset P(Q)\Big\}$$

that is ordered by reversing the order as γ_R hits the midpoints of these cubes. The cube $D_0 := \mathrm{int}\frac{9}{8}Q_0$ is the first and $\mathrm{int}\frac{9}{8}Q$ is the last.

It follows from the construction above that the family $\frac{9}{8}\mathcal{W}$ equipped with these Poincaré chains is a $(1,p)$-Poincaré decomposition of G.

For $j,k\in\mathbb{N}$ and $\sigma\ge 1$, we define

$$\mathcal{W}_{j,k,\sigma} := \big\{Q\in\mathcal{W}_j : 2^{-(j-k)n} \le \big|S(Q)\big| \le \sigma\cdot 2^{-(j-k-1)n}\big\}.$$

The following lemma gives crucial estimates for the cardinality of such a family of cubes.

LEMMA 4.3. *Let $s>1$ and G be an s-John domain in $\mathbb{R}^n$ such that $\dim_{\mathcal{M}}(\partial G)<\lambda$, where $\lambda\in[n-1,n)$. Then there is $\sigma\ge 1$ such that*

$$\mathcal{W}_j = \bigcup_{k=0}^{[j-j/s]} \mathcal{W}_{j,k,\sigma} \quad \textit{for every } j\in\mathbb{N}. \tag{4.3}$$

Furthermore, if $k\in\{0,1,\dots,[j-j/s]\}$, we have

$$\sharp\mathcal{W}_{j,k,\sigma} \le c2^{-kn}2^{j(n+1+(\lambda-n-1)/s)}. \tag{4.4}$$

The positive constant c depends on s, n, ∂G, and the s-John constant of the domain G.

Proof. Let us fix $j\in\mathbb{N}$ and begin with a covering argument. The $5r$-covering theorem, see, for example, [14, p. 23], implies that there is a finite family

$$\mathcal{F}\subset\big\{B^n\big(x,2^{-j/s}\big) : x\in\partial G\big\}$$

of disjoint balls such that

$$\partial G\subset\bigcup_{B\in\mathcal{F}} 5B. \tag{4.5}$$

We claim that, if $Q \in \mathcal{W}_j$, then there exists $B \in \mathcal{F}$ such that $Q \subset c_1 B$. Here c_1 is a constant depending on n only. To verify this, let $y \in \partial G$ be a closest point in ∂G to the midpoint x_Q of Q. Using the covering property (4.5) yields a point x in ∂G such that $B^n(x, 2^{-j/s}) \in \mathcal{F}$ and $y \in B^n(x, 5 \cdot 2^{-j/s})$. Now, if $z \in Q$, we have

$$|z-x| \le |z-x_Q| + |x_Q - y| + |y-x| \le c2^{-j} + c2^{-j} + 5 \cdot 2^{-j/s} < c_1 2^{-j/s}.$$

It follows that $Q \subset B^n(x, c_1 2^{-j/s}) = c_1 B^n(x, 2^{-j/s})$ as required.

Next, we fix $Q \in \mathcal{W}_j$ and any ball $B := B^n(x, 2^{-j/s})$ in $\mathcal{F}$ such that $Q \subset c_1 B$. We claim that

$$S(Q) \subset B^n\big(x, c_2 2^{-j/s}\big), \tag{4.6}$$

where $c_2 > c_1$ is a constant depending on s, n and the s-John constant of G. To show this, we let $R \in \mathcal{W}$ be a cube for which $Q \subset P(R)$. Consider the path γ_R which connects x_R to x_{Q_0} and satisfies (4.2). Because $Q \subset P(R)$, we find that $\gamma_R(t) = x_Q$ for some t. Using the properties of Whitney cubes and (4.2), we obtain

$$|x_R - x_Q|^s \le t^s \le c\,\mathrm{dist}\big(\gamma_R(t), \partial G\big) = c\,\mathrm{dist}(x_Q, \partial G) \le c2^{-j}.$$

It follows that

$$\begin{aligned}\mathrm{diam}(R) &\le c\,\mathrm{dist}(x_R, \partial G)\\ &\le c|x_R - x_Q| + c\,\mathrm{dist}(x_Q, \partial G) \le c2^{-j/s} + c2^{-j} \le c2^{-j/s}.\end{aligned}$$

Hence, if $y \in R$, we have

$$\begin{aligned}|y-x| &\le |y-x_R| + |x_R - x_Q| + |x_Q - x|\\ &\le c2^{-j/s} + c2^{-j/s} + c_1 2^{-j/s} < c_2 2^{-j/s}.\end{aligned}$$

The inclusion (4.6) follows.

As a consequence of (4.6), we have

$$2^{-jn} = |Q| \le \big|S(Q)\big| \le \sigma \cdot 2^{-jn/s}$$

for a constant $\sigma \ge 1$ depending on s, n, and the s-John constant of G. In particular, we see that (4.3) is valid with this constant.

It remains to prove the estimate (4.4). In order to do this, we establish the following auxiliary estimate

$$\sharp\big\{Q \in \mathcal{W}_j : Q \subset P(R)\big\} \le c_3 2^{j(1-1/s)} \quad \text{if } R \in \mathcal{W}. \tag{4.7}$$

Here the constant c_3 depends on s, n, and the s-John constant of G. In order to see this, we fix $R \in \mathcal{W}$ and let γ_R be the path connecting x_R to x_{Q_0}. Let $Q_1, \ldots, Q_M \in \mathcal{W}_j$ be cubes such that $Q_i \subset P(R)$ for every $i \in \{1, 2, \ldots, M\}$. We number these cubes in the same order as γ_R hits their midpoints. In particular, if $\gamma_R(t) = x_{Q_M}$, then $\gamma_R[0,t]$ joins the midpoints of M cubes whose side-length is 2^{-j}. Using (4.2), we obtain

$$(M-1)2^{-j} \le t \le c\,\mathrm{dist}\big(\gamma_R(t), \partial G\big)^{1/s} = c\,\mathrm{dist}(x_{Q_M}, \partial G)^{1/s} \le c2^{-j/s}.$$

It follows that $M \le c_3 2^{j(1-1/s)}$ as required in (4.7).

