

BOUNDS FOR SECTIONAL GENERA OF VARIETIES INVARIANT UNDER PFAFF FIELDS

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ABSTRACT. We establish an upper bound for the sectional genus of varieties which are invariant under Pfaff fields on projective spaces.

1. Introduction

In [20], P. Painlevé asked the following question: “*Is it possible to recognize the genus of the general solution of an algebraic differential equation in two variables which has a rational first integral?*” In [16], Lins Neto has constructed families of foliations with fixed degree and local analytic type of the singularities where foliations with rational first integral of arbitrarily large degree appear. In other words, such families show that Painlevé’s question has a negative answer.

However, one can obtain an affirmative answer to Painlevé’s question provided some additional hypotheses are made. The problem of bounding the genus of an invariant curve in terms of the degree of a foliation on \mathbb{P}^n has been considered by several authors, see for instance [6], [8]. In [3], Campillo, Carnicer and de la Fuente showed that if C is a reduced curve which is invariant by a one-dimensional foliation \mathcal{F} on \mathbb{P}^n then

$$(1) \quad \frac{2p_a(C) - 2}{\deg(C)} \leq \deg(\mathcal{F}) - 1 + a,$$

where $p_a(C)$ is the arithmetic genus of C and a is an integer obtained from the concrete problem of imposing singularities to projective hypersurfaces. For instance, if C has only nodal singularities then $a = 0$, and thus formula

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(1) follows from [11]. This bound has been improved by Esteves and Kleiman in [8].

Painlevé's question is related to the problem posed by Poincaré in [23] of bounding the degree of algebraic solutions of an algebraic differential equation on the complex plane. Nowadays, this problem is known as *Poincaré's Problem*. Many mathematicians have been working on it and on some of its generalizations, see, for instance, the papers by Cerveau and Lins Neto [6], Carnicer [4], Pereira [21], Soares [24], Brunella and Mendes [2], Esteves and Kleiman [8], Cavalier and Lehmann [5], and Zamora [28].

In [8], Esteves and Kleiman extended Jouanolou's work on algebraic Pfaff systems on a nonsingular scheme V . Essentially, an algebraic Pfaff system is a singular distribution. More precisely, an algebraic Pfaff system of rank r on a nonsingular scheme X of pure dimension n is, according to Jouanolou [13, pp. 136–138], a nonzero map $u: E \rightarrow \Omega_X^1$ where E is a locally free sheaf of constant rank r with $1 \leq r \leq n - 1$. Esteves and Kleiman introduced the notion of a *Pfaff field* on V , which is a nontrivial sheaf map $\eta: \Omega_V^k \rightarrow L$, where L is an invertible sheaf on V , and the integer $1 \leq k \leq n - 1$ is called the rank of η . A subvariety $X \subset V$ is said to be invariant under η if the map η factors through the natural map $\Omega_V^k|_X \rightarrow \Omega_X^k$. A Pfaff system on V induces, via exterior powers and the perfect pairing of differential forms, a Pfaff field on V . However, the converse is not true; see [8, Section 3] for more details.

In this paper, we establish new upper bounds for the sectional genera of nonsingular projective varieties which are invariant under Pfaff fields on \mathbb{P}^n .

First, we use the hypothesis of stability (in the sense of Mumford–Takemoto) of the tangent bundle of X to establish an upper bound for the sectional genus in terms of the degree and the rank of a Pfaff field.

More precisely, our first main result is the following. Let $g(X, \mathcal{O}_X(1))$ denote the sectional genus of X with respect to the line bundle $\mathcal{O}_X(1)$ associated to the hyperplane section.

THEOREM 1. *Let X be a nonsingular projective variety of dimension m which is invariant under a Pfaff field \mathcal{F} of rank k on \mathbb{P}^n ; assume that $m \geq k$. If the tangent bundle Θ_X is stable, then*

$$(2) \quad \frac{2g(X, \mathcal{O}_X(1)) - 2}{\deg(X)} \leq \frac{\deg(\mathcal{F}) - k}{\binom{m-1}{k-1}} + m - 1.$$

To the best of our knowledge, this is the first time that the stability of the tangent bundle is used to obtain such bounds. Notice that the left-hand side of inequality (2) does not change when we take generic linear sections $\mathbb{P}^l \subset \mathbb{P}^n$, while the right-hand side gets larger, and so the bound becomes worse. This means that the above result is a truly higher dimensional one.