Then we fix $k \in \{0, 1, \ldots, [j - j/s]\}$ where $[j - j/s]$ is the integer part of $j - j/s$. Fix also $B := B^n(x, 2^{-j/s}) \in \mathcal{F}$. First, we estimate the number of cubes that are included in $c_1 B$. Inclusion (4.6) yields

$$\begin{aligned}
&\sharp\{Q \in \mathcal{W}_{j,k,\sigma} : Q \subset c_1 B\} \\
&\quad \le \sum_{\substack{Q \in \mathcal{W}_{j,k,\sigma} \\ Q \subset c_1 B}} 2^{(j-k)n} \big|S(Q)\big| \le 2^{(j-k)n} \sum_{Q \in \mathcal{W}_{j,k,\sigma}} \big|S(Q) \cap c_2 B\big| \\
&\quad \le 2^{(j-k)n} \sum_{Q \in \mathcal{W}_{j,k,\sigma}} \sum_{\substack{R \in \mathcal{W} \\ Q \subset P(R)}} |R \cap c_2 B| = 2^{(j-k)n} \sum_{R \in \mathcal{W}} \sum_{\substack{Q \in \mathcal{W}_{j,k,\sigma} \\ Q \subset P(R)}} |R \cap c_2 B|.
\end{aligned}$$

Now (4.7) shows that the last term above is bounded by

$$c_3 2^{(j-k)n} 2^{j(1-1/s)} |c_2 B| \le c_4 2^{-kn} 2^{j(n+1-1/s-n/s)}.$$

Here c_4 is a constant depending on s, n, and the s-John constant of G.

From the considerations above, it follows that

$$\sharp\mathcal{W}_{j,k,\sigma} \le \sum_{B \in \mathcal{F}} \sharp\{Q \in \mathcal{W}_{j,k,\sigma} : Q \subset c_1 B\} \le c_4 \sum_{B \in \mathcal{F}} 2^{-kn} 2^{j(n+1-1/s-n/s)}. \tag{4.8}$$

Recall that $\mathcal{F}$ is a family of disjoint balls, each of which is centered in ∂G and whose radius is $2^{-j/s} \in (0,1]$. Therefore, Lemma 4.2 yields $\sharp\mathcal{F} \le c2^{j\lambda/s}$. Combining this estimate with (4.8) allows us to conclude that

$$\sharp\mathcal{W}_{j,k,\sigma} \le c2^{j\lambda/s} 2^{-kn} 2^{j(n+1-1/s-n/s)}.$$

Simplifying the exponents gives us (4.4). □

Proof of Theorem 1.3. By using both Remark 2.1 and (2.2), we obtain $\kappa_{1,p}(D) \le c(n)|D|^{\frac{1}{n}} \le 1$ for every $D \in \frac{9}{8}\mathcal{W}$. Hence, according to Remark 3.3, it suffices to verify the finiteness of

$$\Sigma := \sum_{A \in \frac{9}{8}\mathcal{W}} \Bigg(\sum_{D \in A(\frac{9}{8}\mathcal{W})} |D| |A|^{1/n - 1/p} \Bigg)^{p/(p-1)}.$$

From the definitions and the estimate $|\frac{9}{8}Q| \le c_n |Q|$ it follows that

$$\Sigma \le c \sum_{Q \in \mathcal{W}} \big(\big|S(Q)\big| |Q|^{1/n - 1/p} \big)^{p/(p-1)}.$$

By using (4.3) from Lemma 4.3, we can write

$$\Sigma \le c\sum_{j=0}^{\infty}\sum_{k=0}^{[j-j/s]}\sum_{Q\in\mathcal{W}_{j,k,\sigma}}\big(|S(Q)||Q|^{1/n-1/p}\big)^{p/(p-1)}.$$

Then, by using the definition of $\mathcal{W}_{j,k,\sigma}$ and (4.4) from Lemma 4.3, we obtain the estimate

$$\begin{aligned}\Sigma &\le c\sum_{j=0}^{\infty}\sum_{k=0}^{[j-j/s]} 2^{-kn}2^{j(n+1+(\lambda-n-1)/s)}\cdot\big(2^{-(j-k)n}\cdot 2^{-jn(1/n-1/p)}\big)^{p/(p-1)}\\ &= c\sum_{j=0}^{\infty}\sum_{k=0}^{[j-j/s]} 2^{kn(p/(p-1)-1)}2^{j(n+1+(\lambda-n-1)/s-np/(p-1)-p/(p-1)+n/(p-1))}.\end{aligned}$$

We fix j and k as in the summation above. Then

$$kn\Big(\frac{p}{p-1}-1\Big)\le n(j-j/s)\Big(\frac{p}{p-1}-1\Big)=\frac{jn(1-1/s)}{p-1}.$$

Using also the trivial estimate $[j-j/s]\le j$, we find that

$$\begin{aligned}\Sigma &\le c\sum_{j=0}^{\infty} j\cdot 2^{j(n(1-1/s)/(p-1)+n+1+(\lambda-n-1)/s-np/(p-1)-p/(p-1)+n/(p-1))}\\ &\le c\sum_{j=0}^{\infty} j\cdot 2^{j(ns-s+\lambda p-\lambda-np-p+1)/s(p-1)}.\end{aligned}$$

By (1.2), we see that the last series converges. ☐

5. Failure of a $(1,p)$-Poincaré inequality

Theorem 1.3 states that an s-John domain G in $\mathbb{R}^n$ with $s>1$ is a $(1,p)$-Poincaré domain if $\dim_{\mathcal{M}}(\partial G)\le\lambda\in[n-1,n)$, $p\in(1,\infty)$, and

$$p>\frac{s(n-1)-\lambda+1}{n-\lambda+1}. \tag{5.1}$$

We show that this result is sharp by constructing an s-John domain G_s in $\mathbb{R}^n$ such that $\dim_{\mathcal{M}}(\partial G_s)=\lambda$ and G_s is not a $(1,p)$-Poincaré domain if (5.1) fails.