Examples of projective varieties with stable tangent bundle are Calabi–Yau [27], Fano [9], [12], [22], [25] and complete intersection [22], [26] varieties.

In the critical case when the rank of Pfaff field \mathcal{F} is equal to the dimension of the invariant variety X , we show that one can substitute for the stability condition the conditions of X being Gorenstein and smooth in codimension 1, that is, $\text{codim}(\text{Sing}(X), X) \geq 2$.

THEOREM 2. *Let $X \subset \mathbb{P}^n$ be a Gorenstein projective variety nonsingular in codimension 1, which is invariant under a Pfaff field \mathcal{F} on \mathbb{P}^n whose rank is equal to the dimension of X . Then*

$$(3) \quad \frac{2g(X, \mathcal{O}_X(1)) - 2}{\text{deg}(X)} \leq \text{deg}(\mathcal{F}) - 1.$$

This generalizes a bound obtained by Campillo, Carnicer and de la Fuente in [3, Theorem 4.1(a)]. As an application, we improve upon a bound obtained by Cruz and Esteves [7, Corollary 4.5], see Section 5.

This note is organized as follows. First, in order to make this presentation as self-contained as possible, we provide all the necessary definitions in Section 2. The proofs of our main results along with some further consequences are given in Sections 4 and 3.

2. Background material

We work over the field of complex numbers. Let (X, L) be a Gorenstein projective variety X of dimension n equipped with a very ample line bundle L ; recall that, since X is Gorenstein, the canonical divisor K_X is a Cartier divisor.

DEFINITION 1. The *sectional genus* of X with respect to L , denoted $g(X, L)$, is defined by the formula:

$$2g(X, L) - 2 = (K_X + (\dim(X) - 1)L) \cdot L^{\dim(X)-1}.$$

This quantity has the following geometric interpretation. Suppose that X is nonsingular, and let H_1, \dots, H_{n-1} be general elements in the linear system $|L|$. By Bertini’s Theorem, the curve $X_{n-1} = H_1 \cap \dots \cap H_{n-1}$ is nonsingular. Then $g(X, L)$ coincides with the geometric genus of X_{n-1} , see [10, Remark 2.5].

DEFINITION 2. Let (V, L) be a nonsingular polarized algebraic variety. A Pfaff field \mathcal{F} of rank k on V is a nonzero global section of $\wedge^k \Theta_V \otimes N$, where Θ_V is the tangent bundle and N is a line bundle, where $0 < k < n$. The degree of \mathcal{F} with respect to L is defined by the formula $\text{deg}_L(\mathcal{F}) = \text{deg}_L(N) + k \text{deg}_L(L)$, where the degree of a line bundle N relative to L is given by $\text{deg}_L(N) = N \cdot L^{\dim(V)-1}$.

Since the ambient space is nonsingular, our definition is equivalent to the one introduced in [8, Section 3]. In fact, since $\wedge^k \Theta_V \otimes N \simeq \text{Hom}(\Omega_V^k, N) \simeq \text{Hom}(N^*, \wedge^k \Theta_V)$, a Pfaff field can also be regarded either as a map $\xi_{\mathcal{F}} : N^* \rightarrow \wedge^k \Theta_V$ or as a map $\xi_{\mathcal{F}}^{\vee} : \Omega_V^k \rightarrow N$. The present definition emphasizes the

existence of a global section of $\bigwedge^k \Theta_V \otimes N$, which will play a central role in our arguments.

DEFINITION 3. The *singular set* of \mathcal{F} is given by

$$\begin{aligned} \text{Sing}(\mathcal{F}) &= \{x \in V; \xi_{\mathcal{F}}(x) \text{ is not injective}\} \\ &= \{x \in V; \xi_{\mathcal{F}}^{\vee}(x) \text{ is not surjective}\}. \end{aligned}$$

For instance, a Pfaff field of rank k on \mathbb{P}^n is a section of $\bigwedge^k \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(s)$, and $\text{deg}_{\mathcal{O}_{\mathbb{P}^n}(1)}(\mathcal{F}) = s + k$.

More generally, if $\text{Pic}(V) \simeq \mathbb{Z}$ and $L := \mathcal{O}_V(1)$ is the positive generator of $\text{Pic}(V)$, then a Pfaff field of rank k on V is a section of $\bigwedge^k \Theta_V \otimes \mathcal{O}_V(s)$, for some $s \in \mathbb{Z}$. Thus, $\text{deg}_L(\mathcal{F}) = (s + k) \text{deg}(V)$, where $\text{deg}(V) = \text{deg}_L(L)$. If we define $d_{\mathcal{F}} := s + k$, we have $\text{deg}_L(\mathcal{F}) = d_{\mathcal{F}} \cdot \text{deg}(V)$.

Alternatively, a Pfaff field can also be defined as a global section of $\Omega_V^{n-k} \otimes N'$, where $N' = N \otimes K_V^{-1}$. If V is nonsingular, this definition is equivalent to the one above.

Let $X \subset V$ be a closed subscheme of dimension larger than or equal to the rank of a Pfaff field \mathcal{F} . Following [8, Section 3], we introduce the following definition.

DEFINITION 4. We say X is invariant under \mathcal{F} if $X \not\subset \text{Sing}(\mathcal{F})$ and there exists a morphism of sheaves $\phi : \Omega_X^k \rightarrow N|_X$ such that the following diagram

$$\begin{array}{ccc} \Omega_V^k|_X & \xrightarrow{\xi_{\mathcal{F}}^{\vee}|_X} & N|_X \\ \downarrow & \nearrow \phi & \\ \Omega_X^k & & \end{array}$$

commutes.

Applying the functor $\text{Hom}(\cdot, \mathcal{O}_X)$ to the above diagram, we get the following commutative diagram:

$$\begin{array}{ccc} N^*|_X & & \\ \downarrow \phi^{\vee} & \searrow \xi_{\mathcal{F}}|_X & \\ (\Omega_X^k)^{\vee} & \longrightarrow & \bigwedge^k \Theta_V|_X. \end{array}$$

Therefore, X is invariant under \mathcal{F} if $\xi_{\mathcal{F}}|_X$ induces a nonzero global section of $(\Omega_X^k)^{\vee} \otimes N|_X$.

Our two main results are concerned only with the case when $V = \mathbb{P}^n$; but we would like to conclude this section with two general propositions.

Let E be a torsion-free sheaf on V . The ratio $\mu_L(E) = \text{deg}_L(E)/\text{rk}(E)$ is called the slope of E , where $\text{deg}_L(E) = \text{deg}_L((\Lambda^r E)^{\vee\vee})$ and $r = \text{rk}(E)$. Recall

that E is *semistable* (in the sense of Mumford–Takemoto) if every torsion-free subsheaf E' of E satisfies $\mu_L(E') \leq \mu_L(E)$. Furthermore, E is *stable* if the strict inequality is satisfied for proper subsheaves. Further details can be found in [14, Sections V.6 and V.7].

PROPOSITION 5. *If Θ_V is stable, then the following inequality holds:*

$$\deg_L(\mathcal{F}) \geq \operatorname{rk}(\mathcal{F}) \left(\deg_L(V) + \frac{\deg_L(K_V)}{\dim(V)} \right).$$

If $V = \mathbb{P}^n$ the above inequality becomes $\deg(\mathcal{F}) \geq 0$. Bott’s formula [19, page 8] implies the existence of a rank k Pfaff field of degree 0 for each k , hence in this case the bound given above is sharp.

Proof of Proposition 5. The stability of Θ_V implies that $\bigwedge^k \Theta_V$ is semi-stable with slope equal to $k\mu_L(\Theta_V)$ [1, Corollary 1.6]. As observed above, a Pfaff field \mathcal{F} of rank k induces a map $\xi_{\mathcal{F}} : N^* \rightarrow \bigwedge^k \Theta_V$, so from the semistability of $\bigwedge^k \Theta_V$ we conclude that $-\deg_L(N) \leq k\mu_L(\Theta_V) = -k \deg_L(K_V) / \dim(V)$. The stated inequality follows easily. \square

If D is a divisor on an algebraic variety V with $\operatorname{Pic}(V) \simeq \mathbb{Z}$, then $\mathcal{O}_V(D) = \mathcal{O}_V(d_D)$, for some $d_D \in \mathbb{Z}$. In this case, we denote $\kappa(V) = d_{K_V}$.