The construction is based on modifying a given 1-John domain G such that the resulting domain G_s, known as the s-version of G, is an s-John domain containing multiple copies of rooms and s-passages at every size-scale 2^{-j}. The number of these copies at each scale depends on the upper Minkowski dimension of ∂G or, more precisely, on the number of Whitney cubes at each scale. The modification also preserves the upper Minkowski dimension so that $\dim_{\mathcal{M}}(\partial G)=\dim_{\mathcal{M}}(\partial G_s)$.

Before the modification procedure can take place, we need to find suitable 1-John domains in $\mathbb{R}^n$. Such domains G with

$$\dim_{\mathcal{M}}(\partial G) = \lambda \in [n-1, n)$$

are constructed in the proof of the following proposition.

PROPOSITION 5.1. *Let $n \geq 2$ and $\lambda \in [n-1, n)$. There is a 1-John domain G in $\mathbb{R}^n$ such that* $\dim_{\mathcal{M}}(\partial G) = \lambda$ *and*

$$\limsup_{k\to\infty} 2^{-\lambda k} \cdot \sharp\mathcal{W}_k > 0. \tag{5.2}$$

Here $\sharp\mathcal{W}_k$ denotes the number of those cubes in $\mathcal{W}_G$ whose side-lengths are 2^{-k}.

Proof. We describe the construction in the case $n = 2$. The general case is similar.

Let us denote $Q := [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$, $\kappa \in (0, 1)$, and $r(\kappa) := (1-\kappa)/2 \in (0, 1/2)$. Let us write

$$z_1 := \big(\kappa + r(\kappa), \kappa + r(\kappa)\big),$$

and let z_2, z_3, z_4 stand for the corresponding symmetric points in the three remaining quadrants in any order. Let S_1, S_2, S_3, S_4 be similitudes that are defined by $S_i(x) := r(\kappa)x + z_i$, $i = 1, 2, 3, 4$. Reasoning as in [14, pp. 66–67], we see that there is a non-empty compact set K in Q for which

$$K = S_1(K) \cup S_2(K) \cup S_3(K) \cup S_4(K). \tag{5.3}$$

The similitudes S_1, S_2, S_3, S_4 satisfy an open set condition [14, p. 67]. Hence, we can use both Corollary 5.8 and Theorem 4.14 in [14] to see that

$$\dim_{\mathcal{M}}(K) = \dim_{\mathcal{H}}(K) = -\frac{\log 4}{\log r(\kappa)}.$$

Notice that $-\log 4/\log r(\kappa)$ reaches all the values in $(0, 2)$ if we let κ vary between $(0, 1)$. In particular, there exists $\kappa = \kappa(\lambda) \in (0, 1)$ for which the upper Minkowski dimension of the corresponding compact set $K_\lambda := K$ is λ. We define G to be the open set

$$G := B^n(0, 2) \setminus K_\lambda.$$

Since $\partial G = \partial B^n(0, 2) \cup K_\lambda$, we see that $\dim_{\mathcal{M}}(\partial G) = \lambda$.

We omit the proof of the evident fact that G is a 1-John domain. This proof can be based on that the iterations

$$\bigcup_{i_1=1}^{4} \cdots \bigcup_{i_m=1}^{4} S_{i_1} \circ \cdots \circ S_{i_m}(Q) \tag{5.4}$$

will converge to K_λ in the Hausdorff metric.

The inequality (5.2) is not immediately clear, so let us verify it. For this purpose, we write

$$Q_0^1 := [-\kappa,\kappa]\times[-\kappa,\kappa]\subset Q,$$

where $\kappa=\kappa(\lambda)$ is defined above. For every $m\in\mathbb{N}$, we re-index the 4^m disjoint cubes

$$S_{i_1}\circ\cdots\circ S_{i_m}\big(Q_0^1\big),\quad i_1,i_2,\dots,i_m\in\{1,2,3,4\},$$

by labeling them as Q_m^i, $i=1,\dots,4^m$, in some fixed order. From (5.3), it follows that $\mathrm{int}Q_0^1\subset Q\setminus K_\lambda$. Because (5.4) converges to K_λ in the Hausdorff metric, we see that $Q_0^1\cap K_\lambda$ contains the four corner points of Q_0^1. These facts and (5.3) imply that $\mathrm{int}Q_m^i\subset Q\setminus K_\lambda\subset G$ and the intersection $Q_m^i\cap K_\lambda\subset\partial G$ contains the four corner points of Q_m^i for every $m\in\mathbb{N}$ and $i=1,2,\dots,4^m$.

Let us fix $m\in\mathbb{N}$. The previous observations imply that there are 4^m cubes $R_1,R_2,\dots,R_{4^m}$ in $\mathcal{W}_G$ that are determined by requiring that the midpoint of Q_m^i is in R_i. Using also the properties of Whitney cubes, we find a constant $N\in\mathbb{N}$ such that

$$2^{-N}\ell(R_i)<\ell\big(Q_m^i\big)=2\kappa\left(\frac{1-\kappa}{2}\right)^m\le 2^N\ell(R_i),\quad i=1,2,\dots,4^m.$$

By the pigeonhole principle, there is an index $k(m)\in\mathbb{Z}$ for which we have $\sharp\mathcal{W}_{k(m)}\ge 4^m/2N$ and

$$2^{-N-k(m)}<2\kappa\left(\frac{1-\kappa}{2}\right)^m\le 2^{N-k(m)}.$$

Solving m gives us the inequalities

$$\text{(5.5)}\qquad \frac{k(m)-N+\log_2(2\kappa)}{\log_2(2/(1-\kappa))}\le m<\frac{k(m)+N+\log_2(2\kappa)}{\log_2(2/(1-\kappa))}.$$

By using the first inequality in (5.5) and the identity

$$\lambda=-\frac{\log 4}{\log r(\kappa)}=\frac{2}{\log_2(2/(1-\kappa))},$$

we obtain the estimate

$$\text{(5.6)}\quad \sharp\mathcal{W}_{k(m)}\ge 4^m/2N\ge \underbrace{(2N)^{-1}4^{\frac{-N+\log_2(2\kappa)}{\log_2(2/(1-\kappa))}}}_{=:c_{N,\kappa}}\cdot 2^{\frac{2k(m)}{\log_2(2/(1-\kappa))}}=c_{N,\kappa}2^{k(m)\lambda}.$$