PROPOSITION 6. *Let V be a n -dimensional nonsingular algebraic variety with $\operatorname{Pic}(V) \simeq \mathbb{Z}$. Let X be a k -dimensional nonsingular complete intersection of hypersurfaces D_1, \dots, D_{n-k} on V . If X is invariant under a Pfaff field \mathcal{F} of rank k on V , then*

$$d_{D_1} + \dots + d_{D_{n-k}} \leq d_{\mathcal{F}} - k - \kappa(V).$$

Proof. Since X is invariant by \mathcal{F} we have that $H^0(X, \bigwedge^k \Theta_X \otimes \mathcal{O}_V(d_{\mathcal{F}} - k)|_X) \neq \{0\}$, then $\deg(\bigwedge^k \Theta_X \otimes \mathcal{O}_V(d_{\mathcal{F}} - k)|_X) \geq 0$. Let $\mathcal{O}_V(D_i)$ be the line bundle associated to the hypersurface D_i , $i = 1, \dots, n - k$. We have the following adjunction formula

$$\bigwedge^k \Theta_X = \bigwedge^k \Theta_V|_X \otimes \mathcal{O}_V(-D_1)|_X \otimes \dots \otimes \mathcal{O}_V(-D_{n-k})|_X.$$

Therefore $\bigwedge^k \Theta_X = \mathcal{O}_V(-\kappa(V) - d_{D_1} - \dots - d_{D_{n-k}})|_X$, thus

$$\begin{aligned} \deg(\mathcal{O}_V(d_{\mathcal{F}} - k - \kappa(V) - d_{D_1} - \dots - d_{D_{n-k}})|_X) \\ = \deg\left(\bigwedge^k \Theta_X \otimes \mathcal{O}_V(d_{\mathcal{F}} - k)|_X\right) \geq 0. \end{aligned} \quad \square$$

3. Proof of Theorem 1

We recall that the stability of Θ_X implies that $\bigwedge^k \Theta_X$ is semistable. Since X is invariant under \mathcal{F} , we can conclude that $H^0(X, \bigwedge^k \Theta_X \otimes \mathcal{O}_X(d-k)) \neq \{0\}$, with $d = \text{deg}(\mathcal{F})$. It then follows from the semistability of $\bigwedge^k \Theta_X$ that $\bigwedge^k \Theta_X \otimes \mathcal{O}_X(d-k)$ is also semistable, thus

$$(4) \quad \text{deg} \left(\bigwedge^k \Theta_X \otimes \mathcal{O}_X(d-k) \right) \geq 0.$$

On the other hand, note that

$$(5) \quad \text{deg} \left(\bigwedge^k \Theta_X \right) = - \binom{\dim(X) - 1}{k-1} \text{deg}(K_X).$$

Let $i : X \rightarrow \mathbb{P}^n$ be the embedding, and set, as usual, $\mathcal{O}_X(1) = i^* \mathcal{O}_{\mathbb{P}^n}(1)$. Now, we consider the following difference, using (5):

$$\begin{aligned} & (2g(X, \mathcal{O}_X(1)) - 2) - \left[\frac{\mathcal{O}_X(d-k)}{\binom{m-1}{k-1}} + (m-1)\mathcal{O}_X(1) \right] \cdot \mathcal{O}_X(1)^{m-1} \\ &= - \left(-K_X + \frac{\mathcal{O}_X(d-k)}{\binom{m-1}{k-1}} \right) \cdot \mathcal{O}_X(1)^{m-1} = - \frac{\text{deg}(\bigwedge^k \Theta_X \otimes \mathcal{O}_X(d-k))}{\binom{m-1}{k-1}}. \end{aligned}$$

It follows from (4) that the difference must be less than or equal to zero, hence

$$\begin{aligned} 2g(X, \mathcal{O}_X(1)) - 2 &\leq \left[\frac{\mathcal{O}_X(d-k)}{\binom{m-1}{k-1}} + (m-1)\mathcal{O}_X(1) \right] \cdot \mathcal{O}_X(1)^{m-1} \\ &\leq \text{deg}(X) \left(\frac{d-k}{\binom{m-1}{k-1}} + m-1 \right). \end{aligned}$$

This completes the proof of Theorem 1.