The second inequality in (5.5) implies that $\lim_{m\to\infty}k(m)=\infty$. Hence, using also (5.6), we have

$$\sup\big\{\sharp\mathcal{W}_k\cdot 2^{-\lambda k}:k\ge k_0\big\}\ge c_{N,\kappa}>0\quad\text{if } k_0\in\mathbb{N}.$$

The inequality (5.2) follows by taking the limit as $k_0\to\infty$. □

Let us fix $s > 1$ and let Q in $\mathbb{R}^n$ be a closed cube that is centered at $x = (x_1, \ldots, x_n)$, and whose side-length is $\ell(Q) = \ell \le 1$. That is,

$$Q := \prod_{i=1}^{n} [x_i - \ell/2, x_i + \ell/2].$$

The *room* in Q is the open cube

$$R(Q) := \operatorname{int}\left(\frac{1}{4}Q\right) = \prod_{i=1}^{n} (x_i - \ell/8, x_i + \ell/8)$$

whose center is x and side-length is $\ell/4$. The *s-passage* in Q is the open set

$$P_s(Q) := \left(\prod_{i=1}^{n-1} \big(x_i - (\ell/8)^s, x_i + (\ell/8)^s\big)\right) \times (x_n + \ell/8, x_n + \ell/4).$$

Note that $\ell/8 < 1$ and $s > 1$, so that we have $(\ell/8)^s < \ell/4$. Hence $P_s(Q) \subset \frac{1}{2}Q$. The *long s-passage* in Q is the open set

$$L_s(Q) := \left(\prod_{i=1}^{n-1} \big(x_i - (\ell/8)^s, x_i + (\ell/8)^s\big)\right) \times (x_n, x_n + \ell/2) \subset Q.$$

The *s-apartment* of Q is the set

$$A_s(Q) := L_s(Q) \cup Q \setminus \big(\partial R(Q) \cup \partial P_s(Q)\big) \subset Q, \tag{5.7}$$

see Figure 1.

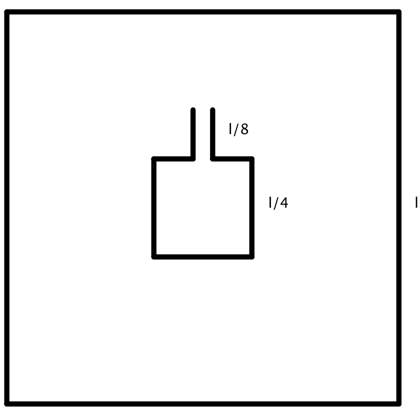

FIGURE 1. The s-apartment $A_s(Q)$.

DEFINITION 5.2. If G in $\mathbb{R}^n$ is a 1-John domain and $s>1$, then the *s-version of* G is the domain

$$G_s := Q_0 \cup \bigcup_{\substack{Q\in\mathcal{W}_G\\ Q\neq Q_0}} A_s(Q).$$

Recall that $\mathcal{W}_G$ is a Whitney decomposition of a bounded domain G, and Q_0 is the Whitney cube containing the 1-John center x_0 of G.

REMARK 5.3. Since the s-apartment in $Q\in\mathcal{W}_G$ is a subset of Q, we have

$$G_s \subset \bigcup_{Q\in\mathcal{W}_G} Q = G.$$

The boundary of the s-version of G is given by

$$\partial G_s = \partial G \cup \bigcup_{\substack{Q\in\mathcal{W}_G\\ Q\neq Q_0}} \partial A_s(Q)\setminus \partial Q.$$

In particular, the countable stability of the Hausdorff dimension implies that $\dim_{\mathcal{H}}(\partial G_s) = \dim_{\mathcal{H}}(\partial G)$.

The upper Minkowski dimension is lacking the countable stability property. Therefore, we need the following computation to verify that the upper Minkowski dimension of the boundary is preserved.

PROPOSITION 5.4. *Let G in $\mathbb{R}^n$ be a 1-John domain. Then* $\dim_{\mathcal{M}}(\partial G) = \dim_{\mathcal{M}}(\partial G_s)$ *for every* $s>1$.

Proof. Because $\partial G\subset \partial G_s$, the upper Minkowski dimension of ∂G is bounded by the upper Minkowski dimension of ∂G_s. Fix $\lambda > \dim_{\mathcal{M}}(\partial G)$. It remains to show that

$$\limsup_{r\to 0+} \mathcal{M}_\lambda(\partial G_s, r) < \infty.$$

Let us fix $r\in(0,1)$ and an integer J such that $2^J < r^{-1} \le 2^{J+1}$. Remark 5.3 yields

$$\big|\partial G_s + B^n(0,r)\big| \le \big|\partial G + B^n(0,r)\big| + \Big|\bigcup_{Q\in\mathcal{W}_G} \big(\partial A_s(Q)\setminus\partial Q\big) + B^n(0,r)\Big|. \tag{5.8}$$

By using the properties of Whitney cubes, we have

$$\begin{aligned}\Big|\bigcup_{\substack{Q\in\mathcal{W}_G\\ \ell(Q)<2^{-J}}} \big(\partial A_s(Q)\setminus\partial Q\big) + B^n(0,r)\Big| \\ \le \Big|\bigcup_{\substack{Q\in\mathcal{W}_G\\ \ell(Q)<2^{-J}}} \big(Q + B^n(0,r)\big)\Big| \le \big|\partial G + B^n(0,cr)\big|.\end{aligned} \tag{5.9}$$

Here the constant $c\ge 1$ is independent of r.

On the other hand, we have

$$\Big| \bigcup_{\substack{Q\in\mathcal{W}_G\\ \ell(Q)\ge 2^{-J}}} \big(\partial A_s(Q)\setminus\partial Q\big) + B^n(0,r)\Big| \le \sum_{j=0}^{J}\sum_{Q\in\mathcal{W}_j} \big|\big(\partial A_s(Q)\setminus\partial Q\big) + B^n(0,r)\big|. \tag{5.10}$$

We bound $\sharp\mathcal{W}_j$ by the number N_j of those cubes whose side-length is 2^{-j} and which belong to the Whitney decomposition of $\mathbb{R}^n\setminus\partial G$. Since $\dim_{\mathcal{M}}(\partial G)<\lambda$ and $|\partial G|=0$, see [13, Corollary 6.4], we can use Theorem 3.12 in [13] to conclude that N_j is bounded by a constant multiple of $2^{j\lambda}$. Also, the Lebesgue measure of $(\partial A_s(Q)\setminus\partial Q)+B^n(0,r)$ is bounded by a constant multiple of $r\cdot\ell(Q)^{n-1}$ if $Q\in\mathcal{W}_j$ and $0\le j\le J$. Combining the estimates above yields

$$\sum_{j=0}^{J}\sum_{Q\in\mathcal{W}_j} \big|\big(\partial A_s(Q)\setminus\partial Q\big) + B^n(0,r)\big| \le cr\cdot\sum_{j=0}^{J} 2^{j(\lambda-n+1)} \le cr2^{J(\lambda-n+1)} = cr^{n-\lambda}. \tag{5.11}$$

In the penultimate step, we used the estimate $\lambda>\dim_{\mathcal{M}}(\partial G)\ge n-1$.

By combining the estimates (5.8), (5.9), (5.10), and (5.11) above, we find that

$$\limsup_{r\to 0+}\mathcal{M}_\lambda(\partial G_s,r)\le \limsup_{r\to 0+}\frac{2\cdot|\partial G+B^n(0,cr)|+cr^{n-\lambda}}{r^{n-\lambda}}<\infty.$$

In the last step, we used the estimate $\lambda>\dim_{\mathcal{M}}(\partial G)$. □

PROPOSITION 5.5. *Let $s>1$ and let G be a 1-John domain in $\mathbb{R}^n$ with 1-John center x_0 in G. Then the s-version of G, denoted by G_s, is an s-John domain with s-John center x_0.*

Proof. Let x be a point in G_s and $\delta:[0,l]\to G$, $l\le c$, be a path parametrized by its arc length such that $\delta(0)=x$, $\delta(l)=x_0$, and

$$\operatorname{dist}\big(\delta(t),\partial G\big)\ge t/c \quad \text{for } t\in[0,l]; \tag{5.12}$$

where the positive constant c is independent of x and $\delta(t)\ne x_0$ if $t<l$.

We will construct a path $\gamma:[0,l_1]\to G_s$ connecting x to x_0 as in the definition of s-John domains. The idea behind the construction is to follow the path δ if this is possible, and to modify it otherwise in a quantitatively controlled manner. Note that the modification may be required since ∂G is a

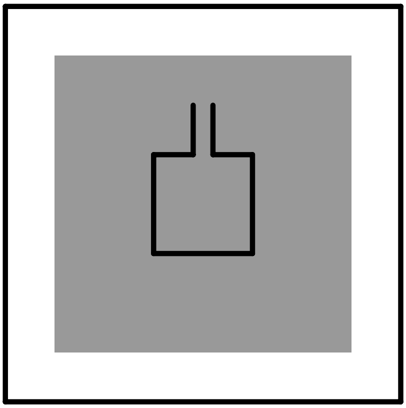

FIGURE 2. $E(Q)$.

proper subset of ∂G_s. To take care of the additional boundary points, we let $Q \in \mathcal{W}_G$, $Q \neq Q_0$, and define

$$E(Q) := \prod_{i=1}^{n} (x_i - 3\ell/8, x_i + 3\ell/8) \subset Q,$$

where $x = (x_1, \ldots, x_n)$ is the center of Q and $\ell = \ell(Q)$, see Figure 2. For later purposes, it is convenient to define $E(Q_0) = \emptyset$.

The following estimates are used while constructing the path γ. Here $\kappa \in (0,1)$ is a constant that is independent of the Whitney cubes. First,

$$\operatorname{dist}(y, \partial G_s) \geq \kappa \ell(Q) \quad \text{for } y \in Q \setminus E(Q) \text{ and } Q \in \mathcal{W}_G. \tag{5.13}$$

A useful property of Whitney cubes is the following:

$$\ell(Q) \geq \kappa \operatorname{dist}(y, \partial G) \quad \text{for } y \in Q \text{ and } Q \in \mathcal{W}_G. \tag{5.14}$$

We also use the following observation: Let $Q \in \mathcal{W}_G$, $Q \neq Q_0$. Then we can join any pair of points $z \in \overline{E(Q)}$ and $\omega \in \partial Q$ by using a rectifiable path parametrized by its arc length $\pi : [0, \rho] \to Q \cap G_s$ such that

$$\ell(Q) \geq \kappa \rho \tag{5.15}$$

and

$$\forall t \in [0, \rho] : \operatorname{dist}\big(\pi(t), \partial G_s\big) \geq \begin{cases} \kappa t^s, & \text{if } z \in E(Q); \\ \kappa \ell(Q), & \text{if } z \in \partial E(Q). \end{cases} \tag{5.16}$$

The construction of γ is based on an iterative algorithm. Hence, it is convenient to introduce the following invariant that allows us to keep track of the partial path that has already been constructed during the previous steps. We say that γ_r satisfies the (r,u)-*invariant* if $r \geq 0$, $u \in [0, l]$, and

$\gamma_r : [0,r] \to G_s$ is a path parametrized by its arc length and satisfying the following conditions (1)–(3):

(1) $r \le 8\kappa^{-1}u$;

(2) $\gamma_r(0) = x$, $\gamma_r(r) = \delta(u)$;

(3) $\operatorname{dist}(\gamma_r(t), \partial G_s) \ge \tau t^s$ if $t \in [0,r]$.

In (3) we have written

$$\tau = \min\{\kappa, 8^{-s}\kappa^{s+2}c^{-s}\} > 0.$$

Our goal is to construct $\gamma = \gamma_{l_1}$ which satisfies the (l_1, l)-invariant. Before the construction, let us introduce the following three steps that are used in the iterative process.