Let us now consider applications of Theorem 1 to a few particular cases. First, specializing to the case when the invariant variety is Fano with Picard number one, i.e., $\text{deg}(K_X) < 0$ and $\rho(X) = \text{rank}(\text{NS}(X)) = 1$, where $\text{NS}(X)$ is the Néron–Severi group of X .

COROLLARY 7. *Let X be a nonsingular Fano variety, with Picard number one, and let $\mathcal{O}_X(1) := K_X^{-1}$. If X is invariant under a Pfaff field \mathcal{F} of rank $k = \dim(X)$, then*

$$\text{deg}_{K_X^{-1}}(X) \leq k^k (\text{deg}(\mathcal{F}) + 2)^k,$$

where $\text{deg}_{K_X^{-1}}(X)$ is the degree of X with respect to the anticanonical polarization.

Proof. Indeed, in this case we have

$$2g(X, K_X^{-1}) - 2 = (k-2) \text{deg}_{K_X^{-1}}(X).$$

Thus, it follows from Theorem 1 that $k \leq \text{deg}(\mathcal{F}) + 1$. On the other hand, it follows from [18] that $d(X) \leq k + 1$ and $\text{deg}_{K_X^{-1}}(X) \leq (d(X)k)^k$, where $d(X)$ is the least positive integer d for which X can be covered by rational curves of (anticanonical) degree at most d , see [18, Section 1.3]. \square

Finally, we also consider the case when the invariant variety is Calabi–Yau, that is, $K_X = 0$.

COROLLARY 8. *If X is Calabi–Yau and invariant by \mathcal{F} , then $\text{rk}(\mathcal{F}) \leq \text{deg}(\mathcal{F})$.*

In other words, Pfaff fields of small degree do not admit invariant Calabi–Yau varieties.

4. Proof of Theorem 2

First, let us briefly recall the construction of the so-called *canonical map* $\gamma_X : \Omega_X^k \rightarrow \omega_X$, where ω_X is the dualizing sheaf of X , as it was done in [7, Section 3].

Let X be a reduced projective variety of pure dimension k , and let X_1, \dots, X_s be its irreducible components. For each $i = 1, \dots, s$, consider Kunz’s sheaf $\tilde{\omega}_{X_i}$ of regular differential forms of X_i , see [15]. By definition, the canonical map γ_X is the composition

$$\Omega_X^k \xrightarrow{\tilde{\tau}} \bigoplus_{i=1}^s \Omega_{X_i}^k \xrightarrow{(\gamma_1, \dots, \gamma_s)} \bigoplus_{i=1}^s \tilde{\omega}_{X_i} \xrightarrow{(\zeta_1, \dots, \zeta_s)} \bigoplus_{i=1}^s \omega_{X_i} \xrightarrow{\tau} \omega_X,$$

where $\tilde{\tau}$ and τ are the maps induced by restriction, for each $i = 1, \dots, s$ the map $\gamma_i : \Omega_{X_i}^k \rightarrow \tilde{\omega}_{X_i}$ is the canonical class of X_i , constructed by Lipman in [17], which is an isomorphism on the nonsingular locus of X_i . Moreover, $\zeta_i : \tilde{\omega}_{X_i} \rightarrow \omega_{X_i}$ is an isomorphism on X_i , since it follows from [17, Theorem 0.2B] that $\tilde{\omega}_{X_i}$ is dualizing. Therefore, γ_X is an isomorphism on the nonsingular locus $X_0 := X - \text{Sing}(X)$. Thus the map

$$\tilde{\gamma}_X = \gamma_X^\vee \otimes \mathbf{1}_{\mathcal{O}_X(d-k)} : \omega_X^\vee \otimes \mathcal{O}_X(d-k) \rightarrow (\Omega_X^k)^\vee \otimes \mathcal{O}_X(d-k)$$

is also an isomorphism when restricted to X_0 .

Now assume that $X \subset \mathbb{P}^n$ is a Gorenstein variety of pure dimension k such that $\text{codim}(\text{Sing}(X), X) \geq 2$. Then the sheaf ω_X^\vee is locally-free, hence, in particular, reflexive. Moreover, from [14, Proposition 5.21], we also conclude that ω_X^\vee is normal.