Step I. Let us assume that

$$\delta(0) = x \in E(Q) \quad \text{for some } Q \in \mathcal{W}_G.$$

Recall that we have defined $E(Q_0) = \emptyset$ and therefore $Q \ne Q_0$. Since δ will reach $x_0 \in Q_0$, there is $u \in (0,l]$ such that $\delta(u) \in \partial Q$. Let us join $z = x \in E(Q)$ to $\omega = \delta(u) \in \partial Q$ by a path $\gamma_\sigma : [0,\sigma] \to Q \cap G_s$ satisfying (5.15) and (5.16) with $\rho = \sigma$. We claim that γ_σ satisfies the (σ, u)-invariant. First, it is a rectifiable path parametrized by its arc length whose trace lies in G_s. The other conditions:

(1) By (5.15) we have $u \ge \operatorname{dist}(\partial Q, E(Q)) = \ell(Q)/8 \ge 8^{-1}\kappa\sigma$.

(2) We have $\gamma_\sigma(0) = x$ and $\gamma_\sigma(\sigma) = \delta(u)$.

(3) If $t \in [0,\sigma]$ we use (5.16) for $\operatorname{dist}(\gamma_\sigma(t), \partial G_s) \ge \kappa t^s \ge \tau t^s$.

Step II. Let us assume that γ_r satisfies the (r,u)-invariant and

$$\gamma_r(r) = \delta(u) \in \partial E(Q) \quad \text{for some } Q \in \mathcal{W}_G.$$

There is a time $\bar{u} \in (u,l]$ such that $\delta(\bar{u}) \in \partial Q$. Join $z = \delta(u) \in \partial E(Q)$ to $\omega = \delta(\bar{u}) \in \partial Q$ by a path $\Pi : [0,\sigma] \to Q \cap G_s$ satisfying both (5.15) and (5.16) with $\rho = \sigma$. Then, we define

$$\gamma_{r+\sigma}(t) = \begin{cases} \gamma_r(t) & \text{for } t \in [0,r]; \\ \Pi(t-r) & \text{for } t \in [r, r+\sigma]. \end{cases}$$

We claim that $\gamma_{r+\sigma}$ satisfies the $(r+\sigma, \bar{u})$-invariant. It is an arc length parametrized path whose trace lies in G_s. The other conditions:

(1) We have $\bar{u} - u \ge \operatorname{dist}(\partial Q, \partial E(Q)) = \ell(Q)/8$. Using also (5.15) yields

$$r + \sigma \le 8\kappa^{-1}u + \kappa^{-1}\ell(Q) \le 8\kappa^{-1}(u + \bar{u} - u) = 8\kappa^{-1}\bar{u}. \tag{5.17}$$

(2) We have $\gamma_{r+\sigma}(0) = \gamma_r(0) = x$ and $\gamma_{r+\sigma}(r+\sigma) = \Pi(\sigma) = \delta(\bar{u})$.

(3) If $t \in [0,r]$ we have $\operatorname{dist}(\gamma_{r+\sigma}(t), \partial G_s) = \operatorname{dist}(\gamma_r(t), \partial G_s) \ge \tau t^s$. If $t \in (r, r+\sigma]$, we use (5.16), (5.14), (5.12), and (5.17) for the estimate

$$\begin{aligned} \operatorname{dist}\big(\gamma_{r+\sigma}(t), \partial G_s\big) &= \operatorname{dist}\big(\Pi(t-r), \partial G_s\big) \\ &\ge \kappa\ell(Q) \ge \kappa^2 \operatorname{dist}\big(\delta(\bar{u}), \partial G\big) \ge \kappa^2 c^{-1}\bar{u} \ge 8^{-1}\kappa^3 c^{-1} t. \end{aligned}$$

Note that again by (5.17), we have $0 < t \le 8\kappa^{-1}\bar{u} \le 8\kappa^{-1}l \le 8\kappa^{-1}c$. Since $1-s \le 0$, we obtain

$$t = t^{1-s}t^s \ge \left(8\kappa^{-1}c\right)^{1-s}t^s = 8^{1-s}\kappa^{s-1}c^{1-s}t^s.$$

Hence, we have the estimate $\operatorname{dist}(\gamma_{r+\sigma}(t), \partial G_s) \ge (8^{-s}\kappa^{s+2}c^{-s})t^s \ge \tau t^s$.

Step III. Let us assume that γ_r satisfies the (r,u)-invariant and

$$\gamma_r(r) = \delta(u) \in Q \setminus \overline{E(Q)} \quad \text{for some } Q \in \mathcal{W}_G.$$

By following δ from time u forwards, we will first arrive either at x_0 or $\partial E(Q)$ for some $Q_0 \ne Q \in \mathcal{W}_G$. Denote by $\bar{u} \in [u,l]$ this time of arrival, and define

$$\gamma_{r+\bar{u}-u}(t) = \begin{cases} \gamma_r(t) & \text{for } t \in [0,r], \\ \delta(t-r+u) & \text{for } t \in [r, r+\bar{u}-u]. \end{cases}$$

We claim that $\gamma_{r+\bar{u}-u}$ satisfies the $(r+\bar{u}-u, \bar{u})$-invariant. It is a path parametrized by its arc length and whose trace lies in G_s. The other properties:

(1) Let $\varepsilon \in [0, \bar{u}-u]$. Since $8\kappa^{-1} > 1$, we have

$$r + \varepsilon \le 8\kappa^{-1}u + \varepsilon \le 8\kappa^{-1}(u+\varepsilon). \tag{5.18}$$

Setting $\varepsilon = \bar{u} - u$ yields $r + \bar{u} - u \le 8\kappa^{-1}\bar{u}$.

(2) We have $\gamma_{r+\bar{u}-u}(0) = \gamma_r(0) = x$ and $\gamma_{r+\bar{u}-u}(r+\bar{u}-u) = \delta(\bar{u})$.

(3) If $t \in [0,r]$ we have $\operatorname{dist}(\gamma_{r+\bar{u}-u}(t), \partial G_s) = \operatorname{dist}(\gamma_r(t), \partial G_s) \ge \tau t^s$.

Assuming that $t \in [r, r+\bar{u}-u]$, we have

$$\operatorname{dist}\bigl(\gamma_{r+\bar{u}-u}(t), \partial G_s\bigr) = \operatorname{dist}\bigl(\delta(t-r+u), \partial G_s\bigr).$$

Let us fix $Q_t \in \mathcal{W}_G$ such that $\delta(t-r+u) \in Q_t \setminus E(Q_t)$. By using (5.13), (5.14), (5.12), and (5.18), we see that

$$\begin{aligned} \operatorname{dist}\bigl(\delta(t-r+u), \partial G_s\bigr) &\ge \kappa\ell(Q_t) \ge \kappa^2 \operatorname{dist}\bigl(\delta(t-r+u), \partial G\bigr) \\ &\ge \kappa^2 c^{-1}(u+t-r) \\ &\ge \kappa^2 c^{-1}\left(8\kappa^{-1}\right)^{-1}(r+t-r) = 8^{-1}\kappa^3 c^{-1} t. \end{aligned}$$

Inequalities (5.18) yield

$$0 < t \le r + \bar{u} - u \le 8\kappa^{-1}\bar{u} \le 8\kappa^{-1}l \le 8\kappa^{-1}c.$$

Proceeding as in the end of Step II, we obtain the estimate

$$\operatorname{dist}\bigl(\gamma_{r+\bar{u}-u}(t), \partial G_s\bigr) \ge \tau t^s.$$

Having introduced these steps, we can now construct the path γ as follows. Let $x \in Q \in \mathcal{W}_G$. If $x \in E(Q)$, we apply Step I and obtain γ_σ satisfying the (σ,u)-invariant. Otherwise we write $\sigma = u = 0$ and define $\gamma_0(0) = x$. In any case, this procedure yields a path γ_σ which satisfies the (σ,u)-invariant and the condition $\gamma_\sigma(\sigma) \in Q \setminus E(Q)$ with $Q \in \mathcal{W}_G$. Assuming that $\gamma_\sigma(\sigma) \ne x_0$, we then proceed by invoking either Step II or Step III, depending on the situation. We keep on iterating these steps in alternating turns until, after a finite number of steps, we obtain a path γ_{l_1} satisfying the (l_1, l)-invariant as required. The process will end because every time we invoke Step II, we make at least

$$\min\big\{\ell(Q)/8 : Q \in \mathcal{W}_G \text{ and } \delta[0,l] \cap Q \neq \emptyset\big\} > 0$$

of progress along the path δ. This is seen by examining the proof of the condition (1) in Step II. □

We can now state one of the main result in this section.

THEOREM 5.6. *Let G in $\mathbb{R}^n$ be a 1-John domain such that*

$$\dim_{\mathcal{M}}(\partial G) = \lambda \in [n-1, n).$$

Then, for every $s > 1$, the s-version of G is an s-John domain with $\dim_{\mathcal{M}}(\partial G_s) = \lambda$ and it is not a (q,p)-Poincaré domain if $1 \leq q \leq p < \infty$ and

$$\frac{(p-q)(\lambda-n)}{pq} + \frac{(s-1)(n-1)}{p} > 1. \tag{5.19}$$

Proof. Let us assume that $s > 1$. The s-version of G is an s-John domain by Proposition 5.5. The upper Minkowski dimension of ∂G_s is λ by Proposition 5.4.

Let us then verify the claim concerning the (q,p)-Poincaré property. Choose $\lambda' \in (0, \lambda)$ so that (5.19) is true with λ replaced by λ'. Hence, by denoting λ' by λ, we may assume that the upper Minkowski dimension of ∂G is strictly greater than $\lambda \in (0,n)$. This fact is used as follows:

By both Theorem 3.12 and Lemma 6.5 in [13], we obtain the estimate

$$1 \leq \limsup_{m\to\infty} 2^{-\lambda m} \cdot N_m \leq c \limsup_{m\to\infty} 2^{-\lambda m} \cdot \left(\sum_{M=m-2}^{m+2} \sharp\mathcal{W}_M\right);$$

where N_m denotes the number of cubes in the Whitney decomposition of $\mathbb{R}^n \setminus \partial G$ whose side-length is 2^{-m} and c is a positive constant depending only on G and n. Choose $k_0 \in \mathbb{N}$ such that

$$\limsup_{m\to\infty} 2^{-\lambda(m+2-k_0)} \cdot \left(\sum_{M=m-2}^{m+2} \sharp\mathcal{W}_M\right) > 10.$$

Let $k \in \mathbb{N}$ and then choose $m := m(k) > \max\{k, k_0, -\log_2 \ell(Q_0)\} + 2$ and $j = j(k) \in \{m-2, \dots, m+2\}$ such that

$$\sharp\mathcal{W}_j \geq \left(\sum_{M=m-2}^{m+2} \sharp\mathcal{W}_M\right)\Big/ 5 \geq 10 \cdot 2^{\lambda(m+2-k_0)}/5 \geq 2 \cdot 2^{\lambda(j-k_0)}. \tag{5.20}$$

Let us write $M_j := 2^{[\lambda(j-k_0)]}$, where $[\lambda(j-k_0)]$ means the integer-part of $\lambda(j-k_0) \geq 0$, and choose cubes

$$Q_j^1, \dots, Q_j^{2M_j} \in \mathcal{W}_j \setminus \{Q_0\}.$$

This can be done because of (5.20).

Let $Q = Q_j^i$ for some i. To the s-apartment $A_s(Q)$ in Q, we associate the function $u_{A_s(Q)} : G_s \to \mathbb{R}$ which has linear decay along the nth variable in $P_s(Q)$ and satisfies

$$u_{A_s(Q)}(x) = \begin{cases} \ell(Q)^{(\lambda-n)/q}, & \text{if } x \in R(Q); \\ 0, & \text{if } x \in G_s \setminus (R(Q) \cup \overline{P_s(Q)}). \end{cases} \tag{5.21}$$

Its partial derivatives in $\mathcal{D}'(G_s)$ are given by

$$\nabla u_{A_s(Q)} = \big(0, \ldots, 0, -8\ell(Q)^{(\lambda-n)/q-1}\chi_{P_s(Q)}\big) \tag{5.22}$$

pointwise almost everywhere.