If X is invariant under a Pfaff field \mathcal{F} on \mathbb{P}^n of rank k and degree d , then we have a nonzero global section $\zeta_{\mathcal{F}}$ of $(\Omega_X^k)^\vee \otimes \mathcal{O}_X(d-k)$; consider its restriction $\zeta_{\mathcal{F},0} = \zeta_{\mathcal{F}}|_{X_0}$ to X_0 . Composing it with the the inverse of $\tilde{\gamma}_X|_{X_0}$, the restriction of the map $\tilde{\gamma}_X$ to X_0 , we obtain a section

$$\tilde{\gamma}_X|_{X_0}(\zeta_{\mathcal{F},0}) \in H^0(X_0, \omega_X^\vee \otimes \mathcal{O}_X(d-k)|_{X_0}).$$

However, $\omega_X^\vee \otimes \mathcal{O}_X(d-k)|_{X_0}$ is a normal sheaf, so the above section extends to a global section of $\omega_X^\vee \otimes \mathcal{O}_X(d-k)$. In particular, $H^0(X, \omega_X^\vee \otimes \mathcal{O}_X(d-k)) \neq \{0\}$, therefore

$$(6) \quad \deg(\omega_X^\vee \otimes \mathcal{O}_X(d-k)) \geq 0.$$

Let K_X be a Cartier divisor such that $\mathcal{O}_X(K_X) = \omega_X$.

Now, consider the following difference

$$\begin{aligned} & (2g(X, \mathcal{O}_X(1)) - 2) - [\mathcal{O}_X(d-k) + (k-1)\mathcal{O}_X(1)] \cdot \mathcal{O}_X(1)^{k-1} \\ &= -(K_X^{-1} + \mathcal{O}_X(d-k)) \cdot \mathcal{O}_X(1)^{k-1} = -\deg(\omega_X^\vee \otimes \mathcal{O}_X(d-k)) \leq 0. \end{aligned}$$

5. Complete intersection invariant varieties

We specialize to the case when the invariant variety X is a complete intersection.

First, we notice that the inequality of Theorem 1 is not sharp in general. To see this, let X be a nonsingular complete intersection variety of dimension m and multidegree (d_1, \dots, d_{n-m}) , which is invariant under a k -dimensional Pfaff field \mathcal{F} on \mathbb{P}^n ; assume that $m \geq k$. It follows from [22, Corollary 1.5] that Θ_X is stable and one can apply Theorem 1 to obtain the following inequality:

$$d_1 + \dots + d_{n-m} \leq \frac{\deg(\mathcal{F}) - k}{\binom{m-1}{k-1}} + n + 1.$$

Setting $m = n - 1$ and $k = 1$, the inequality reduces to $d_1 \leq \deg(\mathcal{F}) + n$. However, Soares has shown, under the same circumstances, that $d_1 \leq \deg(\mathcal{F}) + 1$ [24, Theorem B].

In the critical case $\dim(X) = \text{rank}(\mathcal{F})$, Theorem 2 gives us the following corollary.

COROLLARY 9. *Let X be a k -dimensional complete intersection variety of multidegree (d_1, \dots, d_{n-k}) such that either X is nonsingular in codimension 1. If X is invariant under a Pfaff field \mathcal{F} of rank k on \mathbb{P}^n , then*

$$d_1 + \dots + d_{n-k} \leq \deg(\mathcal{F}) + n - k + 1.$$

Proof. From the adjunction formula for dualizing sheaves, one obtains

$$2g(X, \mathcal{O}_X(1)) - 2 = \deg(X)(d_1 + \dots + d_{n-k} - n + k - 2).$$

By Theorem 2, this is less than or equal to $(\deg(\mathcal{F}) - 1)\deg(X)$, and the desired inequality follows easily. □

It follows from [7, Corollary 4.5] that if X and \mathcal{F} are as above, then

$$d_1 + \dots + d_{n-k} \leq \begin{cases} \deg(\mathcal{F}) + n - k, & \text{if } \rho \leq 0, \\ \deg(\mathcal{F}) + n - k + \rho, & \text{if } \rho > 0, \end{cases}$$

where $\rho := \sigma + n - k + 1 - d_1 - \cdots - d_{n-k}$, with σ denoting the Castelnuovo–Mumford regularity of the singular locus of X . Therefore, Corollary 9 allows us to conclude that if X is nonsingular in codimension 1, then one can take $\rho = 1$, regardless of σ .

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