Let us define

$$u_j := \sum_{i=1}^{M_j} u_{A_s(Q_j^i)} - \sum_{i=M_j+1}^{2M_j} u_{A_s(Q_j^i)} \in W^{1,p}(G_s). \tag{5.23}$$

Note that

$$(u_j)_{G_s} = \frac{1}{|G_s|}\int_{G_s} u_j = 0 \tag{5.24}$$

because the integrals of functions $u_{A_s(Q_j^i)}$ are independent of i. It is also important to realize that the supports of the functions $u_{A_s(Q_j^i)}$ are mutually disjoint as i varies.

Using (5.24) and (5.21), we obtain

$$\begin{aligned} A_j &:= \left(\int_{G_s} \big|u_j - (u_j)_{G_s}\big|^q\right)^{1/q} = \left(\sum_{i=1}^{2M_j}\int_{G_s} |u_{A_s(Q_j^i)}|^q\right)^{1/q} \\ &\geq \big(2 \cdot 2^{\lambda(j-k_0)-1} \cdot 2^{-j(\lambda-n)} \cdot 4^{-n} \cdot 2^{-jn}\big)^{1/q} = c_{n,q,\lambda,k_0}; \end{aligned} \tag{5.25}$$

where $c_{n,q,\lambda,k_0} > 0$ depends on the indicated parameters. On the other hand, by using (5.22), we obtain

$$\begin{aligned} B_j &:= \left(\int_{G_s} |\nabla u_j|^p\right)^{1/p} \\ &= \left(\sum_{i=1}^{2M_j}\int_{G_s} |\nabla u_{A_s(Q_j^i)}|^p\right)^{1/p} \\ &\leq \big(2 \cdot 2^{\lambda(j-k_0)} \\ &\quad \cdot \big(8 \cdot 2^{-j((\lambda-n)/q-1)}\big)^p \cdot \big(2 \cdot \big(2^{-j}/8\big)^s\big)^{n-1} \cdot 2^{-j}/8\big)^{1/p} \\ &= c_{n,s,p,\lambda,k_0} 2^{j(1-(p-q)(\lambda-n)/pq-(s-1)(n-1)/p)}; \end{aligned} \tag{5.26}$$

where $c_{n,s,p,\lambda,k_0} > 0$ depends on the indicated parameters.

By combining the estimates (5.25) and (5.26), we obtain

$$\frac{A_j}{B_j} \geq c_{n,s,p,q,\lambda,k_0} 2^{j(-1+(p-q)(\lambda-n)/pq+(s-1)(n-1)/p)}. \tag{5.27}$$

Recall that $j = j(k) \geq k$. Hence, by using both (5.27) and (5.19), we find that the sequence $(A_{j(k)}/B_{j(k)})_{k=1}^{\infty}$ tends to ∞ as $k \to \infty$. This allows us to conclude that G_s is not a (q,p)-Poincaré domain. □

Under further assumptions, we can replace the inequality in (5.19) by the identity. This is the content of the following theorem which can be used to provide sharp counter-examples if $q < p$.

THEOREM 5.7. *Let G be a 1-John domain in $\mathbb{R}^n$ such that*

$$\limsup_{k\to\infty} 2^{-\lambda k} \cdot \sharp\mathcal{W}_k > 0, \quad \textit{where } \lambda = \dim_{\mathcal{M}}(\partial G) \in [n-1, n).$$

Then, for every $s > 1$, the s-version of G is an s-John domain with $\dim_{\mathcal{M}}(\partial G_s) = \lambda$ and it is not a (q,p)-Poincaré domain if $1 \leq q < p < \infty$ and

$$\frac{(p-q)(\lambda-n)}{pq} + \frac{(s-1)(n-1)}{p} \geq 1. \tag{5.28}$$

Proof. According to Theorem 5.6, we only need to verify that G_s is not a (q,p)-Poincaré domain if the left-hand side of (5.28) is equal to one. To this end, we choose $k_0 \in \mathbb{N}$ such that

$$\limsup_{k\to\infty} 2^{-\lambda(k-k_0)} \cdot \sharp\mathcal{W}_k > 2.$$

This allows us to inductively choose indices $j(k)$, $k \in \mathbb{N}$, such that

$$\max\{k_0, -\log_2 \ell(Q_0)\} < j(1) < j(2) < \cdots$$

and $\sharp\mathcal{W}_{j(k)} \geq 2 \cdot 2^{\lambda(j(k)-k_0)}$ for every $k \in \mathbb{N}$. For every $j = j(k)$, we proceed as in Theorem 5.6; we begin from (5.20) and continue until we reach (5.23). This yields functions $u_{j(k)} \in W^{1,p}(G_s)$. Then, for each $m \in \mathbb{N}$ we define

$$v_m = \sum_{k=1}^{m} u_{j(k)} \in W^{1,p}(G_s).$$

Estimating further as in the proof of Theorem 5.6, we have $(v_m)_{G_s} = 0$ and

$$\begin{aligned} C_m &:= \left(\int_{G_s} \left| v_m - (v_m)_{G_s} \right|^q \right)^{1/q} \\ &= \left(\sum_{k=1}^{m} \sum_{i=1}^{2M_{j(k)}} \int_{G_s} |u_{A_s(Q^i_{j(k)})}|^q \right)^{1/q} \geq c_{n,q,\lambda,k_0} m^{1/q}. \end{aligned}$$

Furthermore, by using (5.28), we have

$$D_m := \left(\int_{G_s} |\nabla v_m|^p\right)^{1/p}$$
$$= \left(\sum_{k=1}^{m}\sum_{i=1}^{2M_{j(k)}} \int_{G_s} |\nabla u_{A_s(Q^i_{j(k)})}|^p\right)^{1/p} \leq c_{n,s,p,\lambda,k_0} m^{1/p}.$$

Concluding from above and using the assumption that $q < p$, we find that

$$\frac{C_m}{D_m} \geq c_{n,s,p,q,k_0,\lambda} m^{1/q-1/p} \xrightarrow{m\to\infty} \infty.$$

This shows that G_s is not a (q,p)-Poincaré domain. □

Acknowledgments. The authors thank the referee for comments which led to improvements of the manuscript.

